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METHODOLOGIES AND APPLICATION

Approaches to multiple attribute group decision making based on triangular cubic linguistic uncertain fuzzy aggregation operators

Fazli Amin[1] · Aliya Fahmi[1] · Muhammad Aslam[2]

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Abstract

In this paper, triangular cubic linguistic uncertain fuzzy averaging (geometric) operator, triangular cubic linguistic uncertain fuzzy weighted averaging operator, triangular cubic linguistic uncertain fuzzy weighted geometric operator, triangular cubic linguistic uncertain fuzzy ordered weighted averaging operator, triangular cubic linguistic uncertain fuzzy ordered weighted geometric operator, triangular cubic linguistic uncertain fuzzy hybrid averaging operator, and triangular cubic linguistic uncertain fuzzy hybrid geometric operator for triangular cubic linguistic uncertain fuzzy numbers have been introduced. Furthermore, by using these aggregation operators an approach to multiple attribute group decision making with triangular cubic linguistic uncertain fuzzy information has been developed. Finally, a numerical example is constructed to validate the established approach.

Keywords Triangular cubic linguistic uncertain fuzzy sets · Aggregation operators on triangular cubic linguistic uncertain fuzzy information · Multiple attribute group decision making · Numerical application

1 Introduction

Multi-criteria decision-making (MCDM) approaches are widely used to find optimal alternatives with multiple criteria. However, carrying out exact evaluations for alternatives is challenging for decision makers when the problems involve uncertainty. Zadeh (1965) introduced the concept of fuzzy set (FS); it had been widely applied to deal with the multiple attribute group decision-making (MAGDM) problems and had been continuously improved and developed. Atanassov (1986) introduced the intuitionistic fuzzy sets (IFSs), which are based on membership and non-membership degrees which describe uncertainty more in detail.

According to Liang et al. (2017), inadequate opinions are then finished with data vaccinated from reliable experts. In adding with respect to Liang et al. (2017), such thoughts are more adopted by simulating their evolution due to social influence. It is also observed that, under certain assumptions, the development of feelings due to effect joins to a final collective opinion. If expectations are not met, standard aggregation approaches are used to select the best alternative. The three widely adopted group decision-making methods are TOPSIS, VIKOR, and GRA (You et al. 2015; Wang et al. 2009; Hashemi et al. 2014). Salmeron et al. (2012) integrated Delphi analysis, FCMs, and TOPSIS methods for scenario-based decision-making problems.

Hesitant fuzzy set (HFS) and the statement of fuzzy set were proposed by Torra (2010) and Torra and Narukawa (2009). Later, it is popular for collaborating variability; HFS is drawn further attentions from investigators. Yu exhibited triangular hesitant fuzzy set and its application to training quality assessment in Yu (2013). Jun et al. (2011) considered cubic sets which include a IVF (Zadeh 1975) with the fuzzy set (Zadeh 1965). Chen et al. (2013) introduced a building which calculations active the possibility

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✉ Fazli Amin
fazliamin@hu.edu.pk

Aliya Fahmi
aliyafahmi@gmail.com

Muhammad Aslam
draslamqau@yahoo.com

[1] Department of Mathematics, Hazara University Mansehra, Mansehra, Pakistan

[2] Department of Mathematics, College of Sciences, King Khalid University Abha, Abha, Saudi Arabia

of HFS (Torra 2010) to IVHFS that type believable the membership assessment of a module into numerous possible interval numbers.

Dong et al. (2013) proposed linguistic computational model based on 2-tuples and intervals, which they writers entitled an interval version of the 2-tuple fuzzy linguistic demonstration classical. The proposed model was comprised of three steps: (1) interval numerical scale; (2) computation based on interval numbers; and (3) a generalized inverse operation of the interval numerical scale. In the first step, the linguistic terms transform into interval numbers, founded on which the second step is executed with output as an interval number. Finally, this number is then mapped into the interval of the linguistic 2-tuples by the generalized opposite action.

Dong et al. (2015) developed consensus issue in the hesitant linguistic group decision-making (GDM) problem. Chen et al. (2015) developed three types of fusion approaches: the indirect approach, the optimization-based approach, and the direct approach for group decision making in a survey. Dong et al. (2016) proposed a consensus reaching model in the complex and dynamic MAGDM problems. Dong et al. (2016) developed a connection between linguistic hierarchy and the numerical scale for the 2-tuple linguistic model, and it is used to deal with hesitant unbalanced linguistic information. Liu et al. (2017) proposed the preference relation with self-confidence by taking multiple self-confidence levels into consideration. Dong et al. (2017) proposed a managing consensus based on leadership in opinion dynamics. Dong et al. (2018) proposed a series of mixed 0-1 linear programming models (MLPMs) to show the process of designing a strategic attribute weight vector. Li et al. (2017) developed personalized individual semantics by means of an interval numerical scale and the 2-tuple linguistic model. Zhang et al. (2017) investigated the 2-rank MAGDM problem under the multigranular linguistic context and proposed a 2-rank consensus reaching framework with the minimum adjustments. Zhang et al. (2017) proposed a novel consensus reaching model for the heterogeneous large-scale GDM with the individual concerns.

Fahmi et al. (2017) developed the Hamming distance for triangular cubic fuzzy number and weighted averaging operator. Fahmi et al. (2017) proposed the cubic TOPSIS method and gray relational analysis set. Fahmi et al. (2018) defined the triangular cubic fuzzy number and operational laws. Amin et al. (2017) defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFWG) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average (GTCLHFOWA) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric (GTCLHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averaging (GTCLHFHA) operator, and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric (GTCLHFHG) operator. Fahmi et al. (2017) developed trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Fahmi et al. (2018) defined aggregation operators for triangular cubic linguistic hesitant fuzzy sets, which include cubic linguistic fuzzy (geometric) operator, triangular cubic linguistic hesitant fuzzy weighted geometric (TCLHFWG) operator, triangular cubic linguistic hesitant fuzzy ordered weighted geometric (TCHFOWG) operator and triangular cubic linguistic hesitant fuzzy hybrid geometric (TCLHFHG) operator. Fahmi et al. (2018) defined the trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined. Fahmi et al. (2018) defined the triangular cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator. Fahmi et al. (2018) developed three arithmetic averaging operators, which is trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating trapezoidal cubic fuzzy information. Fahmi et al. (2018) developed three arithmetic averaging operators, which is cubic fuzzy Einstein weighted averaging (CFEWA) operator, cubic fuzzy Einstein ordered weighted averaging (CFEOWA) operator and cubic fuzzy Einstein hybrid weighted averaging (CFEHWA) operator, for aggregating cubic fuzzy information.

Abualigah and Hanandeh (2015) proposed researcher explored problems embedded process, attempted to find solutions such as the way of choosing mutation probability and fitness function. Abualigah and Khader (2017) proposed the hybrid of particle swarm optimization algorithm with genetic operators for the feature selection problem. Abualigah et al. (2018) proposed the novel feature selection method, namely feature selection method using the particle swarm optimization (PSO) algorithm (FSPSOTC), to solve the feature selection problem by creating a new subset of informative text features. Abualigah et al. (2018) proposed the k-mean clustering algorithm, and the clustering decision is based on two combined objective functions. Abualigah et al. (2018) proposed the purpose of the experiments; six versions are thoroughly investigated to determine the best version for solving the text clustering. Abualigah (2019) proposed the four krill herd algorithms (KHAs), namely the (a) basic KHA, (b) modified KHA, (c) hybrid KHA, and (d) multi-objective hybrid KHA.

Ju and Yang (2015) introduced the top improvements on SFLA for solving multi-objective optimization problems, enhancing local and global exploration, avoiding being trapped into local optima, declining computational time and improving the quality of the initial population. Malik and Shabir (2017) introduced some new aggregation operators, such as intuitionistic trapezoid fuzzy linguistic weighted geometric operator, intuitionistic trapezoid fuzzy linguistic ordered weighted geometric operator, intuitionistic trapezoid fuzzy linguistic hybrid weighted geometric operator, intuitionistic trapezoid fuzzy linguistic generalized weighted averaging operator, intuitionistic trapezoid fuzzy linguistic generalized ordered weighted averaging operator and intuitionistic trapezoid fuzzy linguistic generalized hybrid weighted averaging operator. Ren et al. (2016) introduced the relationship between these two classes of integrals by giving two Newton–Leibniz formulas for SIVIFFs. Ju et al. (2016) introduced the new method for multiple attribute group decision making under a 2-tuple linguistic environment based on the proposed operators. Sarkheyli et al. (2015) defined some structural properties of rough fuzzy bipolar soft sets and studied the effects of the equivalence relation in Pawlak approximation space on the roughness of the fuzzy bipolar soft sets. Alam and Baulkani (2019) introduced the centroid information issue solved by compactness measure, and the OP measure is used to handle the geometric structure of the clustering problem. Additionally, in the proposed clustering approach, the concept of opposition-based generation jumping and opposition-based population initialization is used with the standard GWO to enhance its computational speed and convergence profile. Rajab and Sharma (2019) introduced the efficient and interpretable neuro-fuzzy system for stock price prediction using multiple technical indicators with a focus on interpretability–accuracy tradeoff. Ren and Wang (2019) extended to deal with the fuzzy bi-level linear programming problem through the nearest interval approximation.

Despite having a bulk of related literature on the problem under consideration, the following aspects related to trapezoidal cubic linguistic uncertain fuzzy sets and their aggregation operators motivated the researchers to carry out an in-depth inquiry into the current study.

(1) The main advantages of the proposed operators are these aggregation operators provided more accurate and precious result as compared to the above-mentioned operators.
(2) We generalized the concept of triangular cubic linguistic uncertain fuzzy numbers and triangular intuitionistic linguistic uncertain fuzzy sets and introduce the concept of triangular cubic linguistic uncertain fuzzy numbers. If we have only one element in the membership degree of the triangular cubic linguistic uncertain fuzzy numbers, i.e., instead of interval, then we get triangular intuitionistic linguistic uncertain fuzzy numbers; similarly, if we take membership degree as fuzzy number and non-membership degree equal to zero, then we get triangular linguistic uncertain fuzzy numbers.
(3) The objective of the study is to:

Propose triangular cubic linguistic uncertain fuzzy numbers, operational laws, score value and accuracy value of TrCLUFSs.

Propose six aggregation operators, namely triangular cubic linguistic uncertain fuzzy weighted averaging operator, triangular cubic linguistic uncertain fuzzy weighted geometric operator, triangular cubic linguistic uncertain fuzzy weighted ordered averaging operator, triangular cubic linguistic uncertain fuzzy ordered weighted geometric operator, triangular cubic linguistic uncertain fuzzy hybrid weighted averaging operator, triangular cubic linguistic uncertain fuzzy weighted geometric operator and triangular cubic linguistic uncertain fuzzy hybrid weighted geometric operator.

Establish an MADM program approach based on triangular cubic linguistic uncertain fuzzy information.

Illustrative examples of MADM program are constructed to strengthen our approach.

(4) In order to testify the application of the developed method, we apply the triangular cubic linguistic uncertain fuzzy number in the decision making.
(5) The initial decision matrix is composed of LVs. In order to fully consider the randomness and ambiguity of the linguistic term, we convert LVs into the triangular cubic linguistic uncertain fuzzy information, and the decision matrix is transformed into the triangular cubic linguistic uncertain fuzzy decision matrix.
(6) The operator fully expresses the uncertainty of qualitative concept, and trapezoidal cubic linguistic uncertain fuzzy operators capture the interdependencies among any multiple inputs or attributes by a variable parameter. The aggregation operators take into account the importance of attribute weights. Nevertheless, sometimes, for some MAGDM problems, the weights of attributes are important factors for the decision process.
(7) Moreover, in many multiple attribute group decision-making (MAGDM) problems, considering that the estimations of the attribute values are triangular cubic linguistic uncertain fuzzy sets, it is necessary to give some aggregation techniques to aggregate the triangular cubic linguistic uncertain fuzzy information. However, we are aware of the fact that the

present aggregation techniques have difficulty in coping with group decision-making problems with triangular cubic linguistic uncertain fuzzy information. Therefore, in the current paper we propose a series of aggregation operators for aggregating the triangular cubic linguistic uncertain fuzzy information and investigate some properties of these operators. Then, based on these aggregation operators, we develop an approach to MAGDM with triangular cubic linguistic uncertain fuzzy information. Moreover, we use a numerical example to show the application of the developed approach.

The rest parts of this paper are organized as follows: Section 2, we give some fundamental thought and properties of cubic set. Section 3 exhibits triangular cubic linguistic uncertain fuzzy numbers and operational laws. Section 4 exhibits a series of aggregation operators for triangular cubic linguistic uncertain fuzzy information and observes the associations among these aggregation operators. Section 5 develops an approach to group decision makings with triangular cubic linguistic uncertain fuzzy data. Section 6 shows the application of the developed approach in group decision-making problems, which is shown by an illustrative example. Section 7 discusses the comparison analysis. Finally, we give the conclusions in Sect. 8.

2 Preliminaries

Definition 2.1 (Zadeh 1965) Let H be a universe of discourse. The fuzzy set is defined as follows: $J = \{h, \mu_J(h)|h \in H\}$. A fuzzy set in a set H is defined $\mu_J : H \to I$, which is a membership function $\mu_J(h)$ denoted by the degree of membership of the element h to the set H, where $I = [0, 1]$. The collection of all fuzzy subsets of H is denoted by I^H. A relation on I^H is defined as follows: $(\forall \mu, \eta \in I^H)(\mu \le \eta \Leftrightarrow (\forall h \in H)(\mu(h) \le \eta(h)))$.

Definition 2.2 (Jun et al. 2011) Let H be a non-empty set. By a cubic set in H, we mean a structure $F = \{h, \alpha(h), \beta(h) : h \in H\}$ in which α is an IVF set in H and β is a fuzzy set in H. A cubic set $F = \{h, \alpha(h), \beta(h) : h \in H\}$ is simply denoted by $F = \langle \alpha, \beta \rangle$. The collection of all cubic sets in H is denoted by C^H.

Definition 2.3 (Jun et al. 2011) Let H be a non-empty set. A cubic set $F = (\alpha, \beta)$ in H is said to be an internal cubic set if $\alpha^-(h) \le \beta(h) \le \alpha^+(h)$ for all $h \in H$.

