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METHODOLOGIES AND APPLICATION

Bi-ideal approximation spaces and their applications

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Abstract

The original model of rough sets was advanced by Pawlak, which was mainly involved with the approximation of things using an equivalence relation on the universal set of his approximation space. In this paper, two kinds of approximation operators via ideals which represent extensions of Pawlak's approximation operator have been presented. In both kinds, the definitions of upper and lower approximations based on ideals have been given. Moreover, a new type of approximation spaces via two ideals which is called bi-ideal approximation spaces was introduced for the first time. This type of approximations was analyzed by two different methods, their properties are investigated, and the relationship between these methods is proposed. The importance of these methods was its dependent on ideals which were topological tools, and the two ideals represent two opinions instead of one opinion. At the end of the paper, an applied example had been introduced in the chemistry field by applying the current methods to illustrate the definitions in a friendly way.

Keywords Rough sets · Lower and upper approximations · Ideals

1 Introduction

The theory of rough sets, proposed by Pawlak (1982), is an extension of the set theory for the investigation of intelligent systems identified by insufficient information. The upper and lower approximation operators are introduced by using an equivalence relation on the universe. Using the concepts of the lower and upper approximations from the rough set theory, knowledge hidden in information systems may be fixed and expressed in the form of a decision-making problem (Kryszkiewicz 1998; Ziarko 1993). There are at least two branches for the expansion of the rough set theory, namely the constructive and the axiomatic branches. In the constructive branch, binary relations on the universe, the partitions of the universe, neighborhood systems and Boolean algebras are all the primitive notions. The upper and lower approximation operators are introduced by means of these notions (Chakrabarty et al. 2000; Hosny 2011; Kryszkiewicz 1998; Yao 1998a; Yao and Lin 1996; Yao 2003, 1998b, 1996; Ziarko 1993). The constructive approach is suitable for practical applications of the rough sets. On the other hand, the axiomatic branch, which is appropriate for exploring the structures of rough set algebras, takes the upper and lower approximation operators as primitive notions. In this approach, a set of axioms is used to describe approximation operators that are the same as the ones produced using the constructive approach (Lin and Liu 1994; Yao 1998a). The foundations, as well as a plurality of the modern studies on the rough set theory, are dependent on the constructive branch. The classical rough approximations depended on equivalence relations, but this requirement is not satisfied in some situations. Thus, the classical rough approximations have been extended to the similarity relation-based rough sets (Slowinski and Vanderpooten 2000), the tolerance relation-based rough sets (Skowron and Stepaniuk 1996; Walczak and Massart 1999), the arbitrary binary relation-based rough sets (Yao 1998a, b) and the covering-based rough sets (Hosny and Raafat 2017b; Zhu and Wang 2003). From that time, many researchers were interested in studying the extensions of results and properties of rough set to rough multiset (El-

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Sheikh et al. 2017; Girish and John 2011, 2014; Hosny and Raafat 2017a).

In this paper, two new types of rough set based on ideals are defined to reduce the boundary region and increase the accuracy degree which is the main intention of rough set theory. The concepts of lower and upper approximations based on ideals are presented for both types. Additionally, some vital properties and results of those approximations are instituted. The relationships between the present approximations and the preceding approximations are established. Moreover, comparisons between the present methods and the preceding ones (Allam et al. 2006; Kandil et al. 2013; Kozae et al. 2010; Pawlak 1982; Yao 1996) are presented and shown to be more general. Furthermore, the topology precipitated through the current techniques is finer than the topology brought about through the preceding techniques (Allam et al. 2006; Kandil et al. 2013; Kozae et al. 2010; Pawlak 1982; Yao 1996). Hence, a new technique of approximation spaces via two ideals, called bi-ideal approximation spaces, by two different methods is proposed. The properties of these bi-ideal approximation spaces are presented. The relationships between the two current approximations and the previous ones are analyzed. Moreover, comparisons between the two present methods are introduced and shown to which of them is the best. The importance of this paper is not only that it is introducing a new kind of rough set based on n-ideals, increasing the accuracy measure and reducing the boundary region of the sets which is the main intention of rough set, but also it is introducing an applied example about amino acids in the chemistry field through making use of the current methods to demonstrate the definitions in a pleasant way.

2 Preliminaries

The aim of this section is to illustrate the basic concepts and properties of rough set theory which are needed in the sequel.

2.1 Pawlak's approximation space

Definition 2.1 (Pawlak 1982) Let R be an equivalence relation on a universe X, $[x]_R$ be the equivalence class containing x. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\overline{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}, \tag{2.1}$$
$$\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\}. \tag{2.2}$$

Theorem 2.1 (Yao 2003) *The upper approximation, defined by 2.2, has the following properties:*

1. $\overline{R}(\phi) = \phi$,
2. $A \subseteq \overline{R}(A)$, $\forall A \subseteq X$,
3. $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$, $\forall A, B \subseteq X$,
4. $\overline{R}(\overline{R}(A)) = \overline{R}(A)$, $\forall A \subseteq X$,
5. $\overline{R}(A) = (\underline{R}(A^c))^c$, $\forall A \subseteq X$, *where* A^c *denotes the complement of* A.

Corollary 2.1 (Kandil et al. 2013) *Let R be an equivalence relation on X. Then, the operator $\overline{R}$ on $P(X)$ defined by 2.2 satisfied the Kuratowski's axioms and induced a topology on X denoted by τ_R and defined as*

$$\tau_R = \{A \subseteq X : \overline{R}(A^c) = A^c\}. \tag{2.3}$$

2.2 Yao's approximation space

Definition 2.2 (Yao 1996) Let R be a binary relation on X and A be a subset of X. Then, the pair of lower and upper approximations, $\underline{R}(A)$ *and* $\overline{R}(A)$, are defined by:

$$\underline{R}(A) = \{x \in X : xR \subseteq A\}, \tag{2.4}$$
$$\overline{R}(A) = \{x \in X : xR \cap A \neq \phi\}, \tag{2.5}$$

where xR is called the after set of x and it is defined as $xR = \{y \in X : xRy\}$.

Theorem 2.2 (Allam et al. 2008) *If R is a preorder relation on X (i.e., R is a reflexive and a transitive relation on X), then the upper approximation, defined by 2.5, satisfies the properties in Theorem 2.1.*

2.3 Allam et al.'s approximation space

Definition 2.3 (Allam et al. 2006) Let R be any binary relation on X, a set $< p > R$ is the intersection of all after sets containing p, i.e.,

$$< p > R = \begin{cases} \cap_{p \in xR} xR, & \text{if } \exists x : p \in xR; \\ \phi, & \text{otherwise.} \end{cases}$$

Also, $R < p >$ is the intersection of all fore sets containing p, i.e.,

$$R < p >= \begin{cases} \cap_{p \in Rx} Rx, & \text{if } \exists x : p \in Rx; \\ \phi, & \text{otherwise.} \end{cases}$$

Definition 2.4 (Allam et al. 2005) Let R be a binary relation on X. For any subset A of X, a pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, are defined by:

$$\underline{R}(A) = \{x \in X :< x > R \subseteq A\}, \tag{2.6}$$
$$\overline{R}(A) = \{x \in X :< x > R \cap A \neq \phi\}. \tag{2.7}$$

Lemma 2.1 (Allam et al. 2006) *For any binary relation R on X if $y \in< x > R$, then $< y > R \subseteq< x > R$.*

Theorem 2.3 (Allam et al. 2005) *Let R a reflexive relation on X. Then, the upper approximation, defined by 2.7, satisfies the properties in Theorem 2.1.*

2.4 Kandil et al.'s approximation space

Definition 2.5 (Kandil et al. 2013) Let R be a binary relation on X and I be an ideal on X. For any subset A of X, a pair of R_*-lower and R^*-upper approximations of A are defined by:

$$R_*(A) = \{x \in X :< x > R \cap A^c \in I\}, \tag{2.8}$$

$$R^*(A) = \{x \in X :< x > R \cap A \notin I\}. \tag{2.9}$$

Definition 2.6 (Kandil et al. 2013) Let R be a binary relation on X, I be an ideal on X and A be a subset of X. Then, the pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, the boundary and accuracy measure of A are defined by:

$$\underline{R}(A) = \{x \in A :< x > R \cap A^c \in I\}, \tag{2.10}$$

$$\overline{R}(A) = A \cup R^*(A), \tag{2.11}$$

$$BND_I(A) = \overline{R}(A) - \underline{R}(A), \tag{2.12}$$

$$\mu_I(A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|}, |\overline{R}(A)| \neq 0, \tag{2.13}$$

where $0 \leq \mu_I(A) \leq 1$.

