Soft Computing (2020) 24:7029–7033
https://doi.org/10.1007/s00500-020-04825-8

FOUNDATIONS

Multiplicative derivations and *d*-filters of commutative residuated lattices

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Published online: 9 March 2020

Abstract

In this paper, we consider some properties of multiplicative derivations and d-filters of commutative residuated lattices and show that, for an ideal derivation d of a residuated lattice $L = (L, \wedge, \vee, \odot, \to, 0, 1)$, (1) the set $\mathrm{Fix}_d(L) = (\mathrm{Fix}_d(L), \wedge, \vee, \odot, \mapsto, 0, d1)$ of all fixed points of d forms a residuated lattice and d is a homomorphism from L to $\mathrm{Fix}_d(L)$, (2) for a d-filter F, a map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ is also an ideal derivation of L/F and (3) two quotient residuated lattices $\mathrm{Fix}_{d/F}(L/F)$ and $\mathrm{Fix}_d(L)/d(F)$ are isomorphic as residuated lattices, that is, $\mathrm{Fix}_{d/F}(L/F) \cong \mathrm{Fix}_d(L)/d(F)$.

Keywords (Multiplicative) derivations · Ideal derivations · Monotone · Residuated lattices

1 Introduction

A notion of derivation whose origin is in analysis has been applied to theory of algebras with two operations $+$ and $\cdot$, especially to the theory of rings (Posner 1957). For an algebra $A = (A, +, \cdot)$, a map $f : A \to A$ is called a derivation in Posner (1957) if it satisfies the conditions: For all $x, y \in A$,

$$f(x + y) = f(x) + f(y)$$
$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y).$$

It was applied to the theory of lattices (Ferrari 2001; Szász 1975; Xin et al. 2008), where operations $+$ and $\cdot$ were interpreted as lattice operations $\vee$ and $\wedge$, respectively. In particular, it has been proven in Szász (1975) that if f is a derivation of a bounded lattice L, that is, f satisfies two conditions:

$$f(x \vee y) = f(x) \vee f(y)$$
$$f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y)),$$

then f just has a form $f(x) = x \wedge f(1)$. Since then, every type of derivations is defined as a map $f : X \to X$ satisfying only a condition:

$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y),$$

where $(X, +, \cdot)$ is an algebra. Thus, in the case of lattices, a derivation f of a lattice L is defined a map $f : L \to L$ satisfying the condition:

$$f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y)).$$

By the use of derivations of lattices, characterization theorems of distributive lattices and of modular lattices were proven in [6]:

Let L be a lattice and f a derivation.

(1) The condition that f is monotone $\Leftrightarrow$ $f(f(x) \vee y) = f(x) \vee f(y)$ $(\forall x, y \in L)$ holds if and only if L is a modular lattice.
(2) The condition that f is monotone $\Leftrightarrow$ $f(x \vee y) = f(x) \vee f(y)$ $(\forall x, y \in L)$ holds if and only if L is a distributive lattice.

Further, it is also applied to other algebras, such as MV algebras (Alshehri 2010; Ghorbani et al. 2013; Yazarli 2013), where operations $+$ and $\cdot$ were interpreted as $\oplus$ and $\odot$, respectively. In He et al. (2016), another type of derivations, multiplicative derivations, are defined on residuated

Communicated by A. Di Nola.

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lattices, where operations $+$ and $\cdot$ are interpreted as $\vee$ and $\odot$, respectively. By this definition of derivations, we have several interesting properties about residuated lattices. For example, in He et al. (2016), it is shown that if d is an ideal derivation of a residuated lattice L, then the set $\text{Fix}_d(L)$ of all fixed elements formed a residuated lattice. Thus, it is expectable for residuated lattices with multiplicative derivations to have other deeper properties. Indeed, we show some interesting properties here, for example, for every residuated lattice L with an ideal multiplicative derivation d, the set $\text{Fix}_d(L)$ is isomorphic to $L/\ker(d)$ and hence $\text{Fix}_d(L)$ is a residuated lattice.

