METHODOLOGIES AND APPLICATION

Some new basic operations of probabilistic linguistic term sets and their application in multi-criteria decision making

Na Yue[1] · Jialiang Xie[1] · Shuili Chen[2]

Abstract

This paper is concerned with the operations and methods to tackle the probabilistic linguistic multi-criteria decision making (PL-MCDM) problems where criteria are interactive. To avoid the defects of the existing operations of the probabilistic linguistic term sets (PLTSs) and make the operations easier, we redefine a family of operations for PLTSs and investigate their properties in-depth. Then, based on the probabilistic linguistic group utility measure, the probabilistic linguistic individual regret measure and the probabilistic linguistic compromise measure proposed in this paper, the probabilistic linguistic E-VIKOR method is developed. To make up for the deficiency of the above method, the improved probabilistic linguistic VIKOR method which can not only consider the distances between the alternatives and the positive ideal solution but also consider the distances between the alternatives and the negative ideal solution is developed to solve the correlative PL-MCDM problems. And then a case about the video recommender system is conducted to demonstrate the applicability and effectiveness of the proposed methods. Finally, the improved probabilistic linguistic VIKOR method is compared with the probabilistic linguistic E-VIKOR method, the general VIKOR method and the extended TOPSIS method to show its merits.

Keywords Probabilistic linguistic term set · Operations · Shapley value · Probabilistic linguistic E-VIKOR method · Improved probabilistic linguistic VIKOR method

1 Introduction

In multi-criteria decision making (MCDM) problems, the decision makers (DMs) are asked to offer their evaluations on multiple alternatives over different criteria. However, in practical decision making problems, most DMs tend to provide linguistic information rather than values form as the fuzziness of criteria. For example, when we assess the clarity of a video, the evaluation(s) could be "bad", "medium" or "well". To express the uncertainty of linguistic informa-

✉ Jialiang Xie
  xiejialiang@jmu.edu.cn

  Na Yue
  1328500209@qq.com

  Shuili Chen
  sgzxeagle@sina.com

[1] College of Science, Jimei University, Xiamen 361021, Fujian, China

[2] Chengyi Institute of Applied Technology, Jimei University, Xiamen 361021, Fujian, China

tion, Zadeh (1975) introduced the fuzzy linguistic approach, which uses the linguistic variables to represent the qualitative information and enhance the adaptability and reliability of decision models (Herrera et al. 2008; Herrera-Viedma 2001; Herrera-Viedma and López-Herrera 2010; Herrera-Viedma et al. 2007b, c). Then, an active research field, called computing with words (CWW), has been developed with the development of methods for analyzing linguistic variables.

Different CWW models (Herrera et al. 2009; Herrera-Viedma and Peis 2003; Herrera-Viedma et al. 2006; Morente-Molinera et al. 2018, 2019) have been put forward to solve the decision making problems with linguistic information. Nevertheless, in the processes of making decisions using these models, the DMs are restricted to use the individual and simple linguistic term to represent the value of a linguistic variable, which cannot completely reflect the cognitions and hesitations of DMs. Furthermore, since the information representation models take discrete values in a continuous range, the traditional models of CWW may generate information loss during the calculation processes.

To reduce the linguistic information loss and improve the readability of the processes of CWW, the hesitant fuzzy lin-

guistic term set (HFLTS) (Rodríguez et al. 2012) has been introduced ground on the hesitant fuzzy set (HFS) (Torra 2010) within the context of the linguistic, which can be used to reflect the hesitations of DMs in practical qualitative decision making. However, the HFLTS assigns the same importance or weight to each linguistic term and thus may not represent the true views of the DMs. Thereafter, to preserve all of the original linguistic information provided by the DMs, Pang et al. (2016) introduced probabilistic linguistic term set (PLTS) by adding the corresponding probability to the HFLTS. The PLTS allows the DMs to provide multiple linguistic terms as the values of a linguistic variable, which increases the applicability and accuracy of the DMs in expressing their linguistic evaluations. Furthermore, the PLTS can reflect the true weights of all evaluations over an object, so that comprehensive and accurate evaluation information can be obtained from the DMs. Thus, the research on the PLTS is necessary.

At present, many achievements have been made on PLTSs (Liao et al. 2019a; Mi et al. 2020). These results can be divided into five categories: operations for PLTSs, distance measures of PLTSs (Lin et al. 2019; Lin and Xu 2018; Wang et al. 2019a), possibility degree formula of PLTSs (Bai et al. 2017; Feng et al. 2019; Yu et al. 2019a, b; Xian et al. 2019), probabilistic linguistic preference relation (Gao et al. 2019; Song and Hu 2019; Tian et al. 2019; Zhang et al. 2016, 2018b), and methods for solving the probabilistic linguistic multi-criteria decision making (PL-MCDM) problems.

For instance, for the operations of PLTSs, Pang et al. (2016) firstly provided the operations for PLTSs. For the probabilities of linguistic terms are not reflected in the result, Zhang et al. (2016) redefined some operations for PLTSs to ensure that the results are in the form of PLTSs. With the equivalent transformation functions, Gou and Xu (2016) proposed the operations for PLTSs based on the Algebraic t-conorm and t-norm. However, the Archimedean t-conorm and t-norm (ATT) contain various t-conorms and t-norms, such as the t-conorms and t-norms of Algebraic, Einstein, Hamacher, Frank, Dombi, etc. Therefore, Liu and Teng (2018) proposed the general operations for PLTSs by the ATT and the linguistic scale functions, which can be seen as an extension of the operations introduced in Gou and Xu (2016) and could keep the operations more flexible. Similarly, Mao et al. (2019) deal with the probabilities of linguistic terms and proposed four types of operations for PLTSs based on the ATT, and some desirable properties are discussed. Wu and Liao (2018a) presented new operations of PLTSs based on the adjusted PLTSs and the semantics of linguistic terms. To simplify, Wu et al. (2018) redefine some new operations based on the expected value of PLTSs, Lin and Xu (2018) proposed the new operations of PLTSs that satisfies the sum of the probabilities of all linguistic terms in each PLTS does not have to be equal to one. Inspired by

the literature (Szmidt and Kacprzyk 2001), Tang et al. (2019) firstly defined the union and intersection of the probabilistic linguistic term elements (PLTEs). In addition to defining the operations for PLTSs based on the ATT, the operations of PLTSs can be given from new perspectives, such as the fuzzy linear least absolute regression (Jiang et al. 2018), Einstein product and Einstein sum (Agbodah and Darko 2019), the disparity degrees of linguistic terms (Liao et al. 2019b), the D–S evidence theory (Li and Wei 2019), the 2-tuple linguistic model (Song and Li 2018), and so on.

However, the existing operations for PLTSs by ATT both use the form of probability multiplication, taking the operations proposed by Gou and Xu (2016) as an example; although the probability information of PLTSs remains complete after these operations, there are still some flaws:

(i) Let $L = \{l_t \mid t = -3, -2, -1, 0, 1, 2, 3\}$ be a LTS, and $PL_1(p) = PL_2(p) = \{l_3(0.5), l_{-3}(0.5)\}$ be two PLTSs on the L, then we obtain $PL_1(p) \oplus PL_2(p) = \{l_3(0.75), l_{-3}(0.25)\}$ via using the operations, which is not in line with our intuitive.

(ii) As for the same linguistic term in a PLTS, we can combine them into one linguistic term by calculating the sum of their probabilities. For example, let $PL(p) = \{l_2(0.5), l_2(0.4), l_1(0.1)\}$ be a PLTS, then we get $PL(p) = \{l_2(0.9), l_1(0.1)\}$ via combination method. The result makes sense but is not precise in terms of information fusion, because $l_2(0.5)$ and $l_2(0.4)$ can be regarded as the evaluation information given by two experts before the information fusion, then when we combine $l_2(0.5)$ and $l_2(0.4)$ into a linguistic term l_2, the probability of the linguistic term l_2 should no more than 0.5 via the group aggregation formula given by Wu and Liao (2018b), instead of 0.9 (Gou and Xu 2016).

In addition, many research achievements have been made on the MCDM problems (Delgado et al. 1998; Farhadinia and Herrera-Viedma 2019a, b; Herrera et al. 1995; Herrera-Viedma et al. 2007a, 2017; Lin et al. 2017; Morente-Molinera et al. 2016, 2017; Wang et al. 2019b, c; Wątróbski et al. 2019; Yue et al. 2019), and some results were obtained by using the VIKOR method (Liao et al. 2015a; Wei and Zhang 2014). To the methods for solving the PL-MCDM problems, Krishankumar et al. (2019) proposed a new decision framework for PL-MCDM problems, Liu and Teng (2019) provided an extended probabilistic linguistic TODIM method for assisting potential customers to evaluate alternative products through consumer opinions regarding product performance, Liu et al. (2019) proposed the bidirectional projection method to solve the PL-MCDM problems, Chen et al. (2019) developed the extended MULTIMOORA approach and the probabilistic linguistic Choquet integral operator for the selection of cloud-based ERP, Li et al. (2019) introduced

the failure mode and effect analysis approach to integrating PLTSs and fuzzy Petri nets for the risk assessment and prioritization of failure modes. And the PLTSs have done very good results on issues such as sustainable supplier selection problems (Song and Li 2018), customer relationship management (Zhang et al. 2018a), forecast (Jiang et al. 2018), cluster analysis (Lin et al. 2018), modern medicine (Pan et al. 2018), edge computing problems (Lin et al. 2019), and so on. However, some research results on PLTSs still have two flaws as follows:

(iii) There is a gap in the use of the E-VIKOR method to study the PL-MCDM problems.

(iv) From Table 3, we see that $\rho\left(\mathrm{PL}_1^+(p), \mathrm{PL}_{21}(p)\right) > \rho\left(\mathrm{PL}_1^+(p), \mathrm{PL}_{31}(p)\right)$ and $\rho\left(\mathrm{PL}_1^-(p), \mathrm{PL}_{21}(p)\right) > \rho\left(\mathrm{PL}_1^-(p), \mathrm{PL}_{31}(p)\right)$. In other words, if the distance between the alternative and the probabilistic linguistic positive ideal solution (PL-PIS) is large, it does not necessarily mean that the distance between the alternative and the probabilistic linguistic negative ideal solution (PL-NIS) is small in the probabilistic linguistic decision making process. Therefore, the probabilistic linguistic E-VIKOR method only considers the distances between the alternatives and the positive ideal solution in group utility and individual regret, which may lead to the loss of a large amount of useful information, resulting in inappropriate solutions when solving the PL-MCDM problems.

Based on the above discussions, this paper is committed to making up for the above shortcomings and proposing two PL-MCDM methods to take into account the interactive characteristics among criteria. And the innovations of this paper are summarized as follows:

(I) With the equivalent transformation functions and the adjusted PLTSs which have the same probability set, the operations for PLTSs based on the Algebraic t-conorm and t-norm are redefined to remedy the defects (i) and (ii), and these new operations are relatively simple.

(II) We propose a set of probabilistic linguistic measures, such as the probabilistic linguistic group utility (PLGU) measure, the probabilistic linguistic individual regret (PLIR) measure, and the probabilistic linguistic compromise (PLC) measure, then the probabilistic linguistic E-VIKOR method proposed to tackle the PL-MCDM problems. This fills the gap shown in (iii).

(III) When the E-VIKOR method is applied directly to the PL-MCDM problems, it is found that some results are contrary to common sense, such as (iv). To make up for the deficiency of the probabilistic linguistic E-VIKOR method, we propose the improved probabilistic linguistic VIKOR (PL-VIKOR) method which can not only

consider the distances between alternatives and the PL-PIS, but also consider the distances between alternatives and the PL-NIS. Since the improved PL-VIKOR method uses the Shapley value to calculate and describe the weight of the alone criterion's contribution based on different combinations of criteria set, the interactive characteristics among criteria can be reflected. This makes up for the shortage (iv).

(IV) To demonstrate the advantages of the improved PL-VIKOR method proposed in this paper, some comparative analyses among the improved PL-VIKOR method, the probabilistic linguistic E-VIKOR method, the general VIKOR method, and the extended TOPSIS method are carried out to illustrate the merits of the methods presented in this paper.

The rest of this paper is organized as follows: in Sect. 2, we look back to some necessary basic pieces of knowledge which will be needed. In Sect. 3, we give some new basic operations for PLTSs and then discuss their properties in detail. Section 4 presents a set of probabilistic linguistic measures. With these measures, the probabilistic linguistic E-VIKOR method is introduced to solve the correlative PL-MCDM problems. And then based on this method, we propose the improved PL-VIKOR method for the correlative PL-MCDM problems which can not only consider the distances between alternatives and the PL-PIS, but also consider the distances between alternatives and the PL-NIS. A case about the video recommender system is carried out in Sect. 5. Some comparative analyses among the methods proposed in this paper and the existing methods are made in Sect. 6. The paper finishes in Sect. 7 with some conclusions and the future work about the PL-MCDM problems.

