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FOUNDATIONS

On the initial value problem for fuzzy differential equations of non-integer order $\alpha \in (1, 2)$

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Abstract
The purpose of this paper is to present some basic theories of an initial value problem of fuzzy fractional differential equations involving the Caputo-fuzzy-type concept of fractional derivative in the case of the order $\alpha \in (1, 2)$. The existence and uniqueness results of the solution for the given problem are presented. Finally, some examples are given to illustrate our main results.

Keywords Fractional integral equations · Uniqueness theorem · Fractional fuzzy integral equations · Fractional fuzzy differential equations

1 Introduction

During the past decades, the theory of fuzzy analysis and fuzzy differential equations has attracted a large number of mathematicians, and the study of this field has become one of the most important topics in the uncertain theory and uncertain differential equations. This is due to the intensive development in the theory of fuzzy analysis as well as its applications in many fields of dynamical systems influenced by uncertain factors raised by impreciseness, vagueness, and incomplete information. For the fundamental results and the recent development in the theory of fuzzy differential equations, the readers can refer to the papers (Ahmad et al. 2013; Allahviranloo et al. 2012b; Gasilov et al. 2014; Gomes and Barros 2015; Khastan et al. 2014a; Stefanini and Bede 2009; Stefanini 2010; Bede et al. 2007; Bede and Gal 2005; Bede and Stefanini 2013; Chalco-Cano et al. 2013) and references cited therein. In recent years, the interest in the investigation of fuzzy differential equations of non-integer order lies in the fact that these theories provide a strong tool in describing uncertainty that appears in many fields of dynamical systems influenced by impreciseness vagueness and in exhibiting nonstandard dynamical behaviors with a long memory or with hereditary effects. With this advantage, the fractional order models of fuzzy differential equations become more practical than the classical integer-order models, so the field of fractional fuzzy differential equations has attracted the attention of several researchers. There was a notable development in the fundamental theories of fuzzy fractional calculus and fuzzy fractional differential equations such as the study of fuzzy fractional differential equations in the sense of Riemann–Liouville fractional differentiability based on classical Hukuhara difference; see for instance the papers by Agarwal et al. (2010), in Arshad and Lupulescu (2011), Khastan et al. (2014b) and Allahviranloo et al. (2012a). Particular attention has been given to the fundamental results in theory of fuzzy-type fractional calculus based on the concepts of generalized Hukuhara derivative of fuzzy functions, and to the studies of the qualitative theories of fuzzy fractional differential equations and partial fuzzy differential equations with Caputo fractional derivative concept; see for instance the papers by Allahviranloo et al. (2014), An et al. (2017a, b, 2019), Fard and Salehi (2014), Hoa (2015a, 2018), Hoa et al. (2017, 2018, 2019), Hoa and Ho (2019), Long (2018), Long et al. (2017a, b), Lupulescu (2015), Mazandarani and Najariyan (2014), Noeiaghdam et al. (2019), Prakash et al. (2015), Salahshour et al. (2012), Son (2018), Son and Thao (2019) and the references therein. Very recently, the methods for solving the exact and numerical solutions of fuzzy frac-

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tional differential equations are proposed by Mazandarani and Kamyad (2013) with a modified fractional Euler method, Hoa (2015b); Hoa et al. (2017) with the modified fractional Euler method and the modified Adams–Bashforth–Moulton method, Ahmadian et al. (2017a, b) with the methods based on operational matrix of shifted Chebyshev polynomials and the spectral tau, Allahviranloo et al. (2012c) with the method of fuzzy Laplace transforms. Meanwhile, based on the generalized Hukuhara difference, Allahviranloo and Ghanbari (2020) also proposed a new concept of fuzzy fractional derivative the so-called ABC generalized Hukuhara fractional derivative. Moreover, the authors presented the existence of the unique solution and the numerical method to solve the solution for fuzzy fractional differential equations with ABC fractional derivative.

To the best of our knowledge, most problems studied in the previous works involve only concepts of a fractional derivative with non-integer order $\alpha \in (0, 1)$, and there are only a few remarkable works which provide the basic theories of initial value problems of fuzzy fractional differential equations in the case of a fractional derivative with non-integer $\alpha \in (1, 2)$. Recently, with the concept of fuzzy fractional derivative in the sense of Caputo–Hadamard, which differs from the Riemann–Liouville and Caputo fractional derivatives in the sense that the kernel of the integral in its definitions contains the logarithmic function of arbitrary exponent, in An et al., preprint the authors presented the existence and uniqueness of solutions of the fuzzy fractional differential equations of non-integer $\alpha \in (1, 2)$ by using the Schauder fixed point theorem. Moreover, by employing the reproducing kernel Hilbert space method, in Hasan et al. (2017) the authors proposed a method to solve the approximate solutions of the fuzzy fractional differential equations of non-integer $\alpha \in (1, 2)$ with the concept of Caputo fractional derivative. By continuing the above works, in this paper we introduce a new class of the initial value problem of fuzzy fractional differential equations with the concept of Caputo fractional derivative in the case of non-integer order $\alpha \in (1, 2)$ as follows:

$$
\begin{aligned}
&{}^{C}\mathcal{D}^{\alpha}_{a^+}x(t) = f(t, x, {}^{C}\mathcal{D}^{\alpha-1}_{a^+}x(t)),\\
&x(a) = x_0, \quad x'(a) = x'_0, \quad t \in [a, b], \qquad (1.1)
\end{aligned}
$$

where $a \leq t \leq b$, x_0 and x'_0 are the given fuzzy sets, ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ is the fuzzy Caputo fractional generalized Hukuhara derivative in the case of $\alpha \in (1, 2)$, $f : [a, b] \times E \times E \to E$ is a fuzzy function. The goal of this paper is to introduce the mathematical foundations for studies of fuzzy fractional derivative with non-integer order $\alpha \in (1, 2)$ in the sense of generalized Hukuhara difference, and to show the existence and uniqueness results of solution for the problem (1.1). To be more precise, the main contributions in this paper are as follows: (i) in Sect. 2, we discuss some important and interesting results of Riemann–Liouville and Caputo fractional derivative in the fuzzy setting in the case of the order $\alpha \in (1, 2)$; (ii) by employing the Krasnoselskii–Krein-type condition and the method of successive approximations, in Sect. 3 we will show the existence and uniqueness of the solution for a general form of the fuzzy fractional integral equations in the case of $\alpha \in (1, 2)$, and these results are used to investigate the existence and uniqueness of the solution for the problem (1.1); (iii) in Sect. 4, we propose a standard framework to obtain a formula of the solution for (1.1), which plays an important role in investigating the qualitative theories of (1.1). Moreover, the Banach fixed point theorem is also employed to prove the existence of a unique solution of problem (1.1). Finally, some examples are given to illustrate our main results.

2 Preliminaries

Let E be the class of fuzzy numbers, i.e., normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. For $r \in (0, 1]$, denote $[x]^r = \{u \in \mathbb{R} \mid x(u) \geq r\}$ and $[x]^0 = \overline{\{u \in \mathbb{R} \mid x(u) > 0\}}$. Then it is well-known that the $r-$level set of x, $[x]^r := [\underline{x}(r), \overline{x}(r)]$, is a bounded closed interval, for any $r \in [0, 1]$. For $x_1, x_2 \in E$, and $\lambda \in \mathbb{R}$, the sum $x_1 + x_2$ and the product $\lambda \cdot x_1$ are defined by $[x_1 + x_2]^r = [x_1]^r + [x_2]^r$, $[\lambda \cdot x_1]^r = \lambda[x_1]^r$, $\forall r \in [0, 1]$, where $[x_1]^r + [x_2]^r$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[x_1]^r$ means the usual product between a scalar and a real interval number. For $x \in E$, we define the diameter of the r-level set of x as $d\,([x]^r) = \overline{x}(r) - \underline{x}(r)$. Let $x_1, x_2 \in E$. If there exists $x_3 \in E$ such that $x_1 = x_2 + x_3$, then x_3 is called the Hukuhara difference of x_1 and x_2 and it is denoted by $x_1 \ominus x_2$. We note that $x_1 \ominus x_2 \neq x_1 + (-1)x_2$.

Remark 2.1 Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $x \in E$. If $\lambda_1 > \lambda_2$, then $\lambda_1 x \ominus \lambda_2 x = (\lambda_1 - \lambda_2)x$.

The generalized Hukuhara difference of two fuzzy numbers $x_1, x_2 \in E$ (gH-difference for short) is defined as follows:

$$
x_1 \ominus_{gH} x_2 = x_3 \Leftrightarrow \begin{cases} \text{(i)} \quad x_1 = x_2 + x_3, \\ \text{or} \quad \text{(ii)} \quad x_2 = x_1 + (-1)x_3. \end{cases} \qquad (2.1)
$$

Remark 2.2 Based on the definition of the diameter of the $r-$level set of $x \in E$, from (2.1), we also have the assertion as follows: the condition of the existence of $x_1 \ominus_{gH} x_2$ in the case (i) is $d([x_1]^r) \geq d([x_1]^r)$, and the condition of the existence of $x_1 \ominus_{gH} x_2$ in the case (ii) is $d([x_1]^r) \leq d([x_1]^r)$.

Remark 2.3 (An et al., preprint) Setting $A = (x_1 \ominus_{gH} x_2) \ominus_{gH} x_3$, where $x_1, x_2, x_3 \in E$. According to the definition of (2.1) one has

- If $d([x_1]^r) \geq d([x_2]^r)$ and $d([x_1 \ominus x_2]^r) \geq d([x_3]^r)$ for all $r \in [0, 1]$, then $A = (x_1 \ominus x_2) \ominus x_3$.
- If $d([x_1]^r) \geq d([x_2]^r)$ and $d([x_1 \ominus x_2]^r) \leq d([x_3]^r)$ for all $r \in [0, 1]$, then $A = (-1)[x_3 \ominus (x_1 \ominus x_2)]$.
- If $d([x_1]^r) \leq d([x_2]^r)$ and $d([(-1)(x_2 \ominus x_1)]^r) \geq d([x_3]^r)$ for all $r \in [0, 1]$, then $A = (-1)(x_2 \ominus x_1) \ominus x_3$.
- If $d([x_1]^r) \leq d([x_2]^r)$ and $d([(-1)(x_2 \ominus x_1)]^r) \leq d([x_3]^r)$ for all $r \in [0, 1]$, then $A = (-1)x_3 \ominus (x_2 \ominus x_1)$.

Definition 2.1 Let $x : [a, b] \to E$, then x is said to be d-increasing (d-decreasing) on $[a, b]$ if the function $t \mapsto d([x(t)]^r)$ is nondecreasing (nonincreasing) on $[a, b]$. If x is d-increasing or d-decreasing on $[a, b]$, then we say that x is d-monotone on $[a, b]$.

Definition 2.2 The distance $D_0[x_1, x_2]$ between two fuzzy numbers is defined as

$$D_0[x_1, x_2] = \sup_{0 \leq r \leq 1} H([x_1]^r, [x_2]^r), \quad \forall x_1, x_2 \in E,$$

where $H([x_1]^r, [x_2]^r) = \max\{|\underline{x}_1(r) - \underline{x}_1(r)|, |\overline{x}_1(r) - \overline{x}_1(r)|\}$ is the Hausdorff distance between $[x_1]^r$ and $[x_2]^r$.

Definition 2.3 Bede and Stefanini (2013) Let $x : (a, b) \to E$ and $t \in (a, b)$. The fuzzy function u is said to be generalized Hukuhara differentiable (gH-differentiable) of the first order at t, if there exists an element $x'(t) \in E$ such that

$$x'(t) = \lim_{h \to 0} \frac{x(t + h) \ominus_{gH} x(t)}{h}. \tag{2.2}$$

Definition 2.4 Stefanini and Bede (2009) Let $x : (a, b) \to E$. We say that x is gH-differentiable of the second order at t whenever the function x is gH-differentiable of the first order at t provided that gH-differentiable type has no change, then there exists $x''(t) \in E$ such that

$$x''(t) = \lim_{h \to 0} \frac{x'(t + h) \ominus_{gH} x'(t)}{h}.$$

Denote by $C([a, b], E)$ the set of all continuous fuzzy functions, $C^1([a, b], E)$ the set of all continuously differentiable fuzzy functions, $AC([a, b], E)$ the set of all absolutely continuous fuzzy functions and $AC^1([a, b], E)$ the set of all absolutely continuously differentiable fuzzy functions on the interval $[a, b]$ with values in E. Let $L([a, b], E)$ be the set of all fuzzy functions $x : [a, b] \to E$ such that the functions $t \mapsto D_0[x(t), \hat{0}]$ belong to $L[a, b]$.