In this paper, we show the following results. Let L be a commutative residuated lattice and d be an ideal derivation of it. Then, we have

(1) The set $\text{Fix}_d(L)$ of all fixed points of d forms a residuated lattice and $L/\ker(d) \cong \text{Fix}_d(L)$;
(2) A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ is also an ideal derivation of L/F;
(3) The quotient residuated lattices $\text{Fix}_{d/F}(L/F)$ and $\text{Fix}_d(L)/d(F)$ are isomorphic, namely $\text{Fix}_{d/F}(L/F) \cong \text{Fix}_d(L)/d(F)$.

We also show the characterization theorem of d-filters, which says that for an ideal derivation d and a non-empty subset S, the smallest d-filter containing S is identical with the filter containing $S \cup d(S)$, that is, $[S)_d = [S \cup d(S))$.

2 Derivations of residuated lattices

We recall a definition of bounded integral commutative residuated lattices (Galatos et al. 2007; Ward and Dilworth 1939). An algebraic structure $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a bounded integral commutative residuated lattice (simply called a *residuated lattice*) if

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \odot, 1)$ is a commutative monoid with unit element 1;
(3) For all $x, y, z \in L$, $x \odot y \le z$ if and only if $x \le y \to z$.

For all $x \in L$, by x', we mean $x' = x \to 0$, which is a negation in a sense.

In what follows, let $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. An element $x \in L$ is called *complemented* if there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. By $B(L)$, we mean the set of all complemented elements of L, i.e.,

$$B(L) = \{x \in L \mid \exists y \in L \text{ s.t. } x \wedge y = 0, x \vee y = 1\}.$$

It is easy to show the following results, so we omit their proofs.

Proposition 1 (Galatos et al. 2007) *For a residuated lattice L, we have*

(1) *$x \in B(L)$ if and only if $x \vee x' = 1$;*
(2) *If $x \in B(L)$, then $x \wedge y = x \odot y$ for all $y \in L$;*
(3) *If $x \oplus y = x \vee y$ for all $y \in L$, then $x \vee x' = 1$, where $x \oplus y = (x' \odot y')'$;*
(4) *$B(L)$ is a Boolean subalgebra of L.*

We note that $a \odot (a \to x) = a \odot x$ for all $a \in B(L)$ and $x \in L$, because, since $a \in B(L)$ and $a \odot (a \to x) \le x$, we have $a \odot (a \to x) = a \odot a \odot (a \to x) \le a \odot x$. Conversely, it follows from $x \le a \to x$ that $a \odot x \le a \odot (a \to x)$. Therefore, we get $a \odot (a \to x) = a \odot x$ for all $a \in B(L)$ and $x \in L$.

We have the following basic properties of residuated lattices (Galatos et al. 2007).

Proposition 2 *For all $x, y, z \in L$, we have*

(1) $0' = 1, 1' = 0$;
(2) $x \odot x' = 0$;
(3) $x \le y \iff x \to y = 1$;
(4) $x \odot (x \to y) \le y$;
(5) $x \le y \implies x \odot z \le y \odot z, z \to x \le z \to y, y \to z \le x \to z$;
(6) $1 \to x = x$;
(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$;
(8) $(x \vee y)' = x' \wedge y'$;
(9) $(x'' \odot y'')'' = (x \odot y'')'' = (x \odot y)''$.

We define derivations of residuated lattices according to He et al. (2016). A map $d : L \to L$ is called a *multiplicative derivation* (or simply *derivation* here) of L if it satisfies the condition:

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \quad (\forall x, y \in L).$$

We simply denote dx instead of $d(x)$. A derivation d is called *monotone*; if $x \le y$, then $dx \le dy$, that is, d is order-preserving. A derivation d is *good* when $d1 \in B(L)$. If a derivation d is monotone and good, then it is said to be *ideal*. We note that any multiplicative derivation d is *contractive*, $dx \le x$ for all $x \in L$, because we have $dx = d(x \wedge x) = (dx \odot x) \vee (x \odot dx) = dx \odot x \le x$. This result was not referred in He et al. (2016).

Example Let $X = \{0, a, 1\}$ with $0 < a < 1$ be a residuated lattice if we define $x \wedge y = x \odot y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and

$$x \to y = \begin{cases} 1 & (\text{if } x \le y) \\ y & (\text{otherwise}) \end{cases}.$$

A map d_a defined by $d_a(x) = x \wedge a$ for all $x \in X$ is a monotone derivation, but not good. As another example of derivation, we have $f : X \to X$ defined by $f1 = 0 = f0$, $fa = a$. It is easy to show that f is a good derivation, but it is not monotone.