2 Preliminaries

In this section, we give some concepts related to the equivalent transformation functions, the PLTS and the $\mathrm{SL}_{q,v}$-metric.

2.1 The equivalent transformation functions

The hesitant fuzzy linguistic element (HFLE) is the basic unit of the HFLTS, and its mathematical expression is given by Liao et al. (2015b). With the HFLE and HFS, the equivalent transformation functions are proposed by Gou et al. (2017) to implement the conversion between the operations on linguistic variable and the operations on values.

Definition 1 (Gou et al. 2017) Let $h_L = \left\{l_t \mid t \in [-\delta, \delta]\right\}$ be a HFLE on LTS $L = \left\{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\right\}$, and $H = \left\{\varepsilon \mid \varepsilon \in [0, 1]\right\}$ be a HFS. Then the linguistic

variable l_t and membership degree ε can be transformed into each other by the functions σ and σ^{-1} given as:

$$\sigma: [-\delta, \delta] \rightarrow [0, 1],$$
$$\sigma(h_L) = \left\{ \sigma(l_t) = \varepsilon = \frac{t}{2\delta} + \frac{1}{2} \mid t \in [-\delta, \delta] \right\} = h_\varepsilon;$$
$$\sigma^{-1}: [0, 1] \rightarrow [-\delta, \delta],$$
$$\sigma^{-1}(h_\varepsilon) = \left\{ \sigma^{-1}(\varepsilon) = l_t = l_{(2\varepsilon-1)\delta} \mid \varepsilon \in [0, 1] \right\} = h_L.$$

2.2 The probabilistic linguistic term set

With respect to the shortcoming of HFLTS which cannot express the probabilities of possible linguistic terms, Pang et al. (2016) proposed the probabilistic linguistic term set (PLTS), which includes all possible linguistic terms and their respective probabilities.

Definition 2 (Pang et al. 2016) The probabilistic linguistic term set (PLTS) on LTS L can be defined as:

$$\text{PL}(p) = \left\{ l^{(k)}(p^{(k)}) \mid l^{(k)} \in L, p^{(k)} \right.$$
$$\geq 0, k = 1, 2, \ldots, \#\text{PL}(p), \sum_{k=1}^{\#\text{PL}(p)} p^{(k)} \leq 1 \left. \right\},$$

where $l^{(k)}(p^{(k)})$ is made up of linguistic term $l^{(k)}$ and its associated probability $p^{(k)}$, and the cardinality of $\text{PL}(p)$ is $\#\text{PL}(p)$. If $k = p^{(k)} = 1$, then the PLTS reduces to a linguistic term.

Since the locations of elements in the PLTS can be exchanged at will and the ordered PLTS proposed by Pang et al. (2016) may not be applicable under certain special circumstances, Zhang et al. (2016) redefined the ordered PLTS to make sure that the results of operations between PLTSs can be uniquely determined. Moreover, Pang et al. (2016) presented the associated PLTS to deal with the ignorance of probabilistic information. For the sake of discussion, we assume that the sum of the probabilities of all possible linguistic terms in a PLTS is equal to 1 and the PLTS is ordered in this paper.

In order to compare PLTSs, we propose the score function based on the equivalent transformations functions.

Definition 3 Let $\sigma(l^{(k)})$ is the membership degree of the linguistic term $l^{(k)}$ in PLTS $\text{PL}(p)$, then the score function of $\text{PL}(p)$ is defined as:

$$S(\text{PL}(p)) = \sum_{k=1}^{\#\text{PL}(p)} \sigma(l^{(k)}) p^{(k)}.$$

For two any PLTSs $\text{PL}_1(p)$ and $\text{PL}_2(p)$ with $S(\text{PL}_1(p)) \neq S(\text{PL}_2(p))$, then $\text{PL}_1(p) < \text{PL}_2(p)$ if $S(\text{PL}_1(p)) < S(\text{PL}_2(p))$; otherwise, $\text{PL}_1(p) > \text{PL}_2(p)$.

Remark 1 If the sum of the probabilities of all possible linguistic terms in PLTS $\text{PL}(p)$ is equal to 1, then the score function of $\text{PL}(p)$ in this paper is consistent with the expected value function of $\text{PL}(p)$ presented by Wu et al. (2018).

Regarding PLTSs $\text{PL}_1(p)$ and $\text{PL}_2(p)$ with $S(\text{PL}_1(p)) = S(\text{PL}_2(p))$, we define the variance function to distinguish them bellow.

Definition 4 Suppose that $\sigma(l^{(k)})$ is the membership degree of the linguistic term $l^{(k)}$ in PLTS $\text{PL}(p)$, and $S(\text{PL}(p))$ is the score value of the PLTS $\text{PL}(p)$. Then the variance function of $\text{PL}(p)$ is defined as:

$$D(\text{PL}(p)) = \left(\sum_{k=1}^{\#\text{PL}(p)} \left(\left(\sigma(l^{(k)}) - S(\text{PL}(p)) \right) p^{(k)} \right)^2 \right)^{1/2}.$$

Thus, for two PLTSs $\text{PL}_1(p)$ and $\text{PL}_2(p)$ with $S(\text{PL}_1(p)) = S(\text{PL}_2(p))$, then $\text{PL}_1(p) < \text{PL}_2(p)$ if $D(\text{PL}_1(p)) > D(\text{PL}_2(p))$; otherwise, $\text{PL}_1(p) > \text{PL}_2(p)$.

2.3 The Shapley value-based L_q-metric

Definition 5 (Sugeno and Terano 1977; Wang and Klir 2009) Given $v(\{b_j\})$ is the weight or the importance value of the element $b_j \in B = \{b_1, b_2, \ldots, b_m\}$, then the fuzzy measure f_λ called the λ-measure if it satisfying:

(1) $E, F \in 2^B, E \cap F = \emptyset, E \cup F \in 2^B \Rightarrow f_\lambda(E \cup F) = f_\lambda(E) + f_\lambda(F) + \lambda f_\lambda(E) f_\lambda(F)$;
(2) $\frac{1}{\lambda} \left(\prod_{j=1}^m \left(1 + \lambda v(\{b_j\}) \right) - 1 \right) = 1$.

To determine the expected marginal contribution of the specified element to the set B, the Shapley value of element b_j with respect to ϕ_j is defined as follows.

Definition 6 (Shapley and Shubik 1953) Given v is a fuzzy measure on B, then the Shapley value for $b_j \in B(j = 1, 2, \ldots, m)$ is defined by

$$\phi_j = \sum_{T \subset B/b_j} \frac{(m - |T| - 1)! |T|!}{m!} \left[v(T \cup \{b_j\}) - v(T) \right].$$

(1)

The Shapley value ϕ_j can be interpreted as the contribution of element b_j in set B alone, and $\sum_{j=1}^m \phi_j = 1$. Furthermore, it can be introduced into the L_q-metric as follows:

Definition 7 (Wei and Zhang 2014) Let g and ν be positive real-valued function and fuzzy measure on B, respectively. Then the Shapley value-based L_q-metric of g with respective to ν ($SL_{q,\nu}$-metric) is defined by

$$SL_{q,\nu} = \left\{ \sum_{j=1}^{m} \left[\phi_j \frac{(g_j^+ - g_j)}{(g_j^+ - g_j^-)} \right]^q \right\}^{1/q}, 1 \leq q \leq \infty, \quad (2)$$

where the Shapley value for $b_j \in B$ is ϕ_j, and $\sum_{j=1}^{m} \phi_j = 1$.

3 Some new basic operations of PLTSs

In the processes of solving the PL-MCDM problems, the probabilities and their associated linguistic terms of two PLTSs are usually different, and calculating these PLTSs directly may produce improper results. To make up for this shortcoming, Wu et al. (2018) presented the adjustment method to ensure that the probabilistic sets of two PLTSs are the same before operations. With the adjustment method, the Euclidean distance between two adjusted PLTSs $PL_1(p)$ and $PL_2(p)$ has defined in Wu et al. (2018), where $PL_1(p)$ and $PL_2(p)$ have the same probability set and cardinality.

Definition 8 (Wu et al. 2018) Let $PL_1(p) = \left\{ l_1^{(k)}(p^{(k)}) \,|\, k = 1, 2, \ldots, K \right\}$ and $PL_2(p) = \left\{ l_2^{(k)}(p^{(k)}) \,|\, k = 1, 2, \ldots, K \right\}$ be two adjusted PLTSs, and $K = \#PL_1(p) = \#PL_2(p)$. Then the Euclidean distance measure between $PL_1(p)$ and $PL_2(p)$ is defined by:

$$\rho\left(PL_1(p), PL_2(p)\right) = \sqrt{\sum_{k=1}^{K} p^{(k)} \left(\frac{\varphi_1^{(k)} - \varphi_2^{(k)}}{2\delta} \right)^2}, \quad (3)$$

where $\varphi_1^{(k)}$ and $\varphi_2^{(k)}$ are the subscripts of the linguistic terms $l_1^{(k)}$ and $l_2^{(k)}$, respectively.

As we have pointed in Sect. 1, the results obtained via using the operational laws proposed by Gou and Xu (2016) may be an inaccuracy. To compensate for this shortcoming, with the adjusted PLTSs and the equivalent transformation functions, we first put forward some new basic operations of PLTSs based on the Algebraic t-conorm and t-norm. After that, a variety of properties of the new basic operations are discussed.

Definition 9 Let $PL(p)$, $PL_1(p)$ and $PL_2(p)$ be three adjusted PLTSs on LTS L, and λ be a positive real number. Then

(1) $PL_1(p) \oplus PL_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)}$$

$$\left\{ \sigma^{-1} \left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)} \xi_2^{(k)} \right) \left(p^{(k)} \right) \right\},$$

$$k = 1, 2, \ldots, \#PL_1(p) = \#PL_2(p);$$

(2) $PL_1(p) \otimes PL_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1} \left(\xi_1^{(k)} \xi_2^{(k)} \right) \left(p^{(k)} \right) \right\},$$

$$k = 1, 2, \ldots, \#PL_1(p) = \#PL_2(p);$$

(3) $\lambda PL(p) = \bigcup_{\xi^{(k)} \in \sigma(PL)} \left\{ \sigma^{-1} \left(1 - \left(1 - \xi^{(k)} \right)^\lambda \right) \left(p^{(k)} \right) \right\},$

$$k = 1, 2, \ldots, \#PL(p);$$

(4) $(PL(p))^\lambda = \bigcup_{\xi^{(k)} \in \sigma(PL)} \left\{ \sigma^{-1} \left(\left(\xi^{(k)} \right)^\lambda \right) \left(p^{(k)} \right) \right\},$

$$k = 1, 2, \ldots, \#PL(p);$$

(5) $PL_1(p) \ominus PL_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1} \left(\prod \right) \left(p^{(k)} \right) \right\},$$

where

$$\prod = \begin{cases} \frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}} & \text{if } \xi_1^{(k)} \geq \xi_2^{(k)} \text{ and } \xi_2^{(k)} \neq 1, \\ 0 & \text{if otherwise,} \end{cases}$$

and $k = 1, 2, \ldots, \#PL_1(p) = \#PL_2(p);$

(6) $PL_1(p) \oslash PL_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1} \left(\prod \right) \left(p^{(k)} \right) \right\},$$

where

$$\prod = \begin{cases} \frac{\xi_1^{(k)}}{\xi_2^{(k)}} & \text{if } \xi_1^{(k)} \leq \xi_2^{(k)} \text{ and } \xi_2^{(k)} \neq 0, \\ 0 & \text{if otherwise,} \end{cases}$$

and $k = 1, 2, \ldots, \#PL_1(p) = \#PL_2(p);$

(7) $\overline{PL(p)} = \bigcup_{\xi^{(k)} \in \sigma(PL)} \left\{ \sigma^{-1} \left(1 - \xi^{(k)} \right) \left(p^{(k)} \right) \right\},$

$$k = 1, 2, \ldots, \#PL(p).$$

Remark 2 From the above seven basic operations of the adjusted PLTSs, we could draw some conclusions as follows:

(a) The differences between the operations given in Definition 9 and those given by Gou and Xu (2016) are that the number of linguistic terms and their associated probabilities in the PLTS is different. The number of linguistic terms in the results derived by the operational
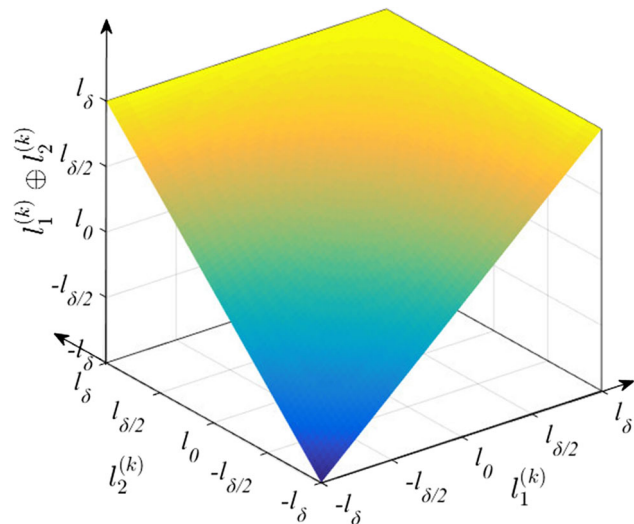
Fig. 1 The region of the addition operation $l_1^{(k)} \oplus l_2^{(k)}$
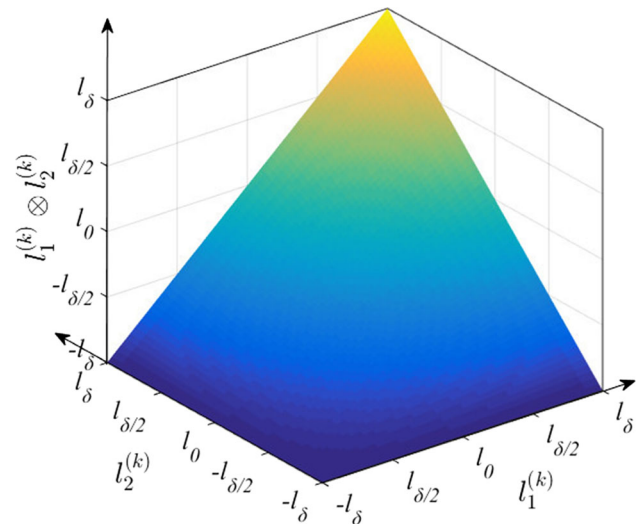

Fig. 2 The region of the multiplication operation $l_1^{(k)} \otimes l_2^{(k)}$

laws presented by Gou and Xu (2016) would be increased exponentially, while via the operations proposed in this paper, the number of linguistic terms in the results is identical to or slightly greater than the number of the original linguistic terms. Normally, the operations proposed in this paper are relatively simple.

