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METHODOLOGIES AND APPLICATION

Archimedean geometric Heronian mean aggregation operators based on dual hesitant fuzzy set and their application to multiple attribute decision making

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Abstract
Fuzzy set, intuitionistic fuzzy set, hesitant fuzzy set can be regarded as a special case of dual hesitant fuzzy set. Therefore, dual hesitant fuzzy set is a more comprehensive set. Further, Archimedean *t*-norm and *t*-conorm provides generalized operational rules for dual hesitant fuzzy set. And geometric Heronian mean have advantages when considering the interrelationship of aggregation arguments. Thus, it is necessary to extend the geometric Heronian mean operator to the dual hesitant fuzzy environment based on Archimedean *t*-norm and *t*-conorm. Comprehensive above, in this paper, the dual hesitant fuzzy geometric Heronian mean operator and dual hesitant fuzzy geometric weighted Heronian mean operator based on Archimedean *t*-norm and *t*-conorm are developed. Their properties and special case are investigated. Moreover, a multiple attribute decision making method is proposed. The effectiveness of our method and the influence of parameters on multiple attribute decision making are studied by an example. The superiority of our method is illustrated by comparing with other existing methods.

Keywords Dual hesitant fuzzy set · Archimedean *t*-norm and *t*-conorm · Geometric Heronian mean · Multiple attribute decision making

1 Introduction

Intuitionistic fuzzy set (IFS) was first introduced by Atanassov (1986) and it is a generalization of fuzzy set (FS) (Zadeh 1965). On the basis of the FS, the non-membership function and hesitant membership function are added in IFS. Torra and Narukawa (2009) developed hesitant fuzzy set (HFS) to express the membership degree of elements as a set of possible values, which is a very useful tool for expressing people's hesitation in our daily life. HFS has attracted more and more attention in multiple attribute decision making (MADM) (Xia and Xu 2011; Zhang and Wei 2013; Wei et al. 2017; Yu 2017; Tang et al. 2018), clustering (Chen et al. 2013; Zhang and Xu 2015; Liu et al. 2017), pattern recognition (Sun et al. 2018; Zhang et al. 2018b) and so on.

Zhu et al. (2012) combined the IFS with HFS, proposed the dual hesitant fuzzy set (DHFS). Similar to IFS, DHFS are also composed of membership function and non-membership function. But different from IFS, their membership function and non-membership function are expressed by several numbers rather than a single number. FS, IFS and HFS can be regarded as special cases of DHFS. Therefore, DHFS has attracted the attention of many scholars. Ye (2014), Tyagi (2015) and Wang et al. (2013) proposed different correlation coefficients for DHFS and their applications. Singh (2015) and Wang et al. (2014) introduced some new distance measures and similarity measures for DHFS and applied to MADM. Zhang et al. (2018a) developed cosine similarity measures which have the ability to model various problems for DHFS in MADM. There are also many achievements in the research of aggregation operators. Yu et al. (2016) introduced dual hesitant fuzzy Heronian mean (DHFHM) operator and dual hesitant fuzzy geometric Heronian mean (DHFGHM) operator and their application to supplier selection. Wang et al. (2014) developed dual hesitant fuzzy weighted average (DHFWA) operator, dual hesitant

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fuzzy weighted geometric (DHFWG) operator, dual hesitant fuzzy ordered weighted average (DHFOWA) operator, dual hesitant fuzzy ordered weighted geometric (DHFOWG) operator, dual hesitant fuzzy hybrid average (DHFHA) operator and dual hesitant fuzzy hybrid geometric (DHFHG) operator and applied to MADM. Ju et al. (2014) proposed dual hesitant fuzzy Choquet ordered geometric (DHFCOG) operator, generalized dual hesitant fuzzy Choquet ordered average (GDHFCOA) operator and generalized dual hesitant fuzzy Choquet ordered geometric (GDHFCOG) operator for MADM. Gao et al. (2018) introduced dual hesitant bipolar fuzzy Hamacher prioritized average (DHBFHPA) operator, dual hesitant bipolar fuzzy Hamacher prioritized geometric (DHBFHPG) operator, dual hesitant bipolar fuzzy Hamacher prioritized weighted average (DHBFHPWA) operator and dual hesitant bipolar fuzzy Hamacher prioritized weighted geometric (DHBFHPWG) operator. Many other aggregation operators are also proposed, such as dual hesitant fuzzy Frank aggregation operators (Tang et al. 2018), generalized dual hesitant fuzzy Choquet ordered aggregation operator (Wang et al. 2014), dual hesitant fuzzy weighted interaction averaging operator, dual hesitant fuzzy weighted interaction geometric operator (Xu et al. 2015), dual hesitant fuzzy aggregation operators Bonferroni means (Tu et al. 2017) and so on.

Based on Archimedean t-norm and t-conorm (Klir and Yuan 1995; Nguyen and Walker 1997; Klement and Mesiar 2005), FS (Tchamova 2006), IFS (Liu and Chen 2017; Das et al. 2017; Xia et al. 2012) and HFS (Xia and Xu 2017; Tan et al. 2015; Xia and Xu 2012) have many researches on their properties and aggregation operators. DHFS also has some researches. Yu (2015) introduced dual hesitant fuzzy weighted averaging (ADHFWA) operator and dual hesitant fuzzy weighted geometric (ADHFWG) operator based on Archimedean t-norm and t-conorm under dual hesitant fuzzy environment. Wang et al. (2016) proposed a wide range of dual hesitant fuzzy power aggregation operators based on Archimedean t-norm and t-conorm. These aggregation operators based on Archimedean t-norm and t-conorm are generalized forms of existing aggregation operators. But these operators do not consider the interrelationship of aggregation arguments. Although Bonferroni mean (BM) (Bonferroni 1950) operator consider the interrelationship of aggregation arguments, Yu and Wu (2012) pointed out that compared with BM operator, Heronian mean (HM) (Beliakov et al. 2007) operator considers the relationship between attribute criteria C_i and itself. And the correlation between criteria C_i and $C_j (i \neq j)$ is equal to the correlation between criteria C_j and $C_i (i \neq j)$, thus, the HM operator avoiding the calculation redundancy. Therefore, the HM operator is better for aggregation. Similar to HM operator, geometric Heronian mean (GHM) (Yu 2013) operator has the same advantage. In our real life, many things are related to each other. Therefore, the GHM operator provides a powerful tool for considering the inter-dependent phenomena among the arguments. But at present it is only used for the operational rules of DHFS based on Algebraic operators. The Archimedean t-norm and t-conorm provides a general rule of operation and more choices for decision makers. Therefore, it is necessary and meaningful to extend the GHM based on Archimedean t-norm and t-conorm and apply to deal with MADM problems under dual hesitant fuzzy environment. So, this paper presents dual hesitant fuzzy geometric Heronian mean and dual hesitant fuzzy geometric weighted Heronian mean based on Archimedean t-norm and t-conorm. When the special case is taken, it will be reduced to the form of an existing operators in studies (Yu et al. 2016; Yu 2013). The validity and superiority of the proposed operators are verified by exploring the influence of parameter values on the ranking results and comparing with other existing operators.

The structure of the paper is as follows: in Sect. 2, some basic notions are reviewed. In Sect. 3, we develop dual hesitant fuzzy geometric Heronian mean and dual hesitant fuzzy geometric weighted Heronian mean based on Archimedean t-norm and t-conorm. We also explore some properties and special cases of the proposed operators. In Sect. 4, we propose a MADM method based on the proposed operators. In Sect. 5, we give an example to explain the application of our method, and also investigate the influence of parameter values on the ranking results and comparison with the methods presented in studies (Yu et al. 2016; Wang et al. 2014; Yu 2015). In Sect. 6, we conclude this paper.

2 Preliminaries

In 2012, Zhu et al. first proposed the concept of dual hesitant fuzzy set (DHFS).