We have fundamental results about derivations of residuated lattices.

Proposition 3 (He et al. 2016) *Let d be a derivation of L. For all $x, y \in L$,*

(1) $d0 = 0$;
(2) $dx \ge x \odot d1$;
(3) $dx^n = x^{n-1} \odot dx$ *for all* $n \ge 1$;
(4) *If* $x \odot y = 0$ *then* $dx \odot y = x \odot dy = dx \odot dy = 0$;
(5) $d(x') \le (dx)'$.

In He et al. (2016), the following characterization theorem about an ideal derivation was proven.

Theorem 1 (He et al. 2016) *Let d be a derivation of L and $d1 \in B(L)$. Then, the following are equivalent: for all $x, y \in L$,*

(1) *d is an ideal derivation;*
(2) $dx \le d1$;
(3) $dx = x \odot d1 = x \wedge d1$;
(4) $d(x \wedge y) = dx \wedge dy$;
(5) $d(x \vee y) = dx \vee dy$;
(6) $d(x \odot y) = dx \odot dy$.

From the above, we see that $dx = d(dx)$ for all $x \in L$ for an ideal derivation d, that is, $d = d^2$, because, since d is the ideal derivation, we have $dx = x \wedge d1$ and thus $d^2x = d(dx) = d(x \wedge d1) = (x \wedge d1) \wedge d1 = x \wedge d1 = dx$ for all $x \in L$. This means that $d^2 = d$.

For a derivation d of L, we denote by $\mathrm{Fix}_d(L)$ the set of all fixed elements of L for d, that is, $\mathrm{Fix}_d(L) = \{x \in L \mid dx = x\}$.

It is easy to show the next result; hence, we omit its proof.

Proposition 4 *For an ideal derivation d of a residuated lattice L, we have $Fix_d(L) = d(L)$.*

Lemma 1 *Let d be an ideal derivation. Then, we have $d(dx \to dy) = d(x \to y)$ for all $x, y \in L$.*

Proof Suppose that d is an ideal derivation. Since $dz = z \odot d1 = z \wedge d1$ for all $z \in L$, we have

$$\begin{aligned} d(dx \to dy) &= (dx \to dy) \odot d1 \\ &= (x \wedge d1 \to y \wedge d1) \odot d1 \\ &= \{(x \wedge d1 \to y) \wedge (x \wedge d1 \to d1)\} \odot d1 \\ &= (x \wedge d1 \to y) \odot d1 \\ &= d1 \odot (d1 \odot x \to y) \\ &= d1 \odot (d1 \to (x \to y)) \\ &= d1 \odot (x \to y) \quad (d1 \in B(L)) \\ &= d(x \to y). \end{aligned}$$

□

It follows from the characterization theorem about ideal derivations and above that we define some operations in $\mathrm{Fix}_d(L) = d(L)$ by

$$\begin{aligned} dx \sqcap dy &= d(x \wedge y); \\ dx \sqcup dy &= d(x \vee y); \\ dx \boxdot dy &= d(x \odot y); \\ dx \mapsto dy &= d(dx \to dy). \end{aligned}$$

Then, we have

Theorem 2 *Let L be a residuated lattice and d be an ideal derivation of L. Then, $Fix_d(L) = (Fix_d(L), \sqcap, \sqcup, \boxdot, \mapsto, 0, d1)$ is a residuated lattice. However, it is not a subalgebra of L in general.*

Proof We only show that $dx \odot dy \le dz$ if and only if $dx \le dy \mapsto dz$ for all $dx, dy, dz \in d(L) = \mathrm{Fix}_d(L)$. If $dx \odot dy \le dz$, then we have $dx \le dy \to dz$ and thus $dx = d(dx) \le d(dy \to dz) = dy \mapsto dz$. Conversely, suppose that $dx \le dy \mapsto dz = d(dy \to dz)$. Since d is contractive, we have $d(dy \to dz) \le dy \to dz$ and hence $dx \le dy \to dz$. This yields $dx \odot dy \le dz$. □

Remark 1 We note that the theorem above was already proven as Theorem 3.15 in He et al. (2016), where the meet operation $\sqcap$ is defined by $d(dx \wedge dy)$.