(b) In the final result derived from the operations presented in this paper, the sum of all the probabilities in a PLTS is still equal to 1. Therefore, the probabilities information will not be lost after the above operations.

(c) In addition, since the equivalent transformation functions given by Gou et al. (2017) are only concerned with the linguistic terms in the PLTS and have no influence on the probabilities, the operations proposed in this paper can effectively avoid the low probability of the linguistic terms in the result caused by the direct multiplication of probabilities.
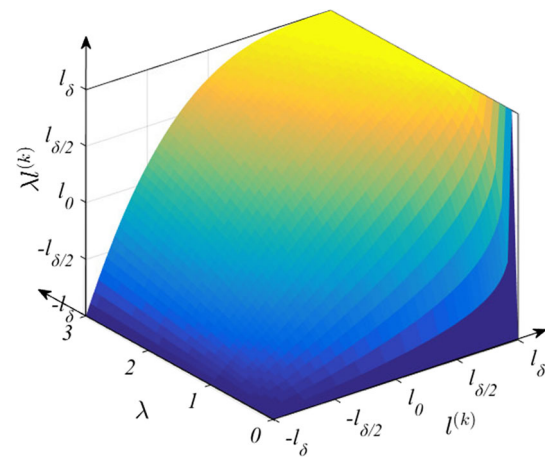
For two adjusted PLTSs $\mathrm{PL}_1(p) = \left\{ l_1^{(k)}\left(p^{(k)}\right) \mid k = 1, 2, \ldots, K \right\}$ and $\mathrm{PL}_2(p) = \left\{ l_2^{(k)}\left(p^{(k)}\right) \mid k = 1, 2, \ldots, K \right\}$, their PLTEs are $l_1^{(k)}(p^{(k)})$ and $l_2^{(k)}(p^{(k)})$, respectively. Since their probabilistic sets are the same, that is to say, the same position has the same probability, each probabilistic in this probabilistic set is only related to location, regardless of linguistic term. Therefore, in the process of calculation, each PLTE can be divided into two parts, one is a linguistic term and other is a probability, and these two parts are "independent". It should noted that the "independent" here is not completely unrelated, but can be calculated separately.

To understand these operations of PLTEs better, we draw six three-dimensional figures (Figs. 1, 2, 3, 4, 5, 6) and a plane figure (Fig. 7) to show the region of each operation.

Fig. 3 The region of the multiplication operation $\lambda l^{(k)}$
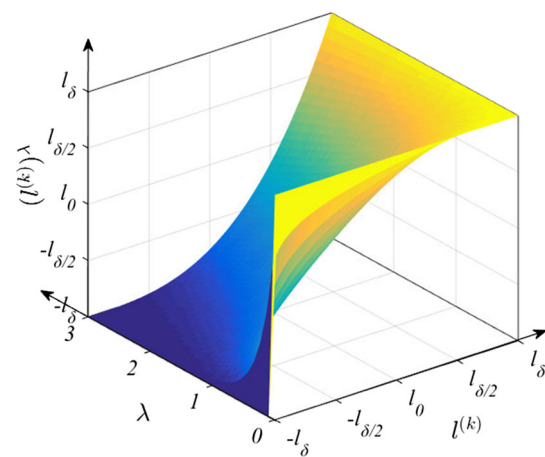

Fig. 4 The region of the power operation $(l^{(k)})^\lambda$

Remark 3 Since the probabilities of two PLTEs are the same in the calculation process, only the region of the operations
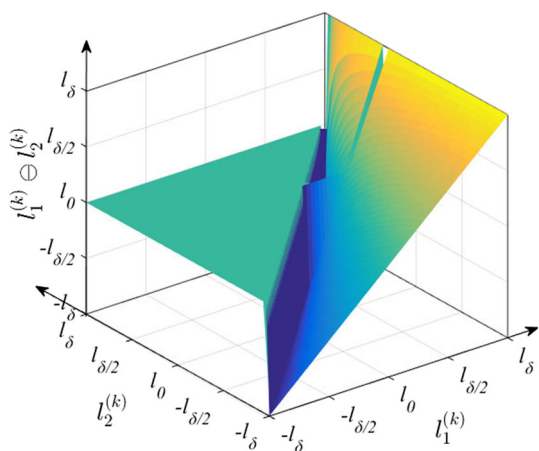
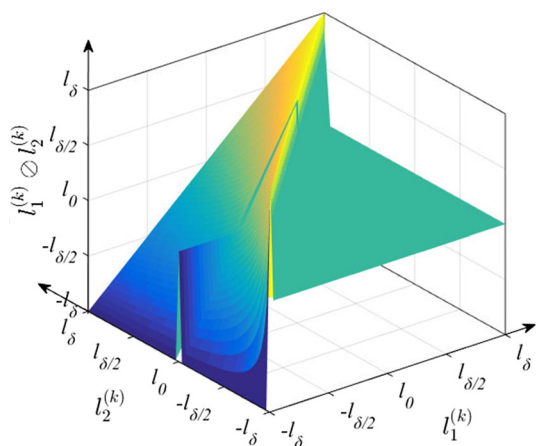**Fig. 5** The region of the subtraction operation $l_1^{(k)} \ominus l_2^{(k)}$

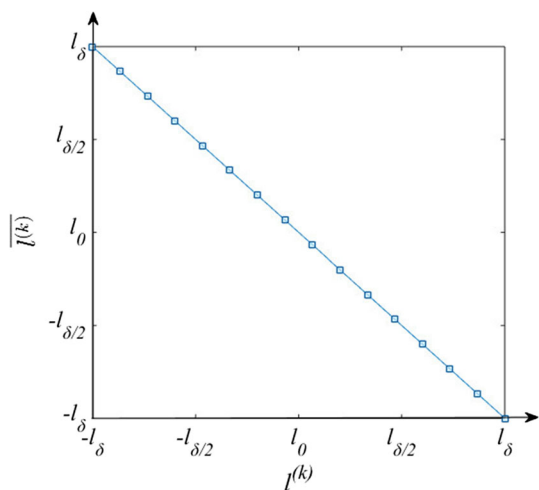**Fig. 6** The region of the division operation $l_1^{(k)} \oslash l_2^{(k)}$

Fig. 7 The region of the supplement operation $\overline{l^{(k)}}$

between the linguistic terms are drawn in Figs. 1, 2, 3, 4, 5, 6 and 7, and the probabilities of these operation results are the probabilities of the original PLTEs.

Based on the example adapted from Gou and Xu (2016), the new basic operations are used to calculate the PLTSs, as follows.

Example 1 For two PLTSs $PL_1(p) = \{l_3(0.5), l_2(0.2), l_1(0.3)\}$ and $PL_2(p) = \{l_0(0.3), l_{-2}(0.2)\}$ on LTS $L = \{l_t \mid t = -3, -2, -1, 0, 1, 2, 3\}$, and $\lambda = 2$. Then the associated PLTS $PL_2(p) = \{l_0\left(\frac{0.3}{0.2+0.3}\right), l_{-2}\left(\frac{0.2}{0.2+0.3}\right)\} = \{l_0(0.6), l_{-2}(0.4)\}$, and the adjusted PLTSs $PL_1^*(p) = \{l_3(0.5), l_2(0.1), l_2(0.1), l_1(0.3)\}$ and $PL_2^*(p) = \{l_0(0.5), l_0(0.1), l_{-2}(0.1), l_{-2}(0.3)\}$.

According to Definition 1, we get $\sigma(PL_1) = \{1, \frac{5}{6}, \frac{5}{6}, \frac{2}{3}\}$ and $\sigma(PL_2) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}\}$. Then,

(1) $PL_1(p) \oplus PL_2(p) \quad (k = 1, 2, 3, 4)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)}$$

$$\left\{\sigma^{-1}\left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \left\{\sigma^{-1}(1)(0.5), \sigma^{-1}\left(\frac{11}{12}\right)(0.1),\right.$$

$$\left.\sigma^{-1}\left(\frac{31}{36}\right)(0.1), \sigma^{-1}\left(\frac{13}{18}\right)(0.3)\right\}$$

$$= \left\{l_3(0.5), l_{2.50}(0.1), l_{2.17}(0.1), l_{1.33}(0.3)\right\}$$

$$= \left\{l_3(0.5), l_{1.33}(0.3), l_{2.50}(0.1), l_{2.17}(0.1)\right\}$$

(2) $PL_1(p) \otimes PL_2(p) \quad (k = 1, 2, 3, 4)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{\sigma^{-1}\left(\xi_1^{(k)}\xi_2^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \left\{\sigma^{-1}\left(\frac{1}{2}\right)(0.5), \sigma^{-1}\left(\frac{5}{12}\right)(0.1), \sigma^{-1}\left(\frac{5}{36}\right)(0.1),\right.$$

$$\left.\sigma^{-1}\left(\frac{1}{9}\right)(0.3)\right\}$$

$$= \left\{l_0(0.5), l_{-0.5}(0.1), l_{-2.17}(0.1), l_{-2.33}(0.3)\right\}$$

(3) $2PL_1(p) \quad (k = 1, 2, 3)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1)} \left\{\sigma^{-1}\left(1 - \left(1 - \xi_1^{(k)}\right)^2\right)\left(p^{(k)}\right)\right\}$$

$$= \left\{\sigma^{-1}(1)(0.5), \sigma^{-1}\left(\frac{35}{36}\right)(0.2), \sigma^{-1}\left(\frac{8}{9}\right)(0.3)\right\}$$

$$= \left\{l_3(0.5), l_{2.83}(0.2), l_{2.33}(0.3)\right\}$$

$$= \left\{l_3(0.5), l_{2.33}(0.3), l_{2.83}(0.2)\right\}$$

$2PL_2(p) \quad (k = 1, 2)$

$$= \bigcup_{\xi_2^{(k)} \in \sigma(PL_2)} \left\{\sigma^{-1}\left(1 - \left(1 - \xi_2^{(k)}\right)^2\right)\left(p^{(k)}\right)\right\}$$

$$= \left\{ \sigma^{-1}\left(\frac{3}{4}\right)(0.6), \sigma^{-1}\left(\frac{11}{36}\right)(0.4) \right\}$$

$$= \left\{ l_{1.50}(0.6), l_{-1.17}(0.4) \right\}$$

(4) $(PL_1(p))^2$ $(k = 1, 2, 3)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1)} \left\{ \sigma^{-1}\left((\xi_1^{(k)})^2\right)\left(p^{(k)}\right) \right\}$$

$$= \left\{ \sigma^{-1}(1)(0.5), \sigma^{-1}\left(\frac{25}{36}\right)(0.2), \sigma^{-1}\left(\frac{4}{9}\right)(0.3) \right\}$$

$$= \left\{ l_3(0.5), l_{1.17}(0.2), l_{-0.33}(0.3) \right\}$$

$(PL_2(p))^2$ $(k = 1, 2)$

$$= \bigcup_{\xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1}\left((\xi_2^{(k)})^2\right)\left(p^{(k)}\right) \right\}$$