Theorem 2.1 *If $x \in AC^1([a, b], E)$, then x is gH-differentiable of the second order on $[a, b]$ and $x'' \in L([a, b], E)$. Moreover, if x and x' are $d-$monotone on $[a, b]$, then for all $t \in [a, b]$*

$$\big(x(t) \ominus_{gH} x(a)\big) \ominus_{gH} (t - a)x'(a) = \int_a^t \int_a^s x''(s)ds. \tag{2.3}$$

Proof If $x \in AC^1([a, b], E)$ is such that $[x(t)]^r = [\underline{x}(r, t), \overline{x}(r, t)]$, then it follows that $\underline{x}'(r, t), \overline{x}'(r, t)$ are absolutely continuous (see Lupulescu 2015-Proposition 4). So, $\underline{x}''(r, t)$ and $\overline{x}''(r, t)$ exist on $[a, b]$ for every $r \in [0, 1]$, and $\underline{x}''(r, t), \overline{x}''(r, t) \in L[a, b]$. Hence, it implies that $x'' \in L([a, b], E)$. If $x \in AC^1([a, b], E)$ and $x' \in AC([a, b], E)$ are $d-$monotone on $[a, b]$, then for every $r \in [0, 1]$ one has

$$\int_a^t \int_a^s x''(s)\mathrm{d}s = \int_a^t (x'(s) \ominus_{gH} x'(a))\mathrm{d}s. \tag{2.4}$$

In addition, since $d([x'(t)]^r) - d([x'(a)]^r)$ has a constant sign on $[a, b]$, we have

$$\begin{aligned} \int_a^t (x'(s) \ominus_{gH} x'(a))\mathrm{d}s &= \int_a^t x'(s)\mathrm{d}s \ominus_{gH} \int_a^t x'(a)\mathrm{d}s \\ &= \big(x(t) \ominus_{gH} x(a)\big) \ominus_{gH} (t - a)x'(a). \end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we get (2.3). □

For a given fuzzy function $x \in L([a, b], E)$, the Riemann–Liouville fractional integral of order $\alpha > 0$ of the fuzzy function x is defined by (see Agarwal et al. 2010)

$$(\Im^{\alpha}_{a+}x)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} x(s)\mathrm{d}s, \quad t > a,$$

where $\Gamma(\alpha)$ is the well-known Gamma function. In Hoa et al. (2018), the concept of the fuzzy Caputo fractional derivative in the case of $\beta \in (0, 1)$ is given by the following definition.

Definition 2.5 Hoa et al. (2018) Let $x \in AC([a, b], E)$ and $\beta \in (0, 1)$. The fuzzy Caputo fractional derivative in the case of $\beta \in (0, 1)$ is defined

$${}^{C}\mathcal{D}^{\beta}_{a+}x(t) = \frac{1}{\Gamma(1 - \beta)} \int_a^t (t - s)^{-\beta} x'(s)\mathrm{d}s, \quad t \in (a, b].$$

Remark 2.4 If $x, y \in L([a, b], E)$ and $\alpha, \beta > 0$, then it is obvious that:

(i) $\Im^{\alpha}_{a^+}\Im^{\beta}_{a^+}x(t) = \Im^{\alpha+\beta}_{a^+}x(t)$, for $t \in [a, b]$;
(ii) $\Im^{\alpha}_{a^+}(x + y)(t) = \Im^{\alpha}_{a^+}x(t) + \Im^{\alpha}_{a^+}y(t)$, for $t \in [a, b]$.

Definition 2.6 Let $x : (a, b) \to E$. Then, the fuzzy Riemann–Liouville generalized Hukuhara fractional derivative (or Riemann–Liouville gH-fractional derivative) of order $\alpha \in (1, 2)$ of x is defined

$$\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}x\right)(t) := \left(\Im^{2-\alpha}_{a^+}x\right)''(t),$$

if $x''_{2-\alpha}(t)$ exists on (a, b), where

$$x_{2-\alpha}(t) := \left(\Im^{2-\alpha}_{a^+}x\right)(t) = \frac{1}{\Gamma(2-\alpha)}\int_a^t (t-s)^{1-\alpha}x(s)\mathrm{d}s, \quad t > a.$$

Theorem 2.2 *Let $x \in L([a, b], E)$, then for $0 < \alpha \le \beta$ one has that:*

$$\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\Im^{\beta}_{a^+}x\right)(t) = \left(\Im^{(\beta-\alpha)}_{a^+}x\right)(t), \quad t \in (a, b].$$

Note that $\left(\Im^{(\beta-\alpha)}_{a^+}x\right)(t) = x(t)$ *if* $\alpha = \beta$.

Proof With the same manner as in the proof of Proposition 2.1 in An et al., preprint, then by Dirichlet formula, the known formula for the Beta function, and setting $u = (\tau - s)/(t - s)$ one gets

$$\begin{aligned}\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\Im^{\beta}_{a^+}x\right)(t) &= \frac{1}{\Gamma(2-\alpha)\Gamma(\beta)}\left(\int_a^t (t-s)^{1-\alpha}\left(\int_a^s (s-\tau)^{\beta-1}x(\tau)d\tau\right)\mathrm{d}s\right)'' \\ &= \frac{1}{\Gamma(2-\alpha)\Gamma(\beta)}\left(\int_a^t (t-s)^{1+\beta-\alpha}x(s)\mathrm{d}s\int_0^1 (1-u)^{2-\alpha-1}u^{\beta-1}du\right)'' \\ &= \frac{B(2-\alpha,\beta)}{\Gamma(2-\alpha)\Gamma(\beta)}\left(\int_a^t (t-s)^{1+\beta-\alpha}x(s)\mathrm{d}s\right)''. \end{aligned} \tag{2.6}$$

By using Leibniz's rule for differentiation under the integral sign, we have that

$$\begin{aligned}\left(\int_a^t (t-s)^{\beta-\alpha-1}x(s)\mathrm{d}s\right)'' &= \left((1+\beta-\alpha)\int_a^t (t-s)^{\beta-\alpha}x(s)\mathrm{d}s\right)' \\ &= (1+\beta-\alpha)(\beta-\alpha)\int_a^t (t-s)^{\beta-\alpha-1}x(s)\mathrm{d}s.\end{aligned}$$

It follows from $B(2-\alpha, \beta) = \frac{\Gamma(2-\alpha)\Gamma(\beta)}{\Gamma(2-\alpha+\beta)}$ and (2.6) that

$$\begin{aligned}\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\Im^{\beta}_{a^+}x\right)(t) &= \frac{1}{\Gamma(\beta-\alpha)}\int_a^t (t-s)^{\beta-\alpha-1}x(s)\mathrm{d}s \\ &= \left(\Im^{(\beta-\alpha)}_{a^+}x\right)(t).\end{aligned}$$

□

Definition 2.7 Let $x \in L([a, b], E)$ be a fuzzy function such that ${}^{RL}\mathcal{D}^{\alpha}_{a^+}x$ exists on (a, b), where $\alpha \in (1, 2)$. The fuzzy Caputo fractional derivative of order $\alpha \in (1, 2)$ of x is defined

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) := {}^{RL}\mathcal{D}^{\alpha}_{a^+}\left[(x(t) \ominus_{gH} x(a)) \ominus_{gH} (t-a)x'(a)\right].$$

Lemma 2.1 *If $x \in AC^1([a, b], E)$ is a $d-$monotone fuzzy function and $\alpha \in (1, 2)$, then*

$${}^{C}\mathcal{D}^{\alpha}_{a^+}x(t) = (\Im^{2-\alpha}_{a^+}x'')(t) = \frac{1}{\Gamma(2-\alpha)}\int_a^t (t-s)^{1-\alpha}x''(s)ds, \quad t \in (a, b].$$

Proof The proof of this assertion is similar to the proof of Theorem 1 in Hoa et al. (2018). □

Theorem 2.3 *If $x \in AC^1([a, b], E)$ is such that x and x' are $d-$monotone on $[a, b]$, then*

$$\left(\Im^{\alpha}_{a^+}{}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) = (x(t) \ominus_{gH} x(a)) \ominus_{gH} (t-a)x'(a), \quad t \in (a, b]. \tag{2.7}$$

Proof By Remark 2.4-(ii), Theorem 2.1 and Lemma 2.1, we have that

$$\left(\Im^{\alpha}_{a^+}{}^{C}\mathcal{D}^{\alpha}_{a^+}x\right)(t) = \left(\Im^{\alpha}_{a^+}\quad(\Im^{2-\alpha}_{a^+}x'')\right)(t)$$

$$= (\Im^2_{a^+} x'')(t)$$
$$= (x(t) \ominus_{gH} x(a)) \ominus_{gH} (t-a)x'(a). \quad (2.8)$$

□

3 Fuzzy integral fractional equations

In this section of paper, we shall consider a general form of the fuzzy fractional integral equation (FFIE) with the form: for $\alpha \in (1,2)$,

$$x(t) \ominus_{gH} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^C\mathcal{D}^{\alpha-1}_{0^+} x(s)) \mathrm{d}s, \quad (3.1)$$

where ${}^C\mathcal{D}^{\alpha-1}_{0^+} x$ is the Caputo-type generalized Hukuhara derivative of the fuzzy function $x(t)$ given as in Definition 2.5, $g \in C^1([0,b], E)$ satisfies $D_0[g(t), \hat{0}] \le N_1$ and $D_0[{}^C\mathcal{D}^{\alpha-1}_{0^+} g(t), \hat{0}] \le N_2$ on $[0,b]$, $f : [0,b] \times E \times E \to E$ is continuous fuzzy function, $D_0[f(t,x,y), \hat{0}] \le M$ on $\mathbb{D}$:

$$\mathbb{D} = \left\{(t,x,y) : 0 \le t \le b,\, D_0[x, \hat{0}] \le \rho_1,\, D_0[y, \hat{0}] \le \rho_2\right\},$$

for $\rho_1, \rho_2 > 0$ are real numbers. We say that a continuous fuzzy function $x : I \to E$ is a solution to FFIE (3.1) if it satisfies Eq. (3.1). In the below theorem, the result of the uniqueness of solution to (3.1) is presented. Recently, the Krasnoselskii–Krein, Nagumo-type uniqueness results and successive approximations have been extended to the fuzzy fractional differential equations in the case pf fractional order $\alpha \in (0,1)$ (see Allahviranloo et al. 2014, 2012a). In this section, the uniqueness of the solution for the fuzzy fractional integral equation of non-integer order $\alpha \in (1,2)$ is proved. We are going to utilize the ideas presented in Yoruk et al. (2013); Bhaskar et al. (2009); Lakshmikantham and Leela (2009) to extend Krasnoselskii–Krein-type results to the problem (3.1), and these results will be used to investigate the existence of the unique solution of the problem (1.1) in the next section. Suppose that f satisfies the following Krasnoselskii–Krein-type conditions on $\mathbb{D}$:

(C1) $D_0[f(t,x_1,y_1), f(t,x_2,y_2)] \le \dfrac{\Gamma(\alpha)\Big[K + q(\alpha-1)\Big]}{2t^{1-q(\alpha-1)}} \Big[D_0[x_1,x_2] + D_0[y_1,y_2]\Big]$, $t \ne 0$, where $K > 0$ and $q \in (0,1)$,

(C2) $D_0[f(t,x_1,y_1), f(t,x_2,y_2)] \le L\Big[D_0^q[x_1,x_2] + t^{q(\alpha-1)} D_0^q[y_1,y_2]\Big]$, $K(1-q) < 1 + q(\alpha-1)$, and $L > 0$ is a constant.

Set $x_0(t) = g(t)$. We shall now state our main result.

Theorem 3.1 *Assume that the fuzzy function f in (3.1) satisfies the Krasnoselskii–Krein-type conditions (C1) and (C2). Then, the following successive approximation given by*

$$x_{n+1}(t) \ominus_{gH} x_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s), {}^C\mathcal{D}^{\alpha-1}_{0^+} x_n(s)) ds, \quad (3.2)$$

converges uniformly to the unique solution $x(t)$ of (3.1), on $[0,T]$, where

$$T = \min\left\{b, \left(\frac{(\rho_1 - N_1)\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}, \frac{\rho_2 - N_2}{M}\right\}.$$

Proof Let $x(t)$ and $\tilde{x}(t)$ be any two solutions of (3.1) on $[0,T]$ such that $x(0) = \tilde{x}(0)$. Set $w_1(t) = D_0[x(t), \tilde{x}(t)]$ and $w_2(t) = D_0[{}^C\mathcal{D}^{\alpha-1}_{0^+} x(t), {}^C\mathcal{D}^{\alpha-1}_{0^+} \tilde{x}(t))]$. We observe that $w_1(0) = 0$ and $w_2(0) = 0$. Using (3.1) and condition (C2), we have

$$w_1(t) = D_0[x(t), \tilde{x}(t)]$$
$$\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[w_1^q(s) + s^{q(\alpha-1)} w_2^q(s)\Big] \mathrm{d}s$$
$$\le \frac{Lt^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \Big[w_1^q(s) + s^{q(\alpha-1)} w_2^q(s)\Big] \mathrm{d}s.$$

By Theorem 2.2, Definition 2.7 and (3.1), we have for $t \in [0,T]$

$${}^C\mathcal{D}^{\alpha-1}_{0^+} \left(x(t) \ominus_{gH} g(t)\right) = \int_0^t f(s, x(s), {}^C D^{\alpha-1}_{0^+} x(s)) \mathrm{d}s.$$

Based on Theorem 1 in Lupulescu (2015), we get the following estimate

$$w_2(t) \le \int_0^t D_0\Big[f(s, x(s), {}^C D^{\alpha-1}_{0^+} x(s)), f(s, \tilde{x}(s), {}^C D^{\alpha-1}_{0^+} \tilde{x}(s))\Big] \mathrm{d}s$$
$$\le L \int_0^t \Big[w_1^q(s) + s^{q(\alpha-1)} w_2^q(s)\Big] \mathrm{d}s.$$

Define $P(t) = \int_0^t \Big[w_1^q(s) + s^{q(\alpha-1)} w_2^q(s)\Big] \mathrm{d}s$, clearly $P(0) = 0$. In addition, by the above estimate, we get $w_1(t) \le$

$\frac{Lt^{\alpha-1}}{\Gamma(\alpha)} P(t)$ and $w_2(t) \leq LP(t)$. Then,

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \left[w_1^q(t) + t^{q(\alpha-1)} w_2^q(t)\right] \leq t^{q(\alpha-1)} L^q \left(\frac{1}{\Gamma^q(\alpha)} + 1\right) P^q(t). \tag{3.3}$$

Since $P(t) > 0$ for $t > 0$, on multiplying (3.3) by $(1-q)P^{-q}(t)$ on both sides and integrating, we get for $t > 0$,

$$P(t) \leq Ct^{\left(\frac{q}{1-q}\alpha\right)+1},$$

where $C := \left(L^q(1-q)\left(\frac{1}{\Gamma^q(\alpha)}+1\right)/(1-q+q\alpha)\right)^{1/(1-q)}$. This leads to the following new estimate on $w_1(t)$, for $t \in [0, T]$,

$$w_1(t) \leq \frac{LC}{\Gamma(\alpha)} t^{\frac{\alpha}{1-q}}. \tag{3.4}$$

Define the function $Q(t) = t^{-K} \max\{w_1(t), w_2(t)\}$ for $t \in (0, T]$ and $Q(0) = 0$. Then, we have the following two cases:

if $\max\{w_1(t), w_2(t)\} = w_1(t)$, then we have

$$0 \leq Q(t) \leq \frac{LC}{\Gamma(\alpha)} t^{\frac{\alpha}{1-q} - K}.$$

if $\max\{w_1(t), w_2(t)\} = w_2(t)$, then we have

$$0 \leq Q(t) \leq LCt^{\frac{\alpha q + (1-q)(1-K)}{1-q}}.$$

By using the assumption $K(1-q) < 1 + q(\alpha - 1)$, we have that the exponents of t in the above inequalities are positive. In either case, we have $\lim_{t\to 0^+} Q(t) = 0$. Therefore, if we define $Q(0) = 0$, the function $Q(t)$ is continuous in $[0, T]$. To prove uniqueness, we need to show that $Q(t) \equiv 0$ which in turn yields $w_1(t) \equiv 0$ and $w_2(t) \equiv 0$. If it is not true, let $0 < Q(t) < m = \max_{[0,T]} Q(t) = Q(t_1)$.

