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FOUNDATIONS

Single axioms for (*S*, *T*)-fuzzy rough approximation operators with fuzzy product operations

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Abstract

There are two different perspectives to study single axioms for (S, T)-fuzzy rough approximation operators, that is, ordinary fuzzy operations and fuzzy product operations. However, it is too complex and tedious to characterize (S, T)-fuzzy rough approximation operators with ordinary fuzzy operations, such as intersection, union and so on. To remedy these defects, this paper further investigates single axioms for (S, T)-fuzzy rough approximation operators with fuzzy product operations, where fuzzy relation is not limited into either a general fuzzy relation or a symmetric one. Considering a left-continuous t-norm T, we describe T-upper fuzzy rough approximation operators with fuzzy product operations by only one axiom. When t-conorm S is right-continuous and fuzzy negation N is strict, S-lower fuzzy rough approximation operators are characterized with fuzzy product operations by a single axiom.

Keywords Fuzzy relations · Fuzzy rough sets · Triangular norms · Fuzzy product operations

1 Introduction

Pawlak established rough set theory (Pawlak 1982, 1991), which is an excellent tool to model incompleteness in intelligent systems. Meanwhile, there are two different approaches to investigate rough sets, i.e., the constructive approach and axiomatic approach. The constructive approach proposes and discusses lower and upper approximation operators on a binary relation or neighborhoods (Cekik and Telceken 2018; Chebrolu and Sanjeevi 2017; Chen et al. 2017; Dai et al. 2018; D'eer et al. 2016; Yao and Yao 2012; Yu and Zhan 2014; Zhao 2016). Different from the constructive approach, abstract lower and upper approximation operators are investigated by certain axioms in the axiomatic approach. Axiomatic characterizations of rough sets and their extensions have been investigated (Liu 2008; Song et al. 2013; Thiele 2000; Zhang et al. 2010; Zhu and Wang 2003). In particular, it is popular to search the minimal axiom sets to characterize rough sets and their extensions. For example, Liu (2013) characterized rough sets with one axiom, which was further discussed in Ma et al. (2015). Yang and Li (2006) discussed the minimization of axiom sets to characterize generalized approximation operators. Moreover, Zhang and Luo (2011) studied the minimization of axiom sets to describe covering-based approximation operators.

Dubois and Prade (1990) originally proposed fuzzy rough sets, which were further investigated with different fuzzy logic operations and fuzzy relations (D'eer et al. 2015; Li and Cui 2015; Mi et al. 2008; Morsi and Yakout 1998; Radzikowska and Kerre 2002; Wang 2017a, 2018; Wu et al. 2016). In particular, (S, T)-fuzzy rough sets were studied with a continuous t-conorm S and a continuous t-norm T (Li and Cui 2015; Mi et al. 2008; Wu et al. 2016). Liu (2013) also characterized fuzzy rough approximation operators by only one axiom. Wu et al. (2016) investigated single axioms for S-lower and T-upper fuzzy rough approximation operators in two approaches that characterize (S, T)-fuzzy rough approximation operators with ordinary fuzzy operations and fuzzy product operations, respectively. In fact, Wang (2018) had pointed out that the continuity of t-(co)norms is redundant. Meanwhile, Wang discussed single axioms for lower fuzzy rough approximation operators determined by different fuzzy implications. Considering L-fuzzy rough sets (Han and Šostak 2018; Huang et al. 2018; Radzikowska and Kerre 2004; Wang 2017b; Wang et al. 2014), She and Wang (2009) studied them in the axiomatic approach, where L denotes a

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residuated lattice. Moreover, Bao et al. (2018) characterized L-fuzzy rough approximation operators by only one axiom, where residuated lattice L is assumed to be regular. However, it is too complex and tedious to characterize fuzzy rough approximation operators with ordinary fuzzy operations, such as intersection, union and so on. Meanwhile, all of single axioms for fuzzy rough approximation operators with fuzzy product operations (Bao et al. 2018; Wang 2018; Wu et al. 2016) are limited into either a general fuzzy relation or a symmetric one. Although Wang et al. (2020) further discussed the single axioms for L-fuzzy rough approximation operators with L-fuzzy product operations on all kinds of L-fuzzy relations, most of axioms still require the regularity of residuated lattices. To remedy these defects, we further study the axiomatic characterizations of (S, T)-fuzzy rough approximation operators with fuzzy product operations in this paper, where S is a right-continuous t-conorm and T is a left-continuous t-norm.

In this context, we characterize T-upper fuzzy rough approximation operators with fuzzy product operations by only one axiom. When fuzzy negation is strict, we study single axioms for S-lower fuzzy rough approximation operators with fuzzy product operations. In particular, we study single axioms for (S, T)-fuzzy rough approximation operators, when fuzzy relations are special fuzzy relations, such as serial, reflexive, T-transitive and T-Euclidean ones as well as any of their compositions.

The content of this paper is organized as follows. In Sect. 2, we recall some fundamental concepts and related properties of (S, T)-fuzzy rough sets. Section 3 characterizes T-upper fuzzy rough approximation operators by only one axiom with fuzzy product operations. In Sect. 4, we study single axioms for S-lower fuzzy rough approximation operators with fuzzy product operations, when fuzzy negation is strict. In the final section, we present some conclusions and further work.

2 Preliminaries

In this section, we present some basic concepts and terminologies used throughout our paper.

Let $[0, 1]$ be the unit interval and U be a universe. Then a mapping $A : U \to [0, 1]$ is called a *fuzzy set* on U. The family of all fuzzy sets on U is denoted as $\mathcal{F}(U)$. Let $\alpha \in [0, 1]$ and a fuzzy set $A \in \mathcal{F}(U)$ satisfy $A(x) = \alpha$ for all $x \in U$. Then A is a *constant* and denoted as $\widehat{\alpha}$. For convenience, the empty set $\emptyset$ and the universe U are also denoted as $\widehat{0}$ and $\widehat{1}$, respectively. Consider $y \in U$. Then a fuzzy set is denoted as 1_y, if for all $x \in U$,

$$1_y(x) = \begin{cases} 1, & x = y; \\ 0, & x \neq y. \end{cases}$$

For a crisp set $M \subseteq U$, 1_M denotes the characteristic function of M.

A binary mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ (resp. $S : [0, 1] \times [0, 1] \to [0, 1]$) is called a *t-norm* (resp. *t-conorm*) on $[0, 1]$, if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0). A t-norm T is said to be left-continuous, if the following holds for all $\{a_j\}_{j\in J} \subseteq [0, 1]$ and $b \in [0, 1]$,

$$T\left(\bigvee_{j\in J} a_j, b\right) = \bigvee_{j\in J} T(a_j, b),$$

where J is a nonempty index set. Similarly, a t-conorm S is said to be right-continuous, if the following holds for all $\{a_j\}_{j\in J} \subseteq [0, 1]$ and $b \in [0, 1]$,

$$S\left(\bigwedge_{j\in J} a_j, b\right) = \bigwedge_{j\in J} S(a_j, b).$$

In the sequel, t-norms and t-conorms are always assumed to be left-continuous and right-continuous, respectively. Moreover, the symbol J always denotes a nonempty index set, if not otherwise specified.