Definition 1 (Zhu et al. 2012) Let X be a fixed set, then a DHFS D on the set X is described as:

$$D = \{\langle x, \mu(x), \nu(x)\rangle \,|x \in X\},$$

where $\mu(x)$ and $\nu(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D, respectively, with the following conditions:

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1$$

where $\gamma \in \mu(x)$, $\eta \in \nu(x)$, $\gamma^+ \in \mu^+(x) = \cup_{\gamma\in\mu(x)} \max\{\gamma\}$, and $\eta^+ \in \nu^+(x) = \cup_{\eta\in\nu(x)} \max\{\eta\}$ for all $x \in X$. For convenience, the pair $\alpha(x) = (\mu(x), \nu(x))$ is called a dual hesitant fuzzy element (DHFE), denoted by $\alpha = (\mu, \nu)$.

In order to rank the DHFEs, Zhu et al. presented the following score function and accuracy function.

Definition 2 (Zhu et al. 2012) Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2)$ be any two DHFEs. The score function and accuracy function of α_i are defined as follows:

$$S(\alpha_i) = \frac{1}{\#\mu_{\alpha_i}} \sum_{\gamma \in \mu_{\alpha_i}} \gamma - \frac{1}{\#\nu_{\alpha_i}} \sum_{\eta \in \nu_{\alpha_i}} \eta;$$
$$P(\alpha_i) = \frac{1}{\#\mu_{\alpha_i}} \sum_{\gamma \in \mu_{\alpha_i}} \gamma + \frac{1}{\#\nu_{\alpha_i}} \sum_{\eta \in \nu_{\alpha_i}} \eta;$$

where $\#\mu_{\alpha_i}$ and $\#\nu_{\alpha_i}$ are the numbers of the elements in μ_{α_i} and ν_{α_i}, respectively.

By the score function and accuracy function, Zhu et al. gave the following method for ranking DHFEs:

(1) If $S(\alpha_1) > S(\alpha_2)$, then α_1 is superior to α_2, denote by $\alpha_1 \succ \alpha_2$;
(2) If $S(\alpha_1) = S(\alpha_2)$, then
 (i) If $P(\alpha_1) = P(\alpha_2)$, then α_1 is equivalent to α_2, denote by $\alpha_1 \sim \alpha_2$;
 (ii) If $P(\alpha_1) > P(\alpha_2)$, then α_1 is superior to α_2, denote by $\alpha_1 \succ \alpha_2$.

Definition 3 (Zhu et al. 2012) Given a DHFE represented by $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\alpha \neq \varnothing$. Its complement is defined as follows:

$$\alpha^c = \begin{cases} (\nu_\alpha, \mu_\alpha), & \text{if } \mu_\alpha \neq \varnothing, \nu_\alpha \neq \varnothing; \\ (1 - \mu_\alpha, \varnothing), & \text{if } \mu_\alpha \neq \varnothing, \nu_\alpha = \varnothing; \\ (\varnothing, 1 - \nu_\alpha), & \text{if } \mu_\alpha = \varnothing, \nu_\alpha \neq \varnothing. \end{cases}$$

Klir and Yuan proposed the t-norm and t-conorm in 1995.

Definition 4 (Klir and Yuan 1995) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm. If it satisfies the following conditions:

(1) $T(0, 0) = 0, T(1, 1) = 1$;
(2) $T(x, y) = T(y, x)$, for any x and y;
(3) $T(x, T(y, z)) = T(T(x, y), z)$, for any x, y and z;
(4) if $x_1 \leq x_2$, $y_1 \leq y_2$, then $T(x_1, y_1) \leq T(x_2, y_2)$;

Furthermore, for any $a \in [0, 1]$, T is a t-norm if $T(a, 1) = a$, T is t-conorm if $T(0, a) = a$.

It is also usually required that t-norm and t-conorm are continuous functions.

Definition 5 (Klir and Yuan 1995) A t-norm function $T(x, y)$ is called Archimedean t-norm if it is continuous and $T(x, x) < x$ for all $x \in (0, 1)$. An Archimedean t-norm is called strict Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0, 1)$.

Definition 6 (Klir and Yuan 1995) A t-conorm function $S(x, y)$ is called Archimedean t-conorm if it is continuous and $S(x, x) > x$ for all $x \in (0, 1)$. An Archimedean t-conorm is called strict Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0, 1)$.

It is well known (Klement and Mesiar 2005) that a strict Archimedean t-norm is expressed via its additive generator g as $T(x, y) = g^{-1}(g(x) + g(y))$, and similarly, applied to its dual t-conorm $S(x, y) = h^{-1}(h(x) + h(y))$ with $h(t) = g(1-t)$. We notice that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0, 1] \to [0, 1]$ such that $g(1) = 0$.

Following, we denote Archimedean t-norm and t-conorm as (T, S). Listed below are some commonly used Archimedean t-norm and t-conorm.

(1) If additive generators is $h(t) = -\ln(1-t)$, $g(t) = -\ln t$, then $T(x, y) = xy$, $S(x, y) = x + y - xy$ are called Algebraic t-norm and t-conorm;
(2) If additive generators is $h(t) = \ln\frac{1+t}{1-t}$, $g(t) = \ln\frac{2-t}{t}$, then $T(x, y) = \frac{xy}{1+(1-x)(1-y)}$, $S(x, y) = \frac{x+y}{1+xy}$ are called Einstein t-norm and t-conorm;
(3) If additive generators is $h(t) = \ln\frac{\theta+(1-\theta)(1-t)}{1-t}$, $g(t) = \ln\frac{\theta+(1-\theta)t}{t}$, then $T(x, y) = \frac{xy}{\theta+(1-\theta)(x+y-xy)}$, $S(x, y) = \frac{x+y-xy-(1-\theta)xy}{1-(1-\theta)xy} (\theta \in (0, +\infty))$ are called Hamacher t-norm and t-conorm;
(4) If additive generators is $h(t) = -\ln\frac{\theta-1}{\theta^{1-t}-1}$, $g(t) = -\ln\frac{\theta-1}{\theta^t-1}$, then $T(x, y) = \log_\theta\left(1 + \frac{(\theta^x-1)(\theta^y-1)}{\theta-1}\right)$, $S(x, y) = 1-\log_\theta\left(1+\frac{(\theta^{1-x}-1)(\theta^{1-y}-1)}{\theta-1}\right) (\theta \in (1, +\infty))$ are called Frank t-norm and t-conorm.

Based on Archimedean t-norm and t-conorm, Wang et al. introduced the following operational rules for DHFEs.

Definition 7 (Wang et al. 2016) Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2)$ be any two DHFEs, $\lambda > 0$. Then