If $\max\{w_1(t), w_2(t)\} = w_1(t_1)$, then from (C1) we get

$$\begin{aligned} m = Q(t_1) &= t_1^{-K} w_1(t_1) \\ &\leq \frac{t_1^{-K}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} D_0[f(s, x(s), \\ &\quad {}^C D_{0^+}^{\alpha-1} x(s)), f(s, \tilde{x}_2(s), {}^C D_{0^+}^{\alpha-1} \tilde{x}(s))]\mathrm{d}s \\ &\leq \frac{[K + q(\alpha-1)]t_1^{-K}}{2} \int_0^{t_1} (t_1 - s)^{\alpha-1} \frac{[w_1(s) + w_2(s)]}{s^{1-q(\alpha-1)}} \mathrm{d}s \\ &= [K + q(\alpha-1)] t_1^{-K+\alpha-1} \int_0^{t_1} s^{K-1+q(\alpha-1)} Q(s)\mathrm{d}s \\ &< m t_1^{(\alpha-1)(q+1)} \leq m, \end{aligned}$$

which is a contradiction.

On the other hand, if $\max\{w_1(t), w_2(t)\} = w_2(t_1)$, then we can show that $m = Q(t_1) = t_1^{-K} w_2(t_1) < m$, which is a contradiction. This implies $Q(t) \equiv 0$ and hence the uniqueness of the solution is established.

We shall now show that the successive approximations $\{x_{n+1}(t)\}$, $n = 0, 1, 2, \ldots$, given by (3.2) are well defined and continuous, uniformly bounded and equicontinuous on $[0, T]$. In view of (3.2) and from the conditions of the fuzzy function $g(t)$, we have

$$D_0[x_{n+1}(t), \hat{0}] \leq N_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_0\left[f(s, x_n(s), {}^C D_{0^+}^{\alpha-1} x_n(s)), \hat{0}\right]\mathrm{d}s,$$

and

$$D_0[{}^C D_{0^+}^{\alpha-1} x_{n+1}(t), \hat{0}] \leq N_2 + \int_0^t D_0\left[f(s, x_n(s), {}^C D_{0^+}^{\alpha-1} x_n(s)), \hat{0}\right]\mathrm{d}s.$$

Then, it yields

$$D_0[x_{n+1}(t), \hat{0}] \leq N_1 + \frac{Mt^\alpha}{\Gamma(\alpha+1)} \leq \rho_1 \quad \text{and}$$
$$D_0[{}^C D_{0^+}^{\alpha-1} x_{n+1}(t), \hat{0}] \leq N_2 + Mt \leq \rho_2.$$

This shows, by induction, that the family $\{x_{n+1}(t)\}$ is well defined and uniformly bounded on $[0, T]$. Further, it may be verified that $\{x_{n+1}(t)\}$ is equicontinuous families of functions on $[0, T]$. Hence, by Arzela–Ascoli theorem, there exists subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges uniformly on $[0, T]$. In addition, if $\lim_{n\to\infty} \sup D_0[x_{n+1}(t), x_n(t)] \to 0$, then (3.2) implies that the limit of any such subsequence is the unique solution $x(t)$ of (3.1). Therefore, for proving Theorem, we have to show that:

$$v_1(t) = \lim_{n\to\infty} \sup D_0[x_{n+1}(t), x_n(t)] \equiv 0, \tag{3.5}$$

and

$$v_2(t) = \lim_{n\to\infty} \sup D_0[{}^C D_{0^+}^{\alpha-1} x_{n+1}(t), {}^C D_{0^+}^{\alpha-1} x_n(t)] \equiv 0. \tag{3.6}$$

For $0 \le t_1 \le t_2$, consider the difference

$$|D_0[x_{n+1}(t_1), x_n(t_1)] - D_0[x_{n+1}(t_2), x_n(t_2)]|$$
$$\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} (t_1 - s)^{\alpha-1} F(s)\mathrm{d}s - \int_0^{t_2} (t_2 - s)^{\alpha-1} F(s)\mathrm{d}s\right|$$
$$\le \frac{2M}{\Gamma(\alpha)}\left|\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]\mathrm{d}s - \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\mathrm{d}s\right|$$
$$\le \frac{4M}{\Gamma(\alpha+1)}(t_2 - t_1)^{\alpha},$$

where $F(s) = D_0[f(s, x_n(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_n(s)), f(s, x_{n-1}(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_{n-1}(s))] \le 2M$ on $\mathbb{D}$. This result proves the continuous of $v_1(t)$ on $[0, T]$. Also, it can be verified that $v_2(t)$ is continuous on $[0, T]$. Now, by using the condition (C2) and (3.2), we get:

$$D_0[x_{n+1}(t), x_n(t)]$$
$$\le Lt^{\alpha-1}\int_0^t \Big(D_0^q[x_n(s), x_{n-1}(s)]$$
$$+ s^{q(\alpha-1)} D_0^q\Big[{}^C\mathcal{D}_{0+}^{\alpha-1}x_n(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_{n-1}(s)\Big]\Big)\mathrm{d}s.$$

For a fixed $t \in [0, T]$, there is a sequence of integers $n_1 < n_2 < \ldots$ and $p_1 < p_2 < \ldots$ such that $D_0[x_{n_{k+1}}(t), x_{n_k}(t)] \to v_1(t)$ and $D_0\Big[{}^C\mathcal{D}_{0+}^{\alpha-1}x_{p_{k+1}}(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_{p_k}(s)\Big] \to v_2(t)$ as $k \to \infty$. Setting $v_1^*(t) = \lim_{n=n_k\to\infty} D_0[x_n(t), x_{n-1}(t)]$ and $v_2^*(t) = \lim_{n=n_k\to\infty} D_0\Big[{}^C\mathcal{D}_{0+}^{\alpha-1}x_n(t), {}^C\mathcal{D}_{0+}^{\alpha-1}x_{n-1}(t)\Big]$ exist uniformly on $[0, T]$. Therefore, we obtain the following result using the fact that $v_1^*(t) \le v_1(t)$ and $v_2^*(t) \le v_2(t)$:

$$v_1(t) \le Lt^{\alpha-1}\int_0^t \Big[(v_1^*(s))^q + s^{q(\alpha-1)}(v_2^*(s))^q\Big]\mathrm{d}s$$
$$\le Lt^{\alpha-1}\int_0^t \Big[(v_1(s))^q + s^{q(\alpha-1)}(v_2(s))^q\Big]\mathrm{d}s.$$

It yields:

$$t^{-1}v_1(t) \to 0, \quad t \to 0^+, \qquad t^{-1}v_2(t) \to 0, \quad t \to 0^+,$$

and, also, we infer that

$$t^{-K}v_1(t) \to 0, \quad t \to 0^+, \qquad t^{-K}v_2(t) \to 0, \quad t \to 0^+.$$

We can define, as before, $Q^*(t) = t^{-K}\max\{v_1(t), v_2(t)\}$. We observe that $Q^*(0) \equiv 0$. We shall prove that $Q^*(t) \equiv 0$ for all $t \in [0, T]$ for demonstrating the convergence of $\{x_{n+1}(t)\}$. If $Q^*(t) > 0$ at any point in $[0, T]$, then there exists a point $t_1 > 0$ such that $0 < \lambda = Q^*(t_1) = \max_{[0,T]} Q^*(t)$. Then, from the condition (C1), we have, as before,

$$\frac{D_0[x_{n+1}(t_1), x_n(t_1)]}{t^K}$$
$$\le \frac{t^{-K}}{\Gamma(\alpha)}\int_0^{t_1} (t_1 - s)^{\alpha-1} D_0\Big[f(s, x_n(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_n(s)),$$
$$f(s, x_{n-1}(s), {}^C\mathcal{D}_{0+}^{\alpha-1}x_{n-1}(s))\Big]\mathrm{d}s$$
$$\le \frac{t^{-K}[K + q(\alpha-1)]}{2}\int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{s^{1-q(\alpha-1)}}$$
$$\Big(D_0[x_n(s), x_{n-1}(s)] + D_0[{}^C\mathcal{D}_{0+}^{\alpha-1}x_n(s),$$
$${}^C\mathcal{D}_{0+}^{\alpha-1}x_{n-1}(s)]\Big)\mathrm{d}s.$$

Proceeding as before, we obtain

$$\lambda = Q^*(t_1) = \frac{v_1(t_1)}{t^K} \le \frac{t^{-K}[K + q(\alpha-1)]}{2\Gamma(\alpha)}$$
$$\int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{s^{1-q(\alpha-1)}}\Big[v_1(s) + v_2(s)\Big]\mathrm{d}s$$
$$< \lambda[K + q(\alpha-1)]t_1^{-K+q-1}\int_0^{} s^{K-1+q(\alpha-1)}\mathrm{d}s \le \lambda.$$

This contradiction proves $Q^*(t) \equiv 0$ for the case $\max\{v_1(t_1), v_2(t_1)\} = v_1(t_1)$. On the other hand, if $\max\{v_1(t_1), v_2(t_1)\} = v_2(t_1)$ then from the condition (C1), again using an argument similar to the above, we can show that $\lambda = Q^*(t_1) = t_1^{-K}v_2(t_1) < \lambda$, which is again a contradiction. So $Q^*(t) \equiv 0$. Therefore, the successive approximations given $\{x_{n+1}(t)\}$ converge uniformly to the unique solution $x(t)$ of (3.1) on $[0, T]$. □

Example 3.2 Let $\alpha \in (1, 2)$, $\beta \ge 0$, $t \in [0, 1]$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $x_0 \in E$. We consider the following fuzzy fractional integral equation

$$x(t) \ominus_{gH} (1 + t^{\beta})x_0$$
$$= \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}x(s)\mathrm{d}s, \quad t \in [0, 1]. \tag{3.7}$$

According to Theorem 3.1, we can easy to check that the hypotheses of fuzzy functions $g(t) := (1 + t^{\beta})x_0$ and

$f(s, x_n(s), {}^C\mathcal{D}_{0^+}^{\alpha-1}x_n(s)) := \lambda x$ are satisfied. Based on the definition of generalized Hukuhara difference, we will solve the problem (3.7) in the below two cases:

Case 1 If $d([x(t)]^r) \geq (1+t^\beta)d([x_0]^r)$ and $\lambda > 0$, for every $r \in [0, 1]$ and for all $t \in [0, 1]$, then we observe that the fuzzy function

$$x(t) = \Big[E_{\alpha,1}\Big(\lambda t^\alpha\Big) + \Gamma(\beta+1)t^\beta E_{\alpha,\beta+1}\Big(\lambda t^\alpha\Big)\Big]x_0$$

is a unique solution of the problem (3.7). Indeed, based on (3.7), making the substitution $z = s/t$ and using the definition of Beta function, we get the right-hand side (RHS) of the problem (3.7) as follows:

$$\begin{aligned}
\text{RHS-(3.7)} &= x_0\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[E_{\alpha,1}\Big(\lambda s^\alpha\Big) \\
&\quad + \Gamma(\beta+1)s^\beta E_{\alpha,\beta+1}\Big(\lambda s^\alpha\Big)\Big]\mathrm{d}s \\
&= x_0\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\Gamma(\alpha)\Gamma(k\alpha+1)}\int_0^t (t-s)^{\alpha-1}s^{k\alpha}\mathrm{d}s \\
&\quad + x_0\sum_{k=0}^{\infty}\frac{\Gamma(\beta+1)\lambda^{k+1}}{\Gamma(\alpha)\Gamma(k\alpha+\beta+1)}\int_0^t (t-s)^{\alpha-1}s^{k\alpha+\beta}\mathrm{d}s \\
&= x_0\sum_{k=0}^{\infty}\frac{[\lambda t^\alpha]^{k+1}}{\Gamma((k+1)\alpha+1)} \\
&\quad + x_0\Gamma(\beta+1)t^\beta\sum_{k=0}^{\infty}\frac{[\lambda t^\alpha]^{k+1}}{\Gamma((k+1)\alpha+\beta+1)} \\
&= x_0\sum_{j=1}^{\infty}\frac{[\lambda t^\alpha]^j}{\Gamma(j\alpha+1)} + x_0\Gamma(\beta+1)t^\beta\sum_{j=1}^{\infty}\frac{[\lambda t^\alpha]^j}{\Gamma(j\alpha+\beta+1)} \\
&= x_0\left(E_{\alpha,1}(\lambda t^\alpha)-1\right) + x_0\Gamma(\beta+1)t^\beta\Big(E_{\alpha,\beta+1}(\lambda t^\alpha) \\
&\quad - \frac{1}{\Gamma(\beta+1)}\Big). \qquad (3.8)
\end{aligned}$$