We have proved above that $\mathrm{Fix}_d(L)$ is a residuated lattice for an ideal derivation d. Moreover, we see that any ideal derivation d is a homomorphism from L to $\mathrm{Fix}_d(L)$, since $d(x \to y) = d(dx \to dy) = dx \mapsto dy$. The other cases such as $d(x \wedge y)$ also can be proved easily. Therefore, d is the homomorphism from L to $\mathrm{Fix}_d(L)$. It follows from the homomorphism theorem of residuated lattices that $L/\ker(d) \cong \mathrm{Fix}_d(L)$, where $\ker(d) = \{(x, y) \mid dx = dy\}$.

Theorem 3 *For every ideal derivation d, it is a homomorphism from L to $Fix_d(L)$ and hence*

$$L/\ker(d) \cong Fix_d(L).$$

3 Galois connection of derivations

In this section, we consider Galois connections of derivations. Let P, Q be partially ordered sets and f, g be maps, $f : P \to Q$, $g : Q \to P$. A pair (f, g) of maps is called a *Galois connection* if

$$f(x) \leq_Q y \Leftrightarrow x \leq_P g(y) \quad (\forall x \in P, \forall y \in Q).$$

A Galois connection (f, g) is simply denoted by $f \dashv g$.

Let L be a residuated lattice and d be an ideal derivation of L. By the characterization theorem of ideal derivations, d has the form $dx = x \odot d1 = x \wedge d1$. In this case, we may ask

"Is there a map g such that $d \dashv g$?"

We have an affirmative solution as follows.

Theorem 4 *Let d be an ideal derivation. There exists an ideal derivation $g : L \to L$ such that $d \dashv g$. Moreover, g is idempotent.*

Proof We define $gx = d1 \to x$ for all $x \in L$. It is obvious that $dx = x \odot d1 \leq y$ if and only if $x \leq d1 \to y = gy$, that is, $dx \leq y \Leftrightarrow x \leq gy$ for all $x, y \in L$. Moreover, since $g(gx) = d1 \to (d1 \to x) = d1 \odot d1 \to x = d1 \to x = gx$, g is idempotent. □

From the general theory of idempotent Galois connection, that is, $f \dashv g$, $f^2 = f$ and $g^2 = g$ that two subsets $F_f = \{x \in L \mid fx = x\}$ and $F_g = \{x \in L \mid gx = x\}$ are isomorphic as partially ordered sets. Hence, we have the following result.

Theorem 5 *For an ideal derivation d, $Fix_d(L) = F_d(L) \cong F_g(L) = Fix_g(L)$ as partially ordered sets.*

4 *d*-filter and its characterization

We define a filter of a residuated lattice which plays an important role in this paper. Let F be a non-empty subset of L. We call F a *filter* of L if it satisfies the following conditions: For all $x, y \in L$,

(F1) if $x, y \in F$, then $x \odot y \in F$;
(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

Let d be an ideal derivation of L. A filter F of L is called an *ideal derivation filter* (simply *d-filter* here) if $x \in F$ implies $dx \in F$ for all $x \in L$. By $\mathcal{F}(L)$ (or $\mathcal{F}_d(L)$), we mean the set of all filters (or d-filters) of L. By $[S)$ (or $[S)_d$), we mean the generated filter (or generated d-filter, respectively) by S. At first, we provide a characterization theorem about d-filters.

Theorem 6 *Let d be an ideal derivation of L. For a non-empty subset S of L, we have $[S)_d = [S \cup d(S))$.*