(1) $$\alpha_1 \oplus \alpha_2 = \cup_{\gamma_{\alpha_1} \in \mu_{\alpha_1}, \eta_{\alpha_1} \in \nu_{\alpha_1}, \gamma_{\alpha_2} \in \mu_{\alpha_2}, \eta_{\alpha_2} \in \nu_{\alpha_2}} \left\{\left\{S\left(\gamma_{\alpha_1}, \gamma_{\alpha_2}\right)\right\}, \left\{T\left(\eta_{\alpha_1}, \eta_{\alpha_2}\right)\right\}\right\}$$
$$= \left\{\cup_{\gamma_{\alpha_1} \in \mu_{\alpha_1}, \gamma_{\alpha_2} \in \mu_{\alpha_2}} \left\{h^{-1}\left(h\left(\gamma_{\alpha_1}\right) + h\left(\gamma_{\alpha_2}\right)\right)\right\}, \cup_{\eta_{\alpha_1} \in \nu_{\alpha_1}, \eta_{\alpha_2} \in \nu_{\alpha_2}} \left\{g^{-1}\left(g\left(\eta_{\alpha_1}\right) + g\left(\eta_{\alpha_2}\right)\right)\right\}\right\};$$
(2) $$\alpha_1 \otimes \alpha_2 = \cup_{\gamma_{\alpha_1} \in \mu_{\alpha_1}, \eta_{\alpha_1} \in \nu_{\alpha_1}, \gamma_{\alpha_2} \in \mu_{\alpha_2}, \eta_{\alpha_2} \in \nu_{\alpha_2}} \left\{\left\{T\left(\gamma_{\alpha_1}, \gamma_{\alpha_2}\right)\right\}, \left\{S\left(\eta_{\alpha_1}, \eta_{\alpha_2}\right)\right\}\right\}$$
$$= \left\{\cup_{\gamma_{\alpha_1} \in \mu_{\alpha_1}, \gamma_{\alpha_2} \in \mu_{\alpha_2}} \left\{g^{-1}\left(g\left(\gamma_{\alpha_1}\right) + g\left(\gamma_{\alpha_2}\right)\right)\right\}, \cup_{\eta_{\alpha_1} \in \nu_{\alpha_1}, \eta_{\alpha_2} \in \nu_{\alpha_2}} \left\{h^{-1}\left(h\left(\eta_{\alpha_1}\right) + h\left(\eta_{\alpha_2}\right)\right)\right\}\right\};$$
(3) $\lambda\alpha = \left\{\cup_{\gamma_\alpha \in \mu_\alpha} \left\{h^{-1}\left(\lambda h\left(\gamma_\alpha\right)\right)\right\}, \cup_{\eta_\alpha \in \nu_\alpha} \left\{g^{-1}\left(\lambda g\left(\eta_\alpha\right)\right)\right\}\right\}$;

(4) $\alpha^{\lambda}=\left\{\cup_{\gamma_\alpha\in\mu_\alpha}\left\{g^{-1}\left(\lambda g\left(\gamma_\alpha\right)\right)\right\},\cup_{\eta_\alpha\in\nu_\alpha}\left\{h^{-1}\left(\lambda h\left(\eta_\alpha\right)\right)\right\}\right\}$.

3 Dual hesitant fuzzy geometric Heronian mean operator based on Archimedean *t*-norm and *t*-conorm

In this section, based on the operational rules of the Archimedean t-norm and t-conorm of DHFEs, we propose the (T, S)-based dual hesitant fuzzy geometric Heronian mean ((T, S)-DHFGHM) operator and (T, S)-based dual hesitant fuzzy geometric weighted Heronian mean ((T, S)-DHFGWHM) operator. Furthermore, we also discuss their properties and some special cases.

According to Definition 7, we have the following operational properties for DHFEs.

Theorem 1 *Let $\alpha_1, \alpha_2, \alpha_3$ be any three DHFEs and $\lambda_1, \lambda_2 > 0$. Then*

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
(3) $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$;
(4) $(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 = \alpha_1 \otimes (\alpha_2 \otimes \alpha_3)$;
(5) $\lambda_1\alpha_1 \oplus \lambda_1\alpha_2 = \lambda_1(\alpha_1 \oplus \alpha_2)$;
(6) $\alpha_1^{\lambda_1} \otimes \alpha_2^{\lambda_1} = (\alpha_1 \otimes \alpha_2)^{\lambda_1}$;
(7) $(\alpha_1^{\lambda_1})^{\lambda_2} = \alpha_1^{\lambda_1\lambda 2}$.

Proof The proof is simple, which is omitted here. □

Aggregation operation is an important tool of multiple attribute decision making (MADM). For existing operator, geometric Heronian mean (GHM) can reflect inter-dependent phenomena among arguments. It can better deal with MADM problem. The definition as follows.

Definition 8 (Yu 2013) Let $I = [0, 1]$, $p, q \geq 0$, $H^{p,q} : I^n \to I$. The GHM operator is defined by the following formula:

$$\mathrm{GHM}^{p,q}(a_1, a_2, \ldots, a_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2}{n(n+1)}}.$$

According to Definition 8, we have the following definition under dual hesitant fuzzy environment.

Definition 9 Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2, \ldots, n)$ be a collection of DHFEs, a (T, S)-based dual hesitant fuzzy geometric Heronian mean((T, S)-DHFGHM) is defined as follows:

$$\begin{aligned}&(T, S) - \mathrm{DHFGHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(p\alpha_i \oplus q\alpha_j\right)^{\frac{2}{n(n+1)}}\end{aligned}$$

where $p, q \geq 0$.

Based on the operational laws of the DHFEs shown in Definition 7, we can get Theorem 2.

Theorem 2 *Let $p, q \geq 0$ and $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2, \ldots, n)$ be a collection of DHFEs , then*

$$\begin{aligned}&(T, S) - \mathrm{DHFGHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(p\alpha_i \oplus q\alpha_j\right)^{\frac{2}{n(n+1)}}\\&=\left\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{h^{-1}\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\right.\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.\left.\left(\frac{2}{n(n+2)}g\left(h^{-1}(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j}))\right)\right)\right)\right)\right)\right\},\\&\quad\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{g^{-1}\left(\frac{1}{p+q}g\left(h^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}\right.\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.\left.\left.h\left(g^{-1}(pg(\eta_{\alpha_i})+qg(\eta_{\alpha_j}))\right)\right)\right)\right)\right)\right\}\right\}.\end{aligned}$$

Proof According to the operations defined in Definition 7, we get

$$\begin{aligned}&p\alpha_i \oplus q\alpha_j\\&=\Big\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\\&\left\{h^{-1}\left(h\left(h^{-1}\left(ph\left(\gamma_{\alpha_i}\right)\right)\right)+h\left(h^{-1}\left(qh\left(\gamma_{\alpha_j}\right)\right)\right)\right)\right\},\\&\quad\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\\&\left\{g^{-1}\left(g\left(g^{-1}\left(pg\left(\eta_{\alpha_i}\right)\right)\right)+g\left(g^{-1}\left(qg\left(\eta_{\alpha_j}\right)\right)\right)\right)\right\}\Big\}\\&=\Big\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{h^{-1}\left(ph\left(\gamma_{\alpha_i}\right)+qh\left(\gamma_{\alpha_j}\right)\right)\right\},\\&\quad\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{g^{-1}\left(pg\left(\eta_{\alpha_i}\right)+qg\left(\eta_{\alpha_j}\right)\right)\right\}\Big\}.\end{aligned}$$

Further, we have

$$\begin{aligned}&\bigotimes_{i=1,j=i}^{n}\left(p\alpha_i \oplus q\alpha_j\right)^{\frac{2}{n(n+1)}}\\&=\left\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{g^{-1}\left(\sum_{i=1,j=i}^{n}g\left(g^{-1}\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.\left(\frac{2}{n(n+2)}g\left(h^{-1}\left(ph\left(\gamma_{\alpha_i}\right)+qh\left(\gamma_{\alpha_j}\right)\right)\right)\right)\right)\right)\right\},\\&\quad\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{h^{-1}\left(\sum_{i=1,j=i}^{n}h\left(h^{-1}\left(\frac{2}{n(n+2)}\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.\left.h\left(g^{-1}\left(pg\left(\eta_{\alpha_i}\right)+qg\left(\eta_{\alpha_j}\right)\right)\right)\right)\right)\right)\right\}\right\}\\&=\left\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{g^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.g\left(h^{-1}\left(ph\left(\gamma_{\alpha_i}\right)+qh\left(\gamma_{\alpha_j}\right)\right)\right)\right)\right)\right\}\end{aligned}$$

$$\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{h^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}\right.\right.\right.$$
$$\left.\left.\left.h\left(g^{-1}\left(pg(\eta_{\alpha_i})+(qg(\eta_{\alpha_j}))\right)\right)\right)\right)\right\}\Bigg\}.$$

Thus, we get

$$\frac{1}{p+q}\overset{n}{\underset{i=1,j=i}{\otimes}}(p\alpha_i\oplus q\alpha_j)^{\frac{2}{n(n+1)}}$$
$$=\Bigg\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{h^{-1}\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left(\frac{2}{n(n+2)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right)\right)\right\},$$
$$\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{g^{-1}\left(\frac{1}{p+q}g\left(h^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.h\left(g^{-1}\left(pg(\eta_{\alpha_i})+qg(\eta_{\alpha_j})\right)\right)\right)\right)\right)\right)\right\}\Bigg\}.$$

□

Theorem 3 *The aggregated value by using the (T,S)-DHFGHM operator is also a DHFE.*

Proof Let

$$\gamma=\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\left\{h^{-1}\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right)+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right)\right\}$$

$$\eta=\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}}\left\{g^{-1}\left(\frac{1}{p+q}g\left(h^{-1}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+2)}h\left(g^{-1}\left(pg(\eta_{\alpha_i})\right)+qg(\eta_{\alpha_j})\right)\right)\right)\right)\right)\right\}.$$

We need to prove the membership degree γ and non-membership degree η satisfies the following conditions:

(a) $0\le\gamma,\eta\le1$;
(b) $0\le\gamma^{+}+\eta^{+}\le1$.