$$= \left\{ \sigma^{-1}\left(\frac{1}{4}\right)(0.6), \sigma^{-1}\left(\frac{1}{36}\right)(0.4) \right\}$$

$$= \left\{ l_{-1.50}(0.6), l_{-2.83}(0.4) \right\}$$

(5) Since $\xi_2^{(k)} \le \xi_1^{(k)}$ and $\xi_2^{(k)} \ne 1$ for $\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2), k = 1, 2, 3, 4$, then

$$PL_1(p) \ominus PL_2(p)$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1}\left(\frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}}\right)\left(p^{(k)}\right) \right\},$$

$$= \left\{ \sigma^{-1}(1)(0.5), \sigma^{-1}\left(\frac{2}{3}\right)(0.1), \right.$$
$$\left. \sigma^{-1}\left(\frac{4}{5}\right)(0.1), \sigma^{-1}\left(\frac{3}{5}\right)(0.3) \right\}$$

$$= \left\{ l_3(0.5), l_1(0.1), l_{1.80}(0.1), l_{0.60}(0.3) \right\}$$

$$= \left\{ l_3(0.5), l_{0.60}(0.3), l_{1.80}(0.1), l_1(0.1) \right\}$$

(6) Since $\xi_2^{(k)} \le \xi_1^{(k)}$ and $\xi_1^{(k)} \ne 0$ for $\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2), k = 1, 2, 3, 4$, then

$$PL_2(p) \oslash PL_1(p)$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1), \xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1}\left(\frac{\xi_2^{(k)}}{\xi_1^{(k)}}\right)\left(p^{(k)}\right) \right\}$$

$$= \left\{ \sigma^{-1}\left(\frac{1}{2}\right)(0.5), \sigma^{-1}\left(\frac{3}{5}\right)(0.1), \right.$$
$$\left. \sigma^{-1}\left(\frac{1}{5}\right)(0.1), \sigma^{-1}\left(\frac{1}{4}\right)(0.3) \right\}$$

$$= \left\{ l_0(0.5), l_{0.60}(0.1), l_{-1.80}(0.1), l_{-1.50}(0.3) \right\}$$

$$= \left\{ l_{0.60}(0.1), l_0(0.5), l_{-1.80}(0.1), l_{-1.50}(0.3) \right\}$$

(7) $\overline{PL_1(p)}$ $(k = 1, 2, 3)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(PL_1)} \left\{ \sigma^{-1}\left(1 - \xi_1^{(k)}\right)\left(p^{(k)}\right) \right\}$$

$$= \left\{ \sigma^{-1}(0)(0.5), \sigma^{-1}\left(\frac{1}{6}\right)(0.2), \sigma^{-1}\left(\frac{1}{3}\right)(0.3) \right\}$$

$$= \left\{ l_{-3}(0.5), l_{-2}(0.2), l_{-1}(0.3) \right\}$$

$$= \left\{ l_{-1}(0.3), l_{-2}(0.2), l_{-3}(0.5) \right\}$$

$\overline{PL_2(p)}$ $(k = 1, 2)$

$$= \bigcup_{\xi_2^{(k)} \in \sigma(PL_2)} \left\{ \sigma^{-1}\left(1 - \xi_2^{(k)}\right)\left(p^{(k)}\right) \right\}$$

$$= \left\{ \sigma^{-1}\left(\frac{1}{2}\right)(0.6), \sigma^{-1}\left(\frac{5}{6}\right)(0.4) \right\}$$

$$= \left\{ l_0(0.6), l_2(0.4) \right\}$$

$$= \left\{ l_2(0.4), l_0(0.6) \right\}$$

Example 2 Let $PL_1(p) = PL_2(p) = \left\{ l_3(0.5), l_{-3}(0.5) \right\}$ be two PLTSs on LTS $L = \left\{ l_t \mid t = -3, -2, -1, 0, 1, 2, 3 \right\}$, then $PL_1(p) \oplus PL_2(p) = \left\{ \sigma^{-1}(1 + 1 - 1 \times 1)(0.5), \sigma^{-1}(0 + 0 - 0 \times 0)(0.5) \right\} = \{ l_3(0.5), l_{-3}(0.5) \}$.

Obviously, the results in this paper are more reasonable than the results in Gou and Xu (2016), as well as the calculations of these operations are simpler than the operational laws given by Gou and Xu (2016). Thus, it is rational to calculate on the PLTSs with the same probability set.

In the following, we propose some properties of the above operations.

Theorem 1 *The new operations of PLTSs in Definition* 9 *are closed.*

Proof To prove the closure of these operations, it is only necessary to prove the operational results of PLTSs by these new operations are still PLTSs.

(1) For any $\xi_1^{(k)} \in \sigma(PL_1) \subset [0, 1], \xi_2^{(k)} \in \sigma(PL_2) \subset [0, 1]$, it holds that $\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)} \in [0, 1], \sigma^{-1}\left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)}\right) \in [-\delta, \delta]$ since $\sigma: [-\delta, \delta] \to [0, 1]$ and $\sigma^{-1}: [0, 1] \to [-\delta, \delta]$.
Therefore, the linguistic terms $\sigma^{-1}\left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)}\right)$ belong to PLTSs.
Also, the probabilities information of the adjusted PLTSs does not change during the calculation. In other words, the sum of all probabilities in a result obtained by the addition operation is still equal to 1.
Hence, the addition operation for PLTSs in Definition 9 is closed.
Similarly, it is easy to prove that the operations (2), (3), (4) and (7) are also closed.

(5) For any $\xi_1^{(k)} \in \sigma(\mathrm{PL}_1) \subset [0,1], \xi_2^{(k)} \in \sigma(\mathrm{PL}_2) \subset$ [0, 1], $\xi_1^{(k)} \geq \xi_2^{(k)}$ and $\xi_2^{(k)} \neq 1$, it holds that $\frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}} \in$ [0, 1], $\sigma^{-1}\left(\frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}}\right) \in [-\delta, \delta]$ since $\sigma : [-\delta, \delta] \rightarrow$ [0, 1] and $\sigma^{-1} : [0, 1] \rightarrow [-\delta, \delta]$.

Therefore, the linguistic terms $\sigma^{-1}\left(\frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}}\right)$ belong to PLTSs. And the sum of all probabilities in a result obtained by the subtraction operation is equal to 1.

Hence, the subtraction operation for PLTSs in Definition 9 is closed.

Analogously, the operation (6) in Definition 9 can be easily proven to be closed.

In summary, the new operations of PLTSs in Definition 9 are closed. In other words, the operational results of PLTSs by these new operations are still PLTSs where the linguistic terms still belong to the LTS and the sum of all the corresponding probabilities is equal to 1. □

Theorem 2 *Suppose that* $\mathrm{PL}(p), \mathrm{PL}_1(p)$ *and* $\mathrm{PL}_2(p)$ *are three adjusted PLTSs, and* λ, λ_1 *and* λ_2 *are three positive real numbers. Then*

(1) $\mathrm{PL}_1(p) \oplus \mathrm{PL}_2(p) = \mathrm{PL}_2(p) \oplus \mathrm{PL}_1(p)$;

(2) $\mathrm{PL}_1(p) \otimes \mathrm{PL}_2(p) = \mathrm{PL}_2(p) \otimes \mathrm{PL}_1(p)$;

(3) $\lambda\left(\mathrm{PL}_1(p) \oplus \mathrm{PL}_2(p)\right) = \lambda \mathrm{PL}_2(p) \oplus \lambda \mathrm{PL}_1(p)$;

(4) $\lambda_1 \mathrm{PL}(p) \oplus \lambda_2 \mathrm{PL}(p) = (\lambda_1 + \lambda_2)\mathrm{PL}(p)$;

(5) $\left(\mathrm{PL}_1(p) \otimes \mathrm{PL}_2(p)\right)^\lambda = \left(\mathrm{PL}_1(p)\right)^\lambda \otimes \left(\mathrm{PL}_2(p)\right)^\lambda$;

(6) $\left(\mathrm{PL}(p)\right)^{\lambda_1} \otimes \left(\mathrm{PL}(p)\right)^{\lambda_2} = \left(\mathrm{PL}(p)\right)^{\lambda_1 + \lambda_2}$;

(7) $\lambda\left(\mathrm{PL}_1(p) \ominus \mathrm{PL}_2(p)\right) = \lambda \mathrm{PL}_1(p) \ominus \lambda \mathrm{PL}_2(p)$, *if* $\xi_1^{(k)} \geq \xi_2^{(k)}$ *and* $\xi_2^{(k)} \neq 1$ *for* $\xi_j^{(k)} \in \sigma(\mathrm{PL}_j)$, $k = 1, 2, \ldots, \#\mathrm{PL}_1(p) = \#\mathrm{PL}_2(p)$, $j = 1, 2$;

(8) $\lambda_1 \mathrm{PL}(p) \ominus \lambda_2 \mathrm{PL}(p) = (\lambda_1 - \lambda_2)\mathrm{PL}(p)$, *if* $\lambda_1 \geq \lambda_2$ *and* $\xi^{(k)} \neq 1$ *for* $\xi^{(k)} \in \sigma(\mathrm{PL})$, $k = 1, 2, \ldots, \#\mathrm{PL}(p)$;

(9) $\left(\mathrm{PL}_1(p) \oslash \mathrm{PL}_2(p)\right)^\lambda = \left(\mathrm{PL}_1(p)\right)^\lambda \oslash \left(\mathrm{PL}_2(p)\right)^\lambda$, *if* $\xi_1^{(k)} \leq \xi_2^{(k)}$ *and* $\xi_2^{(k)} \neq 0$ *for* $\xi_j^{(k)} \in \sigma(\mathrm{PL}_j)$, $k = 1, 2, \ldots, \#\mathrm{PL}_1(p) = \#\mathrm{PL}_2(p)$, $j = 1, 2$;

(10) $\left(\mathrm{PL}(p)\right)^{\lambda_1} \oslash \left(\mathrm{PL}(p)\right)^{\lambda_2} = \left(\mathrm{PL}(p)\right)^{\lambda_1 - \lambda_2}$, *if* $\lambda_1 \geq \lambda_2$ *and* $\xi^{(k)} \neq 0$ *for* $\xi^{(k)} \in \sigma(\mathrm{PL})$, $k = 1, 2, \ldots, \#L(p)$.

Proof

(1) $\mathrm{PL}_1(p) \oplus \mathrm{PL}_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(\xi_2^{(k)} + \xi_1^{(k)} - \xi_2^{(k)}\xi_1^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \mathrm{PL}_2(p) \oplus \mathrm{PL}_1(p)$$

(2) $\mathrm{PL}_1(p) \otimes \mathrm{PL}_2(p)$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(\xi_1^{(k)}\xi_2^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(\xi_2^{(k)}\xi_1^{(k)}\right)\left(p^{(k)}\right)\right\}$$

$$= \mathrm{PL}_2(p) \otimes \mathrm{PL}_1(p)$$

(3) $\lambda\left(\mathrm{PL}_1(p) \oplus \mathrm{PL}_2(p)\right)$

$$= \lambda\left(\bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(\xi_1^{(k)} + \xi_2^{(k)} - \xi_1^{(k)}\xi_2^{(k)}\right)\left(p^{(k)}\right)\right\}\right)$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(1 - \left(1 - \xi_1^{(k)} - \xi_2^{(k)} + \xi_1^{(k)}\xi_2^{(k)}\right)^\lambda\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(1 - \left((1 - \xi_1^{(k)})(1 - \xi_2^{(k)})\right)^\lambda\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{\sigma^{-1}\left(1 - (1 - \xi_2^{(k)})^\lambda\right)\left(p^{(k)}\right)\right\}$$

$$\oplus \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1)} \left\{\sigma^{-1}\left(1 - (1 - \xi_1^{(k)})^\lambda\right)\left(p^{(k)}\right)\right\}$$

$$= \lambda \mathrm{PL}_2(p) \oplus \lambda \mathrm{PL}_1(p)$$

(4) $\lambda_1 \mathrm{PL}(p) \oplus \lambda_2 \mathrm{PL}(p)$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{\sigma^{-1}\left(1 - (1 - \xi^{(k)})^{\lambda_1}\right)\left(p^{(k)}\right)\right\}$$

$$\oplus \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{\sigma^{-1}\left(1 - (1 - \xi^{(k)})^{\lambda_2}\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{\sigma^{-1}\left(1 - (1 - \xi^{(k)})^{\lambda_1}(1 - \xi^{(k)})^{\lambda_2}\right)\left(p^{(k)}\right)\right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{\sigma^{-1}\left(1 - (1 - \xi^{(k)})^{\lambda_1 + \lambda_2}\right)\left(p^{(k)}\right)\right\}$$

$$= (\lambda_1 + \lambda_2)\mathrm{PL}(p)$$

(5) $\left(\mathrm{PL}_1(p) \otimes \mathrm{PL}_2(p)\right)^{\lambda}$

$$= \left(\bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\big(\xi_1^{(k)}\xi_2^{(k)}\big)\big(p^{(k)}\big) \right\} \right)^{\lambda}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\big((\xi_1^{(k)})^{\lambda}(\xi_2^{(k)})^{\lambda}\big)\big(p^{(k)}\big) \right\}$$

$$= \left(\bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1)} \left\{ \big(\sigma^{-1}(\xi_1^{(k)})(p^{(k)})\big) \right\} \right)^{\lambda}$$

$$\otimes \left(\bigcup_{\xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \big(\sigma^{-1}(\xi_1^{(k)})(p^{(k)})\big) \right\} \right)^{\lambda}$$

$$= \big(\mathrm{PL}_1(p)\big)^{\lambda} \otimes \big(\mathrm{PL}_2(p)\big)^{\lambda}$$

