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FOUNDATIONS

Set operations of fuzzy sets using gradual elements

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Abstract
The conventional set operations of fuzzy sets are based on the membership functions using the max and min functions. In this paper, we shall consider the set operations of fuzzy sets based on the concepts of gradual sets and gradual elements. When the fuzzy sets can be formulated as consisting of gradual elements like the usual set consisting of usual elements, the intersection and union of fuzzy sets can be defined as the same way as the intersection and union of usual sets. In this case, the set operations of fuzzy sets will be similar to the set operations of crisp sets.

Keywords Decomposition theorem · Gradual elements · Gradual sets · Interval range · Normal fuzzy sets

1 Introduction

Let A be a (crisp) subset of a universal set U. The concept of element in A is well known by writing $x \in A$. Suppose that $\tilde{A}$ is a fuzzy subset of U. The main focus of this paper is to consider the elements of fuzzy set $\tilde{A}$. The gradual element will play the role to be the element of $\tilde{A}$.

The concepts of gradual elements and gradual sets based on a universal set U were introduced by Dubois and Prade (2008) and Fortin et al. (2008), which were inspired by Goetschel (1997) and Herencia and Lamata (1999). The gradual element is a function from $(0, 1]$ into U, and the gradual set is a set-valued function from $(0, 1]$ into the hyperspace that consists of all subsets of U. When the universal set U is taken to be $\mathbb{R}$, the gradual element is also called gradual numbers. Boukezzoula et al. (2014) used gradual numbers to define the so-called gradual intervals in which the end-points are assumed to be gradual numbers. More motivated argument can also refer to Dubois and Prade (2012).

Let U be a universal set. We denote by $\mathcal{P}(U)$ the collection of all subsets of U. Dubois and Prade (2008) considered the gradual set G defined by an assignment function $\mathcal{A}_G : (0, 1] \to \mathcal{P}(U)$ that does not consider the assignment at 0, where the assignment function $\mathcal{A}_G$ is a set-valued function.

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The gradual set G with assignment function $\mathcal{A}_G$ can induce a fuzzy set F_G with membership function given by

$$\xi_{F_G}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\mathcal{A}_G(\alpha)}(x), \tag{1}$$

where $\chi_{\mathcal{A}_G(\alpha)}$ is a characteristic function given by

$$\chi_{\mathcal{A}_G(\alpha)}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A}_G(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Sanchez et al. (2012) considered the fuzzy concept A defined by an ordered pair (Λ_A, ρ_A), where $\Lambda_A = \{1 = \alpha_1, \alpha_2, \ldots, \alpha_m = 0\}$ is a finite subset of the unit interval $[0, 1]$ satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and ρ_A is a set-valued function $\rho_A : \Lambda_A \to \mathcal{P}(U)$ defined on the finite set Λ_A. In this paper, we shall consider the set-valued function $\mathfrak{G} : I \to \mathcal{P}(U)$ from I into $\mathcal{P}(U)$, where I is any subset of $[0, 1]$. This set-valued function $\mathfrak{G}$ will also be called as a gradual set (or an extended gradual set). The set-valued function $\mathfrak{G}$ will extend the set-valued function $\mathcal{A}_G$.

- In Dubois and Prade (2008), let G_1 and G_2 be two gradual sets. The intersection and union of G_1 and G_2 are defined by the assignment functions

$$\mathcal{A}_{G_1 \cup G_2}(\alpha) = \mathcal{A}_{G_1}(\alpha) \cup \mathcal{A}_{G_2}(\alpha)$$
$$\text{and } \mathcal{A}_{G_1 \cap G_2}(\alpha) = \mathcal{A}_{G_1}(\alpha) \cap \mathcal{A}_{G_2}(\alpha).$$

The gradual sets $G_1 \cup G_2$ and $G_1 \cap G_2$ can induce two fuzzy sets $F_{G_1 \cup G_2}$ and $F_{G_1 \cap G_2}$ with membership functions $\xi_{F_{G_1 \cup G_2}}$ and $\xi_{F_{G_1 \cap G_2}}$ according to (1), respectively. Dubois and Prade (2008) also claim that

$$\xi_{F_{G_1 \cup G_2}}(x) = \max\left\{\xi_{F_{G_1}}(x), \xi_{F_{G_2}}(x)\right\}$$

and

$$\xi_{F_{G_1 \cap G_2}}(x) = \min\left\{\xi_{F_{G_1}}(x), \xi_{F_{G_2}}(x)\right\}.$$