Because the values of $\left(E_{\alpha,1}(\lambda t^\alpha)-1\right)$ and $\Gamma(\beta+1)t^\beta\left(E_{\alpha,\beta+1}(\lambda t^\alpha) - \frac{1}{\Gamma(\beta+1)}\right)$ are positive-defined on $[0, 1]$ with the hypothesis $\lambda > 0$, it follows from the right-hand side (RHS) of the problem (3.7) that

$$\begin{aligned}
\text{RHS-(3.7)} &= \Big(E_{\alpha,1}(\lambda t^\alpha) + \Gamma(\beta+1)t^\beta \\
&\qquad E_{\alpha,\beta+1}(\lambda t^\alpha) - (1+t^\beta)\Big)x_0.
\end{aligned}$$

On the other hand, because $\lambda > 0$, $d([x(t)]^r) \geq (1+t^\beta)d([x_0]^r)$ and Remark 2.1, one also gets the left-hand side (LHS) of the problem (3.7) as follows:

$$\begin{aligned}
\text{LHS-(3.7)} \\
&= \Big[E_{\alpha,1}\Big(\lambda t^\alpha\Big)+\Gamma(\beta+1)t^\beta E_{\alpha,\beta+1}\Big(\lambda t^\alpha\Big)\Big]x_0 \ominus (1+t^\beta)x_0 \\
&= \text{RHS-(3.7)}.
\end{aligned}$$

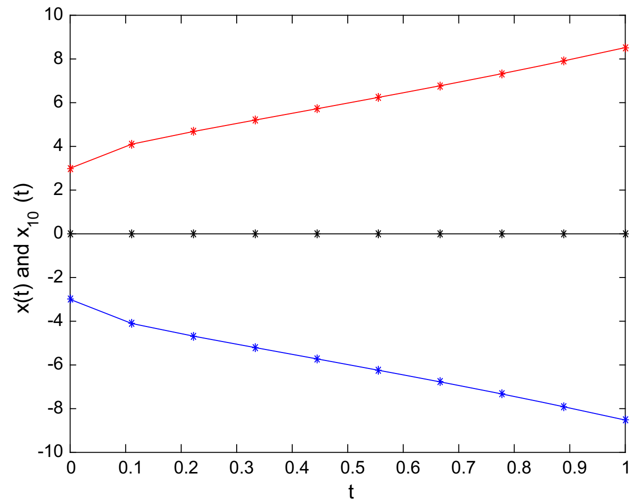

Fig. 1 The graphs of the exact solution (solid line) and the approximate solution (line "*") of the problem 3.2 in Case 1 with $\alpha = 1.75$

It implies that $x(t)$ is a unique solution of the problem (3.7). Now, we shall use the successive approximations defined as in Theorem 3.1 to construct the approximate solution and to give an estimation for the error bound for the problem (3.7). In this case, we recall the following approximate sequence defined as in the proof of Theorem 3.1: for $n = 1, 2, \ldots,$

$$x_n(t) \ominus x_0(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_{n-1}(s)\mathrm{d}s, \qquad (3.9)$$

where $\alpha \in (1, 2)$, $\lambda > 0$, $x_0(t) = (1+t^\beta)x_0$. Then, one has

$$\begin{aligned}
x_1(t) \ominus (1+t^\beta)x_0 &= \lambda\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \Gamma(\beta+1)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)x_0, \\
x_2(t) \ominus (1+t^\beta)x_0 &= \lambda\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \Gamma(\beta+1)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)x_0 \\
&\quad + \lambda^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \Gamma(\beta+1)\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\right)x_0, \\
&\vdots \\
x_n(t) \ominus (1+t^\beta)x_0 &= x_0\sum_{j=1}^{n}\frac{\lambda^j}{\Gamma(j\alpha+1)}t^{j\alpha} \\
&\quad + x_0\Gamma(\beta+1)\sum_{j=1}^{n}\frac{\lambda^j}{\Gamma(j\alpha+\beta+1)}t^{j\alpha+\beta}.
\end{aligned}$$

The result obtained by using the method of successive approximations proposed in this case is shown in Fig. 1. The errors between the analytical and approximate solutions are listed in Tab. 1 with $n = 10$. In our numerical simulations,

Table 1 Error of the exact and approximate solutions of the problem 3.2 in Case 1

t\α	1.25	1.5	1.75
0.2	1.15441E−06	3.43903E−11	0
0.4	2.83996E−05	1.64392E−08	2.9905E−12
0.6	0.000129527	2.9742E−07	2.157E−10
0.8	0.000356247	2.02812E−06	3.64391E−09
1	0.000763889	8.57375E−06	3.03005E−08

we use the parameters as follows: $\lambda = 0.5$; $\beta = 0.5$; $x_0 = (-3, 0, 3)$.

Case 2 If $d([x(t)]^r) \le (1+t^\beta)d([x_0]^r)$ and $\lambda < 0$, for every $r \in [0,1]$ and for all $t \in [0,1]$, then we also recognize that

$$x(t) = \Big[E_{\alpha,1}\Big(\lambda t^\alpha\Big) + \Gamma(\beta+1)t^\beta E_{\alpha,\beta+1}\Big(\lambda t^\alpha\Big)\Big]x_0$$

is a unique solution of the problem (3.7). With the same manner as in Case 1, we also obtain

$$\text{RHS-(3.7)} = x_0\left(E_{\alpha,1}(\lambda t^\alpha) - 1\right) + x_0\Gamma(\beta+1)t^\beta\Big(E_{\alpha,\beta+1}(\lambda t^\alpha) - \frac{1}{\Gamma(\beta+1)}\Big). \tag{3.10}$$

Because the values of $\left(E_{\alpha,1}(\lambda t^\alpha) - 1\right)$ and $\Gamma(\beta+1)t^\beta\Big(E_{\alpha,\beta+1}(\lambda t^\alpha) - \dfrac{1}{\Gamma(\beta+1)}\Big)$ are negative-defined on $[0,1]$ with the hypothesis $\lambda < 0$, it yields that

$$\text{RHS-(3.7)} = (-1)\left((1+t^\beta) - E_{\alpha,1}(\lambda t^\alpha) + \Gamma(\beta+1)t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha)\right)x_0.$$

Otherwise, because $\lambda < 0$, $d([x(t)]^r) \le (1+t^\beta)d([x_0]^r)$ and Remark 2.1, one also gets the left-hand side (LHS) of the problem (3.7) as follows:

$$\begin{aligned}\text{LHS-(3.7)} &= (-1)\Big[(1+t^\beta)x_0 \ominus \Big(E_{\alpha,1}\Big(\lambda t^\alpha\Big) + \Gamma(\beta+1)t^\beta E_{\alpha,\beta+1}\Big(\lambda t^\alpha\Big)\Big)x_0\Big] \\ &= \text{RHS-(3.7)}.\end{aligned}$$

It implies that $x(t)$ is a unique solution of the problem (3.7). In this case, we recall the following approximate sequence defined as in the proof of Theorem 3.1: for $n = 1, 2, \ldots$,

$$(-1)(x_0(t) \ominus x_n(t)) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_{n-1}(s)\mathrm{d}s, \tag{3.11}$$

where $\alpha \in (1,2)$, $\lambda > 0$, $x_0(t) = (1+t^\beta)x_0$. We observe that Eq. (3.11) is similar to

$$x_n(t) = x_0(t) \ominus (-1)\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_{n-1}(s)\mathrm{d}s,$$

and it can be represented by the r−level sets as follows:

$$[x_n(t)]^r = [x_0(t)]^r \ominus (-1)\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[x_{n-1}(s)]^r\mathrm{d}s,$$

where $r \in [0,1]$. Then, one has

$$\begin{cases}\underline{x}_n(r,t) = \underline{x}_0(r,t) + \dfrac{\lambda}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\underline{x}_{n-1}(r,s)\mathrm{d}s \\ \overline{x}_n(r,t) = \overline{x}_0(r,t) + \dfrac{\lambda}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\overline{x}_{n-1}(r,s)\mathrm{d}s,\end{cases}$$

where $r \in [0,1]$, $n = 1, 2, \ldots$, $\lambda < 0$, $\underline{x}_0(r,t) = (1+t^\beta)\underline{x}_0(r)$, $\overline{x}_0(r,t) = (1+t^\beta)\overline{x}_0(r)$. Similar to Case 1, we also obtain

$$\begin{aligned}\underline{x}_n(r,t) &= \underline{x}_0(r)\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+1)}t^{j\alpha} + \underline{x}_0(r)\Gamma(\beta+1)\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+\beta+1)}t^{j\alpha+\beta}, \\ \overline{x}_n(r,t) &= \overline{x}_0(r)\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+1)}t^{j\alpha} + \overline{x}_0(r)\Gamma(\beta+1)\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+\beta+1)}t^{j\alpha+\beta}.\end{aligned}$$

Hence, we deduce that

$$x_n(t) = x_0\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+1)}t^{j\alpha} + x_0\Gamma(\beta+1)\sum_{j=0}^n \frac{\lambda^j}{\Gamma(j\alpha+\beta+1)}t^{j\alpha+\beta}.$$

The result obtained by using the method of successive approximations proposed in this case is shown in Fig. 2. The errors between the analytical and approximate solutions are listed in Table 2 with $n = 10$. In our numerical simulations, we use the parameters as follows: $\lambda = -0.5$; $\beta = 0$; $x_0 = (-3, 0, 3)$.

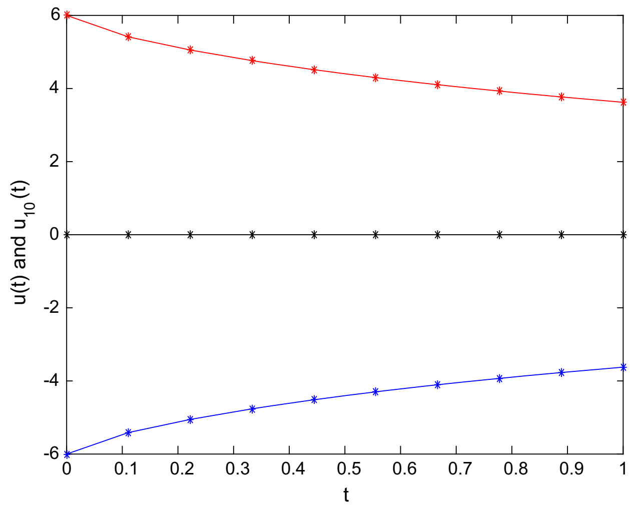

Fig. 2 The graphs of the exact solution (solid line) and the approximate solution (line "*") of the problem 3.2 in Case 2 with $\alpha = 1.75$

Table 2 Error of the exact and approximate solutions of the problem 3.2 in Case 2

t\α	1.25	1.5	1.75
0.2	1.29691E−06	5.38503E−11	0
0.4	2.52009E−05	2.16679E−08	4.71001E−12
0.6	9.96928E−05	3.49138E−07	3.1289E−10
0.8	0.000246235	2.16974E−06	4.94039E−09
1	0.000483352	8.47193E−06	3.86911E−08

4 Fuzzy fractional differential equations

In this section, for $\alpha \in (1, 2)$, we consider the following initial-type problem:

$$
\begin{aligned}
&{}^{C}\mathcal{D}^{\alpha}_{a+}x(t)\\
&\quad= f(t, x(t), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(t)), \qquad x(a) = x_0, \quad x'(a) = x'_0 \quad t \in [a, b].
\end{aligned} \tag{4.1}
$$

A function $x : [a, b] \to E$ is said to be a solution of (4.1) if x is continuous, $x(a) = x_0$, $x'(a) = x'_0$ and ${}^{C}\mathcal{D}^{\alpha}_{a+}x(t) = f(t, x, {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(t))$. Next, we denote by $C^{\alpha,F}([a, b], E)$ the space of the fuzzy functions that are continuous Caputo fractional differentiable of order $\alpha \in (1, 2)$ on $[a, b]$. The following lemma shows the equivalence between a fuzzy fractional differential equation and a fuzzy fractional integral equation.