(6) $\left(\mathrm{PL}(p)\right)^{\lambda_1} \otimes \left(\mathrm{PL}(p)\right)^{\lambda_2}$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_1}\big)\big(p^{(k)}\big) \right\}$$

$$\otimes \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_2}\big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_1}(\xi^{(k)})^{\lambda_2}\big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_1+\lambda_2}\big)\big(p^{(k)}\big) \right\}$$

$$= \big(\mathrm{PL}(p)\big)^{\lambda_1+\lambda_2}$$

(7) If $\xi_1^{(k)} \geq \xi_2^{(k)}$ and $\xi_2^{(k)} \neq 1$ for $\xi_j^{(k)} \in \sigma(\mathrm{PL}_j)$, $k = 1, 2, \ldots, \#\mathrm{PL}_1(p) = \#\mathrm{PL}_2(p)$; $j = 1, 2$, then

$$\lambda\big(\mathrm{PL}_1(p) \ominus \mathrm{PL}_2(p)\big)$$

$$= \lambda \left(\bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\Big(\frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}}\Big)\big(p^{(k)}\big) \right\} \right)$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)}$$

$$\left\{ \sigma^{-1}\Big(1 - \Big(1 - \frac{\xi_1^{(k)} - \xi_2^{(k)}}{1 - \xi_2^{(k)}}\Big)^{\lambda}\Big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)}$$

$$\left\{ \sigma^{-1}\Big(\frac{(1 - \xi_2^{(k)})^{\lambda} - (1 - \xi_1^{(k)})^{\lambda}}{(1 - \xi_2^{(k)})^{\lambda}}\Big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1)} \left\{ \sigma^{-1}\Big(1 - (1 - \xi_1^{(k)})^{\lambda}\Big)\big(p^{(k)}\big) \right\}$$

$$\ominus \bigcup_{\xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\Big(1 - (1 - \xi_2^{(k)})^{\lambda}\Big)\big(p^{(k)}\big) \right\}$$

$$= \lambda\mathrm{PL}_1(p) \ominus \lambda\mathrm{PL}_2(p)$$

(8) If $\lambda_1 \geq \lambda_2$ and $\xi^{(k)} \neq 1$ for $\xi^{(k)} \in \sigma(PL)$, $k = 1, 2, \ldots, \#\mathrm{PL}(p)$, then

$$\lambda_1\mathrm{PL}(p) \ominus \lambda_2\mathrm{PL}(p)$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\Big(1 - (1 - \xi^{(k)})^{\lambda_1}\Big)\big(p^{(k)}\big) \right\}$$

$$\ominus \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\Big(1 - (1 - \xi^{(k)})^{\lambda_2}\Big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\Big(\frac{(1 - \xi^{(k)})^{\lambda_2} - (1 - \xi^{(k)})^{\lambda_1}}{(1 - \xi^{(k)})^{\lambda_2}}\Big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\Big(1 - (1 - \xi^{(k)})^{\lambda_1-\lambda_2}\Big)\big(p^{(k)}\big) \right\}$$

$$= \big(\lambda_1 - \lambda_2\big)\mathrm{PL}(p)$$

(9) If $\xi_1^{(k)} \leq \xi_2^{(k)}$ and $\xi_2^{(k)} \neq 0$ for $\xi_j^{(k)} \in \sigma(\mathrm{PL}_j)$, $k = 1, 2, \ldots, \#\mathrm{PL}_1(p) = \#\mathrm{PL}_2(p)$; $j = 1, 2$, then

$$\left(\big(\mathrm{PL}_1(p) \oslash \mathrm{PL}_2(p)\big)\right)^{\lambda}$$

$$= \left(\bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\Big(\frac{\xi_1^{(k)}}{\xi_2^{(k)}}\Big)\big(p^{(k)}\big) \right\} \right)^{\lambda}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1), \xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\Big(\frac{(\xi_1^{(k)})^{\lambda}}{(\xi_2^{(k)})^{\lambda}}\Big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi_1^{(k)} \in \sigma(\mathrm{PL}_1)} \left\{ \sigma^{-1}\big((\xi_1^{(k)})^{\lambda}\big)\big(p^{(k)}\big) \right\}$$

$$\oslash \bigcup_{\xi_2^{(k)} \in \sigma(\mathrm{PL}_2)} \left\{ \sigma^{-1}\big((\xi_2^{(k)})^{\lambda}\big)\big(p^{(k)}\big) \right\}$$

$$= \big(\mathrm{PL}_1(p)\big)^{\lambda} \oslash \big(\mathrm{PL}_2(p)\big)^{\lambda}$$

(10) If $\lambda_1 \geq \lambda_2$ and $\xi^{(k)} \neq 0$ for $\xi^{(k)} \in \sigma(PL)$, $k = 1, 2, \ldots, \#\mathrm{PL}(p)$, then

$$\big(\mathrm{PL}(p)\big)^{\lambda_1} \oslash \big(\mathrm{PL}(p)\big)^{\lambda_2}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_1}\big)\big(p^{(k)}\big) \right\}$$

$$\oslash \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1}\big((\xi^{(k)})^{\lambda_2}\big)\big(p^{(k)}\big) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1} \left(\frac{(\xi^{(k)})^{\lambda_1}}{(\xi^{(k)})^{\lambda_2}} \right) \left(p^{(k)} \right) \right\}$$

$$= \bigcup_{\xi^{(k)} \in \sigma(\mathrm{PL})} \left\{ \sigma^{-1} \left((\xi^{(k)})^{\lambda_1 - \lambda_2} \right) \left(p^{(k)} \right) \right\}$$

$$= \left(\mathrm{PL}(p) \right)^{\lambda_1 - \lambda_2}$$

$\square$

Remark 4 From the above ten operations of PLTSs, we can see that the properties of the operations in this paper are in line with the operational laws given by Gou and Xu (2016).

4 The methods for the PL-MCDM problems

In this section, we mainly study the probabilistic linguistic E-VIKOR method and the improved PL-VIKOR method to tackle the PL-MCDM problems. First, we make the following provisions for the symbols that appear in the PL-MCDM problems.

4.1 Description and symbols clarification of the PL-MCDM problems

Suppose that a general PL-MCDM problem contains a set of alternatives $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ where ψ_i is the ith alternative, a set of criteria $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$ with the weight vector $\Omega = (\omega_1, \omega_2, \ldots, \omega_m)$, where $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{m} \omega_j = 1$. And the assessment of alternative ψ_i with respect to criterion θ_j is denoted as a PLTS $\mathrm{PL}_{ij}(p)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ via using the LTS L. Finally, a probabilistic linguistic decision matrix which consists of all PLTSs can be given, denoted by $\left(\mathrm{PL}_{ij}(p) \right)_{n \times m}$.

4.2 The probabilistic linguistic E-VIKOR method

In this subsection, we apply the E-VIKOR method directly to the probabilistic linguistic circumstance and propose the probabilistic linguistic E-VIKOR method.

The general VIKOR method (Opricovic 1998) aims to rank and select from a group of alternatives in which the criteria conflict with each other. It introduces a multiple criteria ranking index based on the L_q-metric which can take into account the "closeness" to the ideal solution. With the fuzzy measures (Shapley and Shubik 1953; Sugeno 1974; Sugeno and Terano 1977), in what follows, we consider using $\mathrm{SL}_{q,v}$-metric to rank the alternatives in the PL-MCDM problems.

Definition 10 Given $\left(\mathrm{PL}_{ij}(p) \right)_{n \times m}$ is probabilistic linguistic decision matrix, then the probabilistic linguistic positive

ideal solution (PL-PIS) is determined as follows:

$$\mathrm{PL}^+(p) = \left(\mathrm{PL}_1^+(p), \mathrm{PL}_2^+(p), \ldots, \mathrm{PL}_m^+(p) \right),$$

and the probabilistic linguistic negative ideal solution (PL-NIS) is identified as:

$$\mathrm{PL}^-(p) = \left(\mathrm{PL}_1^-(p), \mathrm{PL}_2^-(p), \ldots, \mathrm{PL}_m^-(p) \right),$$

where $\mathrm{PL}_j^+(p) = \max_{1 \leq i \leq n} \mathrm{PL}_{ij}(p)$, $\mathrm{PL}_j^-(p) = \min_{1 \leq i \leq n} \mathrm{PL}_{ij}(p)(j = 1, 2, \ldots, m)$ can be determined by Definitions 3 and 4.

Definition 11 Suppose that $\left(\mathrm{PL}_{ij}(p) \right)_{n \times m}$ is probabilistic linguistic decision matrix which contains a group of alternatives $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ and the criteria $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$, then the probabilistic linguistic $\mathrm{SL}_{q,v}$-metric over alternative $\psi_i (i = 1, 2, \ldots, n)(\mathrm{PLSL}_{q,v,i})$ is defined by

$$\mathrm{PLSL}_{q,v,i}$$
$$= \left\{ \sum_{j=1}^{m} \left[\phi_j \frac{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_{ij}(p) \right)}{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_j^-(p) \right)} \right]^q \right\}^{1/q}, 1 \leq q \leq \infty,$$

where v is a fuzzy measure on Θ, ϕ_j is the Shapley value of criteria $\theta_j (j = 1, 2, \ldots, m)$ satisfying $0 \leq \phi_j \leq 1, \sum_{j=1}^{m} \phi_j = 1$, $\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_{ij}(p) \right)$ and $\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_j^-(p) \right)$ are the probabilistic linguistic Euclidean distance measure, which can be calculated via Eq. (3).

Based on the $\mathrm{PLSL}_{q,v,i}$-metric, we propose the probabilistic linguistic group utility (PLGU) measure and the probabilistic linguistic individual regret (PLIR) measure for the alternative $\psi_i (i = 1, 2, \ldots, n)$ as follows.

Definition 12 The PLGU measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ is defined by:

$$\mathrm{PLGU}_i = \mathrm{PLSL}_{1,v,i} = \sum_{j=1}^{m} \phi_j \frac{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_{ij}(p) \right)}{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_j^-(p) \right)}.$$

Definition 13 The PLIR measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ is determined as:

$$\mathrm{PLIR}_i = \mathrm{PLSL}_{\infty,v,i} = \max_{1 \leq j \leq m} \left(\phi_j \frac{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_{ij}(p) \right)}{\rho \left(\mathrm{PL}_j^+(p), \mathrm{PL}_j^-(p) \right)} \right).$$

Obviously, $\min_{1 \leq i \leq n} \mathrm{PLGU}_i$ means that the result has the maximum group utility, and $\min_{1 \leq i \leq n} \mathrm{PLIR}_i$ indicates that the result has the minimum individual regret. As the purpose of the VIKOR method is to sort and select from a group of alternatives based on the idea of the "closest" ideal

solution, we give the definition of the probabilistic linguistic compromise (PLC) measure via minimizing PLGU_i and PLIR_i simultaneously to implement the goal of the VIKOR method.

Definition 14 The PLC measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ is defined by:

$$\text{PLC}_i = \eta \frac{\text{PLGU}_i - \text{PLGU}^+}{\text{PLGU}^- - \text{PLGU}^+} + (1 - \eta) \frac{\text{PLIR}_i - \text{PLIR}^+}{\text{PLIR}^- - \text{PLIR}^+},$$

where $\text{PLGU}^+ = \min_{1 \leq i \leq n} \text{PLGU}_i$, $\text{PLGU}^- = \max_{1 \leq i \leq n} \text{PLGU}_i$, $\text{PLIR}^+ = \min_{1 \leq i \leq n} \text{PLIR}_i$, $\text{PLIR}^- = \max_{1 \leq i \leq n} \text{PLIR}_i$, and the parameter $\eta \in [0, 1]$. The larger the value of η, the more average the DM's preference for different criteria. Without loss of generality, we assume $\eta = 0.5$.