Theorem 2.4 (Kandil et al. 2013) *Let R be a binary relation on X. Then, the upper approximation, defined by 2.11, satisfied Kuratowski's axioms and induced a topology on X denoted by τ_R^* and defined as:*

$$\tau_R^* = \{A \subset X : \overline{R}(A^c) = A^c\}. \tag{2.14}$$

In such case, interior of A, $int_R^*(A)$, is identical with $\underline{R}(A)$ defined in 2.10 and closure of A, $cl_R^*(A)$, is identical with $\overline{R}(A)$ defined in 2.11.

2.5 Kozae et al.'s approximation space

Definition 2.7 (Kozae et al. 2010) Let R be a binary relation on X. For any subset A of X, a pair of lower and upper approximations of A are defined by:

$$\underline{R}(A) = \{x \in X : R < x > R \subseteq A\}, \tag{2.15}$$

$$\overline{R}(A) = \{x \in X : R < x > R \cap A \neq \phi\}, \tag{2.16}$$

where $R < x > R =< x > R \cap R < x >$.

Lemma 2.2 (Kozae et al. 2010) *Let R be a binary relation on a non-empty set X. If $y \in R < x > R$, then $R < y > R \subseteq R < x > R$.*

Proposition 2.1 (Hosny 2011) *Let R be any binary relation on a non-empty set X and $A \subseteq X$. Then,*

1. $\underline{R}(A)|_{Allam} \subseteq \underline{R}(A)|_{Kozae}$,
2. $\overline{R}(A)|_{Kozae} \subseteq \overline{R}(A)|_{Allam}$.

3 Generalization of rough sets based on ideals

The purpose of this section is to present a new kind of rough sets based on ideals by using the notation $R < x > R$. The main properties of the current method are studied and compared to the previous methods Allam et al. (2005), Kandil et al. (2013), Kozae et al. (2010).

Definition 3.1 Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. A pair of R^{**}-upper and R_{**}-lower approximations, $R^{**}(A)$ and $R_{**}(A)$, are defined, respectively, as

$$R^{**}(A) = \{x \in X : R < x > R \cap A \notin I\}, \tag{3.1}$$

$$R_{**}(A) = \{x \in X : R < x > R \cap A^c \in I\}. \tag{3.2}$$

Example 3.1 Let $X = \mathbb{R}$, $I = P(\mathbb{N})$ and $R = \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$. Then, $< a > R = [a, \infty[$ and $R < a >=] - \infty, a]$ for all $a \in \mathbb{R}$. Thus, $R < a > R = \{a\}$ for all $a \in \mathbb{R}$. Hence, $R^{**}(\mathbb{Q}) = \mathbb{Q} - \mathbb{N}$ and $R_{**}(\mathbb{Q}) = \mathbb{Q}$.

The following theorem studies the main properties of the current upper approximations.

Theorem 3.1 *Let R be a binary relation on a non-empty set X, I, J be two ideals on X and A, and B be two subsets of X. Then, the following properties hold:*

1. $R^{**}(A) = [R_{**}(A^c)]^c$.
2. $R^{**}(\phi) = \phi$.
3. $A \subseteq B \Rightarrow R^{**}(A) \subseteq R^{**}(B)$.
4. $R^{**}(A \cap B) \subseteq R^{**}(A) \cap R^{**}(B)$.
5. $R^{**}(A \cup B) = R^{**}(A) \cup R^{**}(B)$.
6. $R^{**}(R^{**}(A)) \subseteq R^{**}(A)$.
7. $A \in I \Rightarrow R^{**}(A) = \phi$.
8. $I \subseteq J \Rightarrow R_J^{**}(A) \subseteq R_I^{**}(A)$.

Proof 1.

$$\begin{aligned}[R_{**}(A^c)]^c &= (\{x \in X : R < x > R \cap A \in I\})^c \\ &= \{x \in X : R < x > R \cap A \notin I\} \\ &= R^{**}(A).\end{aligned}$$

2.

$$\begin{aligned}R^{**}(\phi) &= \{x \in X : R < x > R \cap \phi \notin I\} \\ &= \phi.\end{aligned}$$

3. Let $x \in R^{**}(A)$. Then, $R < x > R \cap A \notin I$. Since $A \subseteq B$ and I is an ideal, $R < x > R \cap B \notin I$. Therefore, $x \in R^{**}(B)$. Hence, $R^{**}(A) \subseteq R^{**}(B)$.
4. Immediately by part (3).
5. $R^{**}(A) \cup R^{**}(B) \subseteq R^{**}(A \cup B)$ by part (3). Let $x \in R^{**}(A \cup B)$. Then, $R < x > R \cap (A \cup B) \notin I$. It follows that $(R < x > R \cap A) \cup (R < x > R \cap B) \notin I$. Therefore, $R < x > R \cap A \notin I$ or $R < x > R \cap B \notin I$; this means $x \in R^{**}(A)$ or $x \in R^{**}(B)$. Then, $x \in R^{**}(A) \cup R^{**}(B)$. Thus, $R^{**}(A \cup B) \subseteq R^{**}(A) \cup R^{**}(B)$. Hence, $R^{**}(A \cup B) = R^{**}(A) \cup R^{**}(B)$.
6. Let $x \in R^{**}(R^{**}(A))$. Then, $R < x > R \cap R^{**}(A) \notin I$. Therefore, $R < x > R \cap R^{**}(A) \neq \phi$. Thus, there exists $y \in R < x > R \cap R^{**}(A)$. This means that $R < y > R \subseteq R < x > R$ (by Lemma 2.2) and $R < y > R \cap A \notin I$. Then, $R < x > R \cap A \notin I$. Hence, $x \in R^{**}(A)$. This completes the proof.
7. Straightforward by Definition 3.1.
8. Let $x \in R^{**}_J(A)$. Then, $R < x > R \cap A \notin J$. Since $I \subseteq J$, $R < x > R \cap A \notin I$. Therefore, $x \in R^{**}_I(A)$. Hence, $R^{**}_J(A) \subseteq R^{**}_I(A)$.

□

Remark 3.1 Let R be a binary relation on a non-empty set X, I, J be two ideals on X, and A, B be two subsets of X, the following examples show that in general:

1. $R^{**}(A) \subseteq R^{**}(B) \nRightarrow A \subseteq B$.
2. $R^{**}(A \cap B) \neq R^{**}(A) \cap R^{**}(B)$.
3. $R^{**}(A) = \phi \nRightarrow A \in I$.
4. $R^{**}_J(A) \subseteq R^{**}_I(A) \nRightarrow I \subseteq J$.