Definition 2.4 (Jun et al. 2011) Let H be a non-empty set. A cubic set $F = (\alpha, \beta)$ in H is said to be an external cubic set if

$\beta(h) \notin (\alpha^-(h), \alpha^+(h))$ for all $h \in H$.

3 Triangular cubic linguistic uncertain fuzzy numbers

In this section, the TCLUFNs, operational laws, score, and accuracy values are introduced.

Definition 3.1 Let $\tilde{b}$ be the triangular cubic linguistic uncertain fuzzy number on the set of real numbers, and its interval value triangular linguistic uncertain fuzzy number is defined as follows:

$$\lambda_{\tilde{b}}(h) = \begin{cases} [s_{\theta(b)}, s_{t(b)}], \left\{\frac{(h-a^-)}{(b^- - a^-)}, \frac{(h-a^+)}{(b^+ - a^+)}\right\} & (b^-, a^-) \le h < (b^+, a^+) \\ [s_{\theta(b)}, s_{t(b)}], \left\{\frac{(c^- - h)}{(c^- - b^-)}, \frac{(c^+ - h)}{(c^+ - b^+)}\right\} & (b^-, c^-) \le h < (c^+, c^+) \\ 0 & \text{otherwise} \end{cases}$$

and its triangular linguistic uncertain fuzzy number is

$$\Gamma_{\tilde{b}}(h) = \begin{cases} \left\{[s_{\theta(b)}, s_{t(b)}], \frac{(b-h)}{(b-a)}\right\} & a \le h < b \\ \left\{[s_{\theta(b)}, s_{t(b)}], \frac{(c-h)}{(c-b)}\right\} & b < h \le c \\ 0 & \text{otherwise} \end{cases}$$

where $0 \le \lambda_{\tilde{b}}(h) \le 1$, $0 \le \Gamma_{\tilde{b}}(h) \le 1$. Then, the TCLUFN $b^\sim$ is basically denoted by: $\tilde{b} = \left\{ \begin{array}{c} [s_{\theta(b)}, s_{t(b)}], \langle [(a^-, b^-, c^-)], \\ [(a^+, b^+, c^+)], \\ [(a, b, c)] \rangle \end{array} \right\}$. Then, $\tilde{b}$ is called triangular cubic linguistic uncertain fuzzy number.

Definition 3.2 Let $h = \left\{ \begin{array}{c} \langle [s_{\theta(h)}, s_{t(h)}], [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c \rangle \end{array} \right\}$ $h_1 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)}, s_{t(h_1)}, [a_1^-, b_1^-, c_1^-], \\ [a_1^+, b_1^+, c_1^+], \\ a_1, b_1, c_1 \rangle \end{array} \right\}$ and $h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_2)}, s_{t(h_2)}, [a_2^-, b_2^-, c_2^-], \\ [a_2^+, b_2^+, c_2^+], \\ a_2, b_2, c_2 \rangle \end{array} \right\}$ be three TCLUFNs; then,

$$h^c = \{\alpha^c | \alpha \in h\} = \langle a, b, c, [a^-, b^-, c^-], [a^+, b^+, c^+], s_\theta, s_t \rangle,$$

$$h_1 \cup h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_1) \cup \theta(h_2)}, s_{t(h_1) \cup t(h_2)} \cdot [a_1^- \cup a_2^-, b_1^- \cup b_2^-, c_1^- \cup c_2^-], \\ [a_1^+ \cup a_2^+, b_1^+ \cup b_2^+, c_1^+ \cup c_2^+], \\ a_1 \cap a_2, b_1 \cap b_2, c_1 \cap c_2 \rangle \end{array} \right\};$$

$$h_1 \cap h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_1) \cap \theta(h_2)}, s_{t(h_1) \cap t(h_2)}, [a_1^- \cap a_2^-, b_1^- \cap b_2^-, c_1^- \cap c_2^-], \\ [a_1^+ \cap a_2^+, b_1^+ \cap b_2^+, c_1^+ \cap c_2^+], \\ a_1 \cup a_2, b_1 \cup b_2, c_1 \cup c_2 \rangle \end{array} \right\};$$

$$h_1 \oplus h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)+\theta(h_2)},\, s_{t(h_1)+t(h_2)}, [a_1^- + a_2^- - a_1^- a_2^-,\, b_1^- + b_2^- - b_1^- b_2^-,\, c_1^- + c_2^- - c_1^- c_2^-, \\ a_1^+ + a_2^+ - a_1^+ a_2^+,\, b_1^+ + b_2^+ - b_1^+ b_2^+,\, c_1^+ + c_2^+ - c_1^+ c_2^+], \\ a_1 a_2,\, b_1 b_2,\, c_1 c_2 \rangle \end{array} \right\};$$

$$\ddot{h}_1 \otimes \ddot{h}_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)\times\theta(h_2)},\, s_{t(h_1)\times t(h_2)}, [a_1^- a_2^-,\, b_1^- b_2^-,\, c_1^- c_2^-,\, a_1^+ a_2^+,\, b_1^+ b_2^+,\, c_1^+ c_2^+], \\ a_1 + a_2 - a_1 a_2,\, b_1 + b_2 - b_1 b_2,\, c_1 + c_2 - c_1 c_2 \rangle \end{array} \right\};$$

$$\lambda h = \{\lambda h | a \in h\} = \left\{ \begin{array}{c} \langle s_{\theta^\lambda(h)},\, s_{t^\lambda(h)}, [1-(1-a^-)^\lambda], [1-(1-b^-)^\lambda], [1-(1-c^-)^\lambda], \\ [1-(1-a^+)^\lambda], [1-(1-b^+)^\lambda], \\ [1-(1-c^+)^\lambda], (a)^\lambda, (b)^\lambda, (c)^\lambda \rangle \end{array} \right\};$$

$$h^\lambda = \{\alpha^\lambda | a \in h\} = \left\{ \begin{array}{c} \langle s_{\lambda\times\theta(h)},\, s_{\lambda\times t(h)}, [(a^-)^\lambda, (b^-)^\lambda, (c^-)^\lambda, (a^+)^\lambda, (b^+)^\lambda, (c^+)^\lambda], \\ 1-(1-a)^\lambda,\, 1-(1-b)^\lambda,\, 1-(1-c)^\lambda] \rangle \end{array} \right\}.$$

Theorem 3.3 *Let* $h = \left\{ \begin{array}{c} \langle s_\theta,\, s_t, [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c \rangle \end{array} \right\}$, $h_1 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)},\, s_{t(h_1)}, [a_1^-,\, b_1^-,\, c_1^-], \\ [a_1^+,\, b_1^+,\, c_1^+], \\ a_1,\, b_1,\, c_1 \rangle \end{array} \right\}$ *and* $h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_2)},\, s_{t(h_2)}, [a_2^-,\, b_2^-,\, c_2^-], \\ [a_2^+,\, b_2^+,\, c_2^+], \\ a_2,\, b_2,\, c_2 \rangle \end{array} \right\}$ *be three TCLUFNs. Then score function* $S(\alpha) = \left\{ \left\langle \frac{s_{\theta(\alpha)}, s_{t(\alpha)}]\{[a^- + b^- + c^-] + [a^+ + b^+ + c^+] - [a+b+c]\}}{9} \right\rangle \right\}$, *Accuracy function* $G(\alpha) = \left\{ \left\langle \frac{s_{\theta(\alpha)}, s_{t(\alpha)}\{[a^- + b^- + c^-] + [a^+ + b^+ + c^+] + [a+b+c]\}}{9} \right\rangle \right\}$. *Give an order relation between two TCLUFNs* α_1 *and* α_2 *as follows:*

(1) If $s(\alpha_1) > s(\alpha_2)$, then $s(\alpha_1) > s(\alpha_1)$.
(2) If $s(\alpha_1) = s(\alpha_2)$, then the following hold.
(a) If $g(\alpha_1) > g(\alpha_2)$, then $\alpha_1 > \alpha_2$.
(b) If $g(\alpha_1) = g(\alpha_2)$, then $\alpha_1 = \alpha_2$.
(c) If $g(\alpha_1) > g(\alpha_1)$, then $\alpha_1 > \alpha_2$.

Theorem 3.4 *Let* $h = \left\{ \begin{array}{c} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c \rangle \end{array} \right\}$, $h_1 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)},\, s_{t(h_1)}, \\ [a_1^-,\, b_1^-,\, c_1^-], \\ [a_1^+,\, b_1^+,\, c_1^+], \\ a_1,\, b_1,\, c_1 \rangle \end{array} \right\}$ *and* $h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_2)},\, s_{t(h_2)}, \\ [a_2^-,\, b_2^-,\, c_2^-], \\ [a_2^+,\, b_2^+,\, c_2^+], \\ a_2,\, b_2,\, c_2 \rangle \end{array} \right\}$ *are three TCLUFNs and* $\lambda,\, \lambda_1,\, \lambda_2 \geq 0$. *Then*

(1) $h_1 \cup h_2 = (h_1 \cap h_2)^c$,

(2) $h_1 \cap h_2 = (h_1 \cup h_2)^c$,

(3) $(h^c)^\lambda = (\lambda h)^c$,

(4) $\lambda(h^c) = (h^\lambda)^c$,

(5) $h_1 \oplus h_2 = (h_1 \otimes h_2)^c$,

(6) $h_1 + h_2 = h_2 + h_1$,

(7) $h_1 \otimes h_2 = h_2 \otimes h_1$,

(8) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2,\ \lambda \geq 0$

(9) $\lambda_1 h_1 + h_2 a_1 = (h_1 + h_2) a_1,\ \lambda_1,\ \lambda_2 \geq 0;$

(10) $h_1 \otimes h_2 = (h_1)^{\lambda_1 + \lambda_2},\ \lambda_1,\ \lambda_2 \geq 0;$

(11) $h_1 \otimes h_2 = (h_1 \otimes h_2)^{\lambda_1},\ \lambda_1 \geq 0;$

(12) $h_1 \otimes \ddot{h}_2 = (h_1 \oplus h_2)^c$.

Proof
(1):

$$h_1 \cup h_2 = \left\{ \begin{array}{c} \langle \{a_1,\, b_1,\, c_1, [a_1^-,\, b_1^-,\, c_1^-], [a_1^+,\, b_1^+,\, c_1^+],\, s_{\theta_1},\, s_{t_1}\} \cup \\ \{a_2,\, b_2,\, c_2, [a_2^-,\, b_2^-,\, c_2^-], [a_2^+,\, b_2^+,\, c_2^+],\, s_{\theta_2},\, s_{t_1}\} \\ = \{a_1 \cup a_2,\, b_1 \cup b_2,\, c_1 \cup c_2 [a_1^- \cap a_2^-,\, b_1^- \cap b_2^-,\, c_1^- \cap c_2^-], \\ [a_1^+ \cap a_2^+,\, b_1^+ \cap b_2^+,\, c_1^+ \cap c_2^+],\, s_{\theta(h_1)\cap\theta(h_2)},\, s_{t(h_1)\cap t(h_2)}\} \\ = (h_1 \cap h_2)^c \end{array} \right\}.$$

(2):

$$h_1 \cap h_2 = \left\{ \begin{array}{c} \langle \{a_1,\, b_1,\, c_1, [a_1^-,\, b_1^-,\, c_1^-], [a_1^+,\, b_1^+,\, c_1^+],\, s_{\theta_1},\, s_{t_1}\} \cap \\ \{a_1,\, b_1,\, c_1, [a_1^-,\, b_1^-,\, c_1^-], [a_1^+,\, b_1^+,\, c_1^+],\, s_{\theta_2},\, s_{t_2}\} \\ = \{a_1 \cap a_2,\, b_1 \cap b_2,\, c_1 \cap c_2 [a_1^- \cup a_2^-,\, b_1^- \cup b_2^-,\, c_1^- \cup c_2^-], \\ [a_1^+ \cup a_2^+,\, b_1^+ \cup b_2^+,\, c_1^+ \cup c_2^+], \\ s_{\theta(h_1)\cup\theta(h_2)},\, s_{t(h_1)\cup t(h_2)}\} = (h_1 \cup h_2)^c \end{array} \right\}.$$

(3):

$$(h^c)^\lambda = \left\{ \begin{array}{c} \langle \{\langle \{[a,\, b,\, c], [a^-,\, b^-,\, c^-], [a^+,\, b^+,\, c^+]\},\, s_{\theta(\alpha)},\, s_{t(\alpha)} | \alpha \in \ddot{h}\} \rangle \\ = \langle s_{\theta(\alpha)},\, s_{t(\alpha)},\, 1-(1-a)^\lambda,\, 1-(1-b)^\lambda,\, 1-(1-c)^\lambda \\ [(a^-)^\lambda, (b^-)^\lambda, (c^-)^\lambda], [(a^+)^\lambda, (b^+)^\lambda, (c^+)^\lambda], \\ = (\lambda h)^c \rangle \end{array} \right\}.$$

(4):

$$\lambda(h^c) = \left\{ \begin{array}{c} \lambda \langle \{[a,\, b,\, c], [a^-,\, b^-,\, c^-], [a^+,\, b^+,\, c^+]\},\, s_{\theta(\alpha)},\, s_{t(\alpha)} | \alpha \in \ddot{h}\} \rangle \\ = \langle \lambda [1-(1-a)^\lambda, \\ 1-(1-b)^\lambda,\, 1-(1-c)^\lambda], [(a^-)^\lambda, (b^-)^\lambda, (c^-)^\lambda], \\ [(a^+)^\lambda, (b^+)^\lambda, (c^+)^\lambda],\, s_{(\theta(h))^\lambda},\, s_{(t(h))^\lambda}] \rangle \\ = (h^\lambda)^c \end{array} \right\}.$$

(5):

$$h_1 \oplus h_2 = \left\{ \begin{array}{c} \langle \{a_1, b_1, c_1, [a_1^-, b_1^-, c_1^-\}, [a_1^+, b_1^+, c_1^+], s_{\theta(h_1)}\}\rangle \oplus \\ \langle \{\{a_2, b_2, c_2], [a_2^-, b_2^-, c_2^-\}, [a_2^+, b_2^+, c_2^+], s_{\theta(h_2)}\}\rangle = \\ \langle [a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2], \\ [a_1^- a_2^-, b_1^- b_2^-, c_1^- c_2^-], [a_1^+ a_2^+, b_1^+ b_2^+, c_1^+ c_2^+], s_{\theta(h_1)}, s_{\theta(h_3)}\rangle \\ = (h_1 \otimes h_2)^c. \end{array} \right\}.$$