Lemma 4.1 *Let $f : [a, b] \times E \times E \to E$ such that $t \mapsto f(t, u, v)$ belongs to $C([a, b], E)$, for any $u, v \in E$. Then, a $d-$monotone fuzzy function $x \in C^{\alpha,F}([a, b], E)$ is a solution of initial value problem (4.1) if and only if x satisfies the integral equation*

$$
\begin{aligned}
&(x(t) \ominus_{gH} x(a)) \ominus_{gH} (t - a)x'(a)\\
&= \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s, x(s), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(s))ds, \quad t \in (a, b].
\end{aligned} \tag{4.2}
$$

Proof First, we prove the necessity condition. Let x be a solution of (4.1), then from (4.1) and Theorem 2.3 we have that

$$
\left(\Im^{\alpha}_{a+}\,{}^{C}\mathcal{D}^{\alpha}_{a+}x\right)(t) = (x(t) \ominus_{gH} x(a)) \ominus_{gH} (t - a)x'(a), \tag{4.3}
$$

for $t \in [a, b]$. Since $f(t, u, v) \in C([a, b], \mathbb{R})$ for any $u, v \in \mathbb{R}$, and from (4.1), it follows that

$$
\begin{aligned}
&\left(\Im^{\alpha}_{a+}\,{}^{C}\mathcal{D}^{\alpha}_{a+}x\right)(t)\\
&= \Im^{\alpha}_{a+} f(t, x(t), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(t))\\
&= \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s, x(s), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(s))ds.
\end{aligned} \tag{4.4}
$$

Consequently, combining (4.3) and (4.4) proves the necessity condition. Now, we move to prove the sufficiency. Set $f_\alpha(t, u, v) := \Im^{\alpha}_{a+} f(t, u, v)$. Because of the continuity of the function f, the function $t \mapsto f_\alpha(t, u, v)$ is continuous on $(a, b]$ and $\lim_{t\to a^+} \Im^{\alpha}_{a+} f(t, u, v) = 0$. Then, $x(a) = x_0$ and $x'(a) = x'_0$. Acting on the two sides of (4.2) by the operator ${}^{RL}\mathcal{D}^{\alpha}_{a+}$ and by Theorem 2.2, we obtain

$$
\begin{aligned}
&{}^{RL}\mathcal{D}^{\alpha}_{a+}\left[(x(t) \ominus_{gH} x(a)) \ominus_{gH} x'(a)(t - a)\right]\\
&= f(t, x(t), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(t)).
\end{aligned}
$$

The proof is complete. □

Based on Remark 2.3 and Eq. (4.2), we have the following remark.

Remark 4.1 Based on the $d-$monotonic properties of solution to the fuzzy integral fractional equation (4.2), we have the following four cases.

(i) If $x \in C([a, b], E)$ is such that $d([x(t)]^r) \geq d([x(a)]^r)$ and $d([x'(t)]^r) \geq d([x'(a)]^r)$ for every $r \in [0, 1]$, for all $t \in [a, b]$, then (4.2) can be written as

$$
\begin{aligned}
x(t) &= x(a) + (t - a)x'(a)\\
&\quad+ \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s, x(s), {}^{C}\mathcal{D}^{\alpha-1}_{a+}x(s))ds.
\end{aligned}
$$

(ii) If $x \in C([a, b], E)$ is such that $d([x(t)]^r) \leq d([x(a)]^r)$ and $d([x'(t)]^r) \geq d([x'(a)]^r)$ for every $r \in [0, 1]$, for all $t \in [a,b]$, then (4.2) can be written as

$$x(t) = x(a) \ominus (-1)\left((t-a)x'(a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s), {}^C\mathcal{D}_{a+}^{\alpha-1}x(s))\mathrm{d}s\right).$$

(iii) If $x \in C([a,b], E)$ is such that $d([x(t)]^r) \geq d([x(a)]^r)$ and $d([x'(t)]^r) \leq d([x'(a)]^r)$ for every $r \in [0,1]$, for all $t \in [a,b]$, then (4.2) can be written as

$$x(t) = x(a) + (t-a)x'(a) \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s), {}^C\mathcal{D}_{a+}^{\alpha-1}x(s))\mathrm{d}s.$$

(iv) If $x \in C([a,b], E)$ is such that $d([x(t)]^r) \leq d([x(a)]^r)$ and $d([x'(t)]^r) \leq d([x'(a)]^r)$ for every $r \in [0,1]$, for all $t \in [a,b]$, then (4.2) can be written as

$$x(t) = x(a) \ominus (-1)\left((t-a)x'(a) \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s), {}^C\mathcal{D}_{a+}^{\alpha-1}x(s))\mathrm{d}s\right).$$

Example 4.1 Let $\alpha \in (1,2), \lambda \in \mathbb{R}\setminus\{0\}$. We consider the following fractional integral equation:

$$(x(t) \ominus_{gH} x(0)) \ominus_{gH} tx'(0) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} E_{\alpha,1}(\lambda s^\alpha)(-1,0,1)\mathrm{d}s, \quad t \in (0,1], \tag{4.5}$$

where $x(0) = (1,5,10), x'(0) = (1,3,5)$. Based on Remark 4.1, we consider the following cases:

Case 1 If $d([x(t)]^r) \geq d([x(0)]^r)$ and $d([x'(t)]^r) \geq d([x'(0)]^r)$ for every $r \in [0,1]$, for all $t \in (0,1]$, then Eq. (4.5) becomes

$$\begin{aligned} x(t) &= (1,5,10) + (t,3t,5t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,1}(\lambda s^\alpha)(-1,0,1)\mathrm{d}s \\ &= (1,5,10) + (t,3t,5t) + \frac{\lambda}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\int_0^t (t-s)^{\alpha-1}\frac{\lambda^k s^{\alpha k}}{\Gamma(\alpha k+1)}(-1,0,1)\mathrm{d}s \\ &= (1,5,10) + (t,3t,5t) + \lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)(-1,0,1), \end{aligned}$$

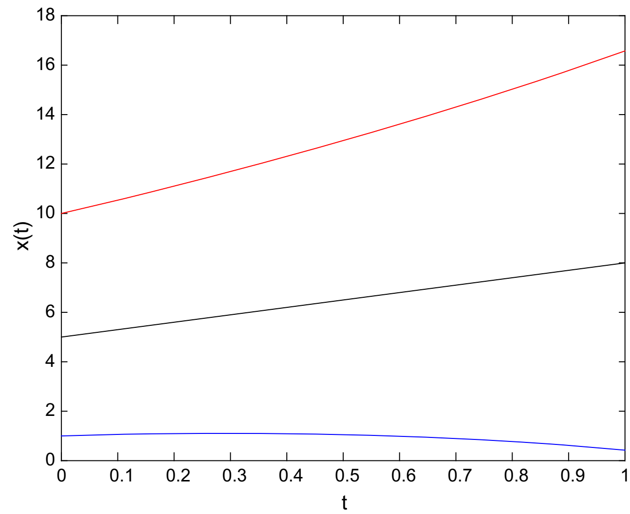

Fig. 3 The graph of $x(t)$ in Case 1

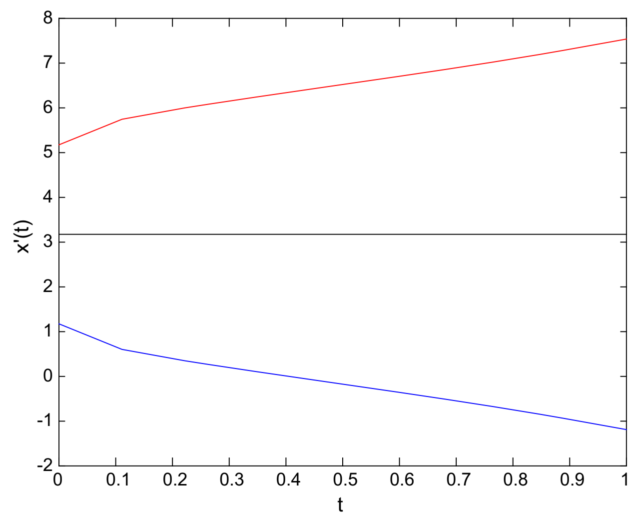

Fig. 4 The graph of $x'(t)$ in Case 1

where we employ the change of variable $z = s/t$ and the definition of the Beta function. The graphs of x and x' are shown in Figs. 3 and 4. In our numerical simulations, we use the parameters as follows: $\alpha = 1.5, \lambda = 1.5$.

Case 2 If $d([x(t)]^r) \leq d([x(0)]^r)$ and $d([x'(t)]^r) \geq d([x'(0)]^r)$ for every $r \in [0,1]$, for all $t \in (0,1]$, then by proceeding similarly as in Case 1 Eq. (4.5) becomes

$$\begin{aligned} x(t) &= (1,5,10) \ominus (-1)\Big((t,3t,5t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,1}(\lambda s^\alpha)(-1,0,1)\mathrm{d}s\Big) \\ &= (1,5,10) \ominus (-1)\Big((t,3t,5t) + \lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)(-1,0,1)\Big). \end{aligned}$$

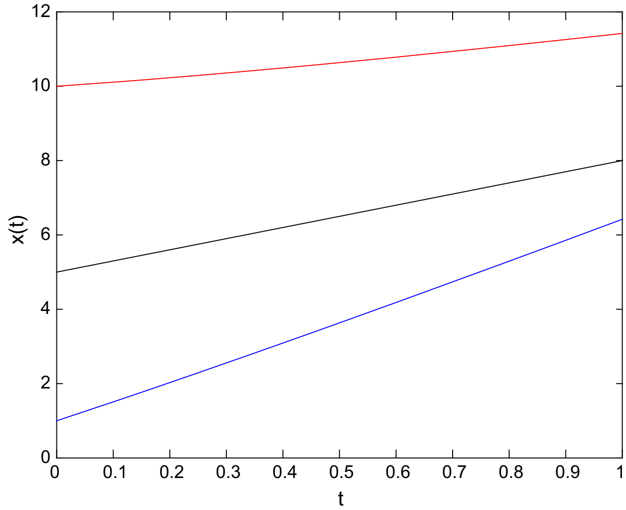

Fig. 5 The graph of $x(t)$ in Case 2

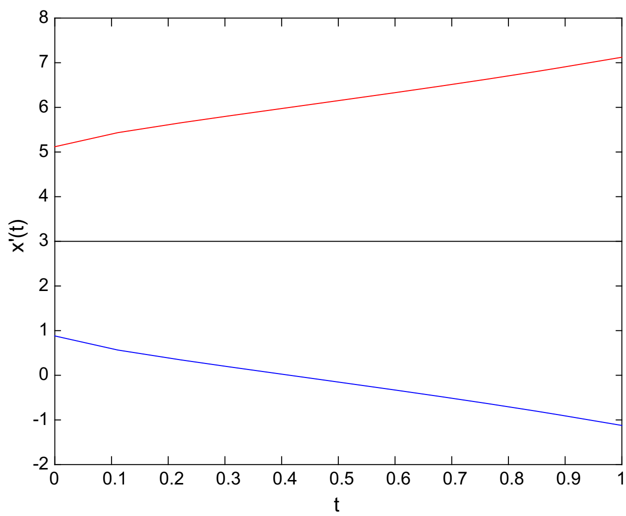

Fig. 6 The graph of $x'(t)$ in Case 2

The graphs of x and x' are shown in Figs. 5 and 6. In our numerical simulations, we use the parameters as follows: $\alpha = 1.75$ and $\lambda = 1.5$.

Case 3 If $d([x(t)]^r) \geq d([x(0)]^r)$ and $d([x'(t)]^r) \leq d([x'(0)]^r)$ for every $r \in [0, 1]$, for all $t \in (0, 1]$, then with the same manner Eq. (4.5) becomes

$$x(t) = (1, 5, 10) + (t, 3t, 5t) \ominus (-1)\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,1}(\lambda s^\alpha)(-1, 0, 1)\mathrm{d}s$$
$$= (1, 5, 10) + (t, 3t, 5t) \ominus (-1)\lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)(-1, 0, 1).$$

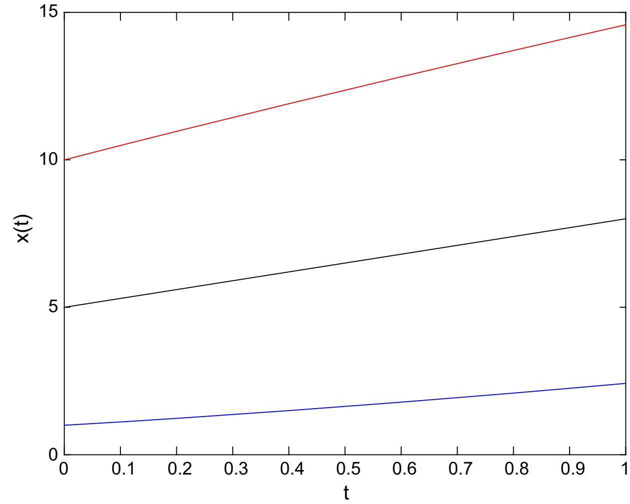

Fig. 7 The graph of $x(t)$ in Case 3

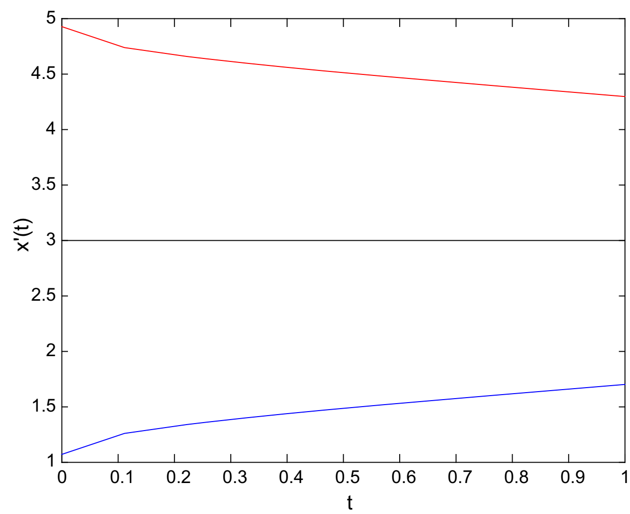

Fig. 8 The graph of $x'(t)$ in Case 3

The graphs of x and x' are shown in Figs. 7 and 8. In our numerical simulations, we use the parameters as follows: $\alpha = 1.5$, $\lambda = 0.5$.