First, we give the proof of (a). Since $h(t)$ is a monotonically increasing function, $h^{-1}(t)$ is also a monotonically increasing function. By $h(0)\le h(\gamma_{\alpha_i})\le h(1)$, $h(0)\le h(\gamma_{\alpha_j})\le h(1)$, we have

$$h^{-1}\Big((p+q)h(0)\Big)\le h^{-1}(ph(\gamma_{\alpha_i})$$
$$+qh(\gamma_{\alpha_j})\le h^{-1}\Big((p+q)h(1)\Big).$$

Similarly, since $g(t)$ is a monotonically decreasing function, $g^{-1}(t)$ is also a monotonically decreasing function, then

$$\sum_{i=1,j=i}^{n}g\Big(h^{-1}((p+q)h(1))\Big)$$
$$\le\sum_{i=1,j=i}^{n}g\Big(h^{-1}\Big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\Big)\Big)$$
$$\le\sum_{i=1,j=i}^{n}g\Big(h^{-1}((p+q)h(0))\Big).$$

That is

$$\frac{n(n+1)}{2}g\Big(h^{-1}((p+q)h(1))\Big)$$
$$\le\sum_{i=1,j=i}^{n}g\Big(h^{-1}\Big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\Big)\Big)$$
$$\le\frac{n(n+1)}{2}g\Big(h^{-1}((p+q)h(0))\Big).$$

Further

$$g^{-1}\Big(g\Big(h^{-1}((p+q)h(0))\Big)\Big)$$
$$\le g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\Big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\Big)\right)\right)$$
$$\le g^{-1}\Big(g\Big(h^{-1}((p+q)h(1))\Big)\Big).$$

Then

$$h\Big(h^{-1}((p+q)h(0))\Big)$$
$$\le h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\Big)\Big)\right)\right)$$
$$\le h\Big(h^{-1}((p+q)h(1))\Big).$$

We have

$$h(0)\le\frac{1}{p+q}\left(h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right)$$
$$\le h(1).$$

Therefore

$$h^{-1}(h(0))$$
$$\le h^{-1}\left(\frac{1}{p+q}\left(h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right.\right.\right.\right.\right.\right.$$

$$+qh(\gamma_{\alpha_j})\Big)\Big)\Big)\Big)\Big)\Big)$$
$$\leq h^{-1}\big(h(1)\big).$$

That is

$$0 \leq h^{-1}\bigg(\frac{1}{p+q}\bigg(h\bigg(g^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph(\gamma_{\alpha_i}) + qh(\gamma_{\alpha_j})\Big)\Big)\bigg)\bigg)\bigg)\bigg) \leq 1.$$

Finally, we have

$$0 \leq \cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}} h^{-1}\bigg(\frac{1}{p+q}\bigg(h\bigg(g^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)} g\Big(h^{-1}\Big(ph(\gamma_{\alpha_i}) + qh(\gamma_{\alpha_j})\Big)\Big)\bigg)\bigg)\bigg)\bigg) \leq 1.$$

That is, $0 \leq \gamma \leq 1$. Similarly, we can prove that $0 \leq \eta \leq 1$.

Next, we prove the condition (b): $0 \leq \gamma^+ + \eta^+ \leq 1$. In the following proof, we will use the following equations:

$$g(t) = h(1-t), h(t) = g(1-t),$$
$$g^{-1}(t) = 1 - h^{-1}(t), h^{-1}(t) = 1 - g^{-1}(t).$$

Since $\gamma^+_{\alpha_i}+\eta^+_{\alpha_i} \leq 1, \gamma^+_{\alpha_j}+\eta^+_{\alpha_j} \leq 1$, we have $\gamma^+_{\alpha_i} \leq 1-\eta^+_{\alpha_i}$, $\gamma^+_{\alpha_j} \leq 1-\eta^+_{\alpha_j}$ and $ph(\gamma^+_{\alpha_i})+qh(\gamma^+_{\alpha_j}) \leq ph(1-\eta^+_{\alpha_i})+qh(1-\eta^+_{\alpha_j}) = pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})$.
Then

$$g\Big(h^{-1}\big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\big)\Big)$$
$$\geq g\Big(1 - g^{-1}\big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\big)\Big)$$
$$= h\Big(g^{-1}\big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\big)\Big).$$

Thus

$$g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(g\Big(h^{-1}\big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\big)\Big)\Big)\right)$$
$$\leq g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg\big(\eta^+_{\alpha_i}\big) + qg\big(\eta^+_{\alpha_j}\big)\Big)\Big)\Big)\right)$$
$$= 1 - h^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\Big)\Big)\Big)\bigg)$$

and

$$h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(g\Big(h^{-1}\big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\big)\Big)\Big)\right)\right)$$
$$\leq h\left(1 - h^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\big(g^{-1}\big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\big)\big)\Big)\right)\right)$$
$$= g\left(h^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\big(g^{-1}\big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\big)\big)\Big)\right)\right).$$

Further

$$\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(g\big(h^{-1}\big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\big)\big)\Big)\right)\right)$$
$$\leq \frac{1}{p+q}g\bigg(h^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)$$

and

$$h^{-1}\bigg(\frac{1}{p+q}h\bigg(g^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(g\Big(h^{-1}\Big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)\bigg)$$
$$\leq h^{-1}\bigg(\frac{1}{p+q}g\bigg(h^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)\bigg)$$
$$= 1 - g^{-1}\bigg(\frac{1}{p+q}g\bigg(h^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)\bigg).$$

Then

$$h^{-1}\bigg(\frac{1}{p+q}h\bigg(g^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(g\Big(h^{-1}\Big(ph(\gamma^+_{\alpha_i}) + qh(\gamma^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)\bigg)$$
$$+g^{-1}\bigg(\frac{1}{p+q}g\bigg(h^{-1}\bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}\Big(h\Big(g^{-1}\Big(pg(\eta^+_{\alpha_i}) + qg(\eta^+_{\alpha_j})\Big)\Big)\Big)\bigg)\bigg)\bigg)$$
$$\leq 1.$$

That is $0 \leq \gamma^+ + \eta^+ \leq 1$. Therefore, the aggregated value of (T, S)-DHFGHM operator is a DHFE. □

Following, we will study some properties of (T,S)-DHFGHM operator.