(6):

$$h_1 \oplus h_2 = \left\{ \begin{array}{c} \langle \{a_1, b_1, c_1, [a_1^-, b_1^-, c_1^-\}, [a_1^+, b_1^+, c_1^+], s_{\theta(h_1)}, s_{t(h_1)}\}\rangle \otimes \\ \langle \{a_2, b_2, c_2], [a_2^-, b_2^-, c_2^-\}, [a_2^+, b_2^+, c_2^+], s_{\theta(h_2)}, s_{t(h_2)}\}\rangle \\ = \langle \{s_{\theta(h_1)+\theta(h_2)}, s_{t(h_1)+t(h_2)}, [a_1^- + a_2^- - a_1^- a_2^-, \\ b_1^- + b_2^- - b_1^- b_2^-, c_1^- + c_2^- - c_1^- c_2^-], [a_1^+ + a_2^+ - a_1^+ a_2^+, \\ b_1^+ + b_2^+ - b_1^+ b_2^+, c_1^+ + c_2^+ - c_1^+ c_2^+], [a_1 a_2, b_1 b_2, c_1 c_2]\}^c\rangle \\ = (h_1 \oplus h_2)^c \end{array} \right\}.$$

(7):

$$h_1 + h_2 = \left\{ \begin{array}{c} \langle s_{\theta(h_1)+\theta(h_2)}, s_{t(h_1)+t(h_2)}, [a_1^- + a_2^- - a_1^- a_2^-, b_1^- + b_2^- - b_1^- b_2^-, \\ c_1^- + c_2^- - c_1^- c_2^-], [a_1^+ + a_2^+ - a_1^+ a_2^+, b_1^+ + b_2^+ - b_1^+ b_2^+, \\ c_1^+ + c_2^+ - c_1^+ c_2^+], [a_1 a_2, b_1 b_2, c_1 c_2]\rangle = \\ \langle s_{\theta(h_1)+\theta(h_2)}, s_{t(h_1)+t(h_2)}, [a_1^- + a_2^- - a_1^- a_2^-, b_1^- + b_2^- \\ -b_1^- b_2^-, c_1^- + c_2^- - c_1^- c_2^-], [a_1^+ + a_2^+ - a_1^+ a_2^+, \\ b_1^+ + b_2^+ \\ -b_1^+ b_2^+, c_1^+ + c_2^+ - c_1^+ c_2^+], [a_1 a_2, b_1 b_2, c_1 c_2]\rangle \end{array} \right\}.$$

Thus, we have that $= h_2 + h_1$.

Definition 3.5 For a TCLUFN h, $s(h) = \sum_{\tilde{\Omega} \in \tilde{h}} \tilde{\Omega}/\#h$ is called the score function of h, where $\#h$ stands for cardinality of h. For two TCLUFNs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$, if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

4 Aggregation triangular cubic linguistic uncertain fuzzy information

In this section, we exhibit a series of operators for aggregating the triangular cubic linguistic uncertain fuzzy information and investigate some desired properties of these operators.

4.1 The TCLUFWA and TCLUFWG operators

In this subsection, the definitions and theorems of TCLUFWA and TCLUFWG operators are introduced.

Definition 4.1.1 Let $h_i = \left\{ \begin{array}{c} \langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle \end{array} \right\}$ $(i = 1, 2, \ldots, n)$ be the collections of TCLUFEs and $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T$ be the weight vector of TCLUFEs $h_i (i = 1, 2, \ldots, n)$ where $\tau_i \in [0, 1]$, $\sum_{i=1}^n \tau_i = 1$. Then, the triangular cubic linguistic uncertain fuzzy weighted averaging operator is a mapping $h^n \to h$

$$\mathrm{TCLUFWA}(h_1, h_2, \ldots, h_n) = \mathop{\oplus}_{i=1}^{n} (\tau_i h_i).$$

If $\tau = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ *and the TCLUFWA operator reduces to the triangular cubic linguistic uncertain fuzzy averaging operator:*

$$TCLUFA(h_1, h_2, \ldots, h_n) = \mathop{\oplus}_{i=1}^{n} \left(\frac{1}{n} h_i \right).$$

Theorem 4.1.2 *Let* $h_i = \left\{ \begin{array}{c} \langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle \end{array} \right\}$ *be a collection of TCLUFEs. Then the aggregated value is calculated by using the TCLUFWA operator which is also a TCLUFE and*

$$\left\{ \begin{array}{c} \mathrm{TCLUFWA}(h_1, h_2, \ldots, h_n) \\ = \mathop{\oplus}_{i=1}^{n} (\tau_i h_i) = \langle \prod_{i=1}^{n} s_{\theta_i}, \prod_{i=1}^{n} s_{t_i}, \\ [1 - \prod_{i=1}^{n} (1 - a_i^-)^{\tau_i}], [1 - \prod_{i=1}^{n} (1 - b_i^-)^{\tau_i}], \\ [1 - \prod_{i=1}^{n} (1 - c_i^-)^{\tau_i}][1 - \prod_{i=1}^{n} (1 - a_i^+)^{\tau_i}], \\ [1 - \prod_{i=1}^{n} (1 - b_i^+)^{\tau_i}], \\ [1 - \prod_{i=1}^{n} (1 - c_i^+)^{\tau_i}], \prod_{i=1}^{n} (a_i)^{\tau_i}, \prod_{i=1}^{n} (b_i)^{\tau_i}, \\ \prod_{i=1}^{n} (c_i)^{\tau_i}\rangle \end{array} \right\}.$$

Proof will be proved by induction on n. The results hold for $n = 1$ because

$$\left\{ \begin{array}{c} \mathrm{TCLUFWA}(h_1, h_2, \ldots, h_n) = \mathop{\oplus}_{i=1}^{1} (\tau_1 h_1) \\ = \langle s_{\theta(\alpha_1)}, s_{t(\alpha_1)}, [1 - (1 - a_{\alpha_i}^-)^{\tau_1}, 1 - (1 - b_{\alpha_i}^-)^{\tau_1}, \\ 1 - (1 - c_{\alpha_i}^-)^{\tau_1}], \\ [1 - (1 - a_{\alpha_i}^+)^{\tau 1}, 1 - (1 - b_{\alpha_i}^+)^{\tau 1}, \\ 1 - (1 - c_{\alpha_i}^+)^{\tau 1}], \\ [(a_{\alpha_i})^{\tau_1}, (b_{\alpha_i})^{\tau_1}, (c_{\alpha_i})^{\tau_1}]\rangle \end{array} \right\}.$$

Let us suppose that it holds for $n = k$, i.e.,

$$\left\{\begin{array}{c} \mathrm{TCLUFWA}(h_1, h_2, \ldots, h_n) = \langle \left(s_{\prod_{i=1}^{k}\theta(\alpha_i)^{\tau_i}}\right), \left(s_{\prod_{i=1}^{k}t(\alpha_i)^{\tau_i}}\right) \\ \left[\begin{array}{c} 1-\prod_{i=1}^{k}(1-a^-_{\alpha_i})^{\tau_i}, 1-\prod_{i=1}^{k}(1-b^-_{\alpha_i})^{\tau_i}, \\ 1-\prod_{i=1}^{k}(1-c^-_{\alpha_i})^{\tau_i} \end{array}\right], \\ \left[1-\prod_{i=1}^{k}(1-a^+_{\alpha_i})^{\tau_i}, 1-\prod_{i=1}^{k}(1-b^+_{\alpha_i})^{\tau_i}, 1-\prod_{i=1}^{k}(1-c^+_{\alpha_i})^{\tau_i}\right], \\ \left[\prod_{i=1}^{k}(a_{\alpha_i})^{\tau_i}, \prod_{i=1}^{k}(b_{\alpha_i})^{\tau_i}, \prod_{i=1}^{k}(c_{\alpha_i})^{\tau_i}\right]\rangle \end{array}\right\}.$$

Now, we prove that the result holds for $n = k + 1$

$$\left\{\begin{array}{c} \mathrm{TCLUFWA}(\ddot{h}_1, \ddot{h}_2, \ldots, \ddot{h}_k, \ddot{h}_{k+1}) = \overset{k+1}{\underset{i=1}{\oplus}}(\tau_1\ddot{h}_i) = \left(\overset{k}{\underset{i=1}{\oplus}}(\tau_1\ddot{h}_i)\right) \\ \oplus(\tau_{k+1}\ddot{h}_{k+1}) \\ = \{(s_{\prod_{i=1}^{k+1}\theta(\alpha_i)^{\tau_i}}), (s_{\prod_{i=1}^{k+1}t(\alpha_i)^{\tau_i}}), [1-\prod_{i=1}^{k+1}(1-a^-_{\alpha_i})^{\tau_i}, 1-\prod_{i=1}^{k+1}(1-b^-_{\alpha_i})^{\tau_i}, \\ 1-\prod_{i=1}^{k+1}(1-c^-_{\alpha_i})^{\tau_i}], [1-\prod_{i=1}^{k+1}(1-a^+_{\alpha_i})^{\tau_i}, 1-\prod_{i=1}^{k+1}(1-b^+_{\alpha_i})^{\tau_i}, \\ 1-\prod_{i=1}^{k+1}(1-c^+_{\alpha_i})^{\tau_i}], \prod_{i=1}^{k+1}(a_{\alpha_i})^{\tau_i}, \prod_{i=1}^{k+1}(b_{\alpha_i})^{\tau_i}, \prod_{i=1}^{k+1}(c_{\alpha_i})^{\tau_i}\} \end{array}\right.$$

Hence, the result follows.

Definition 4.1.3 Let $h_i = \left\{\begin{array}{c}\langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle\end{array}\right\}$ $(i = 1, 2, \ldots, n)$ be a collection of TCLUFNs, and let $\ddot{\Psi} = (\ddot{\Psi}_1, \ddot{\Psi}_2, \ldots, w_n)^T$ be the weight vector of TCLUFEs h_i $(i = 1, 2, \ldots, n)$, where $\ddot{\Psi}_i \in [0, 1]$, $\sum_{i=1}^{n}\ddot{\Psi}_i = 1$. The cubic triangular linguistic uncertain fuzzy weighted geometric operator is a mapping $\ddot{h}^n \to \ddot{h}$ such that: $\mathrm{TCLUFWG}(h_1, h_2, \ldots, h_n) = \otimes_{i=1}^{n}(h_i^{\ddot{\Psi}_i})$. If $\ddot{\Psi} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the TCLUFWG operator reduces to the triangular cubic linguistic uncertain fuzzy averaging operator: $TCLUFWG(h_1, h_2, \ldots, h_n) = \otimes_{i=1}^{n}(\ddot{h}_i^{\frac{1}{n}})$.

Theorem 4.1.4 *Let* $h_i = \left\{\begin{array}{c}\langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle\end{array}\right\}$ $(i = 1, 2, \ldots, n)$ *be a collection of TCLUFEs. Then the aggregated value is calculated by using the TCLUFWG operator which is also a TCLUFE and TCLUFWG*

$$(h_1, h_2, \ldots, h_n) =$$

$$\left\{\begin{array}{c} \langle s_{\prod_{i=1}^{n}\theta(\alpha_i)^{\ddot{\Psi}_i}}, s_{\prod_{i=1}^{n}t(\alpha_i)^{\ddot{\Psi}_i}}, \left[\begin{array}{c}\prod_{i=1}^{n}(a_i^-)^{\ddot{\Psi}_i}[\prod_{i=1}^{n}(b_i^-)^{\ddot{\Psi}_i}, \\ [\prod_{i=1}^{n}(c_i^-)^{\ddot{\Psi}_i}\end{array}\right], \\ \left[\begin{array}{c}\prod_{i=1}^{n}(a_i^+)^{\ddot{\Psi}_i}, \prod_{i=1}^{n}(b_i^+)^{\ddot{\Psi}_i}, \\ \prod_{i=1}^{n}(c_i^+)^{\ddot{\Psi}_i}\end{array}\right], \\ 1-\prod_{i=1}^{n}(1-a_i)^{\ddot{\Psi}_i}, 1-\prod_{i=1}^{n}(1-b_i)^{\ddot{\Psi}_i}, 1-\prod_{i=1}^{n}(1-c_i)^{\ddot{\Psi}_i}\rangle \end{array}\right\}.$$

Example 4.1.5 Suppose that $h_1 = \left\{\begin{array}{c}\langle\{[s_2, s_4], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ 0.5, 0.7, 0.9\}\rangle\end{array}\right\}$; $h_2 = \left\{\begin{array}{c}\langle\{[s_3, s_4], \\ [0.3, 0.5, 0.7], \\ [0.5, 0.7, 0.9], \\ 0.4, 0.6, 0.8\}\rangle\end{array}\right\}$; and $h_3 = \left\{\begin{array}{c}\langle\{[s_1, s_6], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ 0.6, 0.8, 0.10\}\end{array}\right\}$ are three TCLUFEs and $\tau = \{0.25, 0.50, 0.25\}$ is their weight vector. Then, by definition above, we can obtain

$$TCLUFWA_\lambda(h_1, h_2, h_3) =$$

$$\left\{\begin{array}{c}\langle[s_{2.0597}, s_{4.4267}], [0.3808, 0.5838, 0.7941], \\ [0.5838, 0.7941, 0.7008], \\ [0.4681, 0.6700, 0.4898]\rangle\end{array}\right\},$$

$$TCLUFWG(h_1, h_2, h_3)$$

$$= \left\{\begin{array}{c}\langle[s_{2.0597}, s_{4.4267}], [0.3662, 0.5692, 0.7706], \\ [0.5692, 0.7706, 0.3072], \\ [0.4819, 0.6869, 0.7551]\rangle\end{array}\right\}.$$

Theorem 4.1.6 *Let* $h_i = \left\{\begin{array}{c}\langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle\end{array}\right\}$ *be a collection of TCLUFNs having the weight vector* $\ddot{\Psi} = (\ddot{\Psi}_1, \ddot{\Psi}_2, \ldots, \ddot{\Psi}_n)^T$ *such that* $\ddot{\Psi}_i \in [0, 1]$ *and* $\sum_{i=1}^{n}\ddot{\Psi}_i = 1$, $\lambda > 0$.