Case 4 If $d([x(t)]^r) \leq d([x(0)]^r)$ and $d([x'(t)]^r) \leq d([x'(0)]^r)$ for every $r \in [0, 1]$, for all $t \in (0, 1]$, then Eq. (4.5) becomes

$$x(t) = (1, 5, 10) \ominus (-1)(t, 3t, 5t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,1}(\lambda s^\alpha)(-1, 0, 1)\mathrm{d}s$$
$$= (1, 5, 10) \ominus (-1)(t, 3t, 5t) + \lambda t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)(-1, 0, 1).$$

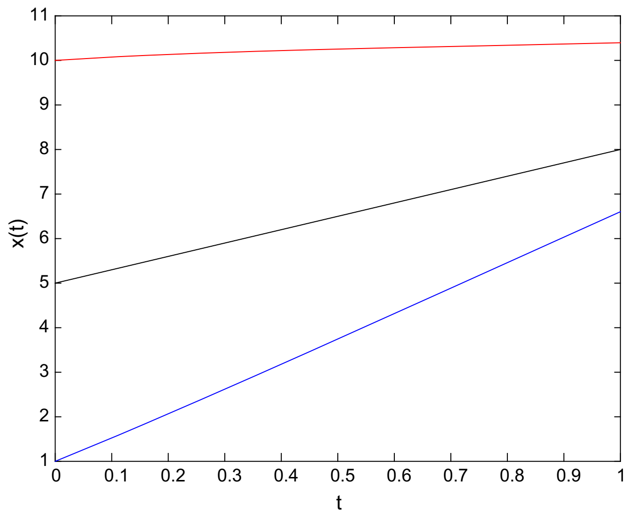

Fig. 9 The graph of the solution $x(t)$ in Case 4

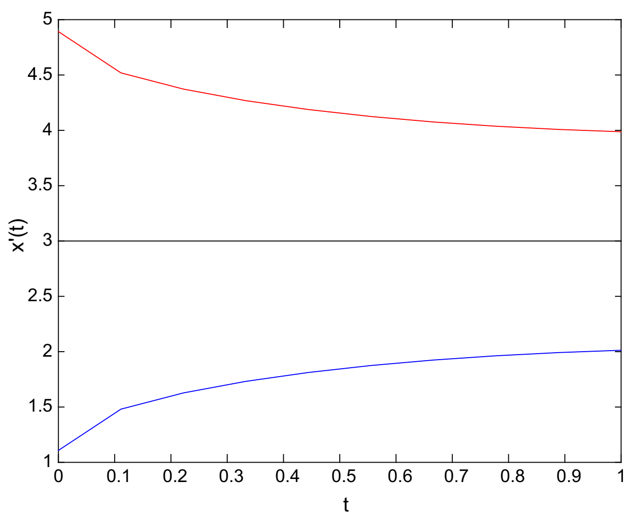

Fig. 10 The graph of $x'(t)$ in Case 4

The graphs of x and x' are shown in Figs. 9 and 10. In our numerical simulations, we use the parameters as follows: $\alpha = 1.5, \lambda = -1$.

Remark 4.2 If we put $g(t) = x(a) \ominus_{gH} (t-a)x'(a)$, then one has $g'(t) = (-1)x'(a) \in E$. In addition, we can infer that $d([x'(t)]^r) - d([g'(t)]^r)$ has a constant sign on $[a, b]$ for every $r \in [0, 1]$. Therefore, based on Theorem 9 in Lupulescu (2015), for $\alpha \in (1, 2)$ we have

$$ {}^C\mathcal{D}^{\alpha-1}_{a^+} \left(x(t) \ominus_{gH} g(t)\right) = {}^C\mathcal{D}^{\alpha-1}_{a^+} x(t) \ominus_{gH} {}^C\mathcal{D}^{\alpha-1}_{a^+} g(t), $$

and ${}^C\mathcal{D}^{\alpha-1}_{a^+} g(t) = \dfrac{(-1)x'(a)}{\Gamma(3-\alpha)}(t-a)^{2-\alpha}$.

We start with a presentation of the existence and uniqueness of the solution to the problem (4.1). In the following, for a chosen constant $\theta > 0$, we consider the set S_θ of all continuous functions for all $t \geq a$ such that $D_0[u(t), \hat{0}] \exp(-\theta t) < \infty$. On S_θ we can define the following metric

$$ D_\theta(u, v) := \sup_{t \geq a} D_0[u(t), v(t)] \exp(-\theta t), \tag{4.6} $$

where $\theta > 0$ is chosen suitably later. In fact, for $\theta > 0$ the metric $D_\theta(u, v)$ is equivalent to the metric $D_0[\cdot, \cdot]$. Indeed, we can see that that $D_0[u(t), v(t)] \exp(-\theta t) \leq D_\theta(u, v) \leq D_0[u(t), v(t)]$, for all $u, v \in \mathbb{E}$. We observe that (S_θ, D_θ) is a complete metric space with the distance (4.6) for any $\theta > 0$. By $\mathbb{S}_\theta$, we denote the set of all functions $u \in S_\theta$ such that ${}^C\mathcal{D}^{\alpha-1}_{a^+} u(t)$ exists as a continuous function on $[a, b]$. For $u, v \in \mathbb{S}_\theta$, we consider the following metric:

$$ \mathbb{D}_\theta(u, v) = D_\theta(u, v) + D_\theta\left({}^C\mathcal{D}^{\alpha-1}_{a^+} u, {}^C\mathcal{D}^{\alpha-1}_{a^+} v\right). \tag{4.7} $$

Theorem 4.2 *Let $\alpha \in (1, 2)$ and $f : [a, b] \times \mathbb{S}_\theta \times \mathbb{S}_\theta \to E$ be continuous. Assume that there exist positive constants $M, 0 < \gamma < \theta, L_1, L_2$ such that $\sup_{t \in [a,b]} |f(t, 0, 0)| \leq M \exp(\gamma t)$ and*

$$ D_0[f(t, u_1, v_1), f(t, u_2, v_2)] \leq L_1 D_0[u_1, u_2] + L_2 D_0[v_1, v_2], $$

for all $u_1, v_1, u_2, v_2 \in \mathbb{S}_\theta$. Then, the initial value problem (4.1) has a unique solution x.

Proof To prove this theorem, we investigate the conditions of the Banach fixed point principle. First, let $g(t) := x(a) \ominus_{gH} (t-a)x'(a)$, $t \in [a, b]$. From Remark 4.2, we have $g'(t) = (-1)x'(a) \in E$. We define the operator $\mathbb{P} : \mathbb{S}_\theta \to \mathbb{S}_\theta$ by

$$ (\mathbb{P}x)(t) \ominus_{gH} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s), {}^C\mathcal{D}^{\alpha-1}_{a^+} x(s)) \mathrm{d}s. \tag{4.8} $$

In addition, from Remark 4.2 we observe that

$$ {}^C\mathcal{D}^{\alpha-1}_{a^+} (\mathbb{P}x)(t) \ominus_{gH} \frac{(-1)x'(a)}{\Gamma(3-\alpha)}(t-a)^{2-\alpha} = \int_a^t f(s, x(s), {}^C\mathcal{D}^{\alpha-1}_{a^+} x(s)) \mathrm{d}s \tag{4.9} $$

Since $x \in C([a, b], E)$ and f satisfies the Lipschitz condition, we deduce that the function f is uniformly continuous and bounded on $[a, b]$. Therefore, there exists M_f such that $D_0\left[f\left(t, x(t), {}^C\mathcal{D}^{\alpha-1}_{a^+} x(t)\right), \hat{0}\right] \leq M_f$. The proof of this theorem is divided into two steps. We first show that $\mathbb{P}$ maps $C([a,b],E)$ into $C([a,b],E)$. Since for $a \le t_1 \le t_2 \le b$,

$$\begin{aligned}
&D_0[(\mathbb{P}x)(t_1), (\mathbb{P}x)(t_2)]\\
&\quad \le |t_1 - t_2| D_0[x'(a), \hat{0}]\\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} D_0[f(s, x(s), {}^C\mathcal{D}_{a^+}^{\alpha-1} x(s)), \hat{0}] \mathrm{d}s\\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} D_0\Big[\left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right) f(s, x(s),\\
&\qquad {}^C\mathcal{D}_{a^+}^{\alpha-1} x(s)), \hat{0}\Big] \mathrm{d}s\\
&\quad \le |t_1 - t_2| D_0[x'(a), \hat{0}] + \frac{M_f}{\Gamma(\alpha+1)} |t_2 - t_1|^\alpha\\
&\qquad + \frac{M_f}{\Gamma(\alpha+1)} \left(|t_2^\alpha - t_1^\alpha| - |t_2 - t_1|^\alpha\right)\\
&\quad \le |t_1 - t_2| D_0[x'(a), \hat{0}]\\
&\qquad + \frac{M_f}{\Gamma(\alpha+1)} \left((t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)\right),
\end{aligned}$$

the last inequality converges to 0 as $t_2 \to t_1$, which yields that the operator $\mathbb{P}$ is a continuous function. Besides, for each $t \in [a,b]$ and $p > 0$ fixed, from Lipschitz condition we have

$$\begin{aligned}
&D_0[(\mathbb{P}x)(t), \hat{0}]\\
&\quad \le D_0[x_0, \hat{0}] + D_0[x'(a), \hat{0}](b - a)\\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} D_0[f(s, 0, 0), \hat{0}] \mathrm{d}s\\
&\qquad + \frac{L}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} \left(D_0[x(s), \hat{0}] + D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} x(s), \hat{0}]\right) \mathrm{d}s\\
&\quad \le D_0[x_0, \hat{0}] + D_0[x'(a), \hat{0}](b - a) + \frac{M(t-a)^\alpha}{\Gamma(\alpha+1)} \exp(\gamma t)\\
&\qquad + \frac{L}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} \left(D_0[x(s), \hat{0}] + D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} x(s), \hat{0}]\right) \mathrm{d}s,
\end{aligned}$$

where $L = \max\{L_1, L_2\}$. Furthermore, we obtain

$$\begin{aligned}
&D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} (\mathbb{P}x)(t), \hat{0}] \le \frac{D_0[x'(a), \hat{0}]}{\Gamma(3-\alpha)} (t-a)^{2-\alpha}\\
&\qquad + \int_a^t D_0[f(s, x(s), {}^C\mathcal{D}_{a^+}^{\alpha-1} x(s)), \hat{0}] \mathrm{d}s\\
&\quad \le \frac{D_0[x'(a), \hat{0}]}{\Gamma(3-\alpha)} (t-a)^{2-\alpha} + \frac{M \exp(\gamma t)}{\gamma}\\
&\qquad + L \int_a^t \left(D_0[x(s), \hat{0}] + D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} x(s), \hat{0}]\right) \mathrm{d}s.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\mathbb{D}_\theta\left(\mathbb{P}x, \hat{0}\right)\\
&\quad = \sup_{t \ge a} \left\{D_0[(\mathbb{P}x)(t), \hat{0}] \exp(-\theta t)\right\}\\
&\qquad + \sup_{t \ge a} \left\{D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} (\mathbb{P}x)(t), \hat{0}] \exp(-\theta t)\right\}\\
&\quad \le \Big(D_0[x_0, \hat{0}] + (b-a) D_0[x'(a), \hat{0}]\\
&\qquad + \frac{M(b-a)^\alpha}{\Gamma(\alpha+1)} \exp((\gamma - \theta)t)\Big)\\
&\qquad + \left(\frac{(b-a)^{2-\alpha}}{\Gamma(3-\alpha)} D_0[x'(a), \hat{0}] + \frac{M}{\gamma} \exp((\gamma - \theta)t\right)\\
&\qquad + L\mathbb{D}_\theta(x, 0) \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + (b-a)\right).
\end{aligned}$$

Since $x \in \mathbb{S}_\theta$, there exists $K > 0$ such that $\mathbb{D}_\theta(x, 0) \le K < \infty$ for all $t \in [a,b]$. Hence, for $t \in [a,b]$ we conclude that $\mathbb{D}_\theta(\mathbb{P}x, 0) < \infty$, and thus $\mathbb{P}x \in \mathbb{S}_\theta$. For the next step, we prove that the operator $\mathbb{P}$ is a contraction map with respect to metric $\mathbb{D}_\theta$. For $x, y \in \mathbb{S}_\theta$, we have that

$$\begin{aligned}
&D_\theta(\mathbb{P}x, \mathbb{P}y)\\
&\quad \le \frac{L}{\Gamma(\alpha)} \sup_{t \in [a,b]} \Bigg\{\int_a^t \frac{1}{(t-s)^{1-\alpha}} \big(D_0[x(s), y(s)]\\
&\qquad + D_0[{}^C\mathcal{D}_{a^+}^{\alpha-1} x(s), {}^C\mathcal{D}_{a^+}^{\alpha-1} y(s)]\big) \mathrm{d}s e^{-\theta t}\Bigg\}\\
&\quad \le \frac{L}{\Gamma(\alpha)} \left(D_\theta(x, y) + D_\theta\left({}^C\mathcal{D}_{a^+}^{\alpha-1} x, {}^C\mathcal{D}_{a^+}^{\alpha-1} y\right)\right)\\
&\qquad \sup_{t \in [a,b]} \left\{\int_a^t (t-s)^{\alpha-1} e^{\theta(s-t)} \mathrm{d}s\right\}\\
&\quad \le \frac{L}{\theta^\alpha} \left(D_\theta(x, y) + D_\theta\left({}^C\mathcal{D}_{a^+}^{\alpha-1} x, {}^C\mathcal{D}_{a^+}^{\alpha-1} y\right)\right).
\end{aligned} \tag{4.10}$$