- In Sanchez et al. (2012), let A and B be two fuzzy concepts with (Λ_A, ρ_A) and (Λ_B, ρ_B). The new fuzzy concepts $A \wedge B$ and $A \vee B$ are defined by

$$\Lambda_{A \wedge B} = \Lambda_{A \vee B} = \Lambda_A \cup \Lambda_B$$

and

$$\rho_{A \wedge B}(\alpha) = \rho_A(\alpha) \cap \rho_B(\alpha) \quad \text{and } \rho_{A \vee B}(\alpha) = \rho_A(\alpha) \cup \rho_B(\alpha),$$

which are similar to the approach of Dubois and Prade (2008) by regarding the fuzzy concept $A \wedge B$ as the intersection and the fuzzy concept $A \vee B$ as the union.

In this paper, we are not going to consider the intersection and union of gradual sets as the case of Dubois and Prade (2008) and Sanchez et al. (2012) presented above. We shall study the intersection and union of fuzzy sets based on the concepts of gradual sets and gradual elements, which will be explained in the next paragraph.

Let A be a subset of U. The element x in A is simply written as $x \in A$. Now we assume that $\tilde{A}$ is a fuzzy subset of U. The purpose is to consider the elements of $\tilde{A}$. In other words, we need to define a quantity $\hat{x}$ such that we can reasonably write $\hat{x} \in \tilde{A}$. Under some suitable settings, we shall see that the element $\hat{x}$ in $\tilde{A}$ is also a gradual element. In this case, the intersection and union of fuzzy sets can be defined using their elements as in the usual sense of intersection and union of crisp sets. For example, let A and B be two (crisp) subsets of U. Then, the intersection $A \cap B$ and union $A \cup B$ are given by

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \quad \text{and } A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Suppose now that $\tilde{A}$ and $\tilde{B}$ are two fuzzy subsets of U. Then, we shall try to explain the intersection $\tilde{A} \cap \tilde{B}$ given by

$$\tilde{A} \cap \tilde{B} = \{\hat{x} : \hat{x} \in \tilde{A} \text{ and } \hat{x} \in \tilde{B}\}, \tag{2}$$

where the form of $\hat{x}$ and the meanings of $\hat{x} \in \tilde{A}$ and $\hat{x} \in \tilde{B}$ will be studied in this paper. We shall also try to investigate the union

$$\tilde{A} \cup \tilde{B} = \{\hat{x} : \hat{x} \in \tilde{A} \text{ or } \hat{x} \in \tilde{B}\}, \tag{3}$$

The relationship with the conventional intersection and union using the aggregation functions will also be established.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of U with membership functions $\xi_{\tilde{A}}$ and $\xi_{\tilde{B}}$, respectively. The usual intersection and union of $\tilde{A}$ and $\tilde{B}$ are defined using the min and max functions as follows:

$$\xi_{\tilde{A} \cap \tilde{B}}(x) = \min\left\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(x)\right\} \quad \text{and } \xi_{\tilde{A} \cup \tilde{B}}(x) = \max\left\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(x)\right\}. \tag{4}$$

The generalization for considering t-norm and t-conorm can refer to Dubois and Prade (1985) and Weber (1983). On the other hand, Tan et al. (1993) proposed a different generalization for intersection and union of fuzzy sets. The alternative definitions for the intersection and union of fuzzy sets are also widely discussed in the literature by referring to Yager (1980, 1991). Klement (1982) considered the axiomatic approach for operations on fuzzy sets. More detailed properties can refer to the monographs (Dubois and Prade (1988) and Klir and Yuan (1995)).