Theorem 4 (Idempotency) *Let* $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}) = \{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}}\{\gamma_{\alpha_i}\}, \cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}}\{\eta_{\alpha_i}\}\}$ $(i=1,2,\ldots,n)$ *be a collection of DHFEs, if* $\alpha_i = \alpha = \{\cup_{\gamma\in\mu}\{\gamma\}, \cup_{\eta\in\nu}\{\eta\}\}$ *for all* i*, then*

$$(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha.$$

Proof Let $(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = (\gamma',\eta')$. We first prove that $\gamma'=\gamma$. Since $\gamma_{\alpha_i}=\gamma, \gamma_{\alpha_j}=\gamma$, we have

$$ph(\gamma_{\alpha_i}) + qh(\gamma_{\alpha_j}) = ph(\gamma) + qh(\gamma) = (p+q)h(\gamma)$$

and

$$h^{-1}\Big(ph\big(\gamma_{\alpha_i}\big) + qh\big(\gamma_{\alpha_j}\big)\Big) = h^{-1}\big((p+q)h(\gamma)\big).$$

Then

$$\begin{aligned}&\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\big)\Big)\\ &= \sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\big((p+q)h(\gamma)\big)\Big)\\ &= \frac{n(n+1)}{2}\cdot\frac{2}{n(n+1)}g\Big(h^{-1}\big((p+q)h(\gamma)\big)\Big)\\ &= g\Big(h^{-1}\big((p+q)h(\gamma)\big)\Big)\end{aligned}$$

and

$$\begin{aligned}&g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\big(ph\big(\gamma_{\alpha_i}\big)+qh\big(\gamma_{\alpha_j}\big)\big)\Big)\right)\\ &= g^{-1}\Big(g\Big(h^{-1}\big((p+q)h(\gamma)\big)\Big)\Big) = h^{-1}\big((p+q)h(\gamma)\big).\end{aligned}$$

Further, we have

$$\begin{aligned}&\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\big(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\big)\Big)\right)\right)\\ &= \frac{1}{p+q}h\Big(h^{-1}\big((p+q)h(\gamma)\big)\Big) = \frac{1}{p+q}(p+q)h(\gamma) = h(\gamma)\end{aligned}$$

and

$$\begin{aligned}&h^{-1}\Bigg(\frac{1}{p+q}h\Bigg(g^{-1}\Bigg(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph\big(\gamma_{\alpha_i}\big)\\ &\quad+qh\big(\gamma_{\alpha_j}\big)\Big)\Big)\Bigg)\Bigg)\Bigg)\\ &= h^{-1}\big(h(\gamma)\big) = \gamma.\end{aligned}$$

Therefore

$$\begin{aligned}&\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}\Bigg\{h^{-1}\Bigg(\frac{1}{p+q}h\Bigg(g^{-1}\Bigg(\sum_{i=1,j=i}^{n}\Bigg(\frac{2}{n(n+2)}\\ &g\Big(h^{-1}\Big(ph\big(\gamma_{\alpha_i}\big)\Big)+h^{-1}\Big(qh\big(\gamma_{\alpha_j}\big)\Big)\Big)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg\} = \gamma.\end{aligned}$$

That is $\gamma'=\gamma$. The proof of $\eta'=\eta$ is similar to $\gamma'=\gamma$.

Thus, we have $(\gamma',\eta')=(\gamma,\eta)$, i.e.,

$$(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha.$$

□

Theorem 5 (Monotonicity) *Let* $\alpha_i = (\mu_{\alpha_i},\nu_{\alpha_i})$ *and* $\alpha_i^{'} = (\mu_{\alpha_i}^{'},\nu_{\alpha_i}^{'})(i=1,2,\ldots,n)$ *be two collections of* DHFEs, $p,q\le 0$*. If* $\gamma_{\alpha_i}^{'}\le\gamma_{\alpha_i}$ *and* $\eta_{\alpha_i}^{'}\ge\eta_{\alpha_i}$ *for all* i*, where* $\gamma_{\alpha_i}\in\mu_{\alpha_i}, \eta_{\alpha_i}\in\nu_{\alpha_i}, \gamma_{\alpha_i}^{'}\in\mu_{\alpha_i}^{'}, \eta_{\alpha_i}^{'}\in\nu_{\alpha_i}^{'}$*, then*

$$\begin{aligned}&(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1^{'},\alpha_2^{'},\ldots,\alpha_n^{'})\\ &\quad\le (T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n).\end{aligned}$$

Proof Let

$$(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1^{'},\alpha_2^{'},\ldots,\alpha_n^{'}) = (\gamma^{'},\eta^{'})$$

and

$$(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n) = (\gamma,\eta).$$

Since $\gamma_{\alpha_i}^{'}\le\gamma_{\alpha_i}$ and $\gamma_{\alpha_j}^{'}\le\gamma_{\alpha_j}(1\le j\le n)$, then we have

$$ph\big(\gamma_{\alpha_i}^{'}\big)+qh\big(\gamma_{\alpha_j}^{'}\big)\le ph\big(\gamma_{\alpha_i}\big)+qh\big(\gamma_{\alpha_j}\big)$$

and

$$h^{-1}\Big(ph\big(\gamma_{\alpha_i}^{'}\big)+qh\big(\gamma_{\alpha_j}^{'}\big)\Big)\le h^{-1}\Big(ph\big(\gamma_{\alpha_i}\big)+qh\big(\gamma_{\alpha_j}\big)\Big).$$

Then, we have

$$\begin{aligned}&\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph\big(\gamma_{\alpha_i}^{'}\big)+qh\big(\gamma_{\alpha_j}^{'}\big)\Big)\Big)\\ &\ge\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph\big(\gamma_{\alpha_i}\big)+qh\big(\gamma_{\alpha_j}\big)\Big)\Big)\end{aligned}$$

and

$$g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\Big(h^{-1}\Big(ph\big(\gamma_{\alpha_i}^{'}\big)+qh\big(\gamma_{\alpha_j}^{'}\big)\Big)\Big)\right)$$

$$\leq g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})+qh(\gamma_{\alpha_j})\right)\right)\right).$$

Further, we have

$$\begin{aligned}&\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i}^{'})\right.\right.\right.\right.\\&\left.\left.\left.\left.+qh(\gamma_{\alpha_j}^{'})\right)\right)\right)\right)\\&\leq\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right.\right.\right.\right.\\&\left.\left.\left.\left.+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\end{aligned}$$

and

$$\begin{aligned}&h\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i}^{'})\right.\right.\right.\right.\right.\\&\left.\left.\left.\left.\left.+qh(\gamma_{\alpha_j}^{'})\right)\right)\right)\right)\right)\\&\leq h\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right.\right.\right.\right.\right.\\&\left.\left.\left.\left.\left.+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right).\end{aligned}$$

Then

$$\begin{aligned}&\cup_{\gamma_{\alpha_i}^{'}\in\mu_{\alpha_i}^{'},\gamma_{\alpha_j}^{'}\in\mu_{\alpha_j}^{'}}h\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i}^{'})\right.\right.\right.\right.\right.\\&\left.\left.\left.\left.\left.+qh(\gamma_{\alpha_j}^{'})\right)\right)\right)\right)\right)\\&\leq\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}}h\left(\frac{1}{p+q}h\left(g^{-1}\left(\sum_{i=1,j=i}^{n}\frac{2}{n(n+1)}g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right.\right.\right.\right.\right.\\&\left.\left.\left.\left.\left.+qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right).\end{aligned}$$

That is $\gamma^{'}\leq\gamma$. The proof of $\eta'\geq\eta$ is similar to $\gamma'\leq\gamma$.