Therefore, the algorithm of the probabilistic linguistic E-VIKOR method can be summarized as follows:

> **Step 1** Determine the $\text{PL}^+(p)$ and $\text{PL}^-(p)$ according to Definition 10.
> **Step 2** Compute the values of PLGU_i and $\text{PLIR}_i (i = 1, 2, \ldots, n)$ via Definitions 12 and 13, respectively.
> **Step 3** Compute the PLC_i values of each alternative $\psi_i (i = 1, 2, \ldots, n)$ by Definition 14.
> **Step 4** Sort the alternatives according to the descending order of PLGU_i, PLIR_i and PLC_i.
> **Step 5** Determine the compromise solution(s). Given the alternative $\psi_{(1)}$ which is ranked the best according to the PLC measure, then $\psi_{(1)}$ is a compromise solution if it meets:

A_1 $\text{PLC}_{\psi_{(2)}} - \text{PLC}_{\psi_{(1)}} \geq \frac{1}{n-1}$, where $\psi_{(2)}$ is the next-best alternative according to the PLC measure.
A_2 In the PLGU_i and PLIR_i rankings, $\psi_{(1)}$ is also the best alternative.

If one of the above two conditions is not satisfied, then the compromise solutions are composed of:

(1) $\psi_{(1)}$ and $\psi_{(2)}$ if only condition A_2 is not satisfied, or
(2) $\psi_{(1)}, \psi_{(2)}, \ldots, \psi_{(f)}$ if condition A_1 is not satisfied, where $\psi_{(f)}$ is established by $\text{PLC}_{\psi_{(f)}} - \text{PLC}_{\psi_{(1)}} < \frac{1}{n-1}$ for the maximum f.

4.3 The improved PL-VIKOR method for the PL-MCDM problems

Based on the above analyses, we can see that the compromise solutions obtained by the probabilistic linguistic E-VIKOR

method have the maximum group utility and the minimum individual regret. However, the probabilistic linguistic E-VIKOR method does not consider the distance between the alternatives and the PL-NIS in the process of determining the compromise solution, which may cause a large amount of useful information to be lost, resulting in inappropriate solutions when solving the PL-MCDM problems.

Therefore, it may not be appropriate to apply the E-VIKOR method directly to the probabilistic linguistic circumstance. Below we propose the improved PL-VIKOR method which can not only consider the distances between alternatives and the PL-PIS but also consider the distances between the alternatives and the PL-NIS to tackle the PL-MCDM problems.

In the steps of solving the PL-MCDM problems, the improved PL-VIKOR method is the same as the probabilistic linguistic E-VIKOR method except for three kinds of measures mentioned in the probabilistic linguistic E-VIKOR method. Thus, the **Steps 2** and **3** in the probabilistic linguistic E-VIKOR method are modified as follows:

Step 2 Calculate the improved PLGU measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ as:

$$\text{PLGU}'_i = \sum_{j=1}^{m} \phi_j \frac{\rho\left(\text{PL}_j^+(p), \text{PL}_{ij}(p)\right) - \rho\left(\text{PL}_j^-(p), \text{PL}_{ij}(p)\right)}{\rho\left(\text{PL}_j^+(p), \text{PL}_j^-(p)\right)},$$

and the improved PLIR measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ as:

$$\text{PLIR}'_i = \max_{1 \leq j \leq m} \left(\phi_j \frac{\rho\left(\text{PL}_j^+(p), \text{PL}_{ij}(p)\right) - \rho\left(\text{PL}_j^-(p), \text{PL}_{ij}(p)\right)}{\rho\left(\text{PL}_j^+(p), \text{PL}_j^-(p)\right)} \right).$$

Step 3 Compute the improved PLC measure over the alternative $\psi_i (i = 1, 2, \ldots, n)$ as:

$$\text{PLC}'_i = \eta \frac{\text{PLGU}'_i - \text{PLGU}'^+}{\text{PLGU}'^- - \text{PLGU}'^+} + (1 - \eta) \frac{\text{PLIR}'_i - \text{PLIR}'^+}{\text{PLIR}'^- - \text{PLIR}'^+},$$

where $\text{PLGU}'^+ = \min_{1 \leq i \leq n} \text{PLGU}'_i$, $\text{PLGU}'^- = \max_{1 \leq i \leq n} \text{PLGU}'_i$, $\text{PLIR}'^+ = \min_{1 \leq i \leq n} \text{PLIR}'_i$, $\text{PLIR}'^- = \max_{1 \leq i \leq n} \text{PLIR}'_i$, and the parameter $\eta \in [0, 1]$. The larger the value of η, the more average the DM's preference for different criteria. We assume that η in the above equation is equal to 0.5 in this paper.

Table 1 The probabilistic linguistic decision matrix

	θ_1	θ_2	θ_3	θ_4
ψ_1	$\{l_0(1)\}$	$\{l_1(0.6), l_2(0.2), l_0(0.2)\}$	$\{l_0(0.8), l_{-1}(0.2)\}$	$\{l_1(0.9), l_0(0.1)\}$
ψ_2	$\{l_0(0.2), l_{-2}(0.1), l_{-1}(0.7)\}$	$\{l_1(0.2), l_0(0.8)\}$	$\{l_1(0.4), l_0(0.6)\}$	$\{l_1(0.2), l_0(0.8)\}$
ψ_3	$\{l_0(0.1), l_{-1}(0.9)\}$	$\{l_0(0.6), l_{-1}(0.4)\}$	$\{l_{-1}(0.4), l_{-2}(0.6)\}$	$\{l_2(0.5), l_1(0.5)\}$
ψ_4	$\{l_{-1}(0.5), l_{-2}(0.5)\}$	$\{l_{-1}(1)\}$	$\{l_1(1)\}$	$\{l_0(0.1), l_{-1}(0.9)\}$
ψ_5	$\{l_{-1}(0.4), l_{-2}(0.6)\}$	$\{l_{-1}(0.1), l_{-2}(0.9)\}$	$\{l_1(0.1), l_0(0.9)\}$	$\{l_0(1)\}$

5 Case study: video recommender system

In this section, a PL-MCDM problem about the video recommender system is applied to clarify the applicability of the proposed methods.

5.1 Case presentation

Suppose that a video development company intends to rate five videos: ψ_1(Tou Tube), ψ_2(Tudou), ψ_3(Youku), ψ_4(Hulu), ψ_5(Netflix) with respect to four criteria: θ_1 (content), θ_2 (length), θ_3 (quality) and θ_4 (diversity), and the weight vector of criteria is $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, where $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{4} \omega_j = 1$, and LTS $L = \{l_{-3} = \text{terrible}, l_{-2} = \text{very bad}, l_{-1} = \text{bad}, l_0 = \text{medium}, l_1 = \text{good}, l_2 = \text{very good}, l_3 = \text{fantastic}\}$.

To obtain more objective assessments, the video development company organizes a decision organization that contains multiple DMs to evaluate the videos. In the process of evaluation, the DMs can consider several linguistic terms simultaneously. For example, half of the DMs may consider that the "θ_4 (diversity)" of the video ψ_1 is good and sometimes medium. We take "good (l_1)" with the probability 0.9 and "medium (l_0)" with the probability 0.1; thus, the PLTS is $\{l_1(0.9), l_0(0.1)\}$. Then, the final assessments of these videos can be obtained and a probabilistic linguistic decision matrix $(\text{PL}_{ij}(p))_{5 \times 4}$ can be constructed, shown as Table 1.

Suppose that the fuzzy measures of criteria are provided as follows:

$$\nu(\emptyset) = 0, \nu(\theta_1) = 0.300, \nu(\theta_2) = 0.350, \nu(\theta_3) = 0.300,$$
$$\nu(\theta_4) = 0.220.$$

For convenience, we use the λ-measure to calculate in this paper. Thus, $\lambda = -0.358$ and the λ-measure values of sub-criteria of Θ are shown in Table 2. And via Eq. (1), the weight vector of criteria is $\Omega = (0.256, 0.302, 0.257, 0.185)$.

Table 2 The f_λ values of each criteria set

Θ	$f_\lambda(\Theta)$	Θ	$f_\lambda(\Theta)$	Θ	$f_\lambda(\Theta)$
$\{\theta_1\}$	0.300	$\{\theta_1, \theta_3\}$	0.568	$\{\theta_1, \theta_2, \theta_3\}$	0.847
$\{\theta_2\}$	0.350	$\{\theta_1, \theta_4\}$	0.496	$\{\theta_1, \theta_2, \theta_4\}$	0.784
$\{\theta_3\}$	0.300	$\{\theta_2, \theta_3\}$	0.612	$\{\theta_1, \theta_3, \theta_4\}$	0.743
$\{\theta_4\}$	0.220	$\{\theta_2, \theta_4\}$	0.542	$\{\theta_2, \theta_3, \theta_4\}$	0.784
$\{\theta_1, \theta_2\}$	0.612	$\{\theta_3, \theta_4\}$	0.496	$\{\theta_1, \theta_2, \theta_3, \theta_4\}$	1.000

Table 3 The values of the $\rho(\text{PL}_j^+(p), \text{PL}_{ij}(p))$ and $\rho(\text{PL}_j^-(p), \text{PL}_{ij}(p))$

$i; j$	$\rho(\text{PL}_j^+(p), \text{PL}_{ij}(p))$				$\rho(\text{PL}_j^-(p), \text{PL}_{ij}(p))$			
	1	2	3	4	1	2	3	4
1	0	0	0.211	0.129	0.279	0.497	0.247	0.307
2	0.175	0.183	0.129	0.230	0.158	0.354	0.333	0.190
3	0.158	0.269	0.441	0	0.139	0.264	0	0.408
4	0.264	0.350	0	0.408	0.053	0.158	0.441	0
5	0.279	0.497	0.158	0.264	0	0	0.293	0.158

5.2 Application of the improved PL-VIKOR method in the PL-MCDM problems

In this subsection, we apply the improved PL-VIKOR method to solve the above PL-MCDM problem.
Step 1 Using Definition 10, the $\text{PL}^+(p)$ and the $\text{PL}^-(p)$ can be calculated as:

$$\text{PL}^+(p) = \Big(\{l_0(1)\}, \{l_1(0.6), l_2(0.2), l_0(0.2)\}, \{l_1(1)\},$$
$$\{l_2(0.5), l_1(0.5)\}\Big);$$

$$\text{PL}^-(p) = \Big(\{l_{-1}(0.4), l_{-2}(0.6)\}, \{l_{-1}(0.1), l_{-2}(0.9)\},$$
$$\{l_{-1}(0.4), l_{-2}(0.6)\}, \{l_0(0.1), l_{-1}(0.9)\}\Big).$$

Then, the $\rho(\text{PL}_j^+(p), \text{PL}_{ij}(p))$ and $\rho(\text{PL}_j^-(p), \text{PL}_{ij}(p))$ can be determined using Eq. (3), and the results are shown in Table 3. And from Table 3, we obtain $\rho(\text{PL}_1^+(p), \text{PL}_1^-(p)) = 0.279$, $\rho(\text{PL}_2^+(p), \text{PL}_2^-(p)) = 0.497$, $\rho(\text{PL}_3^+(p), \text{PL}_3^-(p)) = 0.441$, $\rho(\text{PL}_4^+(p), \text{PL}_4^-(p)) = 0.408$.

Table 4 The rankings derived by the PLGU'_i, PLIR'_i and PLC'_i

i	PLGU'_i	Ranking	PLIR'_i	Ranking	PLC'_i	Ranking
1	-0.660	1	-0.081	1	0	1
2	-0.189	2	0.018	2	0.328	2
3	0.093	3	0.257	4	0.758	4
4	0.238	4	0.194	3	0.737	3
5	0.527	5	0.302	5	1.000	5

Table 5 The ranking lists derived by the PLGU_i, PLIR_i and PLC_i

i	PLGU_i	Ranking	PLIR_i	Ranking	PLC_i	Ranking
1	0.182	1	0.123	1	0	1
2	0.451	2	0.161	2	0.335	2
3	0.565	3	0.257	4	0.700	3
4	0.640	4	0.245	3	0.722	4
5	0.770	5	0.302	5	1.000	5

Step 2 Calculate the values of the PLGU'_i, PLIR'_i and $\text{PLC}'_i (i = 1, 2, 3, 4, 5)$, and the results are listed in Table 4.

Thus, ψ_1(Tou Tube) is the best video for the above assessment problem.

6 Comparative analyses

To verify the superiority of our proposed improved PL-VIKOR method, the comparative analyses among the probabilistic linguistic E-VIKOR method, the general VIKOR method (Opricovic 1998), the extended TOPSIS method (Pang et al. 2016), and the improved PL-VIKOR method proposed by us are proposed below.

6.1 Compared with the probabilistic linguistic E-VIKOR method

In this section, comparisons between our proposed improved PL-VIKOR method and the probabilistic linguistic E-VIKOR method are conducted. The probabilistic linguistic E-VIKOR method obtained by applying the E-VIKOR method under hesitant fuzzy environment (Wei and Zhang 2014) directly to the probabilistic linguistic circumstance is a new method for solving the PL-MCDM problems. And the algorithm of the probabilistic linguistic E-VIKOR method has presented in Sect. 4.2.