Example 3.1
1. Let $X = \{a, b, c, d\}$, $I = \{\phi, \{a\}\}$ and $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, d), (c, a), (c, b), (c, d), (d, d)\}$. Then, $R < a > R = \{a\}$, $R < b > R = \{b\}$, $R < c > R = \{b, c\}$ and $R < d > R = \{d\}$. If $A = \{a, b\}$ and $B = \{b, c\}$, then $R^{**}(A) = \{b, c\} = R^{**}(B)$, but $A \nsubseteq B$.
2. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{r\}\}$ and $R = \Delta \cup \{(x, y), (y, x), (x, z), (z, x), (x, r), (r, x), (y, z), (r, z)\}$. Then, $R < x > R = \{x\}$, $R < y > R = \{x, y\}$, $R < z > R = \{x, z\}$ and $R < r > R = \{x, r\}$. If $A = \{x, r\}$ and $B = \{y, z\}$, then $R^{**}(A) = X$, $R^{**}(B) = \{y, z\}$ and $R^{**}(A \cap B) = \phi$. Hence, $R^{**}(A \cap B) \neq R^{**}(A) \cap R^{**}(B)$.
3. Let $X = \{a, b, c, d\}$, $I = \{\phi, \{a\}\}$ and $R = \{(a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (c, d)\}$. Then, $R < a > R = \{a\}$, $R < b > R = \{b\}$, $R < c > R = \{c\}$ and $R < d > R = \phi$. If $A = \{a, d\}$, then $R^{**}(A) = \phi$, but $A \notin I$.
4. Let $X = \{a, b, c, d\}$, $I = \{\phi, \{a\}\}$, $J = \{\phi, \{d\}\}$ and $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, d), (c, a), (c, b), (c, d), (d, d)\}$. Then, $R < a > R = \{a\}$, $R < b > R = \{b\}$, $R < c > R = \{b, c\}$ and $R < d > R = \{d\}$. If $A = \{d\}$, then $R^{**}_J(A) = \phi$ and $R^{**}_I(A) = \{d\}$. Hence, $R^{**}_J(A) \subseteq R^{**}_I(A)$, but $I \nsubseteq J$.

The following remark represents a deviation between the current approach and the previous one (Allam et al. 2005).

Remark 3.2 Let R be a reflexive relation on a non-empty set X, I be an ideal on X, and $A, B \subseteq X$, the following examples show that in general:

1. $A \nsubseteq R^{**}(A)$.
2. $R^{**}(X) \neq X$.
3. $A \nsubseteq R_{**}(R^{**}(A))$ and $R^{**}(A) \nsubseteq R_{**}(R^{**}(A))$.

Example 3.2
1. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{z\}, \{x, z\}\}$ and $R = \Delta \cup \{(x, y), (x, z), (z, r), (y, z), (y, r)\}$, where Δ is an identity relation on X. Then, $R < x > R = \{x\}$, $R < y > R = \{y\}$, $R < z > R = \{z\}$ and $R < r > R = \{r\}$. If $A = \{x, y\}$, then $R^{**}(A) = \{y\}$. Hence, $A \nsubseteq R^{**}(A)$. Also, $R^{**}(X) = \{y, r\} \neq X$.
2. In Example 3.1 part (2), if $A = \{y\}$, then $R^{**}(A) = \{y\}$ and $R_{**}(R^{**}(A)) = \phi$. Hence, $A \nsubseteq R_{**}(R^{**}(A))$ and $R^{**}(A) \nsubseteq R_{**}(R^{**}(A))$.

The main properties of the lower approximations are presented in the following theorem.

Theorem 3.2 *Let R be a binary relation on a non-empty set X, I, J be two ideals on X and $A, B \subseteq X$. Then, the following properties hold:*

1. $R_{**}(A) = [R^{**}(A^c)]^c$.
2. $R_{**}(X) = X$.
3. $A \subseteq B \Rightarrow R_{**}(A) \subseteq R_{**}(B)$.
4. $R_{**}(A \cap B) = R_{**}(A) \cap R_{**}(B)$.
5. $R_{**}(A \cup B) \supseteq R_{**}(A) \cup R_{**}(B)$.
6. $R_{**}(A) \subseteq R_{**}(R_{**}(A))$.
7. $A^c \in I \Rightarrow R_{**}(A) = X$.
8. $I \subseteq J \Rightarrow R_{**I}(A) \subseteq R_{**J}(A)$.

Proof The proof is similar to that of Theorem 3.1. □

Remark 3.3 By the same way, we can add examples to show that in general:

1. $R_{**}(A) \nsubseteq A$.
2. $R_{**}(\phi) \neq \phi$.
3. $R_{**}(A) \subseteq R_{**}(B) \nRightarrow A \subseteq B$.
4. $R_{**}(A \cup B) \neq R_{**}(A) \cup R_{**}(B)$.
5. $R_{**}(A) = X \nRightarrow A^c \in I$.
6. $R_{**I}(A) \subseteq R_{**J}(A) \nRightarrow I \subseteq J$.

Remark 3.4 If R is a binary relation on a non-empty set X, $A \subseteq X$ and I is an ideal on X such that $I = P(X)$. Then, $R^{**}(A) = \phi$.

Theorem 3.3 *Let R be a binary relation on a non-empty set X, I and J be two ideals on X and $A \subseteq X$. Then, the following assertions hold:*

1. $R^{**}_{(I\cap J)}(A) = R^{**}_{I}(A) \cup R^{**}_{J}(A)$.
2. $R^{**}_{(I\cup J)}(A) = R^{**}_{I}(A) \cap R^{**}_{J}(A)$.

Proof 1.

$$\begin{aligned} R^{**}_{(I\cap J)}(A) &= \{x \in X : R<x>R \cap A \notin I \cap J\} \\ &= \{x \in X : R<x>R \cap A \notin I\} \\ &\quad \text{or } \{x \in X : R<x>R \cap A \notin J\} \\ &= \{x \in X : R<x>R \cap A \notin I\} \\ &\quad \cup \{x \in X : R<x>R \cap A \notin J\} \\ &= R^{**}_{I}(A) \cup R^{**}_{J}(A). \end{aligned}$$

2. Similarly.

□

The following theorem presents the relationships between the current approximations in Definition 3.1 and the previous one in Definition 2.5 (Kandil et al. 2013).

Theorem 3.4 *Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. Then,*

1. $R^{**}(A) \subseteq R^{*}(A)$.
2. $R_{*}(A) \subseteq R_{**}(A)$.

Proof Straightforward from the fact that $R<x>R \subseteq <x>R$. □

Remark 3.5 Example 3.2 (part 2) shows that the inclusion in Theorem 3.4 parts 1 and 2 cannot be replaced by equality relation in general (for part 1, if $A = \{z\}$, then $R^{**}(A) \subsetneq R^{*}(A)$). In a similar way, anyone can add example to part 2.

The following theorem presents the relationships between the current approximations in Definition 3.1 and Definition 2.7 in Kozae et al. (2010).

Theorem 3.5 *Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. Then,*

1. $R^{**}(A) \subseteq \overline{R}(A)$.
2. $\underline{R}(A) \subseteq R_{**}(A)$.

Proof Straightforward. □

Remark 3.6 Example 3.2 (part 1) shows that the inclusion in Theorem 3.5 parts 1 and 2 can not be replaced by equality relation in general (for part 1, if $A = X$, then $R^{**}(A) \subsetneq \overline{R}(A)$). In a similar way, anyone can add example to part 2.

Remark 3.7 It should be noted from Proposition 2.1 and Theorems 3.4, 3.5 that the new Definition 3.1 decreases the upper approximation and increases the lower approximation. This new approach is different from the previous approach (Allam et al. 2005; Kandil et al. 2013; Kozae et al. 2010; Pawlak 1982) and more general. As a special case:

1. if $I = \{\phi\}$, then the present approximations coincide with the previous approximations (Kozae et al. 2010).
2. if R is a symmetric relation, then the present approximations coincide with the previous approximations (Kandil et al. 2013).
3. if $I = \{\phi\}$ and R is a symmetric relation, then the present approximations coincide with the previous approximations (Allam et al. 2005).
4. if $I = \{\phi\}$ and R is a equivalence relation, then the present approximations coincide with the previous approximations (Pawlak 1982).

So, the previous approximations are special cases of the current approximations.

4 New kind of generalized approximations based on ideals

In this section, a new type of generalized approximations based on ideals is presented. Some of their properties are studied. In addition, the relationship between these approximations and our approximations in the previous section is investigated. Comparisons between the present approximations and the previous approximations (Allam et al. 2005; Kandil et al. 2013; Kozae et al. 2010) are presented and shown to be more general. Moreover, it is proved that the topology induced by the current approximations is finer than the topology induced by the previous approximations (Allam et al. 2005; Kandil et al. 2013; Kozae et al. 2010). Finally, some examples are used to explain the current definitions in a friendly way.