On the other hand, for $t \in [a,b]$ we get

$$\begin{aligned}
&D_\theta\left({}^C\mathcal{D}_{a^+}^{\alpha-1} \mathbb{P}x, {}^C\mathcal{D}_{a^+}^{\alpha-1} \mathbb{P}y\right)\\
&\quad \le L\left(D_\theta(x, y) + D_\theta\left({}^C\mathcal{D}_{a^+}^{\alpha-1} x, {}^C\mathcal{D}_{a^+}^{\alpha-1} y\right)\right)\\
&\qquad \sup_{t \in [a,b]} \left\{\int_a^t e^{\theta(s-t)} \mathrm{d}s\right\}\\
&\quad \le \sup_{t \in [a,b]} \left\{\frac{L(1 - \exp(\theta(a-t)))}{\theta}\right\}\\
&\quad \left(D_\theta(x, y) + D_\theta\left({}^C\mathcal{D}_{a^+}^{\alpha-1} x, {}^C\mathcal{D}_{a^+}^{\alpha-1} y\right)\right).
\end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we obtain

$$\begin{aligned}\mathbb{D}_\theta(\mathbb{P}x,\mathbb{P}y) &= D_\theta(\mathbb{P}x,\mathbb{P}y) + D_\theta\left({}^C\mathcal{D}^{\alpha-1}_{a+}\mathbb{P}x, {}^C\mathcal{D}^{\alpha-1}_{a+}\mathbb{P}y\right)\\ &\le L\mathbb{D}_\theta(x,y)\left(\frac{1}{\theta^\alpha}+\frac{1-e^{-\theta(b-a)}}{\theta}\right)\\ &\le L\mathbb{D}_\theta(x,y)\left(\frac{1}{\theta^\alpha}+\frac{1-e^{-\theta b}}{\theta}\right).\end{aligned}$$

In the last inequality, we can choose a value for $\theta > \gamma$ big enough such that $\mathbb{P}$ is a contraction. Indeed, using that

$$\lim_{\theta\to\infty}\left(\frac{1}{\theta^\alpha}+\frac{1-e^{-\theta b}}{\theta}\right)=0$$

for $\alpha \in (1,2)$. Therefore, we can choose $\theta > \gamma$ and $\theta > L$ such that $L\left(\frac{1}{\theta^\alpha}+\frac{1-e^{-\theta b}}{\theta}\right) < 1$, and $\mathbb{P}$ is a contraction on $\mathbb{S}_\theta$. Thanks to the Banach fixed point theorem, we obtain the existence of a unique fixed point for $\mathbb{P}$ and the unique fixed of $\mathbb{P}$ is in the space $\mathbb{S}_\theta$ with distance $\mathbb{D}_\theta$ that is a unique solution of the problem (4.1). □

The existence and uniqueness theorem for the initial value problem (4.1) can be obtained by using the method of successive approximations as in Theorem 3.1. If we work directly with the successive approximations (4.12) in the next corollary, we can prove that the iterations converge and that the initial value problem (4.1) has a unique solution on the interval $[a,b]$ under only assumption $L(t-a)^\alpha < \Gamma(1+\alpha)$.

Corollary 4.1 *Let $f : [a,b]\times E\times E \to E$ is a continuous function. Assume that there exists a positive constant L such that for every $u, v, z, w \in \mathbb{E}$, $D_0[f(t,u,v), f(t,z,w)] \le L(D_0[u,z]+D_0[v,w])$, $t\in(a,b]$. Then, the following successive approximations given by $x_0(t) = x(a)\ominus_{gH}(t-a)x'(a)$ and for $n = 1,2,\dots$*

$$x_n(t)\ominus_{gH}x_0(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x_{n-1}(s), {}^C\mathcal{D}^{\alpha-1}_{a+}x_{n-1}(s))ds,\quad t\in(a,b]. \tag{4.12}$$

converge uniformly to a unique solution of the problem (4.1) on some intervals $[a,b^*]$ for some $b^*\in(a,b]$.

Example 4.3 Let $\alpha\in(1,2)$, $\lambda\in\mathbb{R}\backslash\{0\}$, $t > a \ge 0$. We consider the linear Caputo fractional fuzzy differential equation as follows:

$$\begin{aligned}&{}^C D^\alpha_{a+}x(t)\\ &= \lambda x(t) + h(t),\qquad x(a)=x_0,\quad x'(a)=\xi_0\quad t\in(a,b].\end{aligned}\tag{4.13}$$

We observe that $f(t,u) := \lambda u + h$ satisfies the assumptions of Theorem 4.2. To get an explicit solution formula of (4.13), we apply the method of successive approximations. By employing Corollary 4.3, we consider the following, for $n=1,2,\dots,$

$$\begin{aligned}&(x_n(t)\ominus_{gH}x(a))\ominus_{gH}(t-a)x'(a)\\ &=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda x_{n-1}(s)+h(s))\mathrm{d}s,\quad t\in[a,b],\end{aligned}\tag{4.14}$$

where $x_0(t) := x(a)\ominus_{gH}(t-a)x'(a)$. In this example, based on Remark 4.1 we will find the exact solution of (4.13) in the following cases:

Case 1 We assume that $\lambda > 0$, the solution of the problem (4.13), x, is $d-$increasing, and x' is also $d-$increasing on $(a,b]$. Then, by Remark 4.1 the successive approximation (4.14) becomes

$$\begin{aligned}x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda x_{n-1}(s)\\ + h(s))\mathrm{d}s,\quad t\in(a,b],\end{aligned}\tag{4.15}$$

for $n=1,2,\dots,$ where $x_0(t) = x(a)+(t-a)x'(a)$. It is well-known that (4.15) can be represented with the $r-$level sets as follows:

$$\begin{aligned}[x_n(t)]^r = [x_0(t)]^r + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda[x_{n-1}(s)]^r\\ + [h(s)]^r)\mathrm{d}s,\end{aligned}\tag{4.16}$$

where $r\in[0,1]$, $t\in(a,b]$. Then, because x is $d-$increasing, then we have $[x'(a)]^r = [\underline{x}'(r,a), \overline{x}'(r,a)]$, and this yields that

$$\begin{cases}\underline{x}_n(r,t) = \underline{x}_0(r,t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda\underline{x}_{n-1}(r,s)+\underline{h}(r,s))\mathrm{d}s,\quad t\in(a,b],\\ \overline{x}_n(r,t) = \overline{x}_0(r,t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda\overline{x}_{n-1}(r,s)+\overline{h}(r,s))\mathrm{d}s,\quad t\in(a,b],\end{cases}$$

where $\underline{x}_0(r,t) = \underline{x}(r,a) + (t-a)\underline{x}'(r,a)$ and $\overline{x}_0(r,t) = \overline{x}(r,a) + (t-a)\overline{x}'(r,a)$.
- Now, for $n=1$, one gets

$$\underline{x}_1(r,t) = \underline{x}(r,a)\left[1 + \frac{\lambda(t-a)^\alpha}{\Gamma(\alpha+1)}\right] + \underline{x}'(r,a)(t-a)\left[1 + \frac{\lambda(t-a)^\alpha \Gamma(2)}{\Gamma(\alpha+2)}\right] + (\Im^\alpha_{a+}\underline{h})(r,t).$$

For $n=2$, one also obtains that

$$\underline{x}_2(r,t) = \underline{x}(r,a)\left[1 + \frac{\lambda(t-a)^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2(t-a)^{2\alpha}}{\Gamma(2\alpha+1)}\right] + (\Im^\alpha_{a+}h)(r,t) + \lambda(\Im^{2\alpha}_{a+}\underline{h})(r,t) + \underline{x}'(r,a)(t-a)\left[1+\frac{\lambda(t-a)^\alpha}{\Gamma(\alpha+2)}+\frac{\lambda^2(t-a)^{2\alpha}}{\Gamma(2\alpha+2)}\right].$$

By proceeding inductively and taking $n \to \infty$ one has that

$$\begin{aligned}\underline{x}(r,t) &= \underline{x}(r,a)\sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} + \underline{x}'(r,a)(t-a)\\ &\quad \sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+2)} + \int_a^t \sum_{i=1}^{\infty}\frac{\lambda^{i-1}(t-s)^{i\alpha-1}}{\Gamma(i\alpha)}\underline{h}(r,s)\mathrm{d}s\\ &= \underline{x}(r,a)\sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} + x'(r,a)(t-a)\\ &\quad \sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+2)} + \int_a^t \sum_{i=0}^{\infty}\frac{\lambda^{i}(t-s)^{i\alpha+(\alpha-1)}}{\Gamma(i\alpha+\alpha)}\underline{h}(r,s)\mathrm{d}s\\ &= \underline{x}(r,a)\sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} + \underline{x}'(r,a)(t-a)\\ &\quad \sum_{i=0}^{\infty}\frac{\lambda^i(t-a)^{i\alpha}}{\Gamma(i\alpha+2)}\\ &\quad + \int_a^t (t-s)^{\alpha-1}\sum_{i=0}^{\infty}\frac{\lambda^i(t-s)^{i\alpha}}{\Gamma(i\alpha+\alpha)}\underline{h}(r,s)\mathrm{d}s.\end{aligned}$$

Then, it follows from the definition of the Mittag–Leffler function $E_{\alpha,\beta}(u) = \sum_{i=0}^{\infty}\frac{u^i}{\Gamma(i\alpha+\beta)}$, $\alpha,\beta>0$ that the solution of the problem (4.13) is given by

$$\begin{aligned}\underline{x}(r,t) &= \underline{x}(r,a)E_{\alpha,1}\left(\lambda(t-a)^\alpha\right)\\ &\quad + \underline{x}'(r,a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^\alpha\right)\\ &\quad + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^\alpha\right)\underline{h}(r,s)\mathrm{d}s.\end{aligned}$$

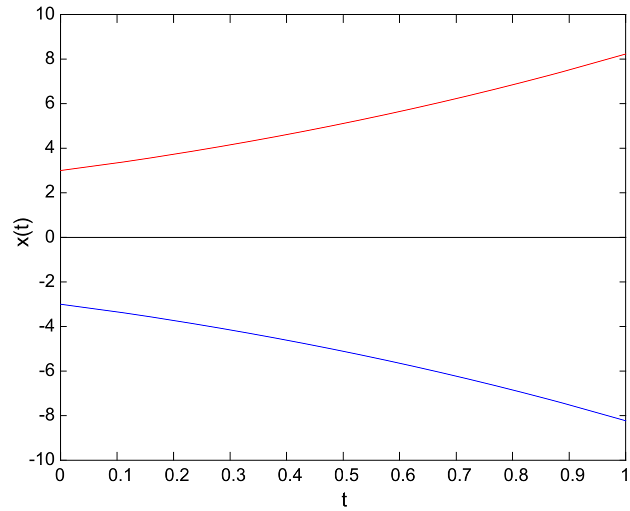

Fig. 11 The graph of the solution $x(t)$ given by (4.18)

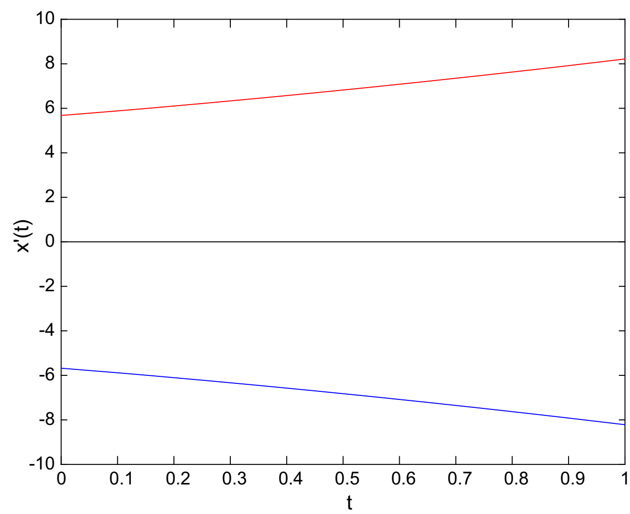

Fig. 12 The graph of $x'(t)$ in Case 1

With the same manner, we also obtain

$$\begin{aligned}\overline{x}(r,t) &= \overline{x}(r,a)E_{\alpha,1}\left(\lambda(t-a)^\alpha\right)\\ &\quad + \underline{x}'(r,a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^\alpha\right)\\ &\quad + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^\alpha\right)\overline{h}(r,s)\mathrm{d}s.\end{aligned}$$

This yields the solution of (4.13) is as follows:

$$x(t) = x(a)E_{\alpha,1}\left(\lambda(t-a)^\alpha\right) + x'(a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^\alpha\right) + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^\alpha\right)h(s)\mathrm{d}s. \quad (4.17)$$

In our numerical simulations, we consider the following parameters: $(a, b] = (0, 1], x(a) = (-3, 0, 3), x'(a) = (-3, 0, 3), \alpha = 1.5, \lambda = 0.5$, and the fuzzy function $h(t) = t^{\beta}(-1, 0, 1)$, where $\beta = 0.5$. Then, the formula (4.17) becomes

$$x(t) = (-3, 0, 3)E_{\alpha,1}(\lambda t^{\alpha}) + (-3t, 0, 3t)E_{\alpha,2}(\lambda t^{\alpha}) + \Gamma(\beta+1)t^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(\lambda t^{\alpha})(-1, 0, 1). \quad (4.18)$$

The graphs of the solution $x(t)$ given by (4.18) and of $x'(t)$ are shown in Figs. 11 and 12.