According to Definition 2, we get

$$\begin{aligned}&(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1^{'},\alpha_2^{'},\ldots,\alpha_n^{'})\\&\leq(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n).\end{aligned}$$

□

Theorem 6 (Boundedness) *Let $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})(i=1,2,\ldots,n)$ be a collection of DHFEs, and $\alpha^-=(min\{\mu_{\alpha_i}\},max\{\nu_{\alpha_i}\})=\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}}min\{\gamma_{\alpha_i}\},\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}}max\{\eta_{\alpha_i}\}\}$, $\alpha^+=(max\{\mu_{\alpha_i}\},min\{\nu_{\alpha_i}\})=\{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}}max\{\gamma_{\alpha_i}\},\cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}}min\{\eta_{\alpha_i}\}\}$, then*

$$\alpha^-\leq(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\alpha^+.$$

Proof According to the proof of Theorems 4 and 5, Theorem 6 can be proved easily. □

Theorem 7 (Permutation) *Let $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})(i=1,2,\ldots,n)$ be a collection of DHFEs. Then*

$$\begin{aligned}&(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&=(T,S)-\mathrm{DHFGHM}^{p,q}(\dot{\alpha}_1,\dot{\alpha}_2,\ldots,\dot{\alpha}_n),\end{aligned}$$

where $(\dot{\alpha}_1,\dot{\alpha}_2,...,\dot{\alpha}_n)$ is any permutation of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Proof By the operations of DHFE in Theorem 1, we have

$$\begin{aligned}&(T,S)-\mathrm{DHFGHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(p\alpha_i\oplus q\alpha_j\right)^{\frac{2}{n(n+1)}}\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(p\dot{\alpha}_i\oplus q\dot{\alpha}_j\right)^{\frac{2}{n(n+1)}}\\&=(T,S)-\mathrm{DHFGHM}^{p,q}(\dot{\alpha}_1,\dot{\alpha}_2,\ldots,\dot{\alpha}_n),\end{aligned}$$

which completes the proof. □

The proposed (T,S)-DHFGHM operator only considers the input parameters p,q, and the relationship between each input data and the importance of input data is not considered. Therefore, in order to consider the importance of each input data and their relationship, we further introduce (T,S)-based dual hesitant fuzzy geometric weighted Heronian mean((T,S)-DHFGWHM) operator.

Definition 10 Let $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})(i=1,2,\ldots,n)$ be a collection of DHFEs, a (T,S)-based dual hesitant fuzzy geometric weighted Heronian mean((T,S)-DHFGWHM) is defined as follows:

$$\begin{aligned}&(T,S)-\mathrm{DHFGWHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left((p\alpha_i)^{\omega_i}\oplus(q\alpha_j)^{\omega_j}\right)^{\frac{2}{n(n+1)}}\end{aligned}$$

where $p,q\geq0$ and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ are the weighted vector of $\alpha_1,\alpha_2,\ldots,\alpha_n$, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$.

Based on the operational laws of the DHFEs shown in Definition 7, we can get Theorem 8.

Theorem 8 *Let $p,q\geq0$ and $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})(i=1,2,\ldots,n)$ be a collection of DHFEs, and*

$$\begin{aligned}&(T,S)-\mathrm{DHFGWHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left((p\alpha_i)^{\omega_i}\oplus(q\alpha_j)^{\omega_j}\right)^{\frac{2}{n(n+1)}}\end{aligned}$$

$$= \left\{ \cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i},\gamma_{\alpha_j}\in\mu_{\alpha_j}} \left\{ h^{-1}\left(\frac{1}{p+q} h\left(g^{-1}\left(\sum_{i=1,j=i}^{n} \left(\frac{2}{n(n+2)} g\left(h^{-1}\left(h\left(g^{-1}\left(\omega_i g\left(h^{-1}\left(ph(\gamma_{\alpha_i})\right)\right)\right)\right) + h\left(g^{-1}\left(\omega_j g\left(h^{-1}\left(qh(\gamma_{\alpha_j})\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right\} \right.$$

$$\left. \cup_{\eta_{\alpha_i}\in\nu_{\alpha_i},\eta_{\alpha_j}\in\nu_{\alpha_j}} \left\{ g^{-1}\left(\frac{1}{p+q} g\left(h^{-1}\left(\sum_{i=1,j=i}^{n} \left(\frac{2}{n(n+2)} h\left(g^{-1}\left(g\left(h^{-1}\left(\omega_i h\left(g^{-1}\left(pg(\eta_{\alpha_i})\right)\right)\right)\right) + g\left(h^{-1}\left(\omega_j h\left(g^{-1}\left(qg(\eta_{\alpha_j})\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right\} \right\}.$$

the aggregated value by using the (T, S)-DHFGWHM operator is also a DHFE.

Proof This proof is similar to Theorems 2 and 3. □

The (T, S)-DHFGWHM operator is also satisfied idempotency, monotonicity, boundedness and permutation.

Theorem 9 (Idempotency) *Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}) = \{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}}\{\gamma_{\alpha_i}\}, \cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}}\{\eta_{\alpha_i}\}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs, if $\alpha_i = \alpha = \{\cup_{\gamma\in\mu}\{\gamma\}, \cup_{\eta\in\nu}\{\eta\}\}$ for all i, then*

$$(T, S) - \mathrm{DHFGWHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha.$$

Theorem 10 (Monotonicity) *Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ and $\alpha_i^{'} = (\mu_{\alpha_i}^{'}, \nu_{\alpha_i}^{'})(i = 1, 2, \ldots, n)$ be two collections of DHFEs, $p, q \le 0$. If $\gamma_{\alpha_i}^{'} \le \gamma_{\alpha_i}$ and $\eta_{\alpha_i}^{'} \ge \eta_{\alpha_i}$ for all i, where $\gamma_{\alpha_i} \in \mu_{\alpha_i}, \eta_{\alpha_i} \in \nu_{\alpha_i}, \gamma_{\alpha_i}^{'} \in \mu_{\alpha_i}^{'}, \eta_{\alpha_i}^{'} \in \nu_{\alpha_i}^{'}$, then*

$$(T, S) - \mathrm{DHFGWHM}^{p,q}(\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_n^{'}) \le (T, S) - \mathrm{DHFGWHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Theorem 11 (Boundedness) *Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2, \ldots, n)$ be a collection of DHFEs, and $\alpha^- = (min\{\mu_{\alpha_i}\}, max\{\nu_{\alpha_i}\}) = \{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}} min\{\gamma_{\alpha_i}\}, \cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}} max\{\eta_{\alpha_i}\}\}$, $\alpha^+ = (max\{\mu_{\alpha_i}\}, min\{\nu_{\alpha_i}\}) = \{\cup_{\gamma_{\alpha_i}\in\mu_{\alpha_i}} max\ \{\gamma_{\alpha_i}\}, \cup_{\eta_{\alpha_i}\in\nu_{\alpha_i}} min\{\eta_{\alpha_i}\}\}$, then*

$$\alpha^- \le (T, S) - \mathrm{DHFGWHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+.$$

Theorem 12 (Permutation) *Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})(i = 1, 2, \ldots, n)$ be a collection of DHFEs. Then*

$$(T, S) - \mathrm{DHFGWHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = (T, S) - \mathrm{DHFGWHM}^{p,q}(\dot\alpha_1, \dot\alpha_2, \ldots, \dot\alpha_n),$$

where $(\dot\alpha_1, \dot\alpha_2, \ldots, \dot\alpha_n)$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

It is obvious that the proposed (T, S)-DHFGHM and (T, S)-DHFGWHM operators provide a general expression with the generators $g(x)$ and $h(x)$. Following, we discuss some special cases of the proposed (T, S)-DHFGHM and (T, S)-DHFGWHM operators.

Case 1 When we adopt algebraic operations, the (T, S)-DHFGHM and (T, S)-DHFGWHM reduce into the dual hesitant fuzzy geometric Heronian mean and dual hesitant fuzzy geometric weighted Heronian mean defined by Yu et al. (2016) in the literature.

Case 2 When we adopt the Einstein operations, the (T, S)-DHFGHM and (T, S)-DHFGWHM reduce into the (T, S)-based dual hesitant fuzzy geometric Einstein Heronian mean operator and (T, S)-based dual hesitant fuzzy Einstein geometric weighted Heronian mean operator.

Case 3 When we adopt the Hamacher operations, the (T, S)-DHFGHM and (T, S)-DHFGWHM reduce into the (T, S)-based dual hesitant fuzzy geometric Hamacher Heronian mean operator and (T, S)-based dual hesitant fuzzy Hamacher geometric weighted Heronian mean operator.