Definition 4.1 Let R be a binary relation on a non-empty set X and I be an ideal on X. For $A \subseteq X$, a pair of lower and upper approximations, $\underline{\underline{R}}(A)$ and $\overline{\overline{R}}(A)$, are defined, respectively, as

$$\underline{\underline{R}}(A) = \{x \in A : R<x>R \cap A^{c} \in I\}, \tag{4.1}$$

$$\overline{\overline{R}}(A) = A \cup R^{**}(A). \tag{4.2}$$

Definition 4.2 Let R be a binary relation on a non-empty set X and I be an ideal on X. For $A \subseteq X$, the boundary and accuracy measure of A are defined, respectively, as

$$BND^{I}(A) = \overline{\overline{R}}(A) - \underline{\underline{R}}(A), \tag{4.3}$$

$$\mu^{I}(A) = \frac{|\underline{\underline{R}}(A)|}{|\overline{\overline{R}}(A)|}, |\overline{\overline{R}}(A)| \neq 0, \tag{4.4}$$

where $0 \leq \mu^{I}(A) \leq 1$.

Example 4.1 Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{z\}, \{x, z\}\}$, $R = \Delta \cup \{(x, y), (x, z), (z, r), (y, z), (y, r)\}$. Then, $< x > R = \{x, y, z\}$, $< y > R = \{y, z\}$, $< z > R = \{z\}$, $< r > R = \{r\}$. Also, $R < x >= \{x\}$, $R < y >= \{y\}$, $R < z >= \{y, z\}$, $R < r >= \{y, z, r\}$. Therefore, $R < x > R = \{x\}$, $R < y > R = \{y\}$, $R < z > R = \{z\}$, $R < r > R = \{r\}$. If $A = \{x, y\}$, then $\underline{\underline{R}}(A) = \overline{\overline{R}}(A) = \{x, y\}$, $BND^{I}(A) = \phi$ and $\mu^{I}(A) = 1$.

The following theorem presents the properties of upper approximation in Definition 4.1.

Theorem 4.1 *Let R be a binary relation on a non-empty set X, I, J be two ideals on X and $A, B \subseteq X$. Then, the following properties hold:*

1. $\overline{\overline{R}}(A) = [\underline{\underline{R}}(A^{c})]^{c}$.
2. $\overline{\overline{R}}(\phi) = \phi$.
3. $A \subseteq B \Rightarrow \overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B)$.
4. $A \subseteq \overline{\overline{R}}(A)$.
5. $\overline{\overline{R}}(A \cap B) \subseteq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$.
6. $\overline{\overline{R}}(A \cup B) = \overline{\overline{R}}(A) \cup \overline{\overline{R}}(B)$.
7. $\overline{\overline{R}}(\overline{\overline{R}}(A)) = \overline{\overline{R}}(A)$.
8. $A \in I \Rightarrow \overline{\overline{R}}(A) = A$.
9. $I \subseteq J \Rightarrow \overline{\overline{R}}_{J}(A) \subseteq \overline{\overline{R}}_{I}(A)$.

Proof The proof is similar to that of Theorem 3.1. □

The following theorem presents the properties of lower approximation in Definition 4.1.

Theorem 4.2 *Let R be a binary relation on a non-empty set X, I, J be two ideals on X and $A, B \subseteq X$. Then, the following properties hold:*

1. $\underline{\underline{R}}(A) = [\overline{\overline{R}}(A^{c})]^{c}$.
2. $\underline{\underline{R}}(X) = X$.
3. $A \subseteq B \Rightarrow \underline{\underline{R}}(A) \subseteq \underline{\underline{R}}(B)$.
4. $\underline{\underline{R}}(A) \subseteq A$.
5. $\underline{\underline{R}}(A \cap B) = \underline{\underline{R}}(A) \cap \underline{\underline{R}}(B)$.
6. $\underline{\underline{R}}(A \cup B) \supseteq \underline{\underline{R}}(A) \cup \underline{\underline{R}}(B)$.
7. $\underline{\underline{R}}(\underline{\underline{R}}(A)) = \underline{\underline{R}}(A)$.
8. $A^{c} \in I \Rightarrow \underline{\underline{R}}(A) = A$.
9. $I \subseteq J \Rightarrow \underline{\underline{R}}_{I}(A) \subseteq \underline{\underline{R}}_{J}(A)$.

Proof Immediately. □

Remark 4.1 For a binary relation R on a non-empty set X, I, J are two ideals on X and $A, B \subseteq X$, the following examples show that in general:

1. $A \subseteq B \Rightarrow \overline{\overline{R}}(A) \subsetneq \overline{\overline{R}}(B)$.
2. $A \neq \overline{\overline{R}}(A)$.
3. $\overline{\overline{R}}(A \cap B) \neq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$.
4. $I \subseteq J \Rightarrow \overline{\overline{R}}_{J}(A) \neq \overline{\overline{R}}_{I}(A)$.
5. $\overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B) \nRightarrow A \subseteq B$.
6. $\overline{\overline{R}}(A) = A \nRightarrow A \in I$.
7. $\overline{\overline{R}}_{J}(A) \subseteq \overline{\overline{R}}_{I}(A) \nRightarrow I \subseteq J$.

Example 4.1
1. Let $X=\{x, y, z, r\}$, $I=\{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta\cup\{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{r\}$ and $B = \{x, r\}$, then $\overline{\overline{R}}(A) = A$ and $\overline{\overline{R}}(B) = B$. Hence, $\overline{\overline{R}}(A) \subsetneq \overline{\overline{R}}(B)$.
2. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta \cup \{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{y\}$, then $\overline{\overline{R}}(A) = \{x, y, r\}$. Hence, $A \subsetneq \overline{\overline{R}}(A)$.
3. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta \cup \{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{y\}$, $B = \{x, z, r\}$, then $\overline{\overline{R}}(A) = \{x, y, r\}$ and $\overline{\overline{R}}(B) = \{x, z, r\}$. Therefore, $\overline{\overline{R}}(A \cap B) \subsetneq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$.
4. Let $X=\{x, y, z, r\}$, $I=\{\phi, \{x\}\}$, $J = \{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta\cup\{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{r\}$, then $\overline{\overline{R}}_{J}(A) \subsetneq \overline{\overline{R}}_{I}(A)$.
5. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta \cup \{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{x, y, z\}$, $B = \{y, z\}$, then $\overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B)$ but $A \nsubseteq B$.
6. Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{y\}, \{x, y\}\}$ and $R = \Delta \cup \{(x, y), (x, r), (y, z), (z, y)\}$. Then, $R < x > R = \{x\}$, $R < y > R = \{y\}$, $R < z > R = \{y, z\}$, $R < r > R = \{r\}$. If $A = \{x, y, r\}$, then $\overline{\overline{R}}(A) = A$ but $A \notin I$.
7. Let $X = \{x, y, z, r\}$, $J=\{\phi, \{x\}\}$, $I=\{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta\cup\{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$. If $A = \{y\}$, then $\overline{\overline{R}}_{J}(A) \subseteq \overline{\overline{R}}_{I}(A)$ but $I \nsubseteq J$.

Remark 4.2 In a similar way, we can add examples to show that the inclusion in Theorem 4.2 parts 3, 4, 6 and 9 cannot be replaced by equality relation in general. Also, the converse of Theorem 4.2 parts 3, 8 and 9 is not true in general.

Remark 4.3 If R is a binary relation on a non-empty set X, $A \subseteq X$ and I is an ideal on X such that $I = P(X)$. Then, $\overline{\overline{R}}(A) = A$.

Proposition 4.1 *Let R be a binary relation on a non-empty set X, I, J be two ideals on X and $A \subseteq X$. Then, the following assertions hold:*

1. $\overline{\overline{R}}_{(I\cap J)}(A) = \overline{\overline{R}}_I(A) \cup \overline{\overline{R}}_J(A)$.
2. $\overline{\overline{R}}_{(I\cup J)}(A) = \overline{\overline{R}}_I(A) \cap \overline{\overline{R}}_J(A)$.