The definitions of intersection and union defined in (4) are based on the membership functions. In this paper, we shall study the intersection and union given in (2) and (3), respectively, which are similar to the usual sense of intersection and union of crisp sets. The advantage for considering the intersection and union in the sense of (2) and (3) is that the fuzzy sets can be formulated as consisting of gradual elements, which will be studied in this paper. In order to claim the consistency for considering gradual elements in fuzzy sets, a more general decomposition theorem will be established, where the basic properties for decomposition theorem can refer to Fullér and Keresztfalvi (1990), Negoita and Ralescu (1975), Nguyen (1978), Ralescu (1992) and Wu (2010).

In Sect. 2, we present a generalized decomposition theorem which will be useful for further discussion. In Sect. 3, we present the concepts of gradual elements and gradual sets and study the relationships with fuzzy sets. In Sect. 4, we study the complement set of fuzzy set using the gradual elements and investigate the relationship with the usual definition of complement set using the membership functions. In Sect. 5, we study the intersection and union of fuzzy sets using the concept of gradual elements. In Sect. 6, the associativity of intersection and union is also thoroughly studied.

2 Non-normal fuzzy sets

Let $\tilde{A}$ be a fuzzy subset of a universal set U with membership function denoted by $\xi_{\tilde{A}}$. For $\alpha \in (0, 1]$, the α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_\alpha = \left\{ x \in U : \xi_{\tilde{A}}(x) \geq \alpha \right\}. \tag{5}$$

We also define

$$\tilde{A}_{\alpha+} = \left\{ x \in U : \xi_{\tilde{A}}(x) > \alpha \right\}.$$

The *support* of a fuzzy set $\tilde{A}$ is the crisp set defined by

$$\tilde{A}_{0+} = \{x \in U : \xi_{\tilde{A}}(x) > 0\}.$$

The definition of 0-level set is an important issue in fuzzy sets theory. If the universal set U is endowed with a topology τ, then the 0-level set $\tilde{A}_0$ can be defined as the closure of the support of $\tilde{A}$, i.e.,

$$\tilde{A}_0 = \mathrm{cl}\left(\tilde{A}_{0+}\right). \tag{6}$$

If U is not endowed with a topological structure, then the intuitive way for defining the 0-level set is to follow the equality (5) for $\alpha = 0$. In this case, the 0-level set of $\tilde{A}$ is the whole universal set U. This kind of 0-level set seems not so useful. Therefore, we always endow a topological structure to the universal set U when the 0-level set should be seriously considered.

The range of membership function $\xi_{\tilde{A}}$ is denoted by $\mathcal{R}(\xi_{\tilde{A}})$ that is a subset of [0, 1]. We see that the range $\mathcal{R}(\xi_{\tilde{A}})$ can be a proper subset of [0, 1] with $\mathcal{R}(\xi_{\tilde{A}}) \neq [0, 1]$. Notice that if $\alpha \notin \mathcal{R}(\xi_{\tilde{A}})$, we still can consider the α-level set $\tilde{A}_\alpha$. Since $\mathcal{R}(\xi_{\tilde{A}}) \neq [0, 1]$, it is possible that the α-level set $\tilde{A}_\alpha$ can be an empty set for some $\alpha \in [0, 1]$. Therefore, when we study the properties that deal with more than two fuzzy sets, we cannot simply present the properties by saying that they hold true for each $\alpha \in [0, 1]$, since some of the α-level sets can be empty. In this case, we need to carefully treat the ranges of membership functions.

Remark 2.1 Let $\tilde{A}$ be a fuzzy set in U with membership function $\xi_{\tilde{A}}$. Define

$$\alpha^* = \sup \mathcal{R}(\xi_{\tilde{A}}) \text{ and } \alpha^\circ = \inf \mathcal{R}(\xi_{\tilde{A}}).$$

Then, we have the following observations.