When using the probabilistic linguistic E-VIKOR method to solve the above PL-MCDM problem, go to **Step 2** directly as the PL-PIS and the PL-NIS have been obtained in Sect. 5.2. Using Definition 12–14, the values of PLGU_i, PLIR_i and $\text{PLC}_i (i = 1, 2, 3, 4, 5)$ can be calculated, and the results are shown in Table 5.

Obviously, ψ_1(Tou Tube) is the best video for the above assessment problem. Since $0.335 - 0 = 0.335 \geq \frac{1}{5-1} = 0.25$ and the video ψ_1 is the best in the orders derived by PLGU_i and $\text{PLIR}_i (i = 1, 2, 3, 4, 5)$.

It is obvious that the compromise solutions derived by these two methods are consistent, but the ranking lists of the specific alternatives are slightly different. From Table 3, we see that $\rho\big(\text{PL}_1^+(p), \text{PL}_{21}(p)\big) > \rho\big(\text{PL}_1^+(p), \text{PL}_{31}(p)\big)$ and $\rho\big(\text{PL}_1^-(p), \text{PL}_{21}(p)\big) > \rho\big(\text{PL}_1^-(p), \text{PL}_{31}(p)\big)$. That is

to say, when we use the probabilistic linguistic E-VIKOR method, which only considers the distances between the alternatives and the PL-PIS, to solve the PL-MCDM problems, the distance between the alternative and the PL-PIS is large, it does not necessarily mean that the distance between the alternative and the PL-NIS is small. But the improved PL-VIKOR method we proposed in this paper can consider the distance between the alternatives and Pl-NIS as well as the distance between the alternatives and PL-PIS.

6.2 Compared with the general VIKOR method

In this section, our proposed improved PL-VIKOR method is compared with the general VIKOR method (Opricovic 1998). The purpose of the general VIKOR method is to sort and select from a group of alternatives based on the idea of the "closest" ideal solution. With this method, first of all, we determine the PL-PIS and the PL-NIS via the normalized Euclidean distance for PLTSs given by Lin and Xu (2018). Then, we obtain the rankings of the PLGU_i, PLIR_i and $\text{PLC}_i (i = 1, 2, 3, 4, 5)$. Finally, we determine the set of compromise solutions or the best solution. Now we use the general VIKOR method (Opricovic 1998) to solve the case about the video recommender system.

Step 1 Determine the $\text{PL}^+(p)$ and the $\text{PL}^-(p)$.

$$\text{PL}^+(p) = \Big(\{l_0(1)\}, \{l_1(0.6), l_2(0.2), l_0(0.2)\}, \{l_1(1)\},$$
$$\{l_2(0.5), l_1(0.5)\}\Big);$$
$$\text{PL}^-(p) = \Big(\{l_{-1}(0.4), l_{-2}(0.6)\}, \{l_{-1}(0.1), l_{-2}(0.9)\},$$
$$\{l_{-1}(0.4), l_{-2}(0.6)\}, \{l_0(0.1), l_{-1}(0.9)\}\Big).$$

Then via the normalized Euclidean distance introduced by Lin and Xu (2018), we obtain $\rho'\big(\text{PL}_1^+(p), \text{PL}_1^-(p)\big) = 0.269$, $\rho'\big(\text{PL}_2^+(p), \text{PL}_2^-(p)\big) = 0.130$, $\rho'\big(\text{PL}_3^+(p), \text{PL}_3^-(p)\big) = 0.384$, $\rho'\big(\text{PL}_4^+(p), \text{PL}_4^-(p)\big) = 0.260$, and the values of $\rho'\big(\text{PL}_j^+(p), \text{PL}_{ij}(p)\big)(i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4)$ are listed in Table 6.

Step 2 Obtain the rankings of the PLGU_i, PLIR_i and $\text{PLC}_i (i = 1, 2, 3, 4, 5)$. The results are given in Table 7.

Table 6 The values of $\rho'\big(\mathrm{PL}_j^+(p), \mathrm{PL}_{ij}(p)\big)$

$i; j$	1	2	3	4
1	0	0	0.194	0.239
2	0.268	0.213	0.354	0.206
3	0.382	0.084	0.384	0
4	0.243	0.119	0	0.260
5	0.269	0.130	0.530	0.243

Table 7 The ranking lists derived by the PLGU_i, PLIR_i and PLC_i

i	PLGU_i	Ranking	PLIR_i	Ranking	PLC_i	Ranking
1	0.230	1	0.170	1	0	1
2	1.133	5	0.495	5	1.000	5
3	0.816	3	0.364	4	0.623	3
4	0.693	2	0.276	2	0.420	2
5	1.086	4	0.355	3	0.789	4

Step 3 Identify the set of compromise solutions or the best solution. From Table 7, we can find that the video ψ_1 (Tou Tube) is the best solution.

Thus, when using the general VIKOR method to solve the PL-MCDM problems, we can find two problems below:

(1) for two PLTSs with different cardinalities, we have to add linguistic terms to the PLTS with a smaller cardinality until two PLTSs have an equal number of linguistic terms. The added linguistic terms would cause information that is not true, resulting in different results.

(2) from Table 6, we can find multiple outliers, for instance, $\rho'\big(\mathrm{PL}_1^+(p), \mathrm{PL}_{31}(p)\big) > \rho'\big(\mathrm{PL}_1^+(p), \mathrm{PL}_1^-(p)\big), \rho'\big(\mathrm{PL}_2^+(p), \mathrm{PL}_{22}(p)\big) > \rho'\big(\mathrm{PL}_2^+(p), \mathrm{PL}_2^-(p)\big)$, and $\rho'\big(\mathrm{PL}_3^-(p), \mathrm{PL}_{53}(p)\big) > \rho'\big(\mathrm{PL}_3^+(p), \mathrm{PL}_3^-(p)\big)$, which are not in accordance with the actual situations.

However, using the improved PL-VIKOR method to solve the PL-MCDM problems, we can avoid calculating the probabilities of the PLTSs, as well as we don't need to add linguistic terms to the PLTS with a smaller cardinality.

6.3 Compared with the existing method

In the following, comparisons between our proposed the improved PL-VIKOR method and the extended TOPSIS method (Pang et al. 2016) are conducted. When using the extended TOPSIS method to solve the PL-MCDM problems, the decision matrix should be determined first, and then the PL-PIS and the PL-NIS determined. Next, the deviation degrees between each video and the PL-PIS/PL-NIS should be calculated. Finally, rank the videos through the closeness coefficient CI. Now we employ the extended TOPSIS method to handle the above problem.

Step 1 Normalize the probabilistic linguistic decision matrix, and the results are listed in Table 8.

Step 2 Determine the PL-PIS and the PL-NIS according to (Pang et al. 2016):

$$\mathrm{PL}^+(p) = \Big(\{l_0, l_0, l_0\}, \{l_{0.6}, l_{0.4}, l_0\}, \{l_1, l_0, l_0\}, \{l_1, l_{0.5}, l_0\}\Big);$$

$$\mathrm{PL}^-(p) = \Big(\{l_{-0.5}, l_{-1.2}, l_{-0.7}\}, \{l_{-1}, l_{-1.8}, l_0\},$$
$$\{l_{-0.4}, l_{-1.2}, l_0\}, \{l_0, l_{-0.9}, l_0\}\Big).$$

Step 3 Calculate the derivation degrees between each video $\psi_i (i = 1, 2, \ldots, n)$ and the PL-PIS/PL-NIS. The results as listed in Table 9.

Where $d_{\min}\big(\psi_i, \mathrm{PL}^+(p)\big)$ is the smallest deviation degree between the video ψ_i and PL-PIS, and $d_{\max}\big(\psi_i, \mathrm{PL}^-(p)\big)$ is the largest deviation degree between the video $\psi_i (i = 1, 2, 3, 4, 5)$ and the PL-NIS.

Step 4 Using the closeness coefficient CI to rank the videos. The ranking result is $\psi_1 > \psi_2 > \psi_3 > \psi_4 > \psi_5$. Thus, ψ_1(Tou Tube) is the best video.

Table 8 The normalized probabilistic linguistic decision matrix

	θ_1	θ_2	θ_3	θ_4
ψ_1	$\{l_0(1), l_0(0), l_0(0)\}$	$\{l_1(0.6), l_2(0.2), l_0(0.2)\}$	$\{l_0(0.8), l_{-1}(0.2), l_{-1}(0)\}$	$\{l_1(0.9), l_0(0.1), l_0(0)\}$
ψ_2	$\{l_0(0.2), l_{-2}(0.1), l_{-1}(0.7)\}$	$\{l_1(0.2), l_0(0.8), l_0(0)\}$	$\{l_1(0.4), l_0(0.6), l_0(0)\}$	$\{l_1(0.2), l_0(0.8), l_0(0)\}$
ψ_3	$\{l_0(0.1), l_{-1}(0.9), l_{-1}(0)\}$	$\{l_0(0.6), l_{-1}(0.4), l_{-1}(0)\}$	$\{l_{-1}(0.4), l_{-2}(0.6), l_{-2}(0)\}$	$\{l_2(0.5), l_1(0.5), l_1(0)\}$
ψ_4	$\{l_{-1}(0.5), l_{-2}(0.5), l_{-2}(0)\}$	$\{l_{-1}(1), l_{-1}(0), l_{-1}(0)\}$	$\{l_1(1), l_1(0), l_1(0)\}$	$\{l_0(0.1), l_{-1}(0.9), l_{-1}(0)\}$
ψ_5	$\{l_{-1}(0.4), l_{-2}(0.6), l_{-2}(0)\}$	$\{l_{-1}(0.1), l_{-2}(0.9), l_{-2}(0)\}$	$\{l_1(0.1), l_0(0.9), l_0(0)\}$	$\{l_0(1), l_0(0), l_0(0)\}$

Table 9 The values of the deviation degrees

	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	Min or max
$d\big(\psi_i, \mathrm{PL}^+(p)\big)$	0.204	0.350	0.581	0.637	0.768	$d_{\min}\big(\psi_i, \mathrm{PL}^+(p)\big) = 0.204$
$d\big(\psi_i, \mathrm{PL}^-(p)\big)$	0.988	0.992	0.618	0.796	0.589	$d_{\max}\big(\psi_i, \mathrm{PL}^-(p)\big) = 0.992$

Table 10 The ranking lists and compromise solutions derived by the above four methods

Methods	Rankings	Compromise solutions
The improved PL-VIKOR method	$\psi_1 > \psi_2 > \psi_4 > \psi_3 > \psi_5$	ψ_1
The probabilistic linguistic E-VIKOR method	$\psi_1 > \psi_2 > \psi_3 > \psi_4 > \psi_5$	ψ_1
The general VIKOR method (Opricovic 1998)	$\psi_1 > \psi_4 > \psi_3 > \psi_5 > \psi_2$	ψ_1
The extended TOPSIS method (Pang et al. 2016)	$\psi_1 > \psi_2 > \psi_3 > \psi_4 > \psi_5$	ψ_1

In this calculation, we find that using fuzzy measures will lead to different sorting results obtained by these two methods. In the process of solving the PL-MCDM problems where criteria are interactive, the extended TOPSIS method that does not use fuzzy measures can only consider the distances from the PL-PIS and PL-NIS, and cannot consider their relative importance. However, the improved PL-VIKOR method can not only take all distances into consideration but also consider the relative importance of these distances. In order to facilitate comparison, the ranking lists, which is based on the consistency of weight information, and compromise solutions derived by the above four methods are listed in Table 10.

In summary, the improved PL-VIKOR method is more precise than the probabilistic linguistic E-VIKOR method, the general VIKOR method, and the extended TOPSIS method in solving the PL-MCDM problems.

7 Conclusions

In this paper, with the adjusted PLTSs and the equivalent transformation functions, we have proposed some new basic operations ground on the Algebraic t-conorm and t-norm. And then a numerical example has been provided to clarify the rationality of these new operations. After that, the improved PL-VIKOR method was introduced to tackle the PL-MCDM problems where criteria are interactive. A case study about the video recommender system has been conducted to illustrate the applicability and effectiveness of the proposed method, as well as the probabilistic linguistic E-VIKOR method, the general VIKOR method, and the extended TOPSIS method have used to tackle this problem. Compared with these existing methods, we have discovered that the improved PL-VIKOR method can not only reduce the loss of evaluation information but also consider the distance between the alternatives and PL-NIS while considering the distance between the alternatives and PL-PIS in the PL-MCDM problems where criteria are interactive.