Proof The proof is similar to that of Theorem 3.3. □

The following theorem presents the relationship between the current approximations in Definitions 3.1 and 4.1. The important of the current approximations that it is achieving the Kuratowski's axioms.

Theorem 4.3 *Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. Then,*

1. $R^{**}(A) \subseteq \overline{\overline{R}}(A)$.
2. $\underline{\underline{R}}(A) \subseteq R_{**}(A)$.

Proof Immediately by using Definitions 3.1 and 4.1. □

The following theorem presents the relationship between the current approximations in Definition 4.1 and Definition 2.6 in Kandil et al. (2013).

Theorem 4.4 *Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. Then,*

1. $\overline{\overline{R}}(A) \subseteq \overline{R}(A)$.
2. $\underline{R}(A) \subseteq \underline{\underline{R}}(A)$.
3. $BND^I(A) \subseteq BND_I(A)$.
4. $\mu_I(A) \leq \mu^I(A)$.

Proof 1. Let $x \in \overline{\overline{R}}(A)$, then $x \in A$ or $x \in R^{**}(A)$. So, $x \in A$ or $R < x > R \cap A \notin I$. Thus, $x \in A$ or $< x > R \cap A \notin I$. Then, $x \in A \cup R^*(A)$. This means that $x \in \overline{R}(A)$. Hence, $\overline{\overline{R}}(A) \subseteq \overline{R}(A)$.
2. Let $x \in \underline{R}(A)$, then $x \in A$ and $< x > R \cap A^c \in I$. Since $R < x > R \subseteq < x > R$, then $R < x > R \cap A^c \in I$. Therefore, $x \in \underline{\underline{R}}(A)$. Hence, $\underline{R}(A) \subseteq \underline{\underline{R}}(A)$.
3. Immediate.
4. Straightforward from part (1) and part (2).

□

Remark 4.4 It is noted from Theorem 4.4 that Definition 4.1 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximation and decreasing the upper approximation with the comparison of the method in Definition 2.6 Kandil et al. (2013).

Theorem 4.5 *Let R be a binary relation on a non-empty set X, I be an ideal on X and $A \subseteq X$. Then, the upper approximation in Definition 4.1 satisfies Kuratowski's axioms and induces a topology on X called τ_R^{**} given by $\tau_R^{**} = \{A \subseteq X : \overline{\overline{R}}(A^c) = A^c\}$.*

Proof Immediately by Theorems 4.1 and 4.2. □

It should be noted that the interior of a set A, $int_R^{**}(A)$, is identical with $\underline{\underline{R}}(A)$ and the closure of a set A, $cl_R^{**}(A)$, is identical with $\overline{\overline{R}}(A)$.

The relationship between the topology which was generated by the previous method in Theorem 2.4 Kandil et al. (2013) and the topology which is generated by the present method is introduced in the following proposition.

Proposition 4.2 *Let R be a binary relation on a non-empty set X and I be an ideal on X. Then, τ_R^{**} is finer than τ_R^*, i.e., $\tau_R^* \subseteq \tau_R^{**}$.*

Proof Immediately by Theorem 4.4 part(1).

From the following example, the lower, upper approximation, boundary region and accuracy measure for subsets of X are computed by using Kozae et al.'s method in Definition 2.7 (Kozae et al. 2010), Kandil et al.'s method in Definition 2.6 (Kandil et al. 2013) and the present method in Definition 4.1. □

Example 4.2 Let $X = \{x, y, z, r\}$, $I = \{\phi, \{x\}, \{y\}, \{x, y\}\}$, $R = \Delta \cup \{(x, y), (x, r), (y, z), (z, y)\}$. Then, $< x > R = \{x, y, r\}$, $< y > R = \{y\}$, $< z > R = \{y, z\}$, $< r > R = \{r\}$. Also, $R < x >= \{x\}$, $R < y >= \{y, z\}$, $R < z >= \{y, z\}$, $R < r >= \{x, r\}$. Therefore, $R < x > R = \{x\}$, $R < y > R = \{y\}$, $R < z > R = \{y, z\}$, $R < r > R = \{r\}$. The comparison between the previous methods and the current method is shown in Table 1.

For example, take $\{x, z\}$, then the boundary and accuracy by Definition 4.1 are ϕ and 1, respectively, whereas the boundary and accuracy by using Kozae et al.'s method in Definition 2.7 (Kozae et al. 2010) are $\{z\}$ and 0.5, respectively. Also, the boundary and accuracy by using Kandil et al.'s method in Definition 2.6 (Kandil et al. 2013) are $\{x\}$ and 0.5, respectively. Additionally, it is clear that Kozae et al.'s method in Definition 2.7 (Kozae et al. 2010) and Kandil et al.'s method in Definition 2.6 (Kandil et al. 2013) are independent methods.

Table 1 Comparison between the boundary region and accuracy measure of a set A by using Kozae et al.'s method in Definition 2.7 (Kozae et al. 2010), Kandil et al.'s method in Definition 2.6 (Kandil et al. 2013) and the present method in Definition 4.1

A	Kozae et al.'s method in Definition 2.7 (Kozae et al. 2010)				Kandil et al.'s method in Definition 2.6 (Kandil et al. 2013)				The present method in Definition 4.1			
	$\underline{R}(A)$	$\overline{R}(A)$	$BND(A)$	$\mu(A)$	$\underline{R}(A)$	$\overline{R}(A)$	$BND(A)$	$\mu(A)$	$\underline{\underline{R}}(A)$	$\overline{\overline{R}}(A)$	$BND^I(A)$	$\mu^I(A)$
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1
$\{x\}$	$\{x\}$	$\{x\}$	ϕ	1	ϕ	$\{x\}$	$\{x\}$	0	$\{x\}$	$\{x\}$	ϕ	1
$\{y\}$	$\{y\}$	$\{y, z\}$	$\{z\}$	1/2	$\{y\}$	$\{y\}$	ϕ	1	$\{y\}$	$\{y\}$	ϕ	1
$\{z\}$	ϕ	$\{z\}$	$\{z\}$	0	$\{z\}$	$\{z\}$	ϕ	1	$\{z\}$	$\{z\}$	ϕ	1
$\{r\}$	$\{r\}$	$\{r\}$	ϕ	1	$\{r\}$	$\{x, r\}$	$\{x\}$	1/2	$\{r\}$	$\{r\}$	ϕ	1
$\{x, y\}$	$\{x, y\}$	$\{x, y, z\}$	$\{z\}$	2/3	$\{y\}$	$\{x, y\}$	$\{x\}$	1/2	$\{x, y\}$	$\{x, y\}$	ϕ	1
$\{x, z\}$	$\{x\}$	$\{x, z\}$	$\{z\}$	1/2	$\{z\}$	$\{x, z\}$	$\{x\}$	1/2	$\{x, z\}$	$\{x, z\}$	ϕ	1
$\{x, r\}$	$\{x, r\}$	$\{x, r\}$	ϕ	1	$\{x, r\}$	$\{x, r\}$	ϕ	1	$\{x, r\}$	$\{x, r\}$	ϕ	1
$\{y, z\}$	$\{y, z\}$	$\{y, z\}$	ϕ	1	$\{y, z\}$	$\{y, z\}$	ϕ	1	$\{y, z\}$	$\{y, z\}$	ϕ	1
$\{y, r\}$	$\{y, r\}$	$\{y, z, r\}$	$\{z\}$	2/3	$\{y, r\}$	$\{x, y, r\}$	$\{x\}$	2/3	$\{y, r\}$	$\{y, r\}$	ϕ	1
$\{z, r\}$	$\{r\}$	$\{z, r\}$	$\{z\}$	1/2	$\{z, r\}$	$\{x, z, r\}$	$\{x\}$	2/3	$\{z, r\}$	$\{z, r\}$	ϕ	1
$\{x, y, z\}$	$\{x, y, z\}$	$\{x, y, z\}$	ϕ	1	$\{y, z\}$	$\{x, y, z\}$	$\{x\}$	2/3	$\{x, y, z\}$	$\{x, y, z\}$	ϕ	1
$\{x, y, r\}$	$\{x, y, r\}$	X	$\{z\}$	3/4	$\{x, y, r\}$	$\{x, y, r\}$	ϕ	1	$\{x, y, r\}$	$\{x, y, r\}$	ϕ	1
$\{x, z, r\}$	$\{x, r\}$	$\{x, z, r\}$	$\{z\}$	2/3	$\{x, z, r\}$	$\{x, z, r\}$	ϕ	1	$\{x, z, r\}$	$\{x, z, r\}$	ϕ	1
$\{y, z, r\}$	$\{y, z, r\}$	$\{y, z, r\}$	ϕ	1	$\{y, z, r\}$	X	$\{x\}$	3/4	$\{y, z, r\}$	$\{y, z, r\}$	ϕ	1

5 Approximations of a set based on ideals by using two approaches

In this section, a new type of approximation spaces via two ideals, called bi-ideal approximation spaces, is presented. This type of approximations is analyzed by two different methods, their properties are investigated, and the relationships between these methods are studied.