A decreasing function $N : [0, 1] \to [0, 1]$ is called a *fuzzy negation*, if it satisfies $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is said to be *strong*, if $N\big(N(a)\big) = a$ holds for all $a \in [0, 1]$. Meanwhile, the strong fuzzy negation $N_s(a) = 1 - a$ for all $a \in [0, 1]$ is referred as the *standard negation*. Moreover, a fuzzy negation N is said to be *strict*, if N is strictly decreasing and continuous. A t-norm T and a t-conorm S are said to be *dual* with respect to (w.r.t., for short) a strong fuzzy negation N, if the following hold for all $a, b \in [0, 1]$,

$$T(a, b) = N\big(S(N(a), N(b))\big),$$
$$S(a, b) = N\big(T(N(a), N(b))\big).$$

In particular, new fuzzy negations are proposed on the basis of t-(co)norms as follows:

Definition 1 (Baczyński and Jayaram 2008) Let T and S be a t-norm and a t-conorm, respectively. Then the mappings $N_T, N_S : [0, 1] \to [0, 1]$ are defined for all $a \in [0, 1]$ as

$$N_T(a) = \sup\{b \in [0, 1] | T(a, b) = 0\} \text{ and}$$
$$N_S(a) = \inf\{b \in [0, 1] | S(a, b) = 1\}, \text{ respectively.}$$

N_T and N_S are called the *natural negations* of t-norm T and t-conorm S, respectively.

The properties of the natural negations of t-(co)norms are listed as follows:

Lemma 2 (Baczyński and Jayaram 2008) *Let t-norm T be left-continuous and t-conorm S be right-continuous. Then the following hold for all $a, b \in [0, 1]$,*

(1) $T(a, b) = 0$ *iff* $N_T(a) \geqslant b$.
(2) $S(a, b) = 1$ *iff* $N_S(a) \leqslant b$.
(3) $T\big(N_T(a), a\big) = 0$ *and* $S\big(N_S(a), a\big) = 1$.

The following conclusions are obvious.

Proposition 3 *Let t-norm T be left-continuous and t-conorm S be right-continuous. Then the following hold.*

(1) *If N_T is strong, then the following holds for all $a, b \in [0, 1]$,*

$$a \leqslant b \text{ iff } T\big(a, N_T(b)\big) = 0.$$

(2) *If N_S is strong, then the following holds for all $a, b \in [0, 1]$,*

$$a \leqslant b \text{ iff } S\big(b, N_S(a)\big) = 1.$$

A binary mapping $I : [0, 1] \times [0, 1] \to [0, 1]$ is called a *fuzzy implication* on $[0, 1]$, if it satisfies the boundary conditions according to the Boolean implication, and is decreasing in the first argument and increasing in the second argument.

Several classes of fuzzy implications have been studied in the literature (Baczyński and Jayaram 2008; Klir and Yuan 1995). We recall the definitions of three main classes of these operations as follows.

Let T, S and N be a t-norm, a t-conorm and a fuzzy negation, respectively. Then a fuzzy implication I is called

- an *S-implication* based on S and N if $I(a, b) = S\big(N(a), b\big)$ for all $a, b \in [0, 1]$;
- an *R-implication* based on T if $I(a, b) = \bigvee\{c \in [0, 1] | T(a, c) \leqslant b\}$ for all $a, b \in [0, 1]$;
- a *QL-implication* based on T, S and N if $I(a, b) = S\big(N(a), T(a, b)\big)$ for all $a, b \in [0, 1]$.

The operations on fuzzy sets are defined as follows: for all $A, B \in \mathcal{F}(U)$ and $x \in U$,

$$N(A)(x) = N\big(A(x)\big);$$
$$(A \cap B)(x) = A(x) \wedge B(x);$$
$$(A \cup B)(x) = A(x) \vee B(x);$$
$$T(A, B)(x) = T\big(A(x), B(x)\big);$$
$$S(A, B)(x) = S\big(A(x), B(x)\big).$$

These operations on fuzzy sets mentioned above are also called *ordinary fuzzy operations* in the sequel. The order relation on fuzzy sets is defined as $A \subseteq B \Leftrightarrow A(x) \leqslant B(x)$ for all $x \in U$.

Liu proposed inner product and outer product operations to discuss the axiomatic characterizations of fuzzy rough approximation operators (Liu 2008, 2013). Based on it, Wu et al. (2016) further proposed fuzzy product operations as follows:

Definition 4 (Wu et al. 2016) Let T and S be a left-continuous t-norm and a right-continuous t-conorm, respectively. Then the *sup T-product operation* $\big(A, B\big)_T$ and the inf *S-product operation* $\big[A, B\big]_S$ are defined for all $A, B \in \mathcal{F}(U)$ as

$$\big(A, B\big)_T = \bigvee_{x \in U} T\big(A(x), B(x)\big), \text{ and}$$
$$\big[A, B\big]_S = \bigwedge_{x \in U} S\big(A(x), B(x)\big), \text{ respectively.}$$

For simplicity, the sup T-product operation and inf S-product operation are called *fuzzy product operations* in this paper. In particular, the sup T-product operation and the inf S-product operation also can be viewed as the compositions of fuzzy sets. Although Wu et al. (2016) proposed fuzzy product operations with continuous t-(co)norms, the following properties still hold, when we consider a left-continuous t-norm and a right-continuous t-conorm.

Proposition 5 (Wu et al. 2016) *Let t-norm T be left-continuous and t-conorm S be right-continuous. Then the following hold for all $A, B, C \in \mathcal{F}(U)$.*

(1) $\big(A, B\big)_T = \big(B, A\big)_T$ *and* $\big[A, B\big]_S = \big[B, A\big]_S$*;*
(2) $\big(\widehat{0}, B\big)_T = 0$ *and* $\big[\widehat{1}, B\big]_S = 1$*;*
(3) $A \subseteq B$ *implies* $\big(A, C\big)_T \leqslant \big(B, C\big)_T$ *and* $\big[A, C\big]_S \leqslant \big[B, C\big]_S$ *for all* $C \in \mathcal{F}(U)$*;*
(4) $\big(A, D\big)_T \leqslant \big(B, D\big)_T$ *for all* $D \in \mathcal{F}(U)$ *implies* $A \subseteq B$*;*
(5) $\big[A, D\big]_S \leqslant \big[B, D\big]_S$ *for all* $D \in \mathcal{F}(U)$ *implies* $A \subseteq B$*;*
(6) $\big(A \cup B, C\big)_T = \big(A, C\big)_T \vee \big(B, C\big)_T$ *and* $\big[A \cap B, C\big]_S = \big[A, C\big]_S \wedge \big[B, C\big]_S$.

Moreover, we obtain the following conclusion.

Proposition 6 *Let t-norm T be left-continuous and t-conorm S be right-continuous. Then the following hold.*

(1) *If N_T is strong, then the following holds for all $A, B \in \mathcal{F}(U)$,*

$$A \subseteq B \text{ iff } \big(A, N_T(B)\big)_T = 0.$$

(2) *If N_S is strong, then the following holds for all $A, B \in \mathcal{F}(U)$,*

$$A \subseteq B \text{ iff } \big[B, N_S(A)\big]_S = 1.$$

Proof (1) If N_T is strong, then it follows from Proposition 3(1) and Definition 4 that the following holds for all $A, B \in \mathcal{F}(U)$,

$$A \subseteq B \Leftrightarrow T\big(A(x), N_T(B(x))\big) = 0 \text{ for all } x \in U$$
$$\Leftrightarrow \big(A, N_T(B)\big)_T = 0.$$

(2) If N_S is strong, then it follows from Proposition 3(2) and Definition 4 that the following holds for all $A, B \in \mathcal{F}(U)$,

$$A \subseteq B \Leftrightarrow S\big(B(x), N_S(A(x))\big) = 1 \text{ for all } x \in U$$
$$\Leftrightarrow \big[B, N_S(A)\big]_S = 1.$$

□

A fuzzy set $R \in \mathcal{F}(U \times W)$ is called a fuzzy relation from U to W. If $\bigvee_{y\in W} R(x, y) = 1$ holds for all $x \in U$, then fuzzy relation R is said to be *serial*. If $U = W$, then R is said to be a fuzzy relation on U. For every fuzzy relation R on U, a fuzzy relation R^{-1} is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in U$.