- For any $0 \leq \alpha < \alpha^*$, even though $\alpha \notin \mathcal{R}(\xi_{\tilde{A}})$, we have $\tilde{A}_\alpha \neq \emptyset$. It is also obvious that $\tilde{A}_\alpha = \emptyset$ for $\alpha > \alpha^*$.
- If the maximum $\max \mathcal{R}(\xi_{\tilde{A}})$ exists, i.e., $\sup \mathcal{R}(\xi_{\tilde{A}}) = \max \mathcal{R}(\xi_{\tilde{A}})$ (we can also say that the supremum of $\mathcal{R}(\xi_{\tilde{A}})$ is attained), then we have $\tilde{A}_{\alpha^*} \neq \emptyset$. If $\max \mathcal{R}(\xi_{\tilde{A}})$ does not exist, then $\tilde{A}_{\alpha^*} = \emptyset$. For example, assume that

$$\xi_{\tilde{A}}(x) = \begin{cases} 1 - \frac{1}{x}, & \text{if } x \geq 1 \\ 0, & \text{if } x < 1. \end{cases}$$

  It is clear to see that $\mathcal{R}(\xi_{\tilde{A}}) = [0, 1)$. In this case, the maximum $\max \mathcal{R}(\xi_{\tilde{A}})$ does not exist. However, we have $\sup \mathcal{R}(\xi_{\tilde{A}}) = 1 = \alpha^*$. In this case, the 1-level set $\tilde{A}_{\alpha^*} = \tilde{A}_1 = \emptyset$, since $\alpha^* = 1 \notin \mathcal{R}(\xi_{\tilde{A}})$.
- For any $0 \leq \alpha < \alpha^\circ$, even though $\alpha \notin \mathcal{R}(\xi_{\tilde{A}})$, we have $\tilde{A}_\alpha = \tilde{A}_{\alpha^\circ} \neq \emptyset$.

Proposition 2.2 *Let $\tilde{A}$ be a fuzzy set in U with membership function $\xi_{\tilde{A}}$. Define $\alpha^* = \sup \mathcal{R}(\xi_{\tilde{A}})$ and*

$$I_{\tilde{A}} = \begin{cases} [0, \alpha^*), & \text{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \text{ does not exist} \\ [0, \alpha^*], & \text{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \text{ exists.} \end{cases} \tag{7}$$

Then, $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$. Moreover, we have $\mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$ and

$$\tilde{A}_{0+} = \bigcup_{\{\alpha \in I_{\tilde{A}} : \alpha > 0\}} \tilde{A}_\alpha = \bigcup_{\{\alpha \in \mathcal{R}(\xi_{\tilde{A}}) : \alpha > 0\}} \tilde{A}_\alpha \tag{8}$$

Proof From Remark 2.1, we see that $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$ immediately. Moreover, for proving the equality (8), assume that $x \in \tilde{A}_{0+}$, i.e., $\alpha^* \equiv \xi_{\tilde{A}}(x) > 0$. Then, $x \in \tilde{A}_{\alpha^*}$. Since $\alpha^* \in \mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$, we have the inclusions

$$\tilde{A}_{0+} \subseteq \bigcup_{\{\alpha \in \mathcal{R}(\xi_{\tilde{A}}) : \alpha > 0\}} \tilde{A}_\alpha \text{ and } \tilde{A}_{0+} \subseteq \bigcup_{\{\alpha \in I_{\tilde{A}} : \alpha > 0\}} \tilde{A}_\alpha.$$

For proving another direction of inclusions, we consider the following cases.

- Given $x \in \tilde{A}_\alpha$ for some $0 < \alpha \in \mathcal{R}(\xi_{\tilde{A}})$, we have $\xi_{\tilde{A}}(x) \geq \alpha > 0$, i.e., $x \in \tilde{A}_{0+}$.
- Given $x \in \tilde{A}_\alpha$ for some $0 < \alpha \in I_{\tilde{A}}$, we have $\xi_{\tilde{A}}(x) \geq \alpha > 0$, i.e., $x \in \tilde{A}_{0+}$.

This proves the desired equalities. □

The interval $I_{\tilde{A}}$ presented in Proposition 2.2 is also called an *interval range* of $\tilde{A}$. We see that the interval range $I_{\tilde{A}}$ contains the actual range $\mathcal{R}(\xi_{\tilde{A}})$. The role of interval range $I_{\tilde{A}}$ can be used to say $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$.