Case 4 When we adopt the Frank operations, the (T, S)-DHFGHM and (T, S)-DHFGWHM reduce into the (T, S)-based dual hesitant fuzzy geometric Frank Heronian mean operator and (T, S)-based dual hesitant fuzzy Frank geometric weighted Heronian mean operator.

Case 5 When DHFS reduce into intuitionistic fuzzy set and we adopt Algebraic operations, the (T, S)-DHFGHM and (T, S)-DHFGWHM reduce into intuitionistic fuzzy geometric Heronian mean and intuitionistic fuzzy geometric weighted Heronian mean defined by Yu (2013) in the literature.

4 Dual hesitant fuzzy multiple attribute decision making method

In this section, we propose a MADM method based on the (T, S)-DHFGWHM operator. Let $\{A_1, A_2, \ldots A_m\}$ be a set of alternatives and $\{C_1, C_2, \ldots, C_n\}$ be a set of attributes which weighting vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$. Suppose that $R = (r_{ij})_{m\times n}$ is the decision matrix given by decision maker, where $r_{ij} = (\mu_{\gamma_{ij}}, \nu_{\eta_{ij}})$ denotes the evaluation value represented by a DHFE of alternative A_i with respect to attribute $C_j (1 \le i \le m, 1 \le j \le n)$. The steps of MADM method are as follows.

Step 1 Transform the dual hesitant fuzzy decision matrix R into the normalized dual hesitant fuzzy decision matrix

Table 1 Dual hesitant fuzzy decision matrix R

	G_1	G_2	G_3	G_4
A_1	$\{\{0.3, 0.4\}, \{0.6\}\}$	$\{\{0.4, 0.5\}, \{0.3, 0.4\}\}$	$\{\{0.2, 0.3\}, \{0.7\}\}$	$\{\{0.4, 0.5\}, \{0.5\}\}$
A_2	$\{\{0.6\}, \{0.4\}\}$	$\{\{0.2, 0.4, 0.5\}, \{0.4\}\}$	$\{\{0.2\}, \{0.6, 0.7, 0.8\}\}$	$\{\{0.5\}, \{0.4, 0.5\}\}$
A_3	$\{\{0.5, 0.7\}, \{0.2\}\}$	$\{\{0.2\}, \{0.7, 0.8\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.6\}\}$	$\{\{0.5, 0.6, 0.7\}, \{0.3\}\}$
A_4	$\{\{0.7\}, \{0.3\}\}$	$\{\{0.6, 0.7, 0.8\}, \{0.2\}\}$	$\{\{0.1, 0.2\}, \{0.3\}\}$	$\{\{0.1\}, \{0.6, 0.7, 0.8\}\}$
A_5	$\{\{0.6, 0.7\}, \{0.2\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.5\}\}$	$\{\{0.4, 0.5\}, \{0.2\}\}$	$\{\{0.2, 0.3, 0.4\}, \{0.5\}\}$

Table 2 Ranking results for different values of p and q based on the Einstein operations

p and q	Score function $A_i (i = 1, 2, 3, 4, 5)$	Ranking result
$p = 1, q = 1$	$S_1 = 0.6297, S_2 = 0.6444, S_3 = 0.6864, S_4 = 0.6084, S_5 = 0.7313$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 2, q = 2$	$S_1 = 0.272, S_2 = 0.284, S_3 = 0.3382, S_4 = 0.2262, S_5 = 0.4334$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 3, q = 2$	$S_1 = 0.1639, S_2 = 0.1697, S_3 = 0.2161, S_4 = 0.1086, S_5 = 0.3409$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 1, q = 5$	$S_1 = 0.0768, S_2 = 0.061, S_3 = 0.1329, S_4 = -0.0367, S_5 = 0.2431$	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$p = 5, q = 1$	$S_1 = 0.0699, S_2 = 0.0601, S_3 = 0.0885, S_4 = 0.0074, S_5 = 0.2633$	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$p = 4, q = 5$	$S_1 = -0.0627, S_2 = -0.0834, S_3 = -0.0378, S_4 = -0.169, S_5 = 0.1257$	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$p = 5, q = 5$	$S_1 = -0.0957, S_2 = -0.1212, S_3 = -0.082, S_4 = -0.2071, S_5 = 0.0947$	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$p = 3, q = 7$	$S_1 = -0.0981, S_2 = -0.1236, S_3 = -0.0736, S_4 = -0.2131, S_5 = 0.0925$	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$p = 10, q = 1$	$S_1 = -0.14, S_2 = -0.1765, S_3 = -0.1679, S_4 = -0.2443, S_5 = 0.0606$	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 10, q = 10$	$S_1 = -0.2597, S_2 = -0.3044, S_3 = -0.2935, S_4 = -0.3965, S_5 = 0.0617$	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 12, q = 13$	$S_1 = -0.2958, S_2 = -0.3427, S_3 = -0.3412, S_4 = -0.4366, S_5 = -0.0962$	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$

$R^{'} = (r^{'}_{ij})_{m\times n}$, where

$$r^{'}_{ij} = \begin{cases} r_{ij}, & \text{for benefit attribute,} \\ (r_{ij})^c, & \text{for cost attribute.} \end{cases}$$

Step 2 Choose appropriate (T, S), utilize the (T, S)-DHFGWHM operator to obtain the aggregation value for the alternatives

$$r_i = (T, S) - \text{DHFGWHM}^{p,q}(r^{'}_{i1}, r^{'}_{i2}, \ldots, r^{'}_{in}),$$

where $1 \leq i \leq m$.

Step 3 According to Definition 2, the best alternative can be obtained by ranking the DHFEs $r_i (i = 1, 2, \ldots, m)$.

5 Illustrative example

In order to demonstrate the application of the proposed MADM method, we adopted the example shown in Wang et al. (2014) for potential evaluation of emerging technology commercialization with dual hesitant fuzzy information. There is a panel with five possible emerging technology enterprises $A_i (i = 1, 2, 3, 4, 5)$ to select. The experts select four attribute to evaluate the five possible emerging technology enterprises: (1) G_1 is the technical advancement; (2) G_2 is the potential market; (3) G_3 is the industrialization infrastructure, human resources and financial conditions; (4) G_4 is the employment creation and the development of science and technology. And the attribute weighted is $\omega = (0.20, 0.15, 0.35, 0.30)^T$. The decision making evaluate five possible emerging technology enterprise $A_i (i = 1, 2, 3, 4, 5)$ under the above four attributes in anonymity, and the decision matrix $R = (r_{ij})_{5\times 4}$ is presented in Table 1, where $r_{ij} (1 \leq i \leq 5, 1 \leq j \leq 4)$ are in the form of DHFEs.

In the following, we use the proposed method to solve this MADM problem.

5.1 Multiple attribute decision making

In order to get the best alternative, the following steps are performed.

Step 1 Normalized the decision matrix. Because all attribute are benefit, this step is skipped.