Definition 7 Let R be a fuzzy relation on U. Then R is said to be

(1) *reflexive* if $R(x, x) = 1$ for all $x \in U$;
(2) *symmetric* if $R(x, y) = R(y, x)$ for all $x, y \in U$;
(3) T-*transitive* if $T\big(R(x, y), R(y, z)\big) \leqslant R(x, z)$ for all $x, y, z \in U$;
(4) T-*Euclidean* if $T\big(R(y, x), R(y, z)\big) \leqslant R(x, z)$ for all $x, y, z \in U$.

A fuzzy relation R is called a *fuzzy tolerance* if it is reflexive and symmetric, and a fuzzy T-*preorder* if it is reflexive and T-transitive. Moreover, if a fuzzy relation R is reflexive, symmetric and T-transitive, then R is called a fuzzy T-*similarity relation*.

Let R be a fuzzy relation from U to W. Then the triple (U, W, R) is called a *fuzzy approximation space*. When $U = W$ and R is a fuzzy relation on U, the pair (U, R) is also called a fuzzy approximation space.

Definition 8 Let (U, W, R) be a fuzzy approximation space and fuzzy negation N be strict. Then the following mappings $\underline{R}, \overline{R} : \mathcal{F}(W) \to \mathcal{F}(U)$ are defined as for all $A \in \mathcal{F}(W)$ and $x \in U$,

$$\underline{R}(A)(x) = \bigwedge_{y\in W} S\big(N(R(x, y)), A(y)\big),$$
$$\overline{R}(A)(x) = \bigvee_{y\in W} T\big(R(x, y), A(y)\big).$$

The mappings $\underline{R}$ and $\overline{R}$ are called S-*lower* and T-*upper fuzzy rough approximation operators* of (U, W, R), respectively. The pair $\big(\underline{R}(A), \overline{R}(A)\big)$ is called an (S, T)-*fuzzy rough set* of A w.r.t. (U, W, R). For simplicity, the mappings $\underline{R}$ and $\overline{R}$ are called the (S, T)-fuzzy rough approximation operators.

The properties of lower fuzzy rough approximation operators determined by continuous t-conorms were studied with the dual properties of t-norm and t-conorm w.r.t. the standard negation (Li and Cui 2015; Mi et al. 2008; Wu et al. 2016). However, fuzzy negation N in Definition 8 is not necessarily strong. In fact, we cannot define how a left-continuous t-norm T and a right-continuous S are dual w.r.t. a strict fuzzy negation N. It is obvious that an S-lower fuzzy rough approximation operator is a special lower fuzzy rough approximation operator determined by an S-implication (Radzikowska and Kerre 2002; Wang 2018). Meanwhile, Radzikowska and Kerre (2002) applied the law of importation ((LI), for short) between a fuzzy implication I and a t-norm T (Türkşen et al. 1998) as

(LI) $I\big(a, I(b, c)\big) = I\big(T(a, b), c\big)$ for all $a, b, c \in [0, 1]$.

While interpreting a fuzzy implication I as an S-implication, that is, $I(a, b) = S\big(N(a), b\big)$ for all $a, b \in [0, 1]$ in (LI), we have the following law of importation for S-implication ((LIS), for short) among a t-conorm S, a t-norm T and a fuzzy negation N as

(LIS) $S\big(N(a), N(b)\big) = N\big(T(a, b)\big)$ for all $a, b \in [0, 1]$.

If (LIS) holds for S, T and N, then we say that S satisfies (LIS) for T w.r.t. N. Obviously, (LIS) condition is a direct generalization of dual properties of t-norm and t-conorm w.r.t. a strong fuzzy negation. Therefore, an (I, T)-fuzzy rough set in Radzikowska and Kerre (2002) is equal to an (S, T)-fuzzy rough set, when fuzzy implication I is an S-implication based on a right-continuous t-conorm S and a strict fuzzy negation N. Meanwhile, (S, T)-fuzzy rough sets have fuzzy rough sets determined by t-(co)norms (Li and Cui 2015; Mi et al. 2008; Wu et al. 2016) as special cases, where t-(co)norms are assumed to be continuous and fuzzy negation is assumed to be strong or the standard negation.

We present some basic properties of (S, T)-fuzzy rough approximation operators (Du et al. 2013; Mi et al. 2008; Radzikowska and Kerre 2002; Wang 2018) as follows.

Proposition 9 *Let (U, W, R) be a fuzzy approximation space. Then the following hold.*

(1) $\underline{R}\big(\bigcap_{j\in J} A_j\big) = \bigcap_{j\in J} \underline{R}(A_j)$ *and* $\overline{R}\big(\bigcup_{j\in J} A_j\big) = \bigcup_{j\in J} \overline{R}(A_j)$ *for all* $\{A_j\}_{j\in J} \subseteq \mathcal{F}(W)$.
(2) $\underline{R}\big(1_{W-\{y\}}\big)(x) = N\big(R(x, y)\big)$ *and* $\overline{R}\big(1_y\big)(x) = R(x, y)$ *for all* $x \in U$ *and* $y \in W$.

(3) *R is serial iff* $\underline{R}(\widehat{0}) = \widehat{0}$ *iff* $\overline{R}(\widehat{1}) = \widehat{1}$.
(4) *If R is a fuzzy relation on U, then R is reflexive iff* $\underline{R}(A) \subseteq A$ *for all* $A \in \mathcal{F}(U)$ *iff* $A \subseteq \overline{R}(A)$ *for all* $A \in \mathcal{F}(U)$.
(5) *If R is a T-transitive fuzzy relation on U and S satisfies (LIS) for T w.r.t. N, then* $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$ *for all* $A \in \mathcal{F}(U)$.
(6) *If R is a fuzzy relation on U, then R is T-transitive iff* $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$ *for all* $A \in \mathcal{F}(U)$.
(7) *If R is a T-Euclidean fuzzy relation on U and S satisfies (LIS) for T w.r.t. N, then* $\underline{R}(A) \subseteq \underline{R^{-1}}(\underline{R}(A))$ *for all* $A \in \mathcal{F}(U)$.
(8) *If R is a fuzzy relation on U, then R is T-Euclidean iff* $\overline{R^{-1}}(\overline{R}(A)) \subseteq \overline{R}(A)$ *for all* $A \in \mathcal{F}(U)$.

3 Axiomatic characterizations of T-upper fuzzy rough approximation operators

Wu et al. (2016) studied single axioms for T-upper fuzzy rough approximation operators in two approaches, which characterize T-upper fuzzy rough approximation operators with ordinary fuzzy operations and fuzzy product operations, respectively. In fact, it is complex and tedious to characterize T-upper fuzzy rough approximation operators with ordinary fuzzy operations. Meanwhile, the axiomatic characterizations of T-upper fuzzy rough approximation operators with fuzzy product operations are limited into general fuzzy relations or symmetric ones in Wu et al. (2016). Although Bao et al. (2018) discussed single axioms for T-upper fuzzy rough approximation operators with fuzzy product operations from the perspective of regular residuated lattices, the fuzzy relations in the axiomatic characterizations of T-upper fuzzy rough approximation operators are still assumed to be either general or symmetric. Wang et al. (2020) further discussed the single axioms for upper L-fuzzy rough approximation operators with L-fuzzy product operation on other types of L-fuzzy relations. However, most of those axioms in Wang et al. (2020) still require the regularity of residuated lattices. In this section, we further characterize T-upper fuzzy rough approximation operators by only one axiom with fuzzy product operations, when fuzzy relations are special types of fuzzy relations, such as serial, reflexive, T-transitive and T-Euclidean ones as well as any of their compositions. First of all, we recall the auxiliary mapping to study the axiomatic characterizations of T-upper fuzzy rough approximation operators as follows:

Definition 10 (Wu et al. 2016) Let $H : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator. Then the mapping $H^{-1} : \mathcal{F}(U) \to \mathcal{F}(W)$ is defined as for all $B \in \mathcal{F}(U)$ and $y \in W$,

$$H^{-1}(B)(y) = \bigvee_{x \in U} T\Big(H(1_y)(x), B(x)\Big).$$

The property of the auxiliary mapping H^{-1} is recalled as follows:

Lemma 11 (Wu et al. 2016) *Let* $H : \mathcal{F}(W) \to \mathcal{F}(U)$ *be a fuzzy operator,* $R \in \mathcal{F}(U \times W)$ *and* $H = \overline{R}$. *Then* $H^{-1} = \overline{R^{-1}}$ *holds.*

Proposition 12 *Let* $H : \mathcal{F}(W) \to \mathcal{F}(U)$ *be a fuzzy operator. Then there exists a unique fuzzy relation* $R \in \mathcal{F}(U \times W)$ *such that* $H = \overline{R}$ *iff H satisfies for all* $A \in \mathcal{F}(U)$ *and* $B \in \mathcal{F}(W)$,

$$\big(A, H(B)\big)_T = \big(H^{-1}(A), B\big)_T. \tag{1}$$

Proof It follows from Theorem 8 in Wu et al. (2016) that there exists a fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $H = \overline{R}$ iff H satisfies Eq. (1). The uniqueness of fuzzy relation R is verified as follows.

Let $Q \in \mathcal{F}(U \times W)$ and $\overline{Q} = H$. Then it follows from Proposition 9(2) that the following holds for all $x \in U$ and $y \in W$,

$$\begin{aligned} R(x, y) &= \overline{R}(1_y)(x) = H(1_y)(x) \\ &= \overline{Q}(1_y)(x) = Q(x, y). \end{aligned}$$

Hence we have $R(x, y) = Q(x, y)$ for all $x \in U$ and $y \in W$. □

Proposition 13 *Let* $H : \mathcal{F}(U) \to \mathcal{F}(U)$ *be a fuzzy operator. Then there exists a unique fuzzy relation R on U such that R is symmetric and* $H = \overline{R}$ *iff H satisfies for all* $A, B \in \mathcal{F}(U)$,

$$\big(A, H(B)\big)_T = \big(H(A), B\big)_T. \tag{2}$$

Proof It follows immediately from Theorem 19 in Wu et al. (2016) and Proposition 12. □

Remark 14 By Proposition 12, we immediately obtain the uniqueness of fuzzy relations in the single axioms for T-upper fuzzy rough approximation operators with fuzzy product operations in Wu et al. (2016) (see Theorems 30, 40 and 36 in Wu et al. (2016)), when fuzzy relations are a fuzzy tolerance, a fuzzy T-similarity relation and a composition of a symmetric fuzzy relation and a T-transitive one, respectively. Here, we do not list them again.

As there are many types of t-norms on the unit interval, we apply a left-continuous t-norm $\mathscr{T}$ with a strong natural negation $N_{\mathscr{T}}$ to study single axioms for T-upper fuzzy rough approximation operators with fuzzy product operations, when fuzzy relations are serial, T-transitive and T-Euclidean ones as well as their compositions. Notice that t-norm $\mathscr{T}$ may have nothing to do with T-upper fuzzy rough approximation operator.

Proposition 15 *Let $H : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{T}$ be a left-continuous t-norm with a strong natural negation $N_{\mathscr{T}}$. Then there exists a unique fuzzy relation $R \in \mathcal{F}(U \times W)$ such that R is serial and $H = \overline{R}$ iff H satisfies for all $A \in \mathcal{F}(U)$ and $B \in \mathcal{F}(W)$,*

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(H^{-1}(A), B\big)_T. \quad (3)$$

Proof **Necessity.** Assume that there exists a serial fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $H = \overline{R}$, then we have $\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} = 0$ by Propositions 6(1) and 9(3). It follows from Proposition 12 that H satisfies Eq. (3).

Sufficiency. Consider H satisfy Eq. (3) and $A = B = \widehat{0}$ in Eq. (3). Then by Proposition 5(2), we have

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee 0 = 0.$$

Hence H satisfies Eq. (1). It follows from Proposition 12 that there exists a unique fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $H = \overline{R}$. Meanwhile, by Proposition 6(1), we have $H\big(\widehat{1}\big) = \widehat{1}$. By Proposition 9(3), R is serial. □

In the following proposition, we characterize T-upper fuzzy rough approximation operators with fuzzy product operations on a reflexive fuzzy relation without applying a left-continuous t-norm $\mathscr{T}$.

Proposition 16 *Let $H : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator. Then there exists a unique fuzzy relation R on U such that R is reflexive and $H = \overline{R}$ iff H satisfies for all $A, B \in \mathcal{F}(U)$,*

$$\big(A, H(B)\big)_T = \big(A \cup H^{-1}(A), B\big)_T. \quad (4)$$

Proof **Necessity.** Assume that there exists a reflexive fuzzy relation R on U such that $H = \overline{R}$, then by Proposition 9(4) and Lemma 11, we have for all $A \in \mathcal{F}(U)$,

$$A \cup H^{-1}(A) = A \cup \overline{R^{-1}}(A) = \overline{R^{-1}}(A) = H^{-1}(A).$$

It follows from Proposition 12 that H satisfies Eq. (4).

Sufficiency. Consider H satisfy Eq. (4). Then it follows from Proposition 5(6) that the following holds for all $A, B \in \mathcal{F}(U)$,

$$\begin{aligned}\big(A, H(B)\big)_T &= \big(A \cup H^{-1}(A), B\big)_T \\ &= \big(A, B\big)_T \vee \big(H^{-1}(A), B\big)_T.\end{aligned}$$

Hence we obtain $\big(A, B\big)_T \leqslant \big(A, H(B)\big)_T$ for all $A, B \in \mathcal{F}(U)$. Thus it follows Proposition 5(4) that $B \subseteq H(B)$ for all $B \in \mathcal{F}(U)$. For arbitrary $y \in U$, let $B = 1_y$. Then it follows from Definition 10 that the following holds for all $y \in U$ and $A \in \mathcal{F}(U)$,

$$\begin{aligned}H^{-1}(A)(y) &= \bigvee_{x \in U} T\Big(H\big(1_y\big)(x), A(x)\Big) \\ &\geqslant \bigvee_{x \in U} T\Big(\big(1_y\big)(x), A(x)\Big) = A(y).\end{aligned}$$

We obtain $A \subseteq H^{-1}(A)$ for all $A \in \mathcal{F}(U)$. Moreover, H satisfies Eq. (1). By Proposition 12, there exists a unique fuzzy relation R on U such that $H = \overline{R}$. As $B \subseteq H(B)$ holds for all $B \in \mathcal{F}(U)$, R is reflexive by Proposition 9(4). □

Proposition 17 *Let $H : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{T}$ be a left-continuous t-norm with a strong natural negation $N_{\mathscr{T}}$. Then there exists a unique fuzzy relation R on U such that R is T-transitive and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\begin{aligned}&\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T \\ &\quad = \big(H^{-1}(A), B\big)_T.\end{aligned} \quad (5)$$

Proof **Necessity.** Assume that there exists a T-transitive fuzzy relation R on U such that $H = \overline{R}$, then it follows from Propositions 6(1) and 9(6) that the following holds for all $C \in \mathcal{F}(U)$,

$$\begin{aligned}\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} &= \big(\overline{R}\big(\overline{R}(C)\big), N_{\mathscr{T}}\big(\overline{R}(C)\big)\big)_{\mathscr{T}} \\ &= 0.\end{aligned}$$

By Proposition 12, H satisfies Eq. (5).