Definition 5.1 Let I_1, I_2 be two ideals on a non-empty set X. The collection generated by I_1, I_2 is denoted by $< I_1, I_2 >$ and defined as:

$$< I_1, I_2 >= \{G \cup F : G \in I_1, F \in I_2\}. \tag{5.1}$$

Proposition 5.1 *If I_1, I_2 are two ideals on a non-empty set X and A, B are two subsets of X. Then, the collection $< I_1, I_2 >$ is satisfied the following conditions:*

1. $< I_1, I_2 >\neq \phi$,
2. $A \in< I_1, I_2 >,\ B \subseteq A \Rightarrow B \in< I_1, I_2 >$,
3. $A, B \in< I_1, I_2 >\Rightarrow A \cup B \in< I_1, I_2 >$.

It means that the collection $< I_1, I_2 >$ is an ideal on X.

Proof Straightforward. □

Definition 5.2 The quadrable (X, R, I_1, I_2) is said to be a bi-ideal approximation space where R is a binary relation on X, I_1, I_2 are two ideals on X and $(X, R, < I_1, I_2 >)$ is said to be an ideal approximations space associated to (X, R, I_1, I_2). A pair of $R^{**}_{<I_1,I_2>}$-upper and $R_{**<I_1,I_2>}$-lower approximations, $R^{**}_{<I_1,I_2>}(A)$ and $R_{**<I_1,I_2>}(A)$, are defined, respectively, as

$$R^{**}_{<I_1,I_2>}(A) = \{x \in X : R < x > R \cap A \notin< I_1, I_2 >\}, \tag{5.2}$$

$$R_{**<I_1,I_2>}(A) = \{x \in X : R < x > R \cap A^c \in< I_1, I_2 >\}. \tag{5.3}$$

The lower and upper approximations in Definition 5.2 coincide with the previous approximations in Definition 3.1 if $I_1 = I_2$. It should be noted that the properties of the current approximations in Definition 5.2 are the same as the previous one in Theorems 3.1 and 3.2.

Definition 5.3 Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. A pair of lower and upper approximations, $\underline{\underline{R}}_{<I_1,I_2>}(A)$ and $\overline{\overline{R}}_{<I_1,I_2>}(A)$, are defined, respectively, as

$$\underline{\underline{R}}_{<I_1,I_2>}(A) = \{x \in A : R < x > R \cap A^c \in< I_1, I_2 >\}, \tag{5.4}$$

$$\overline{\overline{R}}_{<I_1,I_2>}(A) = A \cup R^{**}_{<I_1,I_2>}(A). \tag{5.5}$$

Remark 5.1 It should be noted that the operators $\overline{\overline{R}}_{<I_1,I_2>}(A)$ and $\underline{\underline{R}}_{<I_1,I_2>}(A)$ satisfy the properties as in Theorems 4.1 and 4.2, respectively, so we omitted it here.

Definition 5.4 Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. Then, the pair of lower and upper approximations, $\underline{\underline{R}}_{I_1,I_2}(A)$ and $\overline{\overline{R}}_{I_1,I_2}(A)$, are defined, respectively, as

$$\underline{\underline{R}}_{I_1,I_2}(A) = \underline{\underline{R}}_{I_1}(A) \cup \underline{\underline{R}}_{I_2}(A), \tag{5.6}$$

$$\overline{\overline{R}}_{I_1,I_2}(A) = \overline{\overline{R}}_{I_1}(A) \cap \overline{\overline{R}}_{I_2}(A), \tag{5.7}$$

where $\underline{\underline{R}}_{I_i}(A)$ and $\overline{\overline{R}}_{I_i}(A)$ are the lower and upper approximations of A with respect to $I_i,\ i \in \{1, 2\}$ as in Definition 4.1.

The following proposition studies the main properties of the current upper and lower approximation in Definition 5.4.

Proposition 5.2 *Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A, B \subseteq X$. Then, the following properties hold:*

1. $\overline{\overline{R}}_{I_1,I_2}(A) = [\underline{\underline{R}}_{I_1,I_2}(A^c)]^c,\ \underline{\underline{R}}_{I_1,I_2}(A) = [\overline{\overline{R}}_{I_1,I_2}(A^c)]^c$.
2. $\overline{\overline{R}}_{I_1,I_2}(\phi) = \phi,\ \underline{\underline{R}}_{I_1,I_2}(X) = X$.
3. $A \subseteq B \Rightarrow \overline{\overline{R}}_{I_1,I_2}(A) \subseteq \overline{\overline{R}}_{I_1,I_2}(B)$ *and* $\underline{\underline{R}}_{I_1,I_2}(A) \subseteq \underline{\underline{R}}_{I_1,I_2}(B)$.
4. $\underline{\underline{R}}_{I_1,I_2}(A) \subseteq A \subseteq \overline{\overline{R}}_{I_1,I_2}(A)$.
5. $\overline{\overline{R}}_{I_1,I_2}(A \cap B) \subseteq \overline{\overline{R}}_{I_1,I_2}(A) \cap \overline{\overline{R}}_{I_1,I_2}(B)$.
6. $\overline{\overline{R}}_{I_1,I_2}(A \cup B) \supseteq \overline{\overline{R}}_{I_1,I_2}(A) \cup \overline{\overline{R}}_{I_1,I_2}(B)$.
7. $\underline{\underline{R}}_{I_1,I_2}(A \cap B) \subseteq \underline{\underline{R}}_{I_1,I_2}(A) \cap \underline{\underline{R}}_{I_1,I_2}(B)$.
8. $\underline{\underline{R}}_{I_1,I_2}(A \cup B) \supseteq \underline{\underline{R}}_{I_1,I_2}(A) \cup \underline{\underline{R}}_{I_1,I_2}(B)$.

Proof The proof is straightforward by using Definition 5.4, Theorems 4.1 and 4.2. □

Remark 5.2 Let (X, R, I_1, I_2) be a bi-ideal approximation space and A, B be two subsets of X, the following examples show that in general:

1. $\overline{\overline{R}}_{I_1,I_2}(A \cup B) \neq \overline{\overline{R}}_{I_1,I_2}(A) \cup \overline{\overline{R}}_{I_1,I_2}(B)$.
2. $\underline{\underline{R}}_{I_1,I_2}(A \cap B) \neq \underline{\underline{R}}_{I_1,I_2}(A) \cap \underline{\underline{R}}_{I_1,I_2}(B)$.
3. $A \in I_1 \cup I_2 \nrightarrow \overline{\overline{R}}_{I_1,I_2}(A) = A$.
4. $A^c \in I_1 \cup I_2 \nrightarrow \underline{\underline{R}}_{I_1,I_2}(A) = A$.

Example 5.1 Let $X = \{x, y, z, r\}$, $I_1 = \{\phi, \{x\}\}$, $I_2 = \{\phi, \{r\}\}$, $< I_1, I_2 >= \{\phi, \{x\}, \{r\}, \{x, r\}\}$ and $R = \Delta \cup \{(x, y), (y, x), (x, r), (r, x), (y, r), (r, y)\}$. Then, $R < x > R = \{x, y, r\}$, $R < y > R = \{x, y, r\}$, $R < z > R = \{z\}$, $R < r > R = \{x, y, r\}$.