Proof It is sufficient to show that $[S \cup d(S))$ is the least d-filter including S. It is obvious $S \subseteq [S \cup d(S))$. We show that $[S \cup d(S))$ is a d-filter, that is, if $x \in [S \cup d(S))$, then $dx \in [S \cup d(S))$. Suppose that $x \in [S \cup d(S))$. There exist $a_i, b_j \in S$ such that $a_1 \odot \cdots \odot a_m \odot db_1 \odot \cdots db_n \leq x$. Since d is the ideal derivation, we have $da_1 \odot \cdots \odot da_m \odot db_1 \odot \cdots db_n = da_1 \odot \cdots \odot da_m \odot d(db_1) \odot \cdots d(db_n) = d(a_1 \odot \cdots \odot a_m \odot db_1 \odot \cdots db_n) \leq dx$. It follows from $da_i, db_j \in d(S) \subseteq S \cup d(S)$ that $dx \in [S \cup d(S))$. Therefore, $[S \cup d(S))$ is the d-filter. Lastly, for any d-filter F including S, we have $d(S) \subseteq d(F) \subseteq F$ and hence $S \cup d(S) \subseteq F$. This implies $[S \cup d(S)) \subseteq F$. Therefore, $[S \cup d(S))$ is the least d-filter containing S and thus $[S)_d = [S \cup d(S))$. □

Corollary 1 (Theorem 3.19 He et al. 2016) *The class $\mathcal{F}_d(L)$ of all d-filters forms a complete Heyting subalgebra of the class $\mathcal{F}(L)$ of all filters of a residuated lattice L*

Corollary 2 *If F is a filter, then $[F)_d = [d(F))$.*

Corollary 3 *For any $F \in \mathcal{F}(L)$, F is a d-filter if and only if $F = [d(F))$.*

Proof It is easy to show that if F is a d-filter, then $F = [F)_d = [d(F))$. Conversely, suppose that $x \in F = [d(F))$. There exist $a_i \in F$ such that $da_1 \odot \cdots \odot da_n \leq x$. If we take $a = a_1 \odot \cdots \odot a_n \in F$, then $da \in d(F)$. Moreover, we have $da = d(a_1 \odot \cdots \odot a_n) = da_1 \odot \cdots \odot da_n \leq x$ and thus $da = d(da) \leq dx$. This implies that $dx \in [d(F)) = F$, namely F is the d-filter. □

Proposition 5 *Let d be an ideal derivation of L. Then, we have*

(1) *$F \in \mathcal{F}_d(L) \Rightarrow d(F) \in \mathcal{F}(d(L))$;*
(2) *$G \in \mathcal{F}(d(L)) \Rightarrow d^{-1}(G) \in \mathcal{F}_d(L)$, where $d^{-1}(G) = \{x \in L \mid dx \in G\}$.*

Proof (1) If $dx, dy \in d(F)$, $(x, y \in F)$ then, since $x \odot y \in F$ and d is the ideal derivation, we have $dx \odot dy = d(x \odot y) \in d(F)$. Moreover, if $dx \in d(F)$ for $x \in F$ and $dx \leq dy$, then $dx \in F$ and thus $dy \in F$. Then, $dy = d(dy) \in d(F)$. Therefore, $d(F)$ is a filter of $d(L)$, that is, $d(F) \in \mathcal{F}(d(L))$.

(2) Let G be a filter of $d(L)$. For all $x, y \in d^{-1}(G)$, since $dx, dy \in G$, we get $dx \odot dy = d(x \odot y) \in G$ and $x \odot y \in d^{-1}(G)$. Next, suppose that $x \in d^{-1}(G)$ and $x \leq y$. It follows from $dx \in G$ and $dy \leq dy$ that $dy \in G$ and hence that $y \in d^{-1}(G)$. Lastly, if $x \in d^{-1}(G)$, then $dx \in G$ and $d(dx) = dx \in G$. This implies $dx \in d^{-1}(G)$. Therefore, $d^{-1}(G)$ is the d-filter, that is, $d^{-1}(G) \in \mathcal{F}_d(L)$. □

For a filter F, we define a quotient structure L/F, where $x/F = y/F$ is defined by $x \to y, y \to x \in F$ for all $x, y \in L$. Since the set of all residuated lattices forms a variety, the quotient algebra L/F by a filter F is also a residuated lattice.