Sufficiency. Consider H satisfy Eq. (5) and $A = B = \widehat{0}$ in Eq. (5). Then it follows from Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee 0 = 0.$$

It follows from Proposition 6(1) that $H\big(H(C)\big) \subseteq H(C)$ holds for all $C \in \mathcal{F}(U)$. Hence Eq. (5) turns out to be Eq. (1). By Proposition 12, there exists a unique fuzzy relation R on U such that $H = \overline{R}$. Meanwhile, it follows from Proposition 9(6) that R is T-transitive. □

Proposition 18 *Let $H : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{T}$ be a left-continuous t-norm with a strong natural negation $N_{\mathscr{T}}$. Then there exists a unique fuzzy relation R on U such that R is T-Euclidean and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\begin{aligned}&\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T \\ &\quad = \big(H^{-1}(A), B\big)_T.\end{aligned} \quad (6)$$

Proof **Necessity.** Assume that there exists a T-Euclidean fuzzy relation R on U such that $H = \overline{R}$, then it follows from Propositions 6(1), 9(8) and Lemma 11 that we obtain for all $C \in \mathcal{F}(U)$,

$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0.$$

We have that H satisfies Eq. (6) by Proposition 12.

Sufficiency. Consider H satisfy Eq. (6) and $A = B = \widehat{0}$ in Eq. (6). Then by Proposition 5(2), we obtain for all $C \in \mathcal{F}(U)$,

$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0.$$

It follows from Proposition 6(1) that $H^{-1}\big(H(C)\big) \subseteq H(C)$ holds for all $C \in \mathcal{F}(U)$. Hence L satisfies Eq. (1). It follows from Proposition 12 that there exists a unique fuzzy relation R on U such that $H = \overline{R}$. Meanwhile, it follows from Proposition 9(8) that R is T-Euclidean. □

When a fuzzy relation is a composition of a serial, reflexive, T-transitive and T-Euclidean fuzzy relation, we obtain the following axiomatic characterizations of T-upper fuzzy rough approximation operators with fuzzy product operations.

Proposition 19 *Let $H : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{T}$ be a left-continuous t-norm with a strong natural negation $N_{\mathscr{T}}$. Then there exists a unique fuzzy relation R on U such that R is serial, symmetric and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(H(A), B\big)_T. \qquad (7)$$

Proof **Necessity.** It follows immediately from Lemma 11 and Proposition 15.

Sufficiency. Consider L satisfy Eq. (7) and $A = B = \widehat{0}$ in Eq. (7). Then it follows from Proposition 5(2) that $\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} = 0$ holds. Hence L satisfies Eq. (2). By Proposition 13, there exists a unique fuzzy relation R on U such that R is symmetric and $H = \overline{R}$. Moreover, it follows from Propositions 6(1) and 9(3) that R is serial. □

Proposition 20 *Let $H : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{T}$ be a left-continuous t-norm with a strong natural negation $N_{\mathscr{T}}$. Then*

(1) *There exists a unique fuzzy relation R on U such that R is serial, T-transitive and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee \big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(H^{-1}(A), B\big)_T. \qquad (8)$$

(2) *There exists a unique fuzzy relation R on U such that R is serial, T-Euclidean and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee \big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(H^{-1}(A), B\big)_T. \qquad (9)$$

(3) *There exists a unique fuzzy relation R on U such that R is a fuzzy T-preorder and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(A \cup H^{-1}(A), B\big)_T. \qquad (10)$$

(4) *There exists a unique fuzzy relation R on U such that R is reflexive, T-Euclidean and $H = \overline{R}$ iff H satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee \big(A, H(B)\big)_T = \big(A \cup H^{-1}(A), B\big)_T. \qquad (11)$$

Proof (1) **Necessity.** It follows immediately from Propositions 6(1), 9(3) and 17.

Sufficiency. Consider H satisfy Eq. (8) and $A = B = \widehat{0}$ in Eq. (8). Then it follows from Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} \vee \big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee 0 = 0.$$

Hence, the following hold

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} = 0 \text{ and}$$
$$\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0 \text{ for all } C \in \mathcal{F}(U).$$

Thus H satisfies Eqs. (3) and (5). It follows from Propositions 15 and 17 that there exists a unique fuzzy relation R on U such that R is serial, T-transitive and $H = \overline{R}$.

(2) **Necessity.** It follows immediately from Propositions 6(1), 9(3) and 18.

Sufficiency. Let H satisfy Eq. (9) and $A = B = \widehat{0}$ in Eq. (9). Then by Proposition 5(2), we obtain

$$\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} = 0 \text{ and}$$
$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0 \text{ for all } C \in \mathcal{F}(U).$$

Thus H satisfies Eq. (6) by $\big(\widehat{1}, N_{\mathscr{T}}\big(H\big(\widehat{1}\big)\big)\big)_{\mathscr{T}} = 0$. It follows from Proposition 18 that there exists a unique fuzzy relation R on U such that R is T-Euclidean and $H = \overline{R}$. Moreover, the seriality of fuzzy relation R can be proven in a similar way as for Proposition 15.

(3) **Necessity.** It follows immediately from Propositions 6(1), 9(6) and 16.

Sufficiency. Consider H satisfy Eq. (10) and $A = B = \widehat{0}$ in Eq. (10). Then it follows from Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} \vee 0 = 0.$$

Hence $\big(H\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0$ holds for all $C \in \mathcal{F}(U)$. Thus H satisfies Eq. (4). By Proposition 16, there exists a unique fuzzy relation R on U such that R is reflexive and $H = \overline{R}$. Moreover, it can be proven in a similar way as for Proposition 17 that R is T-transitive.

(4) **Necessity.** It follows immediately from Propositions 9(4), 18 and Lemma 11.

Sufficiency. Let H satisfy Eq. (11) and $A = B = \widehat{0}$ in Eq. (11). Then it follows from 5(2) that we have for all $C \in \mathcal{F}(U)$

$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0.$$

Thus Eq. (11) turns out to be Eq. (4). It follows from Proposition 16 that there exists a unique fuzzy relation R on U such that R is reflexive and $H = \overline{R}$. Moreover, it follows from Propositions 6(1) and 9(8) that R is T-Euclidean by

$$\big(H^{-1}\big(H(C)\big), N_{\mathscr{T}}\big(H(C)\big)\big)_{\mathscr{T}} = 0$$

for all $C \in \mathcal{F}(U)$. □

Remark 21 In this section, we further study single axioms for T-upper fuzzy rough approximation operators with fuzzy product operations, when fuzzy relations are serial, reflexive, T-transitive and T-Euclidean as well as their compositions. We apply a left-continuous t-norm $\mathscr{T}$ with a strong natural negation $N_{\mathscr{T}}$ in the axiomatic characterizations of T-upper fuzzy rough approximation operators. As there are so many t-norms on the unit interval, it is not necessary that t-norms in sup $\mathscr{T}$-product operation and T-upper fuzzy rough approximation operator must be the same. This section can be viewed as the completion of single axioms for T-upper fuzzy rough approximation operators in Bao et al. (2018), Wang et al. (2020) and Wu et al. (2016), which provides much briefer axiomatic characterizations than the single axioms for T-upper fuzzy rough approximation operators with ordinary fuzzy operations in Wu et al. (2016). For readers' convenience, we summarize and compare the axiomatic characterizations of T-upper fuzzy rough approximation operators with fuzzy product operations in this section and Wu et al. (2016) in Table 1 (see "Appendix"). As we apply a left-continuous t-norm $\mathscr{T}$ with a strong natural negation $N_{\mathscr{T}}$ in some of our conclusions, we abbreviate that condition as "t-norm $\mathscr{T}$."