Then TCLUFWG $(h_1, h_2, \ldots, h_n) \leq$ *TCLUFWA* $(h_1, h_2, \ldots, h_n)$.

4.2 The TCLUFOWA and TCLUFOWG operators

In this subsection, the definitions, theorems of TCLUFOWA and TCLUFOWG operators are introduced.

Definition 4.2.1 Let $h_i = \left\{ \begin{array}{c} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{array} \right\}$ $(i = 1, 2, \ldots, n)$ be a collection of TCLUFEs and $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T$ is the aggregation-associated vector of triangular cubic linguistic uncertain fuzzy ordered weighted averaging operator $\tau_i \in [0, 1]$ and $\sum_{i=1}^{n} \tau_i = 1$. $h^n \to h$. The triangular cubic linguistic uncertain fuzzy ordered weighted averaging operator is a mapping $h^n \to h$, such that TCLUFOWA $(h_1, h_2, \ldots, h_n) = \oplus_{i=1}^{n}(\tau_i h_{\sigma(i)})$.

Theorem 4.2.2 *Let* $\ddot{h}_i = \left\{ \begin{array}{c} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{array} \right\}$ *be a collection of TCLUFEs. Then the aggregated value is calculated by using the TCLUFOWA operator which is also a TCLUFE and*

$$\mathrm{TCLUFOWA}\,(h_1, h_2, \ldots, h_n) = \left\{ \begin{array}{c} \langle s_{\left(\prod_{i=1}^{n} \theta(\alpha_i)^{\tau_i}\right)},\ s_{\left(\prod_{i=1}^{n} t(\alpha_i)^{\tau_i}\right)},\ \left[\begin{array}{c} 1 - \prod_{i=1}^{n}(1 - a^-_{\alpha_{\sigma(i)}})^{\tau_i}, \\ 1 - \prod_{i=1}^{n}(1 - b^-_{\alpha_{\sigma(i)}})^{\tau_i}, \\ 1 - \prod_{i=1}^{n}(1 - c^-_{\alpha_{\sigma(i)}})^{\tau_i} \end{array} \right], \\ \left[\begin{array}{c} 1 - \prod_{i=1}^{n}(1 - a^+_{\alpha_{\sigma(i)}})^{\tau_i}, \\ 1 - \prod_{i=1}^{n}(1 - b^+_{\alpha_{\sigma(i)}})^{\tau_i}, \\ 1 - \prod_{i=1}^{n}(1 - c^+_{\alpha_{\sigma(i)}})^{\tau_i} \end{array} \right]; \\ \prod_{i=1}^{n}(a_{\alpha_{\sigma(i)}})^{\tau_i},\ \prod_{i=1}^{n}(b_{\alpha_{\sigma(i)}})^{\tau_i},\ \prod_{i=1}^{n}(c_{\alpha_{\sigma(i)}})^{\tau_i} \\ |\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}}\rangle \end{array} \right\}$$

Definition 4.2.3 *Let* $h_i = \left\{ \begin{array}{c} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{array} \right\}$ $(i = 1, 2, \ldots, n)$ *be a collection of TCLUFNs and* $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T$ *is the aggregation-associated vector of TCLUFNs, such that* $\tau_i \in [0, 1]$ *and* $\sum_{i=1}^{n} \tau_i = 1$. *The triangular cubic linguistic uncertain fuzzy ordered weighted geometric operator is a mapping* $h^n \to h$, *such that TCLUFOWG* $(h_1, h_2, \ldots, h_n) = \otimes_{i=1}^{n}(h^{\tau_i}_{\sigma(i)})$.

Theorem 4.2.4 *Let* $h_i = \left\{ \begin{array}{c} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{array} \right\}$ *be a collection of TCLUFNs. Then the aggregated value is calculated by using the TCLUFOWG operator which is also a TCLUFN and*

$$\mathrm{TCLUFOWG}\,(h_1, h_2, \ldots, h_n) = \left\{ \begin{array}{c} \langle s_{\left(\prod_{i=1}^{n} \theta(\alpha_i)^{\tau_i}\right)},\ s_{\left(\prod_{i=1}^{n} t(\alpha_i)^{\tau_i}\right)}, \\ \left[\begin{array}{c} \prod_{i=1}^{n}(a^-_{\alpha_{\sigma(i)}})^{\tau_i}, \\ \prod_{i=1}^{n}(b^-_{\alpha_{\sigma(i)}})^{\tau_i}, \\ \prod_{i=1}^{n}(c^-_{\alpha_{\sigma(i)}})^{\tau_i} \end{array} \right],\ \left[\begin{array}{c} \prod_{i=1}^{n}(a^+_{\alpha_{\sigma(i)}})^{\tau_i}, \\ \prod_{i=1}^{n}(b^+_{\alpha_{\sigma(i)}})^{\tau_i}, \\ \prod_{i=1}^{n}(c^+_{\alpha_{\sigma(i)}})^{\tau_i} \end{array} \right], \\ \left(1 - \prod_{i=1}^{n}(1 - a_{\alpha_{\sigma(i)}})^{\tau_i}\right),\ \left(1 - \prod_{i=1}^{n}(1 - b_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^{n}(1 - c_{\alpha_{\sigma(i)}})^{\tau_i}\right) |\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}}\rangle \end{array} \right\}.$$

Example 4.2.5 Suppose that $h_1 = \left\{ \begin{array}{c} \langle [s_3,\, s_1], \\ [0.3,\, 0.5, \\ 0.7],\, [0.5, \\ 0.7,\, 0.9], \\ 0.4,\, 0.6, \\ 0.8\rangle \end{array} \right\}$, $h_2 = \left\{ \begin{array}{c} \langle [s_2,\, s_2], \\ [0.6,\, 0.8, \\ 0.10],\, [0.12, \\ 0.14,\, 0.16], \\ 0.8,\, 0.10, \\ 0.12\rangle \end{array} \right\}$ and $h_3 = \left\{ \begin{array}{c} \langle [s_4,\, s_2], \\ [0.24,\, 0.26, \\ 0.28],\, [0.32, \\ 0.34,\, 0.36]; \\ 0.28,\, 0.30, \\ 0.32\rangle \end{array} \right\}$ are three TCLUFNs. Then, we can define score function (Fig. 1)

$$\left\{ \begin{array}{c} s(h_1) = \frac{[s_3, s_1], [\{[0.3, 0.5, 0.7]\} + \{0.5+0.7+0.9\}] - \{0.4+0.6+0.8\}}{9} = \\ \frac{[s_3, s_1], [1.5+2.1] - 1.8}{9} = \frac{[s_3, s_1], [1.8]}{9} = s_{0.6}, \\ s(h_2) = \frac{[s_2, s_2], [\{0.6+0.8+0.10\} + \{0.12+0.14+0.16\}] - \{0.8-0.10-0.12\}}{9} \\ = \frac{[s_2, s_2], [1.5+0.42] - [1.02]}{9} = \frac{[s_2, s_2], [0.9]}{9} = s_{0.4}, \\ s(h_3) = \frac{[s_4, s_2], \{0.24, 0.26, 0.28\} + \{0.32, 0.34, 0.36\} - \{0.28, 0.30, 0.32\}}{9} \\ = \frac{[s_4, s_2], [0.78+1.02] - 0.9}{3} = \frac{[s_4, s_2], [0.9]}{3} = s_{0.8}. \end{array} \right\}.$$

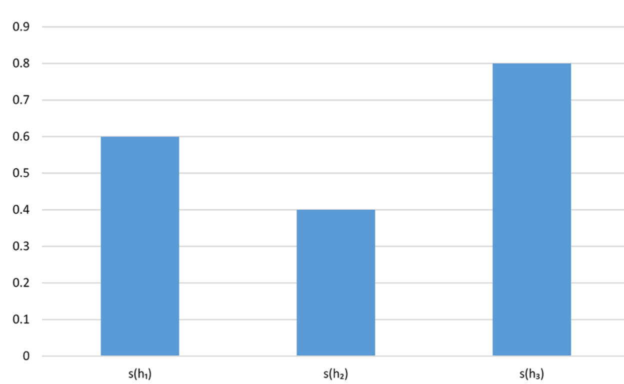

Fig. 1 Different score values of TCLUFOWG operator exhibited

Theorem 4.2.6 *Let* $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ $(i = 1, 2, \ldots, n)$ *be a collection of TCLUFNs, and let* $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ *be the aggregation-associated vector such that* $\tau_i \in [0,\, 1]$ *and* $\sum_{i=1}^n \tau_i = 1$*; then, the TCLUFOWA operator is monotonically increasing with respect to the parameter.*

Theorem 4.2.7 L*et* $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ $(i = 1, 2, \ldots, n)$ *are the collections of TCLUFNs, and let* $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ *be the aggregation-associated vector such that* $\tau_i \in [0,\, 1]$ *and* $\sum_{i=1}^n \tau_i = 1$*,; then, the TCLUFOWG operator is monotonically decreasing.*

Theorem 4.2.8 *Let* $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ $(i = 1, 2, \ldots, n)$ *are the collections of TCLUFNs, and let* $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ *be the aggregation-associated vector such that* $\tau_i \in [0,\, 1]$ *and* $\sum_{i=1}^n \tau_i = 1$ *then*

$$TCLUFOWG(h_1,\, h_2, \ldots,\, h_n) \le TCLUFOWA(h_1,\, h_2, \ldots,\, h_n).$$

4.3 The TCLUFHA and TCLUFHG operators

In this subsection, the definitions, theorems of TCLUFHA and TCLUFHG operators are introduced.

Definition 4.3.1 Let $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ $(i = 1, 2, \ldots, n)$ be the collection of TCLUFNs. The triangular cubic linguistic uncertain fuzzy hybrid averaging operator is defined by the mapping $h^n \to h$ such that, TCLUFHA $(h_1, h_2, \ldots, h_n) = \oplus_{i=1}^n \left(\tau_i h_{\sigma(i)}\right)$, where $h_{\sigma(i)}$ is the largest i of $h_k = n\tau_i h_{\sigma(i)}$ $(k = 1, 2, \ldots, n)$ and where $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ is the weight vector of TCLUFNs, with $\tau_i \in [0,\, 1]$ and $\sum_{i=1}^n \tau_i = 1$, n is the balancing coefficient. Then, we define the following aggregation operators, which are all based on the mapping $h^n \to h$ with an aggregation-associated vector $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ such that $\tau_i \in [0,\, 1]$ and $\sum_{i=1}^n \tau_i = 1$.

Theorem 4.3.2 *Let* $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ *be a collection of TCLUFNs. Then the aggregated value is calculated by using the TCLUFHA operator which is also a TCLUFN and*

$$TCLUFHA\,(h_1,\, h_2, \ldots,\, h_n) = \left\{\begin{matrix} \langle s_{\prod_{i=1}^n \theta(\alpha_i)^{\tau_i}},\, s_{(\prod_{i=1}^n t(\alpha_i)^{\tau_i})},\, \left[\begin{matrix} \left(1 - \prod_{i=1}^n (1 - a^-_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^n (1 - b^-_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^n (1 - c^-_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix}\right], \\ \left[\begin{matrix} \left(1 - \prod_{i=1}^n (1 - a^+_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^n (1 - b^+_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^n (1 - c^+_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix}\right] \\ \left[\begin{matrix} \left(\prod_{i=1}^n (a_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^n (b_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^n (c_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix}\right] |\alpha_{\sigma(1)},\, \alpha_{\sigma(2)}, \ldots,\, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}}\rangle \end{matrix}\right\}.$$

Definition 4.3.3 For a collection of TCLUFNs $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ $(i = 1, 2, \ldots, n)$, $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ is the weight vector of TCLUFNs, with $\tau_i \in [0,\, 1]$ and $\sum_{i=1}^n \tau_i = 1$, and n is the balancing coefficient. Then, we define the following aggregation operators, which are all based on the mapping $h^n \to h$ with an aggregation-associated vector $\tau = (\tau_1,\, \tau_2, \ldots,\, \tau_n)^T$ such that $\tau_i \in [0,\, 1]$ and $\sum_{i=1}^n \tau_i = 1$. The triangular cubic linguistic uncertain fuzzy hybrid geometric operator is TCLUFHG $(h_1,\, h_2, \ldots,\, h_n) = \otimes_{i=1}^n (h_{\sigma(i)})^{\tau_i}$, where $h_{\sigma(i)}$ is the largest ith of $h_k = (h_{\sigma(i)})^{n\tau_k}$ $(k = 1,\, 2, \ldots,\, n)$.