Example 2.3 The membership function of a trapezoidal-like fuzzy number is given by

$$\xi_{\tilde{A}}(x) = \begin{cases} 0.1 + 0.7 \cdot (x-1) & \text{if } 1 \leq x \leq 1.5 \\ 0.2 + 0.7 \cdot (x-1) & \text{if } 1.5 < x < 2 \\ 0.9 & \text{if } 2 \leq x \leq 3 \\ 0.2 + 0.7 \cdot (4-x) & \text{if } 3 < x < 3.5 \\ 0.1 + 0.7 \cdot (4-x) & \text{if } 3.5 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear to see that

$$\mathcal{R}(\xi_{\tilde{A}}) = [0.1, 0.45] \cup (0.55, 0.9] \text{ and } \alpha^* = 0.9,$$

which also says that $I_{\tilde{A}} = [0, 0.9] \neq \mathcal{R}(\xi_{\tilde{A}})$. We see that $0.5 \notin \mathcal{R}(\xi_{\tilde{A}})$. However, we still have the 0.45-level set $\tilde{A}_{0.45}$. In general, we see that $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}} = [0, 0.9]$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}} = [0, 0.9]$. We also see that $\alpha^\circ = \inf \mathcal{R}(\xi_{\tilde{A}}) = 0.1$. Although $0 \leq \alpha < 0.1$ is not in $\mathcal{R}(\xi_{\tilde{A}})$, it is clear to see that $\tilde{A}_\alpha = \tilde{A}_{0.1}$. Therefore, the interval $I_{\tilde{A}}$ plays an important role for considering the α-level sets. In other words, the range $\mathcal{R}(\xi_{\tilde{A}})$ is not helpful for identifying the α-level sets.

Recall that the fuzzy set $\tilde{A}$ is normal if and only if there exists $x \in U$ such that $\xi_{\tilde{A}}(x) = 1$. Suppose that $\tilde{A}$ is a normal fuzzy set in U. Then, the well-known decomposition theorem says that the membership function $\xi_{\tilde{A}}$ can be expressed as

$$\xi_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x), \tag{9}$$

where $\chi_{\tilde{A}_\alpha}$ is the characteristic function of the α-level set $\tilde{A}_\alpha$ defined by

$$\chi_{\tilde{A}_\alpha}(x) = \begin{cases} 1 & \text{if } x \in \tilde{A}_\alpha \\ 0 & \text{if } x \notin \tilde{A}_\alpha, \end{cases}$$

which can refer to Fullér and Keresztfalvi (1990), Negoita and Ralescu (1975), Nguyen (1978), Ralescu (1992) and Wu (2010). The following generalized decomposition theorem based on the interval range $I_{\tilde{A}}$ given in (7) will be used for studying the set operations of fuzzy sets in U.

Theorem 2.4 (Generalized Decomposition Theorem) *Let $\tilde{A}$ be a fuzzy set in U. The membership function $\xi_{\tilde{A}}$ can be expressed as*

$$\begin{aligned} \xi_{\tilde{A}}(x) &= \sup_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) \\ &= \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x), \end{aligned} \tag{10}$$

where $I_{\tilde{A}}$ is given in (7).

Proof Given any fixed $x \in U$, let $\alpha_0 = \xi_{\tilde{A}}(x) \in \mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$, i.e., $\alpha_0 \in I_{\tilde{A}}$. Suppose that $\alpha_0 = 0$. If $x \in \tilde{A}_\alpha \neq \emptyset$ for some $\alpha \in I_{\tilde{A}} \backslash \{0\}$, then $\xi_{\tilde{A}}(x) \geq \alpha > 0$, which contradicts $\xi_{\tilde{A}}(x) = \alpha_0 = 0$. Therefore, we consider the following two cases.

- If $x \notin \tilde{A}_\alpha$ for all $\alpha \in I_{\tilde{A}}$, then $\chi_{\tilde{A}_\alpha}(x) = 0$ for all $\alpha \in I_{\tilde{A}}$. This says that the equalities in (10) are satisfied.
- If $x \in \tilde{A}_0$ and $x \notin \tilde{A}_\alpha$ for $\alpha \in I_{\tilde{A}} \backslash \{0\}$, then $\alpha \cdot \chi_{\tilde{A}_\alpha}(x) = 0$ for all $\alpha \in I_{\tilde{A}}$. This also says that the equalities in (10) are satisfied.