4 Axiomatic characterizations of *S*-lower fuzzy rough approximation operators

Wu et al. (2016) characterized S-lower fuzzy rough approximation operators by only one axiom with fuzzy product operations, when t-conorm S is continuous and fuzzy negation N is strong. Moreover, an S-lower fuzzy rough approximation operator is a special lower fuzzy rough approximation operator in an (I, T)-fuzzy rough set (Radzikowska and Kerre 2002; Wang 2018), when I is an S-implication based on a right-continuous t-conorm and a strict fuzzy negation. Wang (2018) further studied axiomatic characterizations of S-lower fuzzy rough approximation operators, while S is a right-continuous t-conorm and fuzzy negation N is either continuous or strict. Meanwhile, Bao et al. (2018) discussed single axioms for S-lower fuzzy rough approximation operators from the perspective of regular residuated lattices and dual property. However, all the axioms with fuzzy product operations (Bao et al. 2018; Wang 2018; Wu et al. 2016) only hold on either a general fuzzy relation or a symmetric fuzzy relation. Although Wang et al. (2020) further studied the axiomatic characterizations of S-lower fuzzy rough approximation operators on all kinds of L-fuzzy relations from the perspective of residuated lattices, the regularity of residuated lattice is still indispensable. Therefore, we investigate single axioms for S-lower fuzzy rough approximation operators on other types of fuzzy relations, when t-conorm S is right-continuous and fuzzy negation N is strict. Wu et al. (2016) also proposed the following mapping to characterize S-lower fuzzy rough approximation operators.

Definition 22 (Wu et al. 2016) Let $L : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator. Then the mapping $L^{-1} : \mathcal{F}(U) \to \mathcal{F}(W)$ is defined as for all $B \in \mathcal{F}(U)$ and $y \in W$,

$$L^{-1}(B)(y) = \bigwedge_{x \in U} S\Big(L\big(1_{W-\{y\}}\big)(x), B(x)\Big).$$

The property of the mapping L^{-1} is presented as follows.

Lemma 23 (Wu et al. 2016) *Let $L : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator, $R \in \mathcal{F}(U \times W)$ and $L = \underline{R}$. Then $L^{-1} = \underline{R^{-1}}$ holds.*

Wu et al. (2016) studied single axioms for S-lower fuzzy rough approximation operators, where S is a continuous t-conorm and N is a strong fuzzy negation. However, Wu et al. (2016) did not verify the uniqueness of fuzzy relation in the axiomatic characterizations of S-lower fuzzy rough approximation operators. From the perspective of lower fuzzy rough approximation operators determined by fuzzy implications, Wang (2018) further obtained the following axiomatic characterizations of S-lower fuzzy rough approximation operators with fuzzy product operations.

Proposition 24 (Wang 2018; Wu et al. 2016) *Let $L : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator. Then there exists a unique fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $L = \underline{R}$ iff L satisfies for all $A \in \mathcal{F}(U)$ and $B \in \mathcal{F}(W)$,*

$$\left[A, L(B)\right]_S = \left[L^{-1}(A), B\right]_S. \tag{12}$$

Proposition 25 (Wang 2018; Wu et al. 2016) *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator. Then there exists a unique fuzzy relation R on U such that R is symmetric and $L = \underline{R}$ iff L satisfies for all $A, B \in \mathcal{F}(U)$,*

$$\left[A, L(B)\right]_S = \left[L(A), B\right]_S. \tag{13}$$

Remark 26 As the S-lower fuzzy rough approximation operator in Definition 8 is defined with a right-continuous t-conorm S and a strict fuzzy negation N, we cannot directly obtain the single axioms for S-lower fuzzy rough approximation operators by the dual property between t-norm T and t-conorm S w.r.t. a strict fuzzy negation N. Therefore, we give the detailed proofs for these single axioms in the sequel.

Similarly to the axiomatic characterizations of T-upper fuzzy rough approximation operators, we apply a right-continuous t-conorm $\mathscr{S}$ with a natural negation $N_{\mathscr{S}}$ to study the single axioms for S-lower fuzzy rough approximation operators with fuzzy product operations.

Proposition 27 *Let $L : \mathcal{F}(W) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{S}$ be a right-continuous t-conorm with a strong natural negation $N_{\mathscr{S}}$. Then there exists a unique fuzzy relation $R \in \mathcal{F}(U \times W)$ such that R is serial and $L = \underline{R}$ iff L satisfies for all $A \in \mathcal{F}(U)$ and $B \in \mathcal{F}(W)$,*

$$\left[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\right]_{\mathscr{S}} \wedge \left[A, L(B)\right]_S = \left[L^{-1}(A), B\right]_S. \tag{14}$$

Proof **Necessity.** If there exists a serial fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $L = \underline{R}$, then we have $\left[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\right]_{\mathscr{S}} = 1$ by Propositions 6(2) and 9(3). Moreover, it follows from Proposition 24 that L satisfies Eq. (14).

Sufficiency. Let L satisfy Eq. (14) and $A = B = \widehat{1}$ in Eq. (14). Then it follows from Proposition 5(2) that the following holds

$$\left[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\right]_{\mathscr{S}} \wedge 1 = 1.$$

Hence Eq. (14) turns out to be Eq. (12). By Proposition 24, there exists a unique fuzzy relation $R \in \mathcal{F}(U \times W)$ such that $L = \underline{R}$. Moreover, it follows from Proposition 6(2) that $L(\widehat{0}) \subseteq \widehat{0}$ holds. Thus we have $L(\widehat{0}) = \widehat{0}$. It follows from Proposition 9(3) that R is serial. □

Considering a reflexive fuzzy relation, the single axioms for S-lower fuzzy rough approximation operators are obtained without applying a right-continuous t-conorm $\mathscr{S}$.

Proposition 28 *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator. Then there exists a unique fuzzy relation R on U such that R is reflexive and $L = \underline{R}$ iff L satisfies for all $A, B \in \mathcal{F}(U)$,*

$$\left[A, L(B)\right]_S = \left[A \cap L^{-1}(A), B\right]_S. \tag{15}$$

Proof **Necessity.** If there exists a reflexive fuzzy relation R on U such that $L = \underline{R}$, then it follows from Proposition 9(4) and Lemma 23 that the following holds for all $A \in \mathcal{F}(U)$,

$$A \cap L^{-1}(A) = A \cap \underline{R^{-1}}(A) = \underline{R^{-1}}(A) = L^{-1}(A).$$

By Proposition 24, L satisfies Eq. (15).

Sufficiency. Let L satisfy Eq. (15). Then it follows from Proposition 5(6) that for all $A, B \in \mathcal{F}(U)$,

$$\begin{aligned}\left[A, L(B)\right]_S &= \left[A \cap L^{-1}(A), B\right]_S \\ &= \left[A, B\right]_S \wedge \left[L^{-1}(A), B\right]_S.\end{aligned}$$

Hence we obtain for all $A, B \in \mathcal{F}(U)$,

$$\left[A, L(B)\right]_S \leqslant \left[A, B\right]_S \text{ and}$$
$$\left[A, L(B)\right]_S \leqslant \left[L^{-1}(A), B\right]_S.$$

By Proposition 5(5), we have $L(B) \subseteq B$ for all $B \in \mathcal{F}(U)$. For arbitrary $y \in U$, let $B = 1_{U-\{y\}}$. Then it follows from Definition 22 that the following holds for all $y \in U$ and $A \in \mathcal{F}(U)$,

$$\begin{aligned}L^{-1}(A)(y) &= \bigwedge_{x \in U} S\Big(L\big(1_{U-\{y\}}\big)(x), A(x)\Big) \\ &\leqslant \bigwedge_{x \in U} S\Big(\big(1_{U-\{y\}}\big)(x), A(x)\Big) \\ &= A(y).\end{aligned}$$

Thus $L^{-1}(A) \subseteq A$ holds for all $A \in \mathcal{F}(U)$, which implies Eq. (15) turns out to be Eq. (12). It follows from Proposition 24 that there exists a unique fuzzy relation R on U such that $L = \underline{R}$. As $L(B) \subseteq B$ holds for all $B \in \mathcal{F}(U)$, it follows from Proposition 9(4) that R is reflexive. □

Proposition 29 *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator, $\mathscr{S}$ be a right-continuous t-conorm with a strong natural negation $N_{\mathscr{S}}$ and assume that S satisfies (LIS) for T w.r.t. N. Then there exists a unique fuzzy relation R on U such that R is T-transitive and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\left[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\right]_{\mathscr{S}} \wedge \left[A, L(B)\right]_S = \left[L^{-1}(A), B\right]_S. \tag{16}$$

Proof **Necessity.** If there exists a T-transitive fuzzy relation R on U such that $L = \underline{R}$, then it follows from Propositions 6(2) and 9(5) that we have for all $C \in \mathcal{F}(U)$,

$$\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = \big[\underline{R}\big(\underline{R}(C)\big), N_{\mathscr{S}}\big(\underline{R}(C)\big)\big]_{\mathscr{S}} = 1.$$

By Proposition 24, L satisfies Eq. (16).