Theorem 4.3.4 *Let* $h_i = \left\{\begin{matrix} \langle s_\theta,\, s_t \\ [a^-,\, b^-,\, c^-], \\ [a^+,\, b^+,\, c^+], \\ a,\, b,\, c\rangle \end{matrix}\right\}$ *be a collection of TCLUFNs. Then the aggregated value is calculated by using the TCLUFHG operator which is also a TCLUFN and*

$$TCLUFHG(h_1, h_2, \ldots, h_n) = \left\{ \begin{array}{l} \left\langle s_{\prod_{i=1}^{n}\theta(\alpha_i)^{\tau_i}}, s_{\left(\prod_{i=1}^{n} t(\alpha_i)^{\tau_i}\right)}, \begin{bmatrix} \left(\prod_{i=1}^{n}(a^{-}_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n}(b^{-}_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n}(c^{-}_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{bmatrix}, \begin{bmatrix} \left(\prod_{i=1}^{n}(a^{+}_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n}(b^{+}_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n}(c^{+}_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{bmatrix}, \right. \\ \begin{bmatrix} \left(1-\prod_{i=1}^{n}(1-a_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1-\prod_{i=1}^{n}(1-b_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1-\prod_{i=1}^{n}(1-c_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{bmatrix} \left| \alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}} \right\rangle \end{array} \right\}$$

***Example* 4.3.5** *Suppose that* $h_1 = \left\{ \begin{array}{c} \langle s_3, s_2, [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6\rangle; \langle [s_1, s_4], [0.10, 0.12, 0.14], [0.12, 0.14, 0.16], 0.11, 0.13, 0.15\rangle \end{array} \right\}$, $h_2 = \left\{ \begin{array}{c} \langle s_1, s_3, [0.6, 0.8, 0.10], [0.8, 0.10, 0.12], 0.7, 0.9, 0.11\rangle; \langle [s_4, s_6], [0.3, 0.5, 0.9], [0.5, 0.7, 0.9], 0.4, 0.6, 0.8\rangle, \end{array} \right\}$ *and* $h_3 = \left\{ \begin{array}{c} \langle [s_5, s_2], [0.10, 0.12, 0.14], [0.12, 0.14, 0.16], 0.11, 0.13, 0.15\rangle; \langle [s_1, s_6], [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13\rangle, \end{array} \right\}$ *are three TCLUFNs and* $\tau = \{0.25, 0.50, 0.25\}$ *is their weight vector.*

$$TCLUFHA(h_1, h_2, h_3) = \left\{ \begin{array}{c} \langle s_{2.3160}, s_{2.6034}, [0.0513, 0.1141, 0.1902], \\ [0.1141, 0.1902, 0.2914], 0.3851, 0.4775, 0.5477\rangle \end{array} \right\}, \left\{ \begin{array}{c} \langle [s_{3.0000}, s_{4.1815}], [0.4214, 0.6837, 0.7000], \\ [0.6837, 0.7000, 0.7033], 0.5291, 0.7348, 0.2966\rangle \end{array} \right\}, \left\{ \begin{array}{c} \langle [s_{2.4953}, s_{2.7542}], [0.3486, 0.0566, 0.0672], \\ [0.0566, 0.0672, 0.0781], 0.5609, 0.3458, 0.3736\rangle \end{array} \right\}$$

Then, we can define score function (Fig. 2)

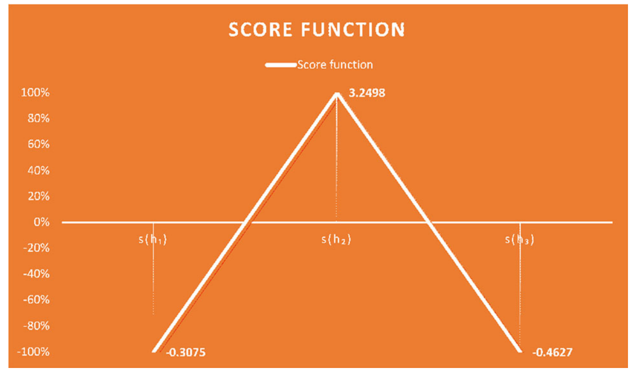

Fig. 2 Different score values of TCLUFHG operator exhibited

$$\left\{ \begin{array}{l} s(h_1) = \dfrac{\begin{array}{c}[s_{2.3160}, s_{2.6034}], [0.0513, 0.1141, 0.1902] \\ +[0.1141, 0.1902, 0.2914] \\ -[0.3851, 0.4775, 0.5477]\end{array}}{9} = \dfrac{[s_{2.3160}, s_{2.6034}], [0.9513-1.4103]}{9} = s_{-0.3075}; \\ s(h_2) = \dfrac{\begin{array}{c}[s_{3.0000}, s_{4.1815}], \{0.4214+0.6837+0.7000\} \\ +\{0.6837+0.7000+0.7033\} \\ -\{0.5291+0.7348+0.2966\}\end{array}}{9} = \dfrac{[s_{3.0000}, s_{4.1815}], [3.8921]-[1.5605]}{9} = s_{3.2498}; \\ s(h_3) = \dfrac{\begin{array}{c}[s_{2.4953}, s_{2.7542}], \{0.3486+0.0566+0.0672\}+ \\ \{0.0566+0.0672+0.0781\} \\ -\{0.5609+0.3458+0.3736\}\end{array}}{9} = \dfrac{[s_{2.4953}, s_{2.7542}], [0.6743]-[1.2803]}{9} = s_{-0.4627}. \end{array} \right\}.$$

5 An approach to multiple attribute group decision making with triangular cubic linguistic uncertain fuzzy information

In this section, we consume the suggested triangular cubic linguistic uncertain fuzzy aggregation operators to develop an approach to multiple attribute group decision making with triangular linguistic uncertain fuzzy information. *To* do this first, a multiple attribute group decision making with triangular cubic linguistic uncertain fuzzy information can be described as follows.

Let $Y = \{Y_1, Y_2, \ldots, Y_m\}$ be a set of m alternatives, $G = \{G_1, G_2, \ldots, G_n\}$ a set of n attributes, whose weight vector is $\ddot{\Psi} = (\ddot{\Psi}_1, \ddot{\Psi}_2, \ldots, \ddot{\Psi}_n)^T$, with $\ddot{\Psi}_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \ddot{\Psi}_i = 1$. Let $D = \{D_1, D_2, \ldots, D_l\}$ be a set of l decision makers, whose weight vector is $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T$, with $\tau_k \in [0, 1]$, $k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} \tau_k = 1$. Let $R^k = (r^k_{ij})_{m\times n}$ be the triangular cubic linguistic uncertain fuzzy decision matrix, where $r^k_{ij} = \{\gamma^k_{ij} | \gamma^k_{ij} \in r^k_{ij}\} = \left\{ \begin{array}{c} \langle s_\theta, s_t \\ [a^-, b^-, c^-], \\ [a^+, b^+, c^+], \\ a, b, c\rangle \end{array} \right\}$ is TCLUFE given by the decision maker $D_k \in D$, where $\left\{ \begin{matrix} s_\theta, s_t, [a^-, b^-, c^-], \\ [a^+, b^+, c^+] \end{matrix} \right\}$ indicates the possible interval value triangular linguistic uncertain fuzzy set range that the alternative $Y_i \in Y$ satisfies the attribute $G_j \in G$, while $[a, b, c]$ indicates the possible triangular linguistic uncertain fuzzy set range that the alternative $Y_i \in Y$ does not satisfy the attribute $G_j \in G$.

In the following, we utilize the proposed operators to develop an approach to multiple attribute group decision making with triangular cubic linguistic uncertain fuzzy information, which includes the following two methods.

5.1 Averaging method

Step 1. Construct the triangular cubic linguistic uncertain fuzzy decision matrix $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{ \begin{matrix} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a^-_{\gamma_{ij}^k}, b^-_{\gamma_{ij}^k}, c^-_{\gamma_{ij}^k}], \\ [a^+_{\gamma_{ij}^k}, b^+_{\gamma_{ij}^k}, c^+_{\gamma_{ij}^k}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{matrix} \right\}$

Step 2. Utilize the TCLUFWA operator

$$\left\{ \begin{matrix} \text{TCLUFWA}(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^l) \overset{n}{\underset{i=1}{\oplus}} (\tau_i h_i) \\ = \langle s_{(\prod_{i=1}^n \theta(\alpha_i)^{\tau_i})}, s_{(\prod_{i=1}^n t(\alpha_i)^{\tau_i})}, [1 - \prod_{i=1}^n (1 - a_i^-)^{\tau_i}], \\ [1 - \prod_{i=1}^n (1 - b_i^-)^{\tau_i}], \\ [1 - \prod_{i=1}^n (1 - c_i^-)^{\tau_i}][1 - \prod_{i=1}^n (1 - a_i^+)^{\tau_i}], \\ [1 - \prod_{i=1}^n (1 - b_i^+)^{\tau_i}], \\ [1 - \prod_{i=1}^n (1 - c_i^+)^{\tau_i}], \prod_{i=1}^n (a_i)^{\tau_i}, \prod_{i=1}^n (b_i)^{\tau_i}, \\ \prod_{i=1}^n (c_i)^{\tau_i} \rangle \end{matrix} \right\}$$

to aggregate all the individual triangular cubic linguistic uncertain fuzzy decision matrix $(A_{ij}^k) = (\alpha_{ij}^k)_{m \times n}$ into the collective triangular cubic linguistic uncertain fuzzy decision matrix $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{ \begin{matrix} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a^-_{\gamma_{ij}^k}, b^-_{\gamma_{ij}^k}, c^-_{\gamma_{ij}^k}], \\ [a^+_{\gamma_{ij}^k}, b^+_{\gamma_{ij}^k}, c^+_{\gamma_{ij}^k}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{matrix} \right\}.$

Step 3. Utilize the TCLUFHA operator

$$\left\{ \begin{matrix} \alpha_i = \text{TCLUFHA}(a_{i1}, a_{i2}, \ldots, a_{in}) = \langle s_{\prod_{j=1}^n \theta(\alpha_i)^{\tau_i}}, s_{(\prod_{i=1}^n t(\alpha_i)^{\tau_i})}, \\ \left[\begin{matrix} \left(1 - \prod_{j=1}^n (1 - a^-_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{j=1}^n (1 - b^-_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{j=1}^n (1 - c^-_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix} \right], \\ \left[\begin{matrix} \left(1 - \prod_{j=1}^n (1 - a^+_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{j=1}^n (1 - b^+_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{j=1}^n (1 - c^+_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix} \right] \\ \left[\begin{matrix} \left(\prod_{j=1}^n (a_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{j=1}^n (b_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(\prod_{j=1}^n (c_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{matrix} \right] |\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}} \rangle \end{matrix} \right\}.$$

to aggregate all the preference values $a_{ij} (j = 1, 2, \ldots, n)$ in the i th line of A and then derive the overall preference value $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{ \begin{matrix} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, \\ [a^-_{\gamma_{ij}^k}, b^-_{\gamma_{ij}^k}, c^-_{\gamma_{ij}^k}], \\ [a^+_{\gamma_{ij}^k}, b^+_{\gamma_{ij}^k}, c^+_{\gamma_{ij}^k}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{matrix} \right\}$ $(i = 1, 2, \ldots, m)$ of the alternative $Y_i (i = 1, 2, \ldots, m)$, where $\ddot{\xi} = (\ddot{\xi}_1, \ddot{\xi}_2, \ldots, \ddot{\xi}_n)^T$ is the associated weight vector of the TCLUFHA operator, with $\ddot{\xi}_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n \xi_j = 1$.

Step 4. Calculate the score values $s(a_i)(i = 1, 2, \ldots, m)$ of $a(i = 1, 2, \ldots, m)$:

$$s(a_i) = \left\{ \frac{\frac{1}{9} \sum_{\alpha_i \in \alpha_i} s_{((\theta(\alpha_{\gamma_{ij}^k}), ([\langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a^-_{\gamma_{ij}^k} + b^-_{\gamma_{ij}^k} + c^-_{\gamma_{ij}^k}] + [a^+_{\gamma_{ij}^k} + b^+_{\gamma_{ij}^k} + c^+_{\gamma_{ij}^k}] - [a_{\gamma_{ij}^k} + b_{\gamma_{ij}^k} + c_{\gamma_{ij}^k}\rangle + \mu^+_{\gamma_{ij}^k}])}}{\#(a_i)} \right\}$$

Step 5. Get the priority of the alternatives $Y_i (i = 1, 2, \ldots, m)$ by ranking $s(a_i)$ $(i = 1, 2, \ldots, m)$. (Fig. 3)

5.2 Geometric method

Step 1. Construct the triangular cubic linguistic uncertain fuzzy decision matrix $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{ \begin{matrix} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a^-_{\gamma_{ij}^k}, b^-_{\gamma_{ij}^k}, c^-_{\gamma_{ij}^k}], \\ [a^+_{\gamma_{ij}^k}, b^+_{\gamma_{ij}^k}, c^+_{\gamma_{ij}^k}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{matrix} \right\}$

Step 2. Utilize the TCLUFWG operator

$$\left\{\begin{array}{l} a_{ij} = \text{TCLUFWG}(h_1, h_2, \ldots, h_n) \\ = \langle s_{\prod_{i=1}^{n} \theta(\alpha_i)^{\ddot{\Psi}_i}}, s_{(\prod_{i=1}^{n} t(\alpha_i)^{\tau_i})}, \\ \left[\begin{array}{c} \prod_{k=1}^{l} (a_{ij}^{-})^{\ddot{\Psi}_i} [\prod_{k=1}^{l} (b_{ij}^{-})^{\ddot{\Psi}_i}, \\ [\prod_{k=1}^{l} (c_{ij}^{-})^{\ddot{\Psi}_i} \end{array}\right], \\ \left[\begin{array}{c} \prod_{k=1}^{l} (a_{ij}^{+})^{\ddot{\Psi}_i}, \prod_{k=1}^{l} (b_{ij}^{+})^{\ddot{\Psi}_i}, \\ \prod_{k=1}^{l} (c_{ij}^{+})^{\ddot{\Psi}_i} \end{array}\right], \\ 1 - \prod_{k=1}^{l} (1 - a_{ij})^{\ddot{\Psi}_i}, \\ 1 - \prod_{k=1}^{l} (1 - b_{ij})^{\ddot{\Psi}_i}, \\ 1 - \prod_{k=1}^{l} (1 - c_{ij})^{\ddot{\Psi}_i} \rangle \end{array}\right\}$$

to aggregate all the individual triangular cubic linguistic uncertain fuzzy decision matrix $(A_{ij}^k) = (\alpha_{ij}^k)_{m\times n}$ into the collective triangular cubic linguistic uncertain fuzzy decision matrix $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{\begin{array}{c} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a_{\gamma_{ij}^k}^{-}, b_{\gamma_{ij}^k}^{-}, c_{\gamma_{ij}^k}^{-}], \\ [a_{\gamma_{ij}^k}^{+}, b_{\gamma_{ij}^k}^{+}, c_{\gamma_{ij}^k}^{+}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{array}\right\}$.