The above two cases show the desired equalities for $\alpha_0 = 0$. Now we assume $\alpha_0 > 0$. Then, $x \in \tilde{A}_{\alpha_0}$. In the sequel, we assume $\alpha \in I_{\tilde{A}}$. For $\alpha > \alpha_0$, if $x \in \tilde{A}_\alpha$, then $\xi_{\tilde{A}}(x) \geq \alpha > \alpha_0$, which contradicts $\alpha_0 = \xi_{\tilde{A}}(x)$. Therefore, we have $x \notin \tilde{A}_\alpha$ for $\alpha > \alpha_0$. If $\alpha \leq \alpha_0$, then $x \in \tilde{A}_{\alpha_0} \subseteq \tilde{A}_\alpha$, which says that $x \in \tilde{A}_\alpha$ for $\alpha \leq \alpha_0$. Then, we obtain

$$\begin{aligned} &\sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) \\ &= \max \left\{ \sup_{\{\alpha \in I_{\tilde{A}} : \alpha \leq \alpha_0\}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x), \sup_{\{\alpha \in I_{\tilde{A}} : \alpha_0 < \alpha\}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) \right\} \\ &= \max \left\{ \sup_{\{\alpha \in I_{\tilde{A}} : \alpha \leq \alpha_0\}} \alpha, 0 \right\} = \alpha_0 = \xi_{\tilde{A}}(x). \end{aligned}$$

Since $\alpha_0 \in I_{\tilde{A}}$, the above supremum is attained. It means that

$$\xi_{\tilde{A}}(x) = \max_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x).$$

The above arguments are still valid when $I_{\tilde{A}}$ is replaced by $\mathcal{R}(\xi_{\tilde{A}})$. Therefore, we obtain the desired equalities. This completes the proof. □

3 Gradual elements and gradual sets

Let U be a universal set. Recall that $\mathcal{P}(U)$ denotes the collection of all subsets of U, which is also called a power set or hyperspace of U. By referring to Dubois and Prade (2008), we propose the slightly different concepts of gradual set and gradual element as follows:

Definition 3.1 Let I be a subset of $[0, 1]$. The *gradual element* g in U is defined to be an assignment function $g : I \to U$ from I into U. If $U = \mathbb{R}$, then the gradual element is also called a *gradual number*. The *gradual set* $\mathfrak{G}$ in U (or gradual subset of U) is defined to be an assignment function $\mathfrak{G} : I \to \mathcal{P}(U)$ from I into $\mathcal{P}(U)$ such that each $\mathfrak{G}(\alpha)$ is a nonempty subset of U for $\alpha \in I$.

Definition 3.2 Let I be a subset of $[0, 1]$. We say that the gradual set $\mathfrak{G}$ defined on I is *nested* if and only if $\mathfrak{G}(\alpha) \subseteq \mathfrak{G}(\beta)$ for $\alpha, \beta \in I$ with $\alpha > \beta$.

The gradual set is in fact a set-valued function from I into U. We remark that the gradual set defined in Dubois and Prade (2008) considers the assignment function $\mathfrak{G} : (0, 1] \to \mathcal{P}(U)$ that does not consider the assignment at 0. In general, the unit interval [0, 1] can be extended to consider as a lattice and consider I to be a sub-lattice.

Let $\tilde{A}$ be a fuzzy set in a topological space U with membership function $\xi_{\tilde{A}}$, and let $\alpha^* = \sup \mathcal{R}(\xi_{\tilde{A}})$. By referring to Proposition 2.2, we can induce a gradual set $\mathfrak{G}_{\tilde{A}}$ from $\tilde{A}$ by defining the assignment function as follows:

- Suppose that the maximum $\max \mathcal{R}(\xi_{\tilde{A}})$ does not exist. Then, the assignment function $\mathfrak{G}_{\tilde{A}} : [0, \alpha^*) \to \mathcal{P}(U)$ is defined on $I \equiv [0, \alpha^*)$ given by $\mathfrak{G}_{\tilde{A}}(\alpha) = \tilde{A}_\alpha$.
- Suppose that the maximum $\max \mathcal{R}(\xi_{\tilde{A}})$ exists. Then, the assignment function $\mathfrak{G}_{\tilde{A}} : [0, \alpha^*] \to \mathcal{P}(U)$ is defined on $I \equiv [0, \alpha^*]$ given by $\mathfrak{G}_{\tilde{A}}(\alpha) = \tilde{A}_\alpha$.