In the future, we will apply our operations of PLTSs to other linguistic decision making methods and the method to solve the PL-MCDM problems where the assessments are completely unknown or partly known is also an issue to be investigated. Furthermore, we may consider establishing the

decision support system or the other recommender system under the probabilistic linguistic circumstance.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this work.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Agbodah K, Darko AP (2019) Probabilistic linguistic aggregation operators based on einstein t-norm and t-conorm and their application in multi-criteria group decision making. Symmetry 11:1–36

Bai CZ, Zhang R, Qian LX, Wu YN (2017) Comparisons of probabilistic linguistic term sets for multi-criteria decision making. Knowl-Based Syst 119:284–291

Chen SX, Wang JQ, Wang TL (2019) Cloud-based ERP system selection based on extended probabilistic linguistic MULTIMOORA method and Choquet integral operator. Comput Appl Math 38:1–32

Delgado M, Herrera F, Herrera-Viedma E, Martínez L (1998) Combining numerical and linguistic information in group decision making. J Inf Sci 107:177–194

Farhadinia B, Herrera-Viedma E (2019a) Multiple criteria group decision making method based on extended hesitant fuzzy sets with unknown weight information. Appl Soft Comput 78:310–323

Farhadinia B, Herrera-Viedma E (2019b) Sorting of decision-making methods based on their outcomes using dominance-vector hesitant fuzzy-based distance. Soft Comput 23:1109–1121

Feng XQ, Liu Q, Wei CP (2019) Probabilistic linguistic QUALIFLEX approach with possibility degree comparison. J Intell Fuzzy Syst 36:719–730

Gao J, Xu ZS, Ren PJ, Liao HC (2019) An emergency decision making method based on the multiplicative consistency of probabilistic linguistic preference relations. Int J Mach Learn Cybern 10:1613–1629

Gou XJ, Xu ZS (2016) Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic term sets. Inf Sci 372:407–427

Gou XJ, Xu ZS, Liao HC (2017) Multiple criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information. Soft Comput 21:6515–6529

Herrera F, Herrera-Viedma E, Verdegay JL (1995) A sequential selection process in group decision making with a linguistic assessment approach. Inf Sci 85:223–239

Herrera F, Herrera-Viedma E, Martínez L (2008) A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. IEEE Trans Fuzzy Syst 16:354–370

Herrera F, Herrera-Viedma E, Alonso S, Chiclana F (2009) Computing with words and decision making. Fuzzy Optim Decis Mak 8:323–324

Herrera-Viedma E (2001) Modeling the retrieval process for an information retrieval system using an ordinal fuzzy linguistic approach. J Am Soc Inf Sci Technol 52:460–475

Herrera-Viedma E, López-Herrera AG (2010) A review on information accessing systems based on fuzzy linguistic modelling. Int J Comput Intell Syst 3:420–437

Herrera-Viedma E, Peis E (2003) Evaluating the informative quality of documents in SGML format from judgements by means of fuzzy linguistic techniques based on computing with words. Inf Process Manag 39:233–249

Herrera-Viedma E, Pasi G, López-Herrera AG, Porcel C (2006) Evaluating the information quality of Web sites: a methodology based on fuzzy computing with words. J Am Soc Inf Sci Technol 57:538–549

Herrera-Viedma E, Alonso S, Chiclana F, Herrera F (2007a) A consensus model for group decision making with incomplete fuzzy preference relations. IEEE Trans Fuzzy Syst 15:863–877

Herrera-Viedma E, López-Herrera AG, Luque M, Porcel C (2007b) A fuzzy linguistic IRS model based on a 2-tuple fuzzy linguistic approach. Int J Uncertain Fuzziness Knowl-Based Syst 15:225–250

Herrera-Viedma E, Peis E, Morales-del-Castillo JM, Alonso S, Anaya K (2007c) A fuzzy linguistic model to evaluate the quality of Web sites that store XML documents. Int J Approx Reason 46:226–253

Herrera-Viedma E, Cabrerizo FJ, Chiclana F, Wu J, Cobo MJ, Samuylov K (2017) Consensus in group decision making and social networks. Stud Inf Control 26:259–268

Jiang LS, Liao HC, Li Z (2018) Probabilistic linguistic linear least absolute regression for fashion trend forecasting. In: Wong W (ed) Artificial intelligence on fashion and textiles. AITA 2018. Advances in intelligent systems and computing, vol 849. Springer, Cham

Krishankumar R, Saranya R, Nethra RP, Ravichandran KS, Kar S (2019) A decision-making framework under probabilistic linguistic term set for multi-criteria group decision-making problem. J Intell Fuzzy Syst 36:5783–5795

Li P, Wei CP (2019) An emergency decision-making method based on D–S evidence theory for probabilistic linguistic term sets. Int J Disaster Risk Reduct 37:1–10

Li XY, Wang ZL, Xiong Y, Liu HC (2019) A novel failure mode and effect analysis approach integrating probabilistic linguistic term sets and fuzzy Petri nets. IEEE Access 7:54918–54928

Liao HC, Xu ZS, Zeng XJ (2015a) Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making. IEEE Trans Fuzzy Syst 23:1343–1355

Liao HC, Xu ZS, Zeng XJ, Merigó JM (2015b) Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. Knowl-Based Syst 76:127–138

Liao HC, Mi XM, Xu ZS (2019a) A survey of decision-making methods with probabilistic linguistic information: bibliometrics, preliminaries, methodologies, applications and future directions. Fuzzy Optim Decis Mak. https://doi.org/10.1007/s10700-019-09309-5

Liao HC, Jiang LS, Lev B, Fujita H (2019b) Novel operations of PLTSs based on the disparity degrees of linguistic terms and their use in designing the probabilistic linguistic ELECTRE III method. Appl Soft Comput J 80:450–464

Lin MW, Xu ZS (2018) Probabilistic linguistic distance measures and their applications in multi-criteria group decision making. Stud Fuzziness Soft Comput 357:411–440

Lin MW, Xu ZS, Zhai YL, Yao ZQ (2017) Multi-attribute group decision-making under probabilistic uncertain linguistic environment. J Oper Res Soc 69:157–170

Lin MW, Wang HB, Xu ZS, Yao ZQ, Huang JL (2018) Clustering algorithms based on correlation coefficients for probabilistic linguistic term sets. Int J Intell Syst 33:2402–2424

Lin MW, Chen ZY, Liao HC, Xu ZS (2019) ELECTRE II method to deal with probabilistic linguistic term sets and its application to edge computing. Nonlinear Dyn. https://doi.org/10.1007/s11071-019-04910-0

Liu PD, Teng F (2018) Some Muirhead mean operators for probabilistic linguistic term sets and their applications to multiple attribute decision-making. Appl Soft Comput 68:396–431

Liu PD, Teng F (2019) Probabilistic linguistic TODIM method for selecting products through online product reviews. Inf Sci 485:441–455

Liu PD, Li Y, Teng F (2019) Bidirectional projection method for probabilistic linguistic multi-criteria group decision-making based on power average operator. Int J Fuzzy Syst. https://doi.org/10.1007/s40815-019-00705-y

Mao XB, Wu M, Dong JY, Wan SP, Jin Z (2019) A new method for probabilistic linguistic multi-attribute group decision making: application to the selection of financial technologies. Appl Soft Comput. https://doi.org/10.1016/j.asoc.2019.01.009

Mi XM, Liao HC, Wu XL, Xu ZS (2020) Probabilistic linguistic information fusion: a survey on aggregation operators in terms of principles, definitions, classifications, applications and challenges. Int J Intell Syst. https://doi.org/10.1002/int.22216

Morente-Molinera JA, Al-Hmouz R, Morfeq A, Balamash AS, Herrera-Viedma E (2016) A decision support system for decision making in changeable and multi-granular fuzzy linguistic contexts. J Mult-Valued Log Soft Comput 26:485–514

Morente-Molinera JA, Kou G, González-Crespo R, Corchado JM (2017) Solving multi-criteria group decision making problems under environments with a high number of alternatives using fuzzy ontologies and multi-granular linguistic modelling methods. Knowl-Based Syst 137:54–64

Morente-Molinera JA, Kou G, Peng Y, Torres-Albero C (2018) Analysing discussions in social networks using group decision making methods and sentiment analysis. Inf Sci 447:157–168

Morente-Molinera JA, Kou G, Samuylov K, Ureña R, Herrera-Viedma E (2019) Carrying out consensual group decision making processes under social networks using sentiment analysis over comparative expressions. Knowl-Based Syst 165:335–345

Opricovic S (1998) Multicriteria optimization of civil engineering systems. Faculty of Civil Engineering, Belgrade

Pan L, Ren PJ, Xu ZS (2018) Therapeutic schedule evaluation for brain-metastasized non-small cell lung cancer with a probabilistic linguistic ELECTRE II method. Int J Environ Res Public Health 15:1–23

Pang Q, Wang H, Xu ZS (2016) Probabilistic linguistic term sets in multi-attribute group decision making. Inf Sci 369:128–143

Rodríguez RM, Martínez L, Herrera F (2012) Hesitant fuzzy linguistic term sets for decision making. IEEE Trans Fuzzy Syst 20:109–119

Shapley LS, Shubik M (1953) Solutions of n-person games with ordinal utilities. Econometrica 21:348–349

Song YM, Hu J (2019) Large-scale group decision making with multiple stakeholders based on probabilistic linguistic preference relation. Appl Soft Comput J 80:721–722

Song YM, Li GX (2018) A large-scale group decision-making with incomplete multi-granular probabilistic linguistic term sets and its application in sustainable supplier selection. J Oper Res Soc 70:827–841

Sugeno M (1974) Theory of fuzzy integrals and applications. Ph.D. thesis, Tokyo Institute of Technology

Sugeno M, Terano T (1977) A model of learning based on fuzzy information. Kybernetes 6:157–166

Szmidt E, Kacprzyk J (2001) Entropy for intuitionistic fuzzy sets. Fuzzy Sets Syst 118:467–477

Tang M, Long YL, Liao HC, Xu ZS (2019) Inclusion measures of probabilistic linguistic term sets and their application in classifying cities in the Economic Zone of Chengdu Plain. Appl Soft Comput J 82:1–16

Tian ZP, Nie RX, Wang JQ (2019) Probabilistic linguistic multi-criteria decision-making based on evidential reasoning and combined ranking methods considering decision-makers' psychological preferences. J Oper Res Soc. https://doi.org/10.1080/01605682.2019.1632752

Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25:529–539

Wang ZY, Klir GJ (2009) Generalized measure theory. Springer Press, New York

Wang XK, Wang JQ, Zhang HY (2019a) Distance-based multicriteria group decision-making approach with probabilistic linguistic term sets. Expert Syst. https://doi.org/10.1111/exsy.12352

Wang XX, Xu ZS, Gou XJ (2019b) Nested probabilistic-numerical linguistic term sets in two-stage multi-attribute group decision making. Appl Intell 49:2582–2602

Wang XX, Xu ZS, Gou XJ, Xu M (2019c) Distance and similarity measures for nested probabilistic-numerical linguistic term sets applied to evaluation of medical treatment. Int J Fuzzy Syst. https://doi.org/10.1007/s40815-019-00625-x

Wątróbski J, Jankowski J, Ziemba P, Karczmarczyk A, Zioło M (2019) Generalised framework for multi-criteria method selection. Omega 86:107–124

Wei GW, Zhang N (2014) A multiple criteria hesitant fuzzy decision making with Shapley value-based VIKOR method. J Intell Fuzzy Syst 26:1065–1075

Wu XL, Liao HC (2018a) A consensus-based probabilistic linguistic gained and lost dominance score method. Eur J Oper Res. https://doi.org/10.1016/j.ejor.2018.07.044

Wu XL, Liao HC (2018b) An approach to quality function deployment based on probabilistic linguistic term sets and ORESTE method for multi-expert multi-criteria decision making. Inf Fusion 43:13–26

Wu XL, Liao HC, Xu ZS, Hafezalkotob A, Herrera F (2018) Probabilistic linguistic MULTIMOORA: a multi-criteria decision making method based on the probabilistic linguistic expectation function and the improved Borda rule. IEEE Trans Fuzzy Syst 26:3688–3702

Xian SD, Chai JH, Yin YB (2019) A visual comparison method and similarity measure for probabilistic linguistic term sets and their applications in multi-criteria decision making. Int J Fuzzy Syst 21:1154–1169

Yu WW, Zhang H, Li BQ (2019a) Comparison and operators based on uncertain probabilistic linguistic term set. J Intell Fuzzy Syst 36:6359–6379

Yu WW, Zhang H, Li BQ (2019b) Operators and comparisons of probabilistic linguistic term sets. Int J Intell Syst 34:1476–1504

Yue N, Xie JL, Chen SL (2019) The method of hesitant fuzzy multiple attribute decision making based on the Sugeno integral form. J Sichuan Normal Univ (Nat Sci) 42:86–91

Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning—I. Inf Sci 8:199–249

Zhang YX, Xu ZS, Wang H, Liao HC (2016) Consistency-based risk assessment with probabilistic linguistic preference relation. Appl Soft Comput 49:817–833

Zhang XF, Xu ZS, Ren PJ (2018a) A novel hybrid correlation measure for probabilistic linguistic term sets and crisp numbers and its application in customer relationship management. Int J Inf Technol Decis Mak. https://doi.org/10.1142/S021962201950007X

Zhang YX, Xu ZS, Liao HC (2018b) An ordinal consistency-based group decision making process with probabilistic linguistic preference relation. Inf Sci 467:179–198