Step 3. Utilize the TCLUFHG operator

$$\left\{\begin{array}{l} \alpha_i = \text{TCLUFHG}(a_{i1}, a_{i2}, \ldots, a_{in}) = \langle s_{\prod_{i=1}^{n} \theta(\alpha_i)^{\tau_i}}, s_{(\prod_{i=1}^{n} t(\alpha_i)^{\tau_i})}, \\ \left[\begin{array}{c} \left(\prod_{i=1}^{n} (a_{\alpha_{\sigma(i)}}^{-})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n} (b_{\alpha_{\sigma(i)}}^{-})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n} (c_{\alpha_{\sigma(i)}}^{-})^{\tau_i}\right) \end{array}\right], \left[\begin{array}{c} \left(\prod_{i=1}^{n} (a_{\alpha_{\sigma(i)}}^{+})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n} (b_{\alpha_{\sigma(i)}}^{+})^{\tau_i}\right), \\ \left(\prod_{i=1}^{n} (c_{\alpha_{\sigma(i)}}^{+})^{\tau_i}\right) \end{array}\right], \\ \left[\begin{array}{c} \left(1 - \prod_{i=1}^{n} (1 - a_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^{n} (1 - b_{\alpha_{\sigma(i)}})^{\tau_i}\right), \\ \left(1 - \prod_{i=1}^{n} (1 - c_{\alpha_{\sigma(i)}})^{\tau_i}\right) \end{array}\right] | \alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)} \in \ddot{h}_{\alpha_{\sigma(n)}} \rangle \end{array}\right\}$$

to aggregate all the preference values $a_{ij} (j = 1, 2, \ldots, n)$ in the i th line of A and then derive the collective overall preference value $A_{ij}^k = a_{ij}^k = \{\alpha_{ij}^k | \alpha_{ij}^k \in A_{ij}^k\} = \left\{\begin{array}{c} \langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, \\ [a_{\gamma_{ij}^k}^{-}, b_{\gamma_{ij}^k}^{-}, c_{\gamma_{ij}^k}^{-}], \\ [a_{\gamma_{ij}^k}^{+}, b_{\gamma_{ij}^k}^{+}, c_{\gamma_{ij}^k}^{+}], \\ a_{\gamma_{ij}^k}, b_{\gamma_{ij}^k}, c_{\gamma_{ij}^k} \rangle \end{array}\right\}$ $(i = 1, 2, \ldots, m)$ of the alternative $Y_i (i = 1, 2, \ldots, m)$, where $\ddot{\xi} = (\ddot{\xi}_1, \ddot{\xi}_2, \ldots, \ddot{\xi}_n)^T$ is the associated weight vector of the TCLUFHG operator, with $\ddot{\xi}_j \in [0, 1], j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \xi_j = 1$.

Step 4. Calculate the score values $s(a_i) (i = 1, 2, \ldots, m)$ of $a (i = 1, 2, \ldots, m)$:

$$s(a_i) = \left\{ \frac{\frac{1}{9} \sum_{\alpha_i \in \alpha_i} s_{((\theta(\alpha_{\gamma_{ij}^k}), ([\langle s_{\theta_{\gamma_{ij}^k}}, s_{t_{\gamma_{ij}^k}}, [a_{\gamma_{ij}^k}^{-} + b_{\gamma_{ij}^k}^{-} + c_{\gamma_{ij}^k}^{-}] + [a_{\gamma_{ij}^k}^{+} + b_{\gamma_{ij}^k}^{+} + c_{\gamma_{ij}^k}^{+}] - [a_{\gamma_{ij}^k} + b_{\gamma_{ij}^k} + c_{\gamma_{ij}^k}\rangle + \mu_{\gamma_{ij}^k}^{+}])}}{\#(a_i)} \right\}$$

Step 5. Get the priority of the alternatives $Y_i (i = 1, 2, \ldots, m)$ by ranking $s(a_i)$ $(i = 1, 2, \ldots, m)$ (Fig. 4)

6 Numerical application

In this section, we construct the two numerical examples of averaging and geometric operators, to illustrate the new approach in the decision-making problem.

6.1 Example

Give us a chance to consider a hospital which means to choose another patient for new structures. Three choices $(C_i = 1, 2, 3)$ are accessible, and the three decision makers $(i = 1, 2, 3)$ consider three criteria to choose which patients to pick: (1) G_1 (kidney patients); (2) G_2 (heat cramps); and (3) G_3 (hernia (abdominal hernia)). The weight vector of the decision makers $D_k (k = 1, 2, 3)$, $\tau = (0.34, 0.26, 0.40)^T$. The DMs assess these alternatives utilizing the linguistic term set $S = f$ $s_0 =$ empty-handed(EH); $s_1 =$ vapor(V); $s_2 =$ up-scale(UP); $s_3 =$ moderate(M); $s_4 =$ super: $s_5 =$ high rise (HR). After the information procurement and factual treatment, the evaluations of the alternatives with respect to attributes can be represented by TCLUVs appeared in Tables 1 and 2. Assume the decision makers (Tables 3, 4 and Figs. 5, 6).

Step 1: We change the triangular cubic linguistic uncertain fuzzy decision matrices 1 and 2.

Step 2: Use TCLUFWA operator $\tau = (0.25, 0.50, 0.25)^T$

Step 3: Utilize the TCLUFHA operator $\xi = (0.5, 0.3, 0.2)^T$

Step 4: Calculate the TCLUFHA score value $s(a_1) = 0.4087$, $s(a_2) = 0.0048$, $s(a_3) = -0.1471$.

Accuracy value

$H(a_1) = 1.6371, H(a_2) = 0.7851, H(a_3) = 0.5854.$

Step 5. Get the priority of the alternatives $Y_i (i = 1, 2, \ldots, m)$ by ranking $s(a_i)$ $(i = 1, 2, \ldots, m)$.

6.1.1 Results and discussion

The results of the TCLUFHA score value and TCLUFHA accuracy value of the numerical example (6.1) are tabulated in Table 5.

Comparison graph of TCLUFHA score ranking and TCLUFHA accuracy ranking is shown in Fig. 7.

6.2 Example

In this subsection, we present an illustrative example of the new approach in a decision-making problem. Suppose that a company wants to invest a sum of money into one of three possible alternatives: A_1 is a car company, A_2 is a computer company, and A_3 is a TV company. The investment company can consider the following three attributes. Consider there are four attributes, and $\tau = (0.5, 0.2, 0.3)^T$ are weighting vector of the attributes, C_1 is the risk analysis, C_2 is the growth analysis and C_3 is the social–political impact analysis. As the environment is very uncertain, the group of experts needs to assess the available information by using TCLUFNs.

Step 1: Calculate the triangular cubic linguistic uncertain fuzzy decision matrices 6 and 7 (Tables 6, 7, 8, 9).

Step 2: Use TCLUFWG operator $\tau = (0.5, 0.2, 0.3)^T$

Step 3: Utilize the TCLUFHG operator $\xi = (0.2, 0.4, 0.4)^T$.

Step 4: Calculate the TCLUFHG score value $s(a_1) = 0.4781$, $s(a_2) = 0.1955$, $s(a_3) = 0.2517$ (Fig. 8).

Accuracy value $H(a_1) = 0.9428, H(a_2) = 1.1302, H(a_3) = 1.2929$ (Figs. 9, 10).

Step 5. Get the priority of the alternatives $Y_i(i = 1, 2, \ldots, m)$ by ranking $s(a_i)$ $(i = 1, 2, \ldots, m)$.

6.2.1 Results and discussion

The results of the TCLUFHG score value and TCLUFHG accuracy value of the numerical example (6.2) are tabulated in Table 10.

7 Comparison analyses

In way to verify the sagacity and efficiency of the proposed approach, a comparative study is driven overshadowing the methods of intuitionistic fuzzy aggregation operator (Xu 2007) and triangular intuitionistic fuzzy aggregation operator (Liang et al. 2014), which are special cases of triangular cubic linguistic uncertain fuzzy numbers (TCLUFNs), to the related expressive example.

7.1 A comparison analysis of the proposed method with intuitionistic fuzzy aggregation operator and the existing MCDM method with intuitionistic fuzzy aggregation operator

The intuitionistic fuzzy aggregation operator can be considered as a special case of triangular cubic linguistic fuzzy numbers (TCLUFNs) when there are only four elements in membership and non-membership degrees. For comparison, the intuitionistic fuzzy aggregation operator can be

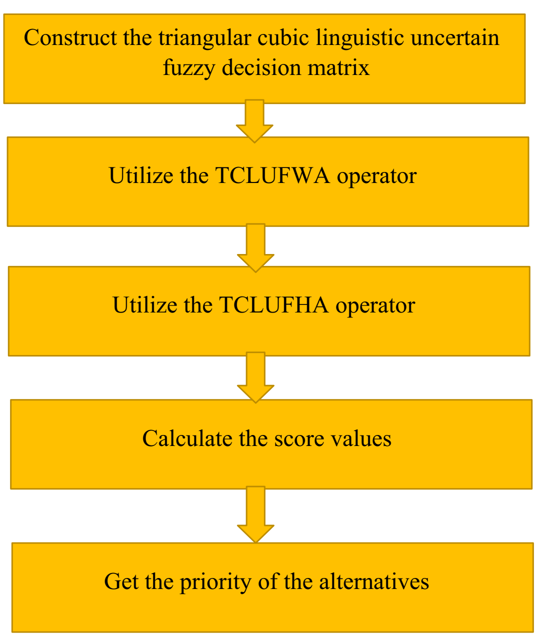

Fig. 3 Graphically, we exhibit 5.1 proposed method

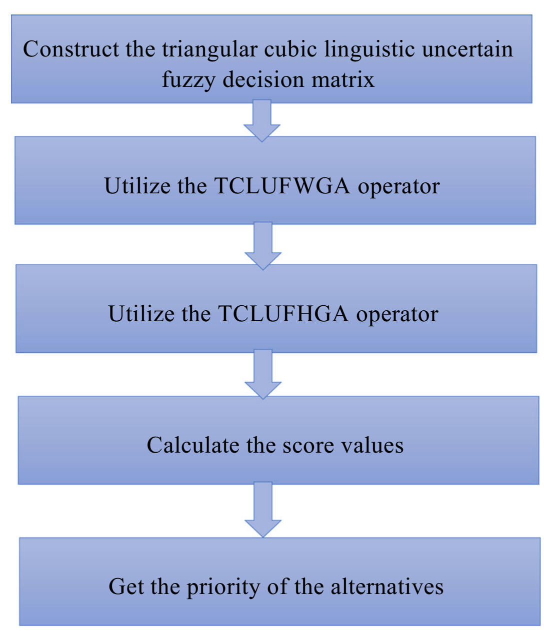

Fig. 4 Graphically, we exhibit 5.2 proposed method

Table 1 Triangular cubic linguistic uncertain fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\left\{ s_2, s_4, \left\langle [0.4, 0.6, 0.8], [0.6, 0.8, 0.10], 0.5, 0.7, 0.9 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$
A_2	$\left\{ s_1, s_3, \left\langle [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_2, s_4, \left\langle [0.4, 0.6, 0.8], [0.6, 0.8, 0.10], 0.5, 0.7, 0.9 \right\rangle \right\}$
A_3	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_1, s_3, \left\langle [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6 \right\rangle \right\}$	$\left\{ s_1, s_3, \left\langle [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6 \right\rangle \right\}$

Table 2 Triangular cubic linguistic uncertain fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_2, s_4, \left\langle [0.4, 0.6, 0.8], [0.6, 0.8, 0.10], 0.5, 0.7, 0.9 \right\rangle \right\}$
A_2	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_1, s_3, \left\langle [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$
A_3	$\left\{ s_2, s_4, \left\langle [0.4, 0.6, 0.8], [0.6, 0.8, 0.10], 0.5, 0.7, 0.9 \right\rangle \right\}$	$\left\{ s_2, s_2, \left\langle [0.8, 0.10, 0.12], [0.10, 0.12, 0.14], 0.9, 0.11, 0.13 \right\rangle \right\}$	$\left\{ s_1, s_3, \left\langle [0.1, 0.3, 0.5], [0.3, 0.5, 0.7], 0.2, 0.4, 0.6 \right\rangle \right\}$

transformed to (TCLUFNs) by calculating the average value of the membership and non-membership degrees. After transformation, intuitionistic fuzzy aggregation operator is given in Table 11.

Step 1: Calculate the IFWA operator $\tau = (0.5,\ 0.2,\ 0.3)^T$

Step 2: Calculate the score function

$s(a_1) = 0.3539,\ s(a_2) = -0.5498,\ s(a_3) = -0.1431.$

Step 3: Find the ranking $s(a_1) > s(a_2) > s(a_3)$. (Fig. 11)

7.2 A comparison analysis with the existing MCDM method triangular intuitionistic fuzzy aggregation operator

The triangular intuitionistic fuzzy aggregation operator can be considered as a special case of triangular cubic linguistic fuzzy numbers (TCLUFNs) when there are only four elements in membership and non-membership degrees. For comparison, the triangular intuitionistic fuzzy aggregation operator can be transformed to (TCLUFNs) by calculating the average value of the membership and non-membership degrees (Table 12). After transformation, triangular intuitionistic fuzzy aggregation operator is given in Table 13.

Table 3 TCLUFWA operator

	C_1	C_2	C_3
A_1	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{1.6817}, \\ [0.4114, 0.2254, \\ 0.3522], [0.2254, \\ 0.3522, 0.0621], \\ 0.8190, 0.5267, \\ 0.5848 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.1, \\ 0.12], [0.1, \\ 0.12, 0.14], \\ 0.9, 0.11, \\ 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{1.6817}, \\ [0.4114, 0.2254, \\ 0.3522], [0.2254, \\ 0.3522, 0.0621], \\ 0.8190, 0.5267, \\ 0.5848 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_{1.1892}, s_{1.5651}, \\ [0.3486, 0.1091, \\ 0.1855], [0.1091, \\ 0.1855, 0.2873], \\ 0.7208, 0.4579, \\ 0.5284 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{2.4494}, \\ [0.5757, 0.2062, \\ 0.3366], [0.2062, \\ 0.3366, 0.4921], \\ 0.4242, 0.2097, \\ 0.2792 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{1.6817}, \\ [0.4114, 0.2254, \\ 0.3522], [0.2254, \\ 0.3522, 0.0621], \\ 0.8190, 0.5267, \\ 0.5848 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{1.6817}, \\ [0.4114, 0.2254, \\ 0.3522], [0.2254, \\ 0.3522, 0.0621], \\ 0.8190, 0.5267, \\ 0.5848 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{2.4494}, \\ [0.5757, 0.2062, \\ 0.3366], [0.2062, \\ 0.3366, 0.4921], \\ 0.4242, 0.2097, \\ 0.2792 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_{1.7321}, \\ [0.0513, 0.1633, \\ 0.2928], [0.1633, \\ 0.2928, 0.4522], \\ 0.4472, 0.6324, \\ 0.7745 \end{array} \right\rangle\right\}$

Table 4 TCLUFHA operator 4

	TCLUFHA operator 4
A_1	$\left\{\left\langle \begin{array}{l} s_{1.9999}, s_{2.3782}, \\ [0.7379, 0.2651, 0.2687], \\ [0.2651, 0.2687, 0.1302], \\ 0.7769, 0.1746, 0.2108 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_{1.2968}, s_{1.7490}, \\ [0.4200, 0.1651, 0.2701], \\ [0.1651, 0.2701, 0.2768], \\ 0.6600, 0.4084, 0.4794 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_{1.1486}, s_{1.4814}, \\ [0.2509, 0.1244, 0.2119], \\ [0.1244, 0.2119, 0.2356], \\ 0.6890, 0.5872, 0.6612 \end{array} \right\rangle\right\}$

Step 1: Calculate the TIFWG operator and $\tau = (0.5, 0.2, 0.3)^T$.