Therefore, we see that

$$\mathfrak{G}_{\tilde{A}}(\alpha) = \begin{cases} \tilde{A}_\alpha \text{ for } \alpha \in [0, \alpha^*), & \text{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \text{ does not exist} \\ \tilde{A}_\alpha \text{ for } \alpha \in [0, \alpha^*], & \text{if the maximum } \max \mathcal{R}(\xi_{\tilde{A}}) \text{ exists,} \end{cases} \tag{11}$$

where $\mathfrak{G}_{\tilde{A}}(0) = \tilde{A}_0$ is defined to be the closure of the support of $\tilde{A}$.

On the other hand, given a gradual set $\mathfrak{G} : I \to \mathcal{P}(U)$, we can induce a fuzzy set $\tilde{A}^{\mathfrak{G}}$ in U by using the form of decomposition theorem. The membership function of $\tilde{A}^{\mathfrak{G}}$ is then defined by

$$\xi_{\tilde{A}^{\mathfrak{G}}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\mathfrak{G}(\alpha)}(x). \tag{12}$$

Remark 3.3 Let $\tilde{A}^{\mathfrak{G}}$ be a fuzzy set in U induced by a gradual set $\mathfrak{G} : I \to \mathcal{P}(U)$. Then, the interval range $I_{\tilde{A}^{\mathfrak{G}}}$ of $\tilde{A}^{\mathfrak{G}}$ has the form of (7). It is clear to see that

$$\sup_{x \in U} \xi_{\tilde{A}^{\mathfrak{G}}}(x) = \sup \mathcal{R}(\xi_{\tilde{A}^{\mathfrak{G}}}) = \sup I.$$

Let $\alpha^* = \sup I$. Then, the domain I of gradual set $\mathfrak{G}$ is not necessarily an interval of the form $[0, \alpha^*]$ or $[0, \alpha^*)$. The domain I can be the disjoint union of more than two intervals. However, if the domain I of gradual set $\mathfrak{G}$ happens to be an interval of the form $[0, \alpha^*]$ or $[0, \alpha^*)$, then the interval range $I_{\tilde{A}^{\mathfrak{G}}}$ of $\tilde{A}^{\mathfrak{G}}$ is equal to I, i.e., $I = I_{\tilde{A}^{\mathfrak{G}}}$.

We also remark that if the gradual set is taken from the sense of Dubois and Prade (2008), then the interval range $I_{\tilde{A}^{\mathfrak{G}}}$ of $\tilde{A}^{\mathfrak{G}}$ cannot be equal to the domain I, since the gradual set proposed by Dubois and Prade (2008) does not consider the assignment at 0.

Let Λ be an index set that can be an uncountable set. We consider a family of gradual elements in U as $\{g_\lambda : \lambda \in \Lambda\}$ that are defined on the same domain $I \subseteq [0, 1]$. This family of gradual elements can induce a gradual set $\mathfrak{G}$ in U defined on I by

$$\mathfrak{G}(\alpha) = \{g_\lambda(\alpha) : \lambda \in \Lambda\} \text{ for } \alpha \in I. \tag{13}$$

Based on this gradual set $\mathfrak{G}$ in U, we can also induce a fuzzy set $\tilde{A}^{\mathfrak{G}}$ in U with membership function defined in (12). In this case, we also say that the family $\{g_\lambda : \lambda \in \Lambda\}$ of gradual elements induces the fuzzy set $\tilde{A}^{\mathfrak{G}}$.