1. If $A = \{x\}$ and $B = \{r\}$, then $\overline{\overline{R}}_{I_1,I_2}(A) \cup \overline{\overline{R}}_{I_1,I_2}(B) = \{x, r\}$ but $\overline{\overline{R}}_{I_1,I_2}(A \cup B) = \{x, y, r\}$.
2. If $A = \{y, r\}$ and $B = \{x, y, z\}$, then $\underline{\underline{R}}_{I_1,I_2}(A) \cap \underline{\underline{R}}_{I_1,I_2}(B) = \{y\}$ but $\underline{\underline{R}}_{I_1,I_2}(A \cap B) = \phi$.
3. If $A = \{x, r\}$, then $A \in I_1 \cup I_2$ but $\overline{\overline{R}}_{I_1,I_2}(A) = \{x, y, r\} \neq A$.
4. If $A = \{y, z\}$, then $A^c \in I_1 \cup I_2$ but $\underline{\underline{R}}_{I_1,I_2}(A) = \{z\} \neq A$.

The following theorem presents the relationships between the two methods of the current approximations via two ideals in Definitions 5.3 and 5.4.

Theorem 5.1 *Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. Then,*

1. $\overline{\overline{R}}_{<I_1,I_2>}(A) \subseteq \overline{\overline{R}}_{I_1,I_2}(A)$.
2. $\underline{\underline{R}}_{I_1,I_2}(A) \subseteq \underline{\underline{R}}_{<I_1,I_2>}(A)$.
3. $BND_{<I_1,I_2>}(A) \subseteq BND_{I_1,I_2}(A)$.
4. $\mu_{I_1,I_2}(A) \leq \mu_{<I_1,I_2>}(A)$.

Proof 1. Let $x \in \overline{\overline{R}}_{<I_1,I_2>}(A)$, then $x \in A$ or $x \in R^{**}_{<I_1,I_2>}(A)$. So, $x \in A$ or $R < x > R \cap A \notin <I_1, I_2>$. The first case, if $x \in A$, then $x \in \overline{R}_{I_1}(A)$ and $x \in \overline{R}_{I_2}(A)$. Thus, $x \in \overline{R}_{I_1,I_2}(A)$. The other case, if $R < x > R \cap A \notin <I_1, I_2>$, then $R < x > R \cap A \notin I_1$ and $R < x > R \cap A \notin I_2$. Therefore, $x \in R^{**}_{I_1}(A)$ and $x \in R^{**}_{I_2}(A)$. Thus, $x \in \overline{\overline{R}}_{I_1}(A)$ and $x \in \overline{\overline{R}}_{I_2}(A)$. This means that $x \in \overline{\overline{R}}_{I_1,I_2}(A)$. Hence, $\overline{\overline{R}}_{<I_1,I_2>}(A) \subseteq \overline{\overline{R}}_{I_1,I_2}(A)$.
2. Let $x \in \underline{\underline{R}}_{I_1,I_2}(A)$. Then, $x \in \underline{\underline{R}}_{I_1}(A)$ or $x \in \underline{\underline{R}}_{I_2}(A)$. Therefore, $R < x > R \cap A^c \in I_1$ or $R < x > R \cap A^c \in I_2$, since $I_1, I_2 \subseteq <I_1, I_2>$, then $R < x > R \cap A^c \in <I_1, I_2>$. Thus, $x \in \underline{\underline{R}}_{<I_1,I_2>}(A)$. Hence, $\underline{\underline{R}}_{I_1,I_2}(A) \subseteq \underline{\underline{R}}_{<I_1,I_2>}(A)$.
3. Immediate.
4. Straightforward from part (1) and part (2).

□

Remark 5.3 It is noted from Theorem 5.1 that Definition 5.3 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximations and decreasing the upper approximations via two ideals with the comparison of the method in Definition 5.4.

Proposition 5.3 *Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. Then,*

1. $\overline{\overline{R}}_{<I_1,I_2>}(A) \subseteq \overline{\overline{R}}_{I_1,I_2}(A) \subseteq \overline{\overline{R}}_{I_i}(A)$, $\forall i \in \{1, 2\}$.
2. $\underline{\underline{R}}_{I_i}(A) \subseteq \underline{\underline{R}}_{I_1,I_2}(A) \subseteq \underline{\underline{R}}_{<I_1,I_2>}(A)$, $\forall i \in \{1, 2\}$.
3. $BND_{<I_1,I_2>}(A) \subseteq BND_{I_1,I_2}(A) \subseteq BND_{I_i}(A)$, $\forall i \in \{1, 2\}$.
4. $\mu_{I_i}(A) \leq \mu_{I_1,I_2}(A) \leq \mu_{<I_1,I_2>}(A)$, $\forall i \in \{1, 2\}$.

Proof Immediately by using Definition 5.4 and Theorem 5.1. □

Theorem 5.2 *Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. Then, upper approximation in Definition 5.3 is satisfied Kuratowski's axioms and induces a topology on X called $\tau_R^{<I_1,I_2>}$ and given by*

$$\tau_R^{<I_1,I_2>} = \{A \subseteq X : \overline{\overline{R}}_{<I_1,I_2>}(A^c) = A^c\}. \tag{5.8}$$

It should be noted that the interior of a set A, $int_R^{<I_1,I_2>}(A)$, is identical with $\underline{\underline{R}}_{<I_1,I_2>}(A)$ and the closure of a set A, $cl_R^{<I_1,I_2>}(A)$, is identical with $\overline{\overline{R}}_{<I_1,I_2>}(A)$.

Proof Immediate. □

Theorem 5.3 *Let (X, R, I_1, I_2) be a bi-ideal approximation space and $A \subseteq X$. Then, the intersection of two topologies on X generated by the upper approximations in Definition 4.1 with respect to I_1, I_2 is satisfied Kuratowski's axioms and induces a topology on X called $\tau_R^{I_1,I_2}$ and given by*

$$\tau_R^{I_1,I_2} = \tau_R^{**I_1} \cap \tau_R^{**I_2},$$

where $\tau_R^{**I_i}$ is a topology which defined as Theorem 4.5 with respect to I_i, $i \in \{1, 2\}$.

Proof Immediate. □

The relationship between the topology which was generated by the two different method in Theorems 5.2 and 5.3 is introduced by the following proposition.

Proposition 5.4 *Let (X, R, I_1, I_2) be a bi-ideal approximation space. Then, $\tau_R^{<I_1,I_2>}$ is finer than $\tau_R^{I_1,I_2}$, i.e., $\tau_R^{I_1,I_2} \subseteq \tau_R^{<I_1,I_2>}$.*

Proof Straightforward. □

From Example 5.1, the lower, upper approximations, boundary region and accuracy measure for subsets of X are computed by using the current methods in Definition 5.3 and 5.4 as shown in Table 2.

From Table 2, the approximation by Definition 5.3 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximation and decreasing the upper approximation via two ideals with the comparison of the approximation by Definition 5.4.

6 Application

Finally in this section, an applied example in chemistry field is introducing by applying the present two approximations in Definitions 5.3 and 5.4 to illustrate the concepts in a friendly way.