Step 2: Calculate the score function

$s(a_1) = 0.0143,\ s(a_2) = 0.2574,\ s(a_3) = 0.0291.$

Step 3: Find the ranking $s(a_2) > s(a_3) > s(a_1)$. (Table 14 and Fig. 12)

The ranking values of the above discussion are given in Table 15 (Fig. 13).

The advantages of our proposed methods can be summarized on the basis of the above comparison analyses. The triangular cubic linguistic fuzzy numbers (TCLUFNs) are very suitable for illustrating uncertain or fuzzy information in MCDM problems because the membership and non-membership degrees can be two sets of several possible values, which cannot be achieved by intuitionistic fuzzy numbers (ITrFNs) and triangular intuitionistic fuzzy number. On the basis of basic operations, aggregation operators and comparison method of triangular cubic linguistic fuzzy numbers (TCLUFNs) can be also used to process intuitionistic fuzzy numbers and triangular intuitionistic fuzzy number after slight adjustments, because triangular cubic linguistic fuzzy numbers (TCLUFNs) can be considered as the generalized form of intuitionistic fuzzy numbers (IFNs) and triangular intuitionistic fuzzy number. The defined operations of triangular cubic linguistic fuzzy numbers (TCLUFNs) give us more accurate than the existing operators.

7.3 Discussion

Compared with other methods, the advantages of the triangular cubic linguistic fuzzy numbers (TCLUFNs) are shown as follows.

Comparing with the intuitionistic fuzzy number by Xu (2007), they are only the special cases of the proposed operators in this paper. The intuitionistic fuzzy number by

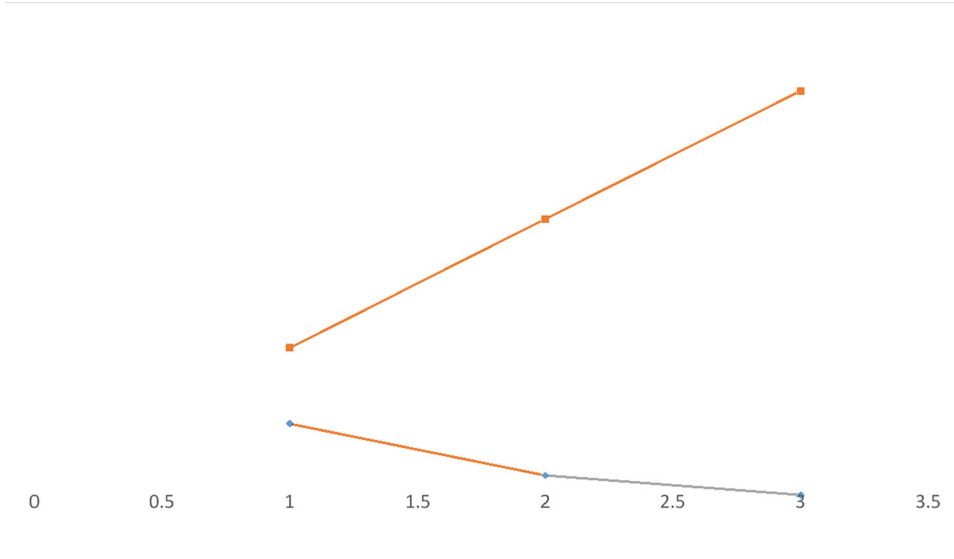

Fig. 5 Graphically, the TCLUFHA score values exhibit

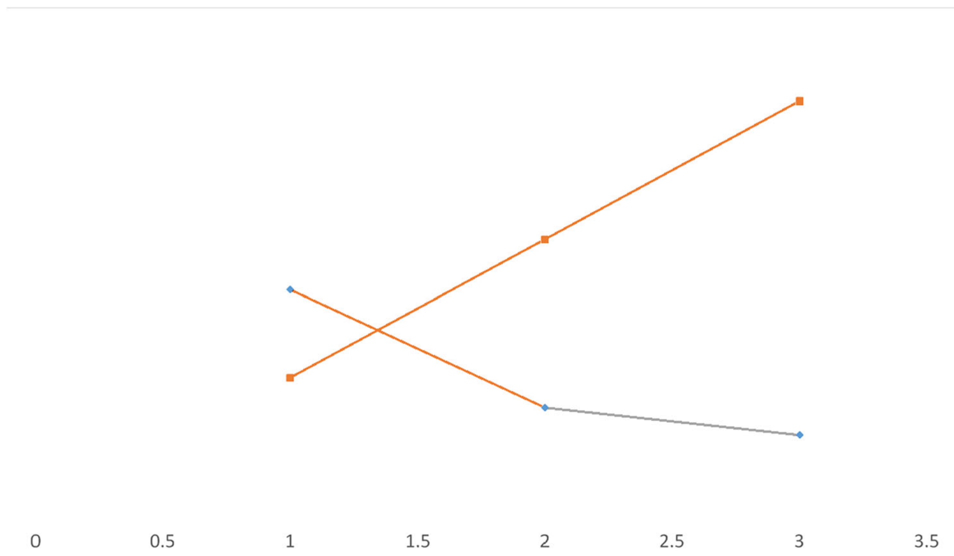

Fig. 6 Graphically, the TCLUFHA accuracy values exhibit

Table 5 TCLUFHA of score value and accuracy value

TCLUFHA	Score values	Ranking 1	TCLUFHA	Accuracy function	Ranking 2	Final ranking
$s(a_1)$	0.4087	1	$H(a_1)$	1.6371	1	1
$s(a_2)$	0.0048	2	$H(a_2)$	0.7850	3	2
$s(a_3)$	− 0.1471	3	$H(a_3)$	0.5854	2	3

Xu (Xu 2007) is based on the membership and non-membership, algebraic operations; the proposed operator in this paper is based on a triangular cubic linguistic fuzzy numbers (TCLUFNs).

The existing decision-making methods based on the prospect theory in the literature (Liang et al. (2014)) only express the preferences of alternatives on criteria with crisp values, fuzzy numbers, and linguistic variables. However, due to the complexity of the socioeconomic environment, there may be hesitation about preferences in decision making. Recently, prospect theories under triangular intuitionistic fuzzy information have been developed, such as Liang et al. (2014), which also consider the hesitation about preferences in decision making. However, IFS and TIF can only express the extent to which a criterion to a fuzzy concept "excellence" or "good" and they only use discrete domains. TCLUFNs method is the extension of IFS which extends the discrete set to continuous set.

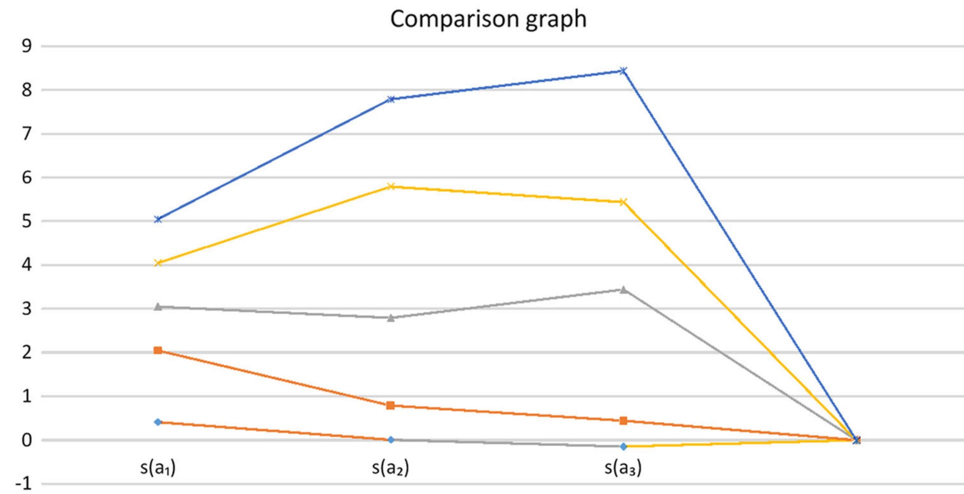

Fig. 7 Comparison graph of TCLUFHA score ranking and TCLUFHAA accuracy ranking

Table 6 Triangular cubic linguistic uncertain fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\left\{\left\langle \begin{array}{l} s_2, s_4, \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ 0.5, 0.7, 0.9 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_1, s_3, \\ [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ 0.2, 0.4, 0.6 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_4, \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ 0.5, 0.7, 0.9 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_3, \\ [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ 0.2, 0.4, 0.6 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_3, \\ [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ 0.2, 0.4, 0.6 \end{array} \right\rangle\right\}$

Table 7 Triangular cubic linguistic uncertain fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_4, \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ 0.5, 0.7, 0.9 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_3, \\ [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ 0.2, 0.4, 0.6 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_2, s_4, \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ 0.5, 0.7, 0.9 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_2, s_2, \\ [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ 0.9, 0.11, 0.13 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_3, \\ [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ 0.2, 0.4, 0.6 \end{array} \right\rangle\right\}$

Table 8 TCLUWG operator

	C_1	C_2	C_3
A_1	$\left\{\left\langle \begin{array}{l} s_2, s_{2.8284}, \\ [0.5656, 0.2449, \\ 0.3098], [0.2449, \\ 0.3098, 0.1183], \\ 0.7763, 0.4832, \\ 0.7051 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.3195}, s_{1.3195}, \\ [0.9146, 0.3981, \\ 0.4282], [0.3981, \\ 0.4282, 0.4554], \\ 0.6018, 0.0455, \\ 0.0541 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.5157}, s_{1.8660}, \\ [0.7104, 0.4299, \\ 0.4950], [0.4299, \\ 0.4950, 0.2778], \\ 0.5920, 0.3270, \\ 0.5193 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_{1.4142}, s_{2.4494}, \\ [0.2828, 0.1732, \\ 0.2449], [0.1732, \\ 0.2449, 0.3130], \\ 0.7171, 0.2692, \\ 0.4100 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.1486}, s_{1.4309}, \\ [0.6034, 0.4959, \\ 0.5696], [0.4959, \\ 0.5696, 0.6284], \\ 0.3965, 0.1179, \\ 0.1903 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.5157}, s_{1.8660}, \\ [0.7104, 0.4299, \\ 0.4950], [0.4299, \\ 0.4950, 0.2778], \\ 0.5920, 0.3270, \\ 0.5193 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_2, s_{2.8284}, \\ [0.5656, 0.2449, \\ 0.3098], [0.2449, \\ 0.3098, 0.1183], \\ 0.7763, 0.4832, \\ 0.7050 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_{1.1486}, s_{1.4309}, \\ [0.6034, 0.4959, \\ 0.5696], [0.4959, \\ 0.5696, 0.6284], \\ 0.3965, 0.1179, \\ 0.1903 \end{array} \right\rangle\right\}$	$\left\{\left\langle \begin{array}{l} s_1, s_{1.9331}, \\ [0.2511, 0.4855, \\ 0.6597], [0.4855, \\ 0.6597, 0.8073], \\ 0.1253, 0.2639, \\ 0.4229 \end{array} \right\rangle\right\}$

Table 9 TCLUFHG operator 9

	TCLUFHG operator 9
A_1	$\left\{\left\langle \begin{array}{l} s_{1.2498}, s_{1.4742}, \\ [0.8185, 0.5302, 0.5800], \\ [0.5302, 0.5800, 0.4315], \\ 0.4846, 0.1979, 0.4527 \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{l} s_{1.4338}, s_{2.1194}, \\ [0.4299, 0.2672, 0.3432], \\ [0.2672, 0.3432, 0.3126], \\ 0.6555, 0.2839, 0.4448 \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{l} s_{1.3946}, s_{2.2769}, \\ [0.3742, 0.3222, 0.4230], \\ [0.3222, 0.4230, 0.3245], \\ 0.5745, 0.3538, 0.5473 \end{array} \right\rangle\right\}$

Compared with IFS and TrIF, TCLUFNs used in our proposed method, by introducing two triangular intuitionistic fuzzy numbers as a reference, can describe and characterize the fuzziness of the objective world meticulously and accurately; it also allows criteria to use different dimensions. Thus, compared with the previous decision-making methods, the proposed method can express more abundant and flexible information and thus have a stronger expression ability to deal with the uncertain information.

7.4 Experimental results

In this work, four classes of aggregation operators, such as TCLUFOWA operator, TCLUFHWG operator, proposed averaging method, and proposed geometric method, are used. Each of which has a number of operators, which is shown in Table 16. Used aggregation operators according to classes.