Proposition 6 *Let d be an ideal derivation and F be a d-filter of L. If we define a map $d/F : L/F \to L/F$ by $(d/F)(x/F) = dx/F$ for all $x/F \in L/F$, then d/F is an ideal derivation of L/F.*

Proof At first, we show that d/F is well defined. Suppose that $x/F = y/F$. Since $x \to y, y \to x \in F (\in \mathcal{F}_d(L))$, we have $d(x \to y), d(y \to x) \in F$. It follows from $d(x \to y) \leq dx \to dy, d(y \to x) \leq dy \to dx$ that $dx \to dy, dy \to dx \in F$. This means that $(d/F)(x/F) = dx/F = dy/F = (d/F)(y/F)$ and hence that d/F is well defined. It is easy to show that d/F is a good derivation of the residuated lattice L/F. It is sufficient to show that d/F is monotone. Suppose that $x/F \leq y/F$. Since $x \to y \in F$ and F is the d-filter, we have $d(x \to y) \in F$ and $dx \to dy \in F$ because of $d(x \to y) \leq dx \to dy$. This implies $dx/F \leq dy/F$ and hence $(d/F)(x/F) = dx/F \leq dy/F = (d/F)(y/F)$, namely d/F is monotone. □

We note that $d1/F = 1/F$, because, since F is the d-filter, the fact $1 \in F$ implies $d1 \in F$, that is, $d1/F = 1/F$. It follows from the above that the quotient structure $(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F) = (\mathrm{Fix}_{d/F}(L/F), \wedge, \vee, \odot, \mapsto, 0/F, 1/F)$ is also a residuated lattice for any ideal derivation d and d-filter F. Moreover, since F is the d-filter, $d(F)$ is the filter of $d(L)$ and thus the quotient structure $d(L)/d(F)$ forms a residuated lattice. It is natural to ask what the relation between two residuated lattices $(d/F)(L/F)$ and $d(L)/d(F)$ is. In order to answer the question, we need the following result.

Lemma 2 *If $F \in \mathcal{F}_d(L)$, then we have $F \cap d(L) = d(F)$.*

Proof For $x \in F \cap d(L)$, since $x \in d(L) = \mathrm{Fix}_d(L)$, we have $x = dx \in d(F)$ and $F \cap d(L) \subseteq d(F)$.

Conversely, suppose $x \in d(F)$. There exists $y \in F$ such that $x = dy$. Since F is the d-filter, we get $x = dy \in F$. On the other hand, it follows from $dx = d(dy) = dy = x$ that $x \in \mathrm{Fix}_d(L) = d(L)$ and $d(F) \subseteq F \cap d(L)$. Therefore, $F \cap d(L) = d(F)$. □

From the above, we have a following result which answers the question.

Theorem 7 *Let d be an ideal derivation and F be a d-filter of L. Then, we have $(d/F)(L/F) = Fix_{d/F}(L/F)$ is isomorphic to $d(L)/d(F)$, that is,*

$$(d/F)(L/F) = Fix_{d/F}(L/F) \cong d(L)/d(F).$$

Proof We define a map $\Phi : (d/F)(L/F) \to d(L)/d(F)$ by $\Phi((d/F)(x/F)) = dx/d(F)$. We show that Φ is an isomorphism. We only show that the map Φ is well defined and injective. At first, Φ is well defined, because we have

$$\begin{aligned}
(d/F)(x/F) &= (d/F)(y/F) \\
&\Rightarrow dx/F = dy/F \\
&\Rightarrow dx \to dy, dy \to dx \in F \\
&\Rightarrow d(dx \to dy), d(dy \to dx) \in d(F) \\
&\Rightarrow dx \mapsto dy, dy \mapsto dx \in d(F) \\
&\Rightarrow dx/d(F) = dy/d(F).
\end{aligned}$$

Hence, Φ is well defined.

Suppose $\Phi((d/F)(x/F)) = \Phi((d/F)(y/F))$. Since $dx/d(F) = dy/d(F)$, we have $dx \mapsto dy, dy \mapsto dx \in d(F)$ and thus $d(dx \to dy), d(dy \to dx) \in d(F) = F \cap d(L)$. This implies $d(dx \to dy), d(dy \to dx) \in F$. Since d is contractive, we also get $d(dx \to dy) \leq dx \to dy, d(dy \to dx) \leq dy \to dx$ and $dx \to dy, dy \to dx \in F$. Therefore, $dx/F = dy/F$ and $(d/F)(x/F) = dx/F = dy/F = (d/F)(y/F)$, namely Φ is injective. □

Acknowledgements This work was partly supported by JSPS KAKENHI Grant Number 15K00024.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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