Case 2 We assume that $\lambda < 0$, the solution of the problem (4.13), x, is $d-$decreasing, and x' is also $d-$increasing on $(a, b]$. Then, by Remark 4.1 the successive approximation (4.14) becomes

$$x_n(t) = x_0(t) \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(\lambda x_{n-1}(s) + h(s))\mathrm{d}s, \quad t \in (a, b], \quad (4.19)$$

for $n = 1, 2, \ldots$, where $x_0(t) = x(a) \ominus (-1)(t-a)x'(a)$. Similar to Case 1, Eq. (4.19) can be represented with the $r-$level sets as follows:

$$[x_n(t)]^r = [x_0(t)]^r \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(\lambda[x_{n-1}(s)]^r + [h(s)]^r)\mathrm{d}s, \quad t \in (a, b]. \quad (4.20)$$

Because x is $d-$decreasing and $\lambda < 0$, we have $[x'(a)]^r = [\overline{x}'(r, a), \underline{x}'(r, a)]$ and

$$\begin{cases} \underline{x}_n(r, t) = \underline{x}_0(r, t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1}(\lambda \underline{x}_{n-1}(r, s) + \overline{h}(r, s))\mathrm{d}s, \quad t \in (a, b], \\ \overline{x}_n(r, t) = \overline{x}_0(r, t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1}(\lambda \overline{x}_{n-1}(r, s) + \underline{h}(r, s))\mathrm{d}s, \quad t \in (a, b], \end{cases}$$

where $\underline{x}_0(r, t) = \underline{x}(r, a) + (t-a)\underline{x}'(r, a)$ and $\overline{x}_0(r, t) = \overline{x}(r, a) + (t-a)\overline{x}'(r, a)$. By proceeding similarly as in Case 1, as $n \to \infty$ we also get

$$\underline{x}(r, t) = \underline{x}(r, a)E_{\alpha,1}\left(\lambda(t-a)^{\alpha}\right) + \underline{x}'(r, a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^{\alpha}\right) + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)\overline{h}(r, s)\mathrm{d}s,$$

$$\overline{x}(r, t) = \overline{x}(r, a)E_{\alpha,1}\left(\lambda(t-a)^{\alpha}\right) + \underline{x}'(r, a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^{\alpha}\right) + \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)\underline{h}(r, s)\mathrm{d}s.$$

This yields the solution of (4.13) is as follows:

$$x(t) = x(a)E_{\alpha,1}\left(\lambda(t-a)^{\alpha}\right) \ominus (-1)x'(a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^{\alpha}\right) \ominus (-1)\int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)h(s)\mathrm{d}s. \quad (4.21)$$

In our numerical simulations, we consider the following parameters: $(a, b] = (0, 1], x(a) = (1, 5, 10), x'(a) = (-1, 0, 1), \alpha = 1.75, \lambda = -1$, and the fuzzy function $h(t) = t^{\beta}(-1/2, 0, 1/2$, where $\beta = 2$. Then, the formula (4.21) becomes

$$x(t) = (1, 5, 10)E_{\alpha,1}(\lambda t^{\alpha}) \ominus (-1)(-t, 0, t)E_{\alpha,2}(\lambda t^{\alpha}) \ominus (-1)\Gamma(\beta+1)t^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(\lambda t^{\alpha})(-1/2, 0, 1/2). \quad (4.22)$$

The graphs of the solution $x(t)$ given by (4.22) and of $x'(t)$ are shown in Figs. 13 and 14.

Case 3 We assume that $\lambda < 0$, the solution of the problem (4.13), x, is $d-$increasing, and x' is also $d-$decreasing on $(a, b]$. Then, by Remark 4.1 the successive approximation (4.14) becomes

$$x_n(t) = x_0(t) \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(\lambda x_{n-1}(s) + h(s))\mathrm{d}s, \quad t \in [a, b], \quad (4.23)$$

for $n = 1, 2, \ldots$, where $x_0(t) = x(a) + (t-a)x'(a)$. By the same manner as the above two cases, the solution of (4.13) in this case is given by

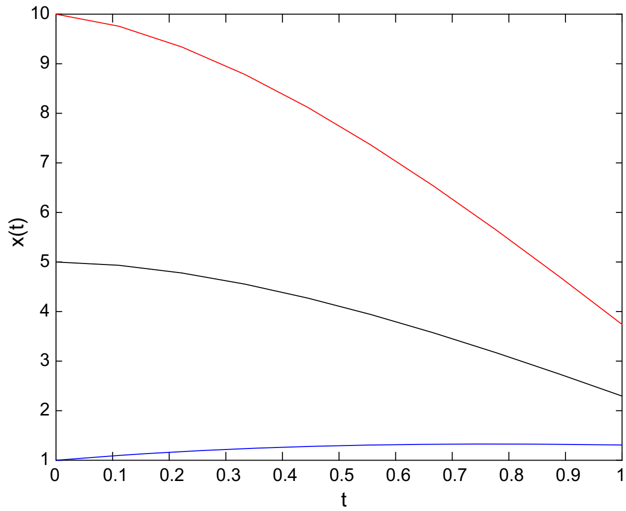

Fig. 13 The graph of the solution $x(t)$ given by (4.22)

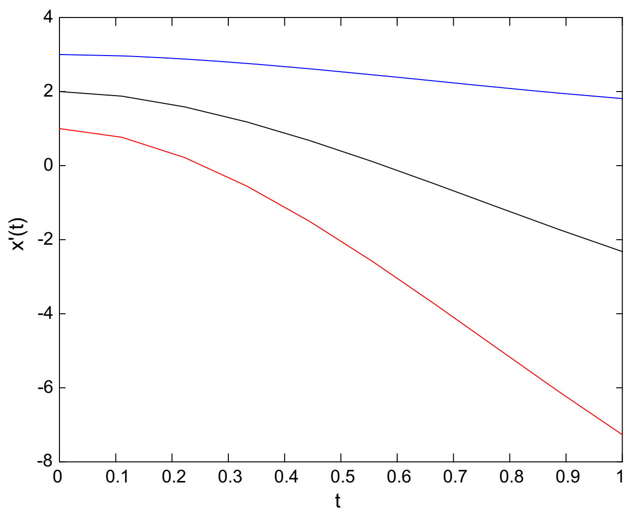

Fig. 14 The graph of $x'(t)$ in Case 2

$$x(t) = x(a)E_{\alpha,1}\left(\lambda\,(t-a)^{\alpha}\right) + x'(a)(t-a)E_{\alpha,2}\left(\lambda\,(t-a)^{\alpha}\right) \ominus (-1)\int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\,(t-s)^{\alpha}\right)h(s)\mathrm{d}s. \tag{4.24}$$

In our numerical simulations, we consider the following parameters: $(a, b] = (0, 1]$, $x(a) = (1, 5, 10)$, $x'(a) = (1, 5, 10)$, $\alpha = 1.75$, $\lambda = -1$, and the fuzzy function $h(t) = t^{\beta}(-1/2, 0, 1/2)$, where $\beta = 2$. Then, the formula (4.24) becomes

$$x(t) = (1, 5, 10)E_{\alpha,1}(\lambda t^{\alpha}) + (t, 5t, 10t)E_{\alpha,2}(\lambda t^{\alpha}) \ominus (-1)\Gamma(\beta+1)t^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(\lambda t^{\alpha})(-1/2, 0, 1/2). \tag{4.25}$$

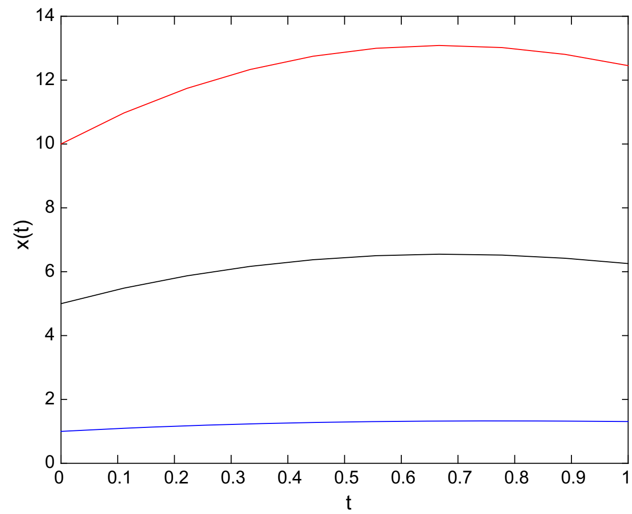

Fig. 15 The graph of the solution $x(t)$ given by (4.25)

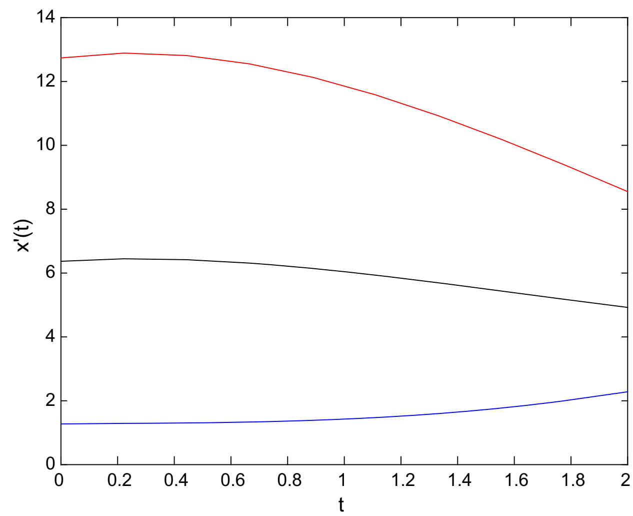

Fig. 16 The graph of $x'(t)$ in Case 3

The graphs of the solution $x(t)$ given by (4.25) and of $x'(t)$ are shown in Figs. 15 and 16.

Case 4 We assume that $\lambda > 0$, the solution of the problem (4.13), x, is $d-$decreasing, and x' is also $d-$decreasing on $(a, b]$. Then, by Remark 4.1 the successive approximation (4.14) becomes

$$x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\lambda x_{n-1}(s) + h(s))\mathrm{d}s, \quad t \in (a, b], \tag{4.26}$$

for $n = 1, 2, \ldots$, where $x_0(t) = x(a) \ominus (-1)(t-a)x'(a)$. Similarly, we also get the solution of (4.13) in this case as follows:

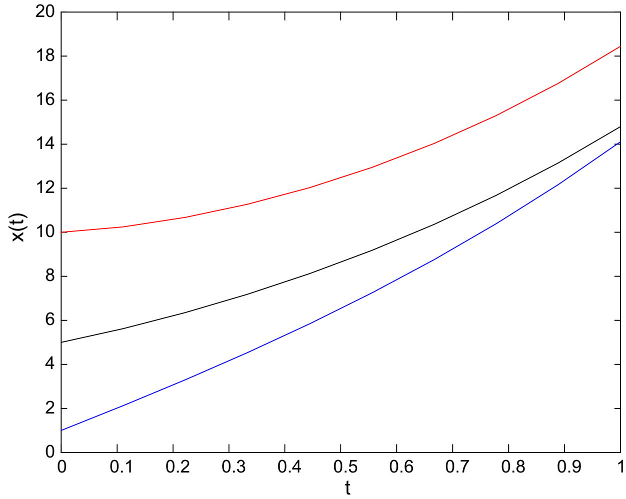

Fig. 17 The graph of the solution $x(t)$ given by (4.29)

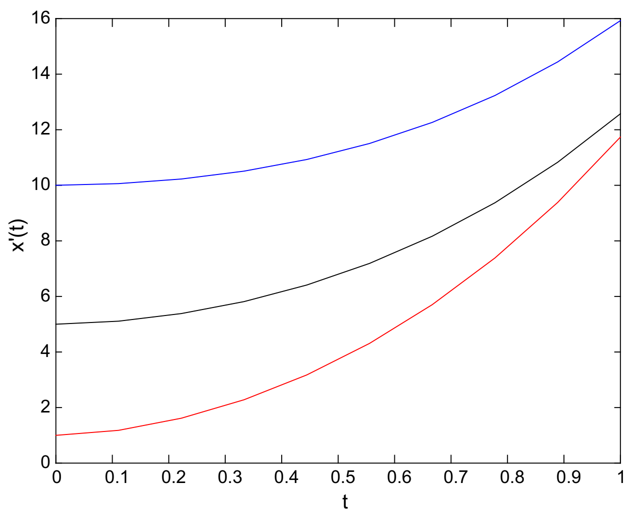

Fig. 18 The graph of $x'(t)$ in Case 4

$$x(t) = x(a)E_{\alpha,1}\left(\lambda(t-a)^{\alpha}\right) \ominus (-1)x'(a)(t-a)E_{\alpha,2}\left(\lambda(t-a)^{\alpha}\right) \quad (4.27)$$

$$+\int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)h(s)\mathrm{d}s. \quad (4.28)$$

In our numerical simulations, we consider the following parameters: $(a, b] = (0, 1]$, $x(a) = (1, 5, 10)$, $x'(a) = (1, 5, 10)$, $\alpha = 1.75$, $\lambda = 0.75$, and the fuzzy function $h(t) = t^{\beta}(-1/2, 0, 1/2)$, where $\beta = 2$. Then, the formula (4.27) becomes

$$x(t) = (1, 5, 10)E_{\alpha,1}(\lambda t^{\alpha}) \ominus (-1)(t, 5t, 10t)E_{\alpha,2}(\lambda t^{\alpha}) + \Gamma(\beta+1)t^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(\lambda t^{\alpha})(-1/2, 0, 1/2). \quad (4.29)$$

The graphs of the solution $x(t)$ given by (4.29) and of $x'(t)$ are shown in Figs. 17 and 18.

5 Conclusions

The study of the fundamental theories and applications of fuzzy fractional integral equations and fuzzy differential equations with the concepts of the fractional derivative have increased over the years, and during the last decade, a large number of investigators have presented interesting and important results in these new fields. In this paper, we propose a standard framework to investigate the existence and uniqueness of the solution for a general form of fractional integral equations and an initial value problem of fractional differential equations in the fuzzy environment with the non-integer order $\alpha \in (1, 2)$. Also, we proposed a formula of the solution for the problem (1.1) in the general case and in the linear case, which plays an essential role in investigating the qualitative theories of the problem (1.1) in future.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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