8 Conclusion

In this paper, the concept of triangular cubic linguistic uncertain fuzzy sets deliberates their basic properties and develops operational laws for triangular cubic linguistic uncertain fuzzy elements, which are introduced. The aggregation operators for triangular cubic linguistic uncertain fuzzy sets which includes cubic linguistic uncertain fuzzy averaging (geometric) operator, triangular cubic linguistic uncertain fuzzy weighted averaging (TCLUFWA) operator, triangular cubic linguistic uncertain fuzzy weighted geometric (TCLUFWG) operator, triangular cubic linguistic uncertain fuzzy ordered weighted

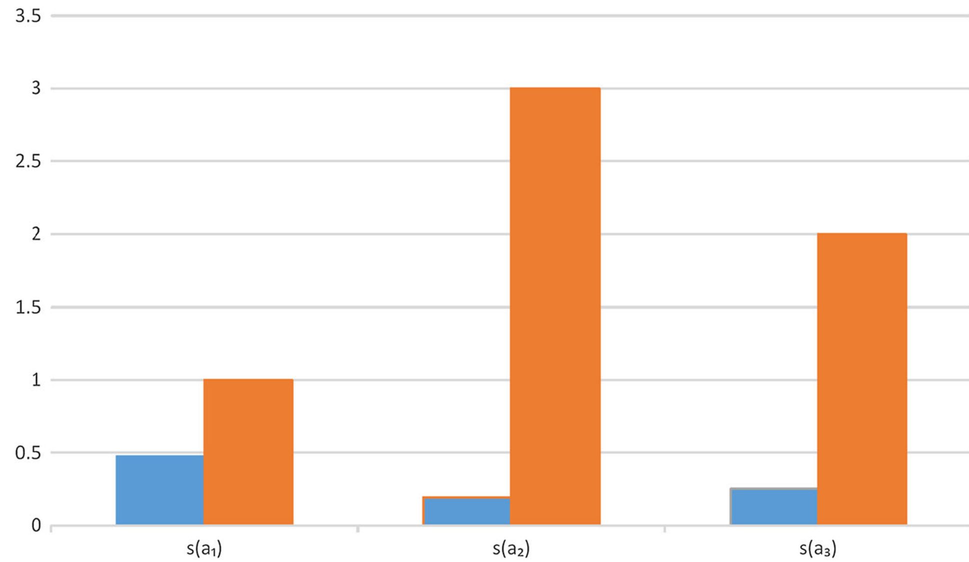

Fig. 8 Graphically, the TCLUFHG score values exhibit

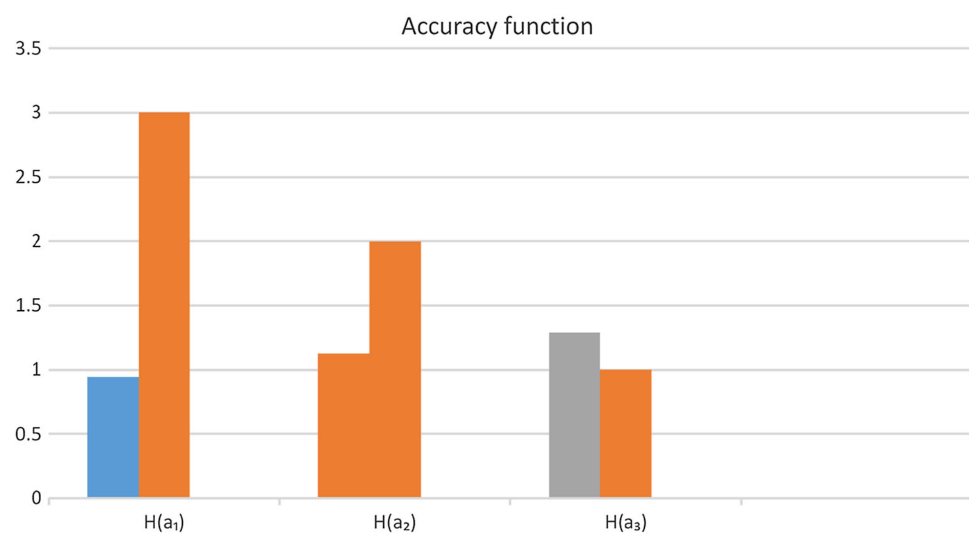

Fig. 9 Graphically, the TCLUFHG accuracy values exhibit

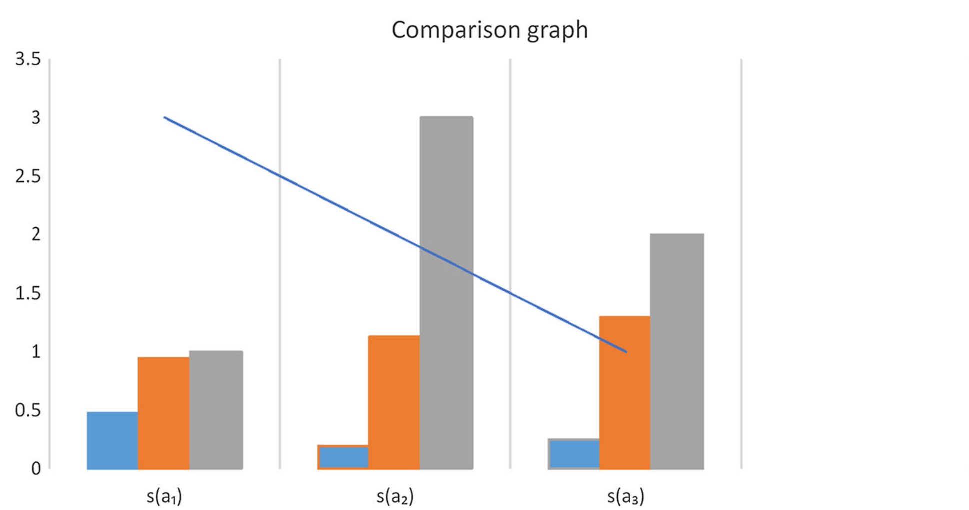

Fig. 10 Comparison graph of TCLUFHG score ranking and TCLUFHG accuracy ranking

Table 10 TCLUFHG of score value and accuracy value

TCLUFHG	Score values	Ranking 1	TCLUFHG	Accuracy function	Ranking 2	Final ranking
$s(a_1)$	0.4781	1	$H(a_1)$	0.9428	3	3
$s(a_2)$	0.1955	3	$H(a_2)$	1.1302	2	2
$s(a_3)$	0.2517	2	$H(a_3)$	1.2929	1	1

Table 11 IF decision matrix

	C_1	C_2	C_3
A_1	[0.4, 0.8]	[0.10, 0.14]	[0.8, 0.9]
A_2	[0.1, 0.5]	[0.4, 0.8]	[0.4, 0.6]
A_3	[0.10, 0.12]	[0.3, 0.5]	[0.1, 0.3]

Table 12 IFWA operator 12

	IFWA operator 12
A_1	$[0.6713, 0.3174]$
A_2	$[0.2018, 0.7516]$
A_3	$[0.1565, 0.2996]$

averaging (TCLUFOWA) operator, triangular cubic linguistic uncertain fuzzy ordered weighted geometric (TCLUFOWG) operator, triangular cubic linguistic uncertain fuzzy hybrid averaging (TCLUFHA) operator and triangular cubic linguistic uncertain fuzzy hybrid geometric (TCLUFHG) operator are developed. Moreover, the developed aggregation operators are applied to multiple attribute group decision making with triangular cubic linguistic uncertain fuzzy information. At last, a numerical example is used to illustrate the validity of the proposed approach in group decision-making problems.

Table 14 TIFWG operator 14

	TIFWG operator 14
A_1	$[0.5059, 0.0774, 0.1073], [0.2324, 0.1700]$
A_2	$[0.3314, 0.4643, 0.5618], [0.7361, 0.1672]$
A_3	$[0.2349, 0.2433, 0.3492], [0.4704, 0.3651]$

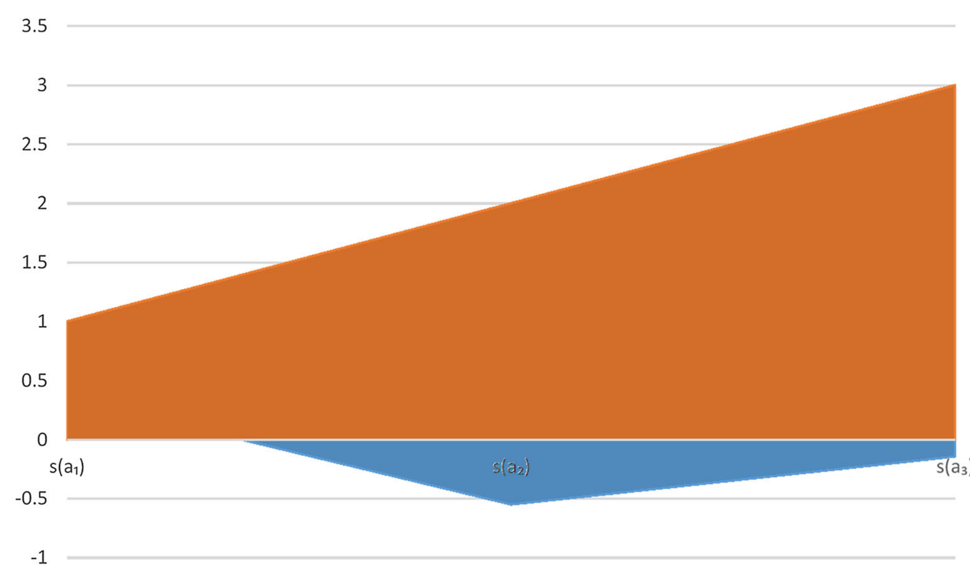

Fig. 11 Graphically, the IF score values exhibit

Table 13 TIF decision matrix

	C_1	C_2	C_3
A_1	$[0.4, 0.6, 0.8], [0.6, 0.10]$	$[0.8, 0.10, 0.12], [0.10, 0.14]$	$[0.8, 0.10, 0.12], [0.8, 0.9]$
A_2	$[0.1, 0.3, 0.5], [0.3, 0.5]$	$[0.10, 0.12, 0.14], [0.9, 0.11]$	$[0.4, 0.6, 0.8], [0.8, 0.10]$
A_3	$[0.8, 0.10, 0.12], [0.9, 0.12]$	$[0.1, 0.3, 0.5], [0.3, 0.5]$	$[0.1, 0.3, 0.5], [0.3, 0.5]$

Fig. 12 Graphically, the TIF score values exhibit

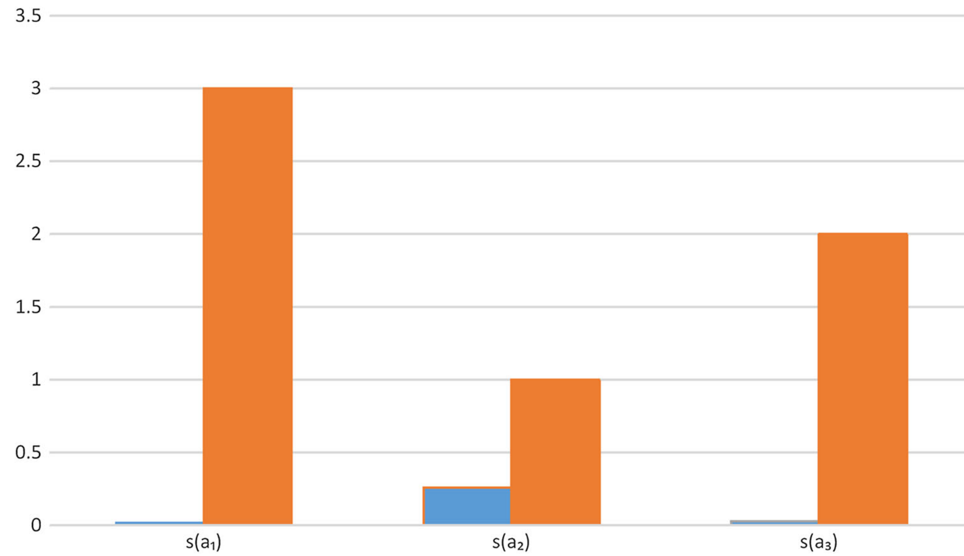

Table 15 Comparison method with the existing methods

Method	Ranking
Proposed averaging method	$s(a_1) > s(a_2) > s(a_3)$
Proposed geometric method	$s(a_1) > s(a_3) > s(a_2)$
Intuitionistic fuzzy number (Xu 2007)	$s(a_1) > s(a_2) > s(a_3)$
Triangular intuitionistic fuzzy aggregation (Jun et al. 2011)	$s(a_2) > s(a_3) > s(a_1)$

Fig. 13 Whole-comparison method of Table 15

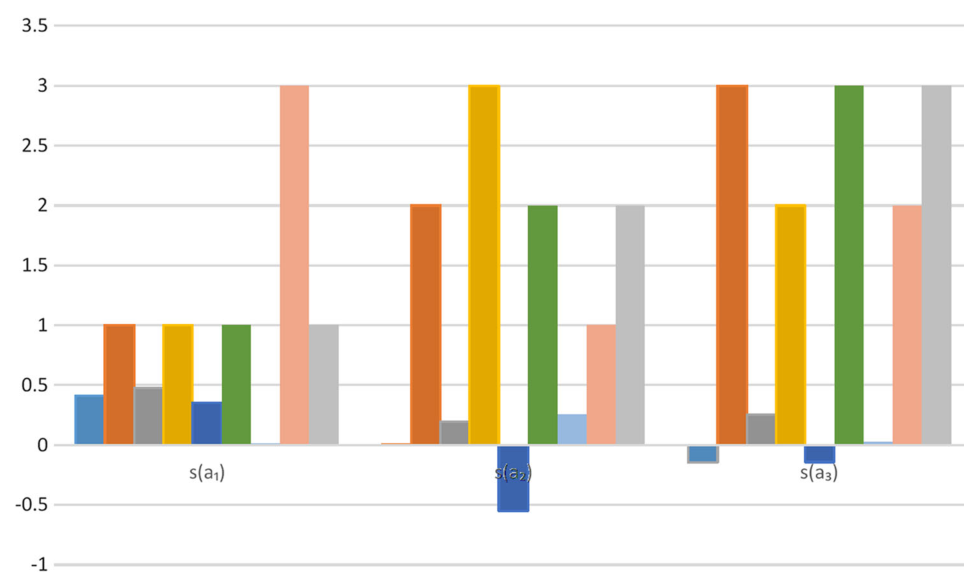

Table 16 Experimental results

Aggregation operator	Ranking
TCLUFOWA operator	$s(h_3) > s(h_1) > s(h_2)$
TCLUFHWA operator	$s(h_2) > s(h_3) > s(h_1)$
Averaging score function	$s(a_1) > s(a_2) > s(a_3)$
Averaging accuracy function	$H(a_1) > H(a_2) > H(a_3)$
Geometric score function	$s(a_1) > s(a_3) > s(a_2)$
Geometric accuracy function	$H(a_3) > H(a_2) > H(a_1)$
Intuitionistic fuzzy number (Xu 2007)	$s(a_1) > s(a_2) > s(a_3)$
Triangular intuitionistic fuzzy aggregation (Liang et al. 2014)	$s(a_2) > s(a_3) > s(a_1)$

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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