Sufficiency. Let L satisfy Eq. (16) and $A = B = \widehat{1}$ in Eq. (16). Then it follows Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge 1 = 1.$$

Hence, we have $\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1$. By Proposition 6(2), $L(C) \subseteq L\big(L(C)\big)$ holds for all $C \in \mathcal{F}(U)$. Thus L satisfies Eq. (12). It follows from Proposition 24 that there exists a unique fuzzy relation R on U such that $L = \underline{R}$. We check the T-transitivity of fuzzy relation R as follows.

Assume that R is not T-transitive, then there exist $x_0, y_0, z_0 \in U$ such that

$$T\big(R(x_0, y_0), R(y_0, z_0)\big) > R(x_0, z_0).$$

Consider $B = U - \{z_0\}$. Then it follows from Proposition 9(2) that $\underline{R}(B)(y) = N\big(R(y, z_0)\big)$ holds for all $y \in U$. Because fuzzy negation N is strict and S satisfies (LIS) for T w.r.t. N, we obtain

$$\begin{aligned}\underline{R}\big(\underline{R}(B)\big)(x_0) &= \bigwedge_{y\in U} S\Big(N\big(R(x_0, y)\big), N\big(R(y, z_0)\big)\Big)\\ &= \bigwedge_{y\in U} N\Big(T\big(R(x_0, y), R(y, z_0)\big)\Big)\\ &\leqslant N\Big(T\big(R(x_0, y_0), R(y_0, z_0)\big)\Big)\\ &< N\big(R(x_0, z_0)\big) = \underline{R}(B)(x_0).\end{aligned}$$

Hence we have a contradiction with the conclusion $\underline{R}(A) \subseteq \underline{R}\big(\underline{R}(A)\big)$ for all $A \in \mathcal{F}(U)$. Therefore, R is T-transitive. □

Proposition 30 *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator, $\mathscr{S}$ be a right-continuous t-conorm with a strong natural negation $N_{\mathscr{S}}$ and assume that S satisfies (LIS) for T w.r.t. N. Then there exists a unique fuzzy relation R on U such that R is T-Euclidean and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[L^{-1}(A), B\big]_S. \tag{17}$$

Proof **Necessity.** If there exists a T-Euclidean fuzzy relation R on U such that $L = \underline{R}$, then we have $\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1$ by Propositions 6(2), 9(7) and Lemma 23. It follows from Proposition 24 that L satisfies Eq. (17).

Sufficiency. Let L satisfy Eq. (17) and $A = B = \widehat{1}$ in Eq. (17). Then by Proposition 5(2), we have $\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1$ for all $C \in \mathcal{F}(U)$. It follows from Proposition 6(2) that $L(C) \subseteq L^{-1}\big(L(C)\big)$ holds for all $C \in \mathcal{F}(U)$. Thus L satisfies Eq. (12). By Proposition 24, there exists a unique fuzzy relation R on U such that $L = \underline{R}$. We verify that R is T-Euclidean as follows.

Assume that R is not T-Euclidean, then there are $x_0, z_0 \in U$ such that

$$\bigvee_{y\in U} T\big(R(y, x_0), R(y, z_0)\big) > R(x_0, z_0).$$

Let $B = U - \{z_0\}$. Then we have $\underline{R}(B)(y) = N\big(R(y, z_0)\big)$ for all $y \in U$ by Proposition 9(2). Hence we obtain

$$\begin{aligned}\underline{R^{-1}}\big(\underline{R}(B)\big)(x_0) &= \bigwedge_{y\in U} S\Big(N\big(R(y, x_0)\big), N\big(R(y, z_0)\big)\Big)\\ &= \bigwedge_{y\in U} N\Big(T\big(R(y, x_0), R(y, z_0)\big)\Big)\\ &= N\Big(\bigvee_{y\in U} T\big(R(y, x_0), R(y, z_0)\big)\Big)\\ &< N\big(R(x_0, z_0)\big) = \underline{R}(B)(x_0),\end{aligned}$$

which implies a contradiction with $L(C) \subseteq L^{-1}\big(L(C)\big)$ for all $C \in \mathcal{F}(U)$. Thus fuzzy relation R is T-Euclidean. □

Considering the compositions of serial, reflexive, T-transitive and T-Euclidean, we obtain the following conclusions.

Proposition 31 *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator and $\mathscr{S}$ be a right-continuous t-conorm with a strong natural negation $N_{\mathscr{S}}$. Then there exists a unique fuzzy relation R on U such that R is serial, symmetric and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[L(A), B\big]_S. \tag{18}$$

Proof **Necessity.** It follows immediately from Lemma 23 and Proposition 27.

Sufficiency. Let L satisfy Eq. (18) and $A = B = \widehat{1}$ in Eq. (18). Then we obtain $\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} = 1$ by Proposition 5(2). Hence Eq. (18) turns out to be Eq. (13). It follows from Proposition 25 that there exists a unique fuzzy relation R on U such that R is symmetric and $L = \underline{R}$. As $\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} = 1$ holds, it follows from Propositions 6(2) and 9(3) that R is serial. □

Proposition 32 *Let $L : \mathcal{F}(U) \to \mathcal{F}(U)$ be a fuzzy operator, $\mathscr{S}$ be a right-continuous t-conorm with a strong natural* *negation $N_{\mathscr{S}}$ and assume that S satisfies (LIS) for T w.r.t. N. Then*

(1) *There exists a unique fuzzy relation R on U such that R is serial, T-transitive and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} \wedge \big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[L^{-1}(A), B\big]_S. \tag{19}$$

(2) *There exists a unique fuzzy relation R on U such that R is serial, T-Euclidean and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} \wedge \big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[L^{-1}(A), B\big]_S. \tag{20}$$

(3) *There exists a unique fuzzy relation R on U such that R is a fuzzy T-preorder and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[A \cap L^{-1}(A), B\big]_S. \tag{21}$$

(4) *There exists a unique fuzzy relation R on U such that R is reflexive, T-Euclidean and $L = \underline{R}$ iff L satisfies for all $A, B, C \in \mathcal{F}(U)$,*

$$\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge \big[A, L(B)\big]_S = \big[A \cap L^{-1}(A), B\big]_S. \tag{22}$$

Proof (1) **Necessity.** It follows immediately from Propositions 6(2), 9(3) and 29.

Sufficiency. Let L satisfy Eq. (19) and $A = B = \widehat{1}$ in Eq. (19). Then it follows from Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} \wedge \big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge 1 = 1.$$

Hence, we have

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} = 1 \text{ and}$$
$$\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1 \text{ for all } C \in \mathcal{F}(U).$$

Thus L satisfies Eqs. (14) and (16). By Propositions 27 and 29, there exists a unique fuzzy relation R on U such that R is serial, T-transitive and $L = \underline{R}$

(2) **Necessity.** It follows immediately from Propositions 6(2), 9(3) and 30.