Table 2 Comparison between the boundary region and accuracy measure by using two different methods in Definitions 5.3 and 5.4

A	First method in Definition 5.3				Second method in Definition 5.4			
	$\underline{\underline{R}}_{<I_1,I_2>}(A)$	$\overline{\overline{R}}_{<I_1,I_2>}(A)$	$BND_{<I_1,I_2>}(A)$	$\mu_{<I_1,I_2>}(A)$	$\underline{\underline{R}}_{I_1,I_2}(A)$	$\overline{\overline{R}}_{I_1,I_2}(A)$	$BND_{I_1,I_2}(A)$	$\mu_{I_1,I_2}(A)$
X	X	X	ϕ	1	X	X	ϕ	1
$\{x\}$	ϕ	$\{x\}$	$\{x\}$	0	ϕ	$\{x\}$	$\{x\}$	0
$\{y\}$	$\{y\}$	$\{x, y, r\}$	$\{x, r\}$	1/3	ϕ	$\{x, y, r\}$	$\{x, y, r\}$	0
$\{z\}$	$\{z\}$	$\{z\}$	ϕ	1	$\{z\}$	$\{z\}$	ϕ	1
$\{r\}$	ϕ	$\{r\}$	$\{r\}$	0	ϕ	$\{r\}$	$\{r\}$	0
$\{x, y\}$	$\{x, y\}$	$\{x, y, r\}$	$\{r\}$	2/3	$\{x, y\}$	$\{x, y, r\}$	$\{r\}$	2/3
$\{x, z\}$	$\{z\}$	$\{x, z\}$	$\{x\}$	1/2	$\{z\}$	$\{x, z\}$	$\{x\}$	1/2
$\{x, r\}$	ϕ	$\{x, r\}$	$\{x, r\}$	0	ϕ	$\{x, y, r\}$	$\{x, y, r\}$	0
$\{y, z\}$	$\{y, z\}$	X	$\{x, r\}$	1/2	$\{z\}$	X	$\{x, y, r\}$	1/4
$\{y, r\}$	$\{y, r\}$	$\{x, y, r\}$	$\{x\}$	2/3	$\{y, r\}$	$\{x, y, r\}$	$\{x\}$	2/3
$\{z, r\}$	$\{z\}$	$\{z, r\}$	$\{r\}$	1/2	$\{z\}$	$\{z, r\}$	$\{r\}$	1/2
$\{x, y, z\}$	$\{x, y, z\}$	X	$\{r\}$	3/4	$\{x, y, z\}$	X	$\{r\}$	3/4
$\{x, y, r\}$	$\{x, y, r\}$	$\{x, y, r\}$	ϕ	1	$\{x, y, r\}$	$\{x, y, r\}$	ϕ	1
$\{x, z, r\}$	$\{z\}$	$\{x, z, r\}$	$\{x, r\}$	1/3	$\{z\}$	X	$\{x, y, r\}$	1/4
$\{y, z, r\}$	$\{y, z, r\}$	X	$\{x\}$	3/4	$\{y, z, r\}$	X	$\{x\}$	3/4

Table 3 Quantitative attributes of five amino acids

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
u_1	−0.11	− 0.22	0.29	335	3.458	− 1.19	127.5
u_2	− 0.51	− 0.64	0.76	311.6	3.243	− 1.43	120.5
u_3	0	0	0	224.9	1.662	0.03	65
u_4	0.15	0.13	− 0.25	337.2	3.856	− 1.06	140.6
u_5	1.2	1.8	− 2.1	322.6	3.35	0.04	131.7

Example 6.1 Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be five amino acids (for short, AAs). The (AAs) are described in terms of seven attributes: a_1 = PIE , a_2 =PIF (two measures of the side chain lipophilicity), a_3 = DGR = ΔG of transfer from the protein interior to water, a_4 =SAC=surface area, a_5 =MR=molecular refractivity, a_6 =LAM=the side chain polarity, and a_7 = Vol=molecular volume. (El-Tayar et al. 1992; Walczak and Massart 1999). Table 3 shows all quantitative attributes of five AAs.

We consider seven reflexive relations on U defined as follows: $R_i = \{(u_i, u_j) : u_i(a_k) - u_j(a_k) < \frac{\sigma_k}{2}, i, j = 1, 2, ..., 5, k = 1, 2, ..., 7\}$ where σ_k represents the standard deviation of the quantitative attributes a_k, $k = 1, 2, ..., 7$. The right neighborhoods for all elements of $U=\{u_1, u_2, u_3, u_4, u_5\}$ with respect to the relations R_k, $k = 1, 2, ..., 7$ are shown in Table 4.

Therefore, we find the intersection of all right neighborhoods of all element $k = 1, 2, ..., 7$ as the following: $u_1R = \cap_{k=1}^{7}(u_1R_k) = \{u_1, u_4\}$, $u_2R = \cap_{k=1}^{7}(u_2R_k) = \{u_1, u_2\}$, $u_3R = \cap_{k=1}^{7}(u_3R_k) = \{u_3\}$, $u_4R = \cap_{k=1}^{7}(u_4R_k) = \{u_1, u_4\}$, $u_5R = \cap_{k=1}^{7}(u_5R_k) = \{u_5\}$. Then, $R = \Delta \cup \{(u_1, u_4), (u_2, u_1), (u_4, u_1)\}$. Hence, anyone can give two ideals to extend an example similar to the example in Table 2 which show that the approximations in Definition 5.3 are better than the other in Definition 5.4 by comparing the resulting accuracy.

7 Conclusions

It is well known that rough set theory has been considered as a generalization of classical set theory in one way. Furthermore, this is a vital mathematical tool to deal with vagueness (uncertainty). The boundary region technique is usually related to vagueness (i.e., existing of things which cannot be determined by the set or its complement) which was first introduced in 1893 by Frege (1893). With respect to Frege "The concept must have an exact boundary. To the concept without an exact boundary, there would correspond an area that did not have an exact boundary line all around," i.e., mathematics must use crisp, not ambiguous definitions, otherwise, it would be impossible to reason punctually. Pawlak introduced the concept of rough sets which have a large area of applications in various fields like artificial intelligence, cognitive sciences, machine learning expert systems, knowledge discovery from databases, and other fields can be found in Jian et al. (2011), Ma et al. (2017), Pal and Mitra (2004) and Zhu and Wang (2003).

The main aim of rough sets is to increase the accuracy measure and reduce the boundary region of sets by increasing the lower approximations and decreasing the upper approx-

Table 4 Right neighborhood of seven reflexive relations

	$u_i R_1$	$u_i R_2$	$u_i R_3$	$u_i R_4$	$u_i R_5$	$u_i R_6$	$u_i R_7$
u_1	$\{u_1, u_3, u_4, u_5\}$	U	$\{u_1, u_2, u_3, u_4\}$	$\{u_1, u_4, u_5\}$	$\{u_1, u_2, u_4, u_5\}$	U	$\{u_1, u_2, u_4, u_5\}$
u_2	U	U	$\{u_1, u_2\}$	$\{u_1, u_2, u_4, u_5\}$	$\{u_1, u_2, u_4, u_5\}$	U	$\{u_1, u_2, u_4, u_5\}$
u_3	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_2, u_3, u_4\}$	U	U	$\{u_3, u_5\}$	U
u_4	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_2, u_3, u_4\}$	$\{u_1, u_4, u_5\}$	$\{u_1, u_4\}$	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_4, u_5\}$
u_5	$\{u_5\}$	$\{u_5\}$	U	$\{u_1, u_2, u_4, u_5\}$	$\{u_1, u_2, u_4, u_5\}$	$\{u_3, u_5\}$	$\{u_1, u_2, u_4, u_5\}$

imations, so in this paper, new two kinds of lower and upper approximations based on ideals are proposed to achieve this aim. The results of these new approximations are studied, compared to the previous ones (Allam et al. 2006; Kandil et al. 2013; Kozae et al. 2010; Pawlak 1982; Yao 1996), and shown to be more general. In Sect. 5, two different methods of approximation spaces based on two ideals in Definitions 5.3 and 5.4 are introduced. Also, the comparison between these methods is presented and we obtain that the approximation by Definition 5.3 is better than the other in Definition 5.4. It should be noted that any one can extend this method similarly by using n-ideals. In the real life, this comparison is represented whether we need to make a decision about a manuscript and send it to two reviewers I_1, I_2. Then, we have two cases:

1. Take two reports separately from the referees and use one of the mathematical methods to find the final decision,
2. Take a combined report from the referees together.

It is clear that the second case is better that the first one like as the approximations in Definition 5.3. Finally, an applied example about amino acids in the chemistry field is introduced to illustrate our methods of approximations. The importance of the current paper is not only that it is introducing a new kind of rough set based on n-ideals, increasing the accuracy measure and reducing the boundary region of the sets which is the main aim of rough set, but also it is introducing a chemical application to explain the concepts.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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