Step 2 Based on decision matrix and the (T, S)-DHFGWHM operator (where we choose Einstein t-norm and t-conorm, $p = 1, q = 1$). Take emerging technology enterprise A_1 for example, we have

$$r_1 = (T, S) - \text{DHFGWHM}^{p,q}(r_{11}, r_{12}, r_{13}, r_{14})$$
$$= \frac{1}{1+1} \underset{j=1,k=j}{\overset{4}{\otimes}} \left((pr_{1j})^{\omega_j} \oplus (qr_{1k})^{\omega_k}\right)^{\frac{2}{4(4+1)}}$$

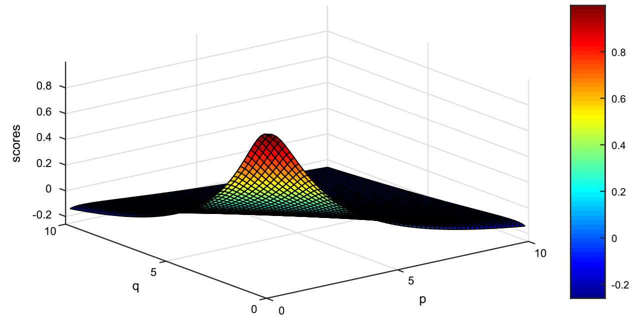

Fig. 1 Score for alternative A_1 obtained by (T, S)-DHFGWHM operator

$$= \{\{0.7787, 0.7883, 0.8013, 0.8114, 0.7828, 0.7924, 0.8056, 0.8157, 0.7861, 0.7957, 0.809, 0.8193, 0.7902, 0.7999, 0.8133, 0.8236\}, \{0.1696, 0.1727\}\}.$$

Step 3 Calculate the score valued $S(r_i)(i = 1, 2, 3, 4, 5)$: $S(r_1) = 0.6297$, $S(r_2) = 0.6444$, $S(r_3) = 0.6864$, $S(r_4) = 0.6084$, $S(r_5) = 0.7313$.
According to Definition 2, we have $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$. Therefore, the best emerging technology enterprise is A_5.

Because p, q are parameter variables, different parameters may have different influences on decision results. In the following, we discuss the influences of the parameters p, q on the ranking results of this example. From Table 2, we can know that different score functions can be obtained when p, q are take different values. The ranking results change from $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ to $A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$ to $A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$. It shows that the ranking result of A_1 is better with the increase of p, q. The best choice is always A_5 and the worst choice is always A_4. When the parameter p and q change, the change of the score function of the five alternatives is shown in Figs. 1, 2, 3, 4 and 5. Therefore, the decision maker can choose the value of p, q according to actual needs.

5.2 Comparative analysis

Next, in order to verify the effectiveness and superiority of our method, we use other existing methods to deal with this example and compare the result with our method.

(1) Comparing with the dual hesitant fuzzy geometric weighted Heronian mean (i.e., when we take Algebraic t-norm and t-conorm in this paper) in the literature (Yu et al. 2016): According to dual hesitant fuzzy geomet-

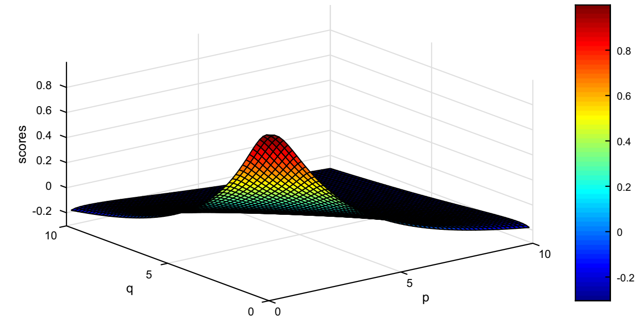

Fig. 2 Score for alternative A_2 obtained by (T, S)-DHFGWHM operator

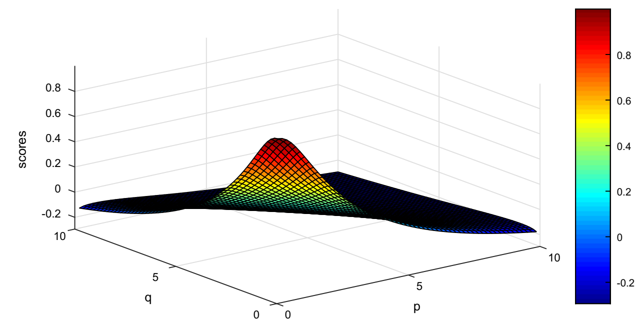

Fig. 3 Score for alternative A_3 obtained by (T, S)-DHFGWHM operator

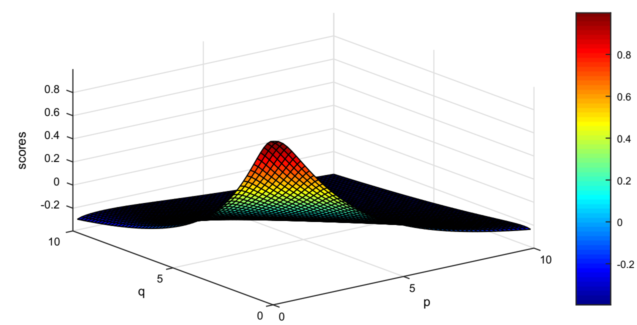

Fig. 4 Score for alternative A_4 obtained by (T, S)-DHFGWHM operator

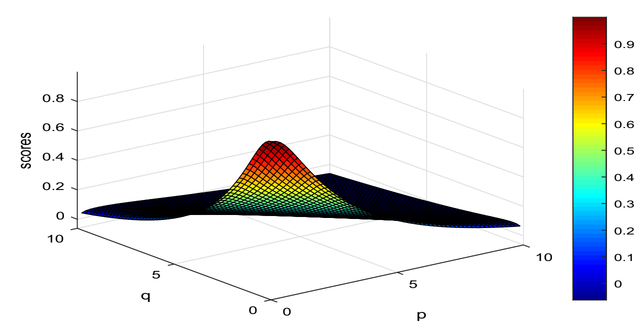

Fig. 5 Score for alternative A_5 obtained by (T, S)-DHFGWHM operator

ric weighted Heronian mean, where $p = 1, q = 1$, we get the ranking result is: $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$. Obviously, the method in the literature (Yu et al. 2016) has the same ranking result as our method. However, our method adopted the general t-norm and t-conorm, and the literature (Yu et al. 2016) uses the Algebraic t-norm and t-conorm. Because the Algebraic t-norm and t-conorm are only a special case of the general t-norm and t-conorm. Therefore, the proposed method in this paper is more general than the method in the literature (Yu et al. 2016).

(2) Comparing with the dual hesitant fuzzy Algebraic weighted geometric operator in the literature (Wang et al. 2014): As the example used in the literature (Wang et al. 2014) is the same as this paper, we can see from the literature (Wang et al. 2014) that the ranking result is: $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$.

(3) Comparing with the dual hesitant fuzzy Einstein weighted geometric operator in the literature (Yu 2015): We get the ranking result is: $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$.

According to (2) and (3), we compare the characteristics of our method with studies (Wang et al. 2014; Yu 2015). The method in the literature (Wang et al. 2014) and the method in the literature (Yu 2015) have the same ranking result. Although the best alternative is the same as (T, S)-DHFGWHM (Einstein t-norm and t-conorm) and dual hesitant fuzzy geometric weighted Heronian mean in the literature (Yu et al. 2016), the ranking results are different. The reason is our method considers the inter-dependent phenomena among the arguments, and studies (Wang et al. 2014; Yu 2015) only can provide the weighted geometric function and do not take into account interrelationship of aggregation arguments.

To sum up, the proposed method in this paper combines the Archimedean t-norm and t-conorm with GHM operator under dual hesitant fuzzy environment. It not only take into account interrelationship of aggregation arguments, but also provides a general and flexible tool to deal with dual hesitant fuzzy MADM problems.

6 Conclusion

Considering the inter-dependent phenomena among the arguments and the generalization of the existing aggregation operators by general generators based on the Archimedean t-norm and t-conorm, this paper combines the GHM and Archimedean t-norm and t-conorm to propose the (T, S)-DHFGHM and (T, S)-DHFGWHM operators. Their some properties have been investigated, such as idempotency, monotonicity, boundedness, and permutation. Meanwhile, some special cases have been studied. Based on the (T, S)-DHFGWHM operator, a method has been developed to deal with a MADM problem under dual hesitant fuzzy environment. Finally, an example has been given to demonstrate the effectiveness of our method. The influence of parameters on the ranking results has been studied, and the superiority of this method has been illustrated by comparing with other existing methods.

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Compliance with ethical standards

Conflict of interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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