Sufficiency. Consider L satisfy Eq. (20) and $A = B = \widehat{1}$ in Eq. (20). Then it follows from Proposition 5(2) that the following hold

$$\big[\widehat{0}, N_{\mathscr{S}}\big(L(\widehat{0})\big)\big]_{\mathscr{S}} = 1 \text{ and}$$
$$\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1 \text{ for all } C \in \mathcal{F}(U).$$

Thus L satisfies Eq. (14). By Propositions 27, there exists a unique fuzzy relation R on U such that R is serial and $L = \underline{R}$. Moreover, it can be proven in a similar way as for Proposition 30 that R is T-Euclidean.

(3) **Necessity.** It follows immediately from Propositions 6(2), 9(5) and 28.

Sufficiency. Let L satisfy Eq. (21) and $A = B = \widehat{1}$ in Eq. (21). Then it follows from Proposition 5(2) that the following holds for all $C \in \mathcal{F}(U)$,

$$\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} \wedge 1 = 1.$$

Hence we have $\big[L\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1$ for all $C \in \mathcal{F}(U)$. Thus, Eq. (21) turns out to be Eq. (15). It follows from Proposition 28 that there exists a unique fuzzy relation R on U such that R is reflexive and $L = \underline{R}$. Moreover, the T-transitivity of fuzzy relation R can be proven in a similar way as for Proposition 29.

(4) **Necessity.** It follows immediately from Propositions 9(4), 30 and Lemma 23.

Sufficiency. Consider L satisfy Eq. (22) and $A = B = \widehat{1}$ in Eq. (22). Then it follows from Proposition 5(2) that $\big[L^{-1}\big(L(C)\big), N_{\mathscr{S}}\big(L(C)\big)\big]_{\mathscr{S}} = 1$ holds for all $C \in \mathcal{F}(U)$. Hence L satisfies Eq. (15). By Proposition 28, there exists a unique fuzzy relation R on U such that R is reflexive and $L = \underline{R}$. Moreover, Eq. (22) turns out to be Eq. (17). It follows from Proposition 30 that R is T-Euclidean. □

Remark 33 In this section, single axioms for S-lower fuzzy rough approximation operators are further studied with fuzzy product operations. We apply a right-continuous t-conorm with a strong natural negation in the axiomatic characterizations of S-lower fuzzy rough approximation operators. This section provides the completion of single axioms for S-lower fuzzy rough approximation operators (Bao et al. 2018; Wang 2018; Wang et al. 2020; Wu et al. 2016). For readers' convenience, Table 2 summarizes and compares single axioms for S-lower fuzzy rough approximation operators with fuzzy product operations in this section, Wang (2018) and Wu et al. (2016) (see "Appendix"). We abbreviate the condition that $\mathscr{S}$ is a right-continuous t-conorm with a strong natural negation $N_{\mathscr{S}}$ as "t-conorm $\mathscr{S}$." The abbreviation "(LIS)" is used to show the condition that S satisfies (LIS) for T w.r.t. N. Moreover, because Wu et al. (2016) applied the dual properties of continuous t-(co)norms w.r.t. the standard negation in the axiomatic characterizations of S-lower fuzzy rough approximation operator, we abbreviate that condition as "Dual."

5 Conclusions and further work

In this paper, we further study the single axioms for (S, T)-fuzzy rough approximation operators with fuzzy product operations, where fuzzy relations are serial, reflexive, T-transitive and T-Euclidean ones as well as any of their compositions. As there are different t-(co)norms on the interval, we apply a left-continuous t-norm and a right-continuous t-conorm with strong natural negations in the axiomatic characterizations of (S, T)-fuzzy rough approximation operators with fuzzy product operations. This paper can be regarded as the completion of single axioms for (S, T)-fuzzy rough approximation operators (Bao et al. 2018; Wang 2018; Wang et al. 2020; Wu et al. 2016), which provides much briefer axiomatic characterizations than the single axioms for (S, T)-fuzzy rough approximation operators with ordinary fuzzy operations in Wu et al. (2016). Moreover, two tables summarize and compare single axioms for (S, T)-fuzzy rough approximation operators with fuzzy product operations in our paper, Wang (2018) and Wu et al. (2016) (see "Appendix").

As one of the goals for the further work, we intend to further study the single axioms for I-fuzzy rough approximation operators with fuzzy product operations, where I is an arbitrary fuzzy implication. The reasons are explained as follows. If I is an R-implication and the natural negation of R-implication I is strong, then R-implication I turns out to be an S-implication based on a right-continuous t-conorm and a strong fuzzy negation. Otherwise, new fuzzy product operations are required.

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Compliance with ethical standards

Conflict of interest The authors confirm that they do not have conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

Ethical approval This article does not contain any studies with animals performed by any of the authors.

Appendix

See Tables 1 and 2.

Table 1 Single axioms for T-upper fuzzy rough approximation operators

Fuzzy relation	Axiom	Condition
General	Theorem 8 (Wu et al. 2016)	–
Serial	Proposition 15	t-norm $\mathscr{T}$
Reflexive	Proposition 16	–
Symmetric	Theorem 19 (Wu et al. 2016)	–
T-transitive	Proposition 17	t-norm $\mathscr{T}$
T-Euclidean	Proposition 18	t-norm $\mathscr{T}$
Serial & symmetric	Proposition 19	t-norm $\mathscr{T}$
Serial & T-transitive	Proposition 20(1)	t-norm $\mathscr{T}$
Serial & T-Euclidean	Proposition 20(2)	t-norm $\mathscr{T}$
Reflexive & T-Euclidean	Proposition 20(4)	t-norm $\mathscr{T}$
Symmetric & T-transitive	Theorem 36 (Wu et al. 2016)	–
A fuzzy tolerance	Theorem 30 (Wu et al. 2016)	–
A fuzzy T-preorder	Proposition 20(3)	t-norm $\mathscr{T}$
A fuzzy T-similarity	Theorem 40 (Wu et al. 2016)	–

Table 2 Single axioms for S-lower fuzzy rough approximation operators

Fuzzy relation	Axiom	Condition(s)
General	Theorem 7 (Wu et al. 2016) [Proposition 3.16 (Wang 2018)]	–
Serial	Proposition 27	t-conorm $\mathscr{S}$
Reflexive	Proposition 28	–
Symmetric	Theorem 18 (Wu et al. 2016) [Proposition 3.17 (Wang 2018)]	–
T-transitive	Proposition 29	(LIS), t-conorm $\mathscr{S}$
T-Euclidean	Proposition 30	(LIS), t-conorm $\mathscr{S}$
Serial & symmetric	Proposition 31	t-conorm $\mathscr{S}$
Serial & T-transitive	Proposition 32(1)	(LIS), t-conorm $\mathscr{S}$
Serial & T-Euclidean	Proposition 32(2)	(LIS), t-conorm $\mathscr{S}$
Reflexive & T-Euclidean	Proposition 32(4)	(LIS), t-conorm $\mathscr{S}$
Symmetric & T-transitive	Theorem 35 (Wu et al. 2016) [Proposition 3.19(2) (Wang 2018)]	Dual [(LIS)]
A fuzzy tolerance	Theorem 29 (Wu et al. 2016) [Proposition 3.19(1) (Wang 2018)]	–
A fuzzy T-preorder	Proposition 32(3)	(LIS), t-conorm $\mathscr{S}$
A fuzzy T-similarity	Theorem 39 (Wu et al. 2016) [Proposition 3.19(3) (Wang 2018)]	Dual [(LIS)]

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