The gradual set $\mathfrak{G}$ is a set-valued function defined on $I \subseteq [0, 1]$ with function value $\mathfrak{G}(\alpha)$ that is a subset of U for each $\alpha \in I$. According to the topic of set-valued analysis, the *selector* (or *selection function*) of $\mathfrak{G}$ is a single-valued function $g : I \to U$ defined on I by $g(\alpha) \in \mathfrak{G}(\alpha)$. In this case, we may write $g \in \mathfrak{G}$. It is clear to see that the selector of gradual set $\mathfrak{G}$ (i.e., set-valued function $\mathfrak{G}$) is a gradual element in U. In some sense, we may say that the gradual set consists of gradual elements, which is similar to say that the (usual) set consists of (usual) elements.

Given a subset A of U, the concept of elements of A can be realized in the usual sense by simply writing $a \in A$ when a is assumed to be an element of A. For the fuzzy subset $\tilde{A}$ of U, we plan to consider the concept of element of $\tilde{A}$ by also simply writing $\hat{a} \in \tilde{A}$, where the definition of $\hat{a}$ will be presented below.

Given a fuzzy set $\tilde{A}$ in a topological space U, we can induce a gradual set $\mathfrak{G}_{\tilde{A}}$ as given in (11). Therefore, we have the selector $\hat{a}$ of $\mathfrak{G}_{\tilde{A}}$ given by $\hat{a}(\alpha) \in \mathfrak{G}_{\tilde{A}}(\alpha) = \tilde{A}_\alpha$ for $\alpha \in I$. We also see that the selector $\hat{a}$ is a gradual element in U. This gradual element $\hat{a}$ can be regarded as an element of $\tilde{A}$ by simply writing $\hat{a} \in \tilde{A}$. The formal definition is given below.

Definition 3.4 Let $\tilde{A}$ be a fuzzy set in a topological space U. We say that an *element* $\hat{a}$ is in $\tilde{A}$ if and only if $\hat{a}$ is a gradual element $\hat{a} : I_{\tilde{A}} \to U$ defined on $I_{\tilde{A}}$ satisfying $\hat{a}(\alpha) \in \tilde{A}_\alpha$ for each $\alpha \in I_{\tilde{A}}$, where $I_{\tilde{A}}$ is given in (7). In this case, we also write $\hat{a} \in \tilde{A}$.

The gradual element $\hat{a}$ in Definition 3.4 is defined on the interval range $I_{\tilde{A}}$ of $\tilde{A}$. Since the interval range $I_{\tilde{A}}$ contains 0, the gradual element $\hat{a}$ must have the assignment at 0. Therefore, the gradual element proposed by Dubois and Prade (2008) cannot be used in Definition 3.4, since the assignment at 0 was not considered by Dubois and Prade (2008).

Example 3.5 Continued from Example 2.3, the α-level set of $\tilde{A}$ is given by

$$\tilde{A}_\alpha = \begin{cases} [1.1, 3.9] & \text{if } 0 \le \alpha < 0.1 \\ [1+\alpha, 4-\alpha] & \text{if } 0.1 \le \alpha \le 0.45 \\ [1.45, 3.55] & \text{if } 0.45 < \alpha < 0.55 \\ [0.9+\alpha, 4.1-\alpha] & \text{if } 0.55 \le \alpha \le 0.9 \\ \emptyset & \text{if } 0.9 < \alpha \le 1 \end{cases} \tag{14}$$

The element $\hat{a} \in \tilde{A}$ must satisfy $\hat{a}(\alpha) \in \tilde{A}_\alpha$ for each $\alpha \in [0, 0.9]$. For example, if we take

$$\hat{a}_1(\alpha) = \begin{cases} 1.1 & \text{if } 0 \le \alpha < 0.1 \\ 1+\alpha & \text{if } 0.1 \le \alpha \le 0.45 \\ 1.45 & \text{if } 0.45 < \alpha < 0.55 \\ 0.9+\alpha & \text{if } 0.55 \le \alpha \le 0.9 \end{cases}$$

$$\text{or } \hat{a}_2(\alpha) = \begin{cases} 3.9 & \text{if } 0 \le \alpha < 0.1 \\ 4-\alpha & \text{if } 0.1 \le \alpha \le 0.45 \\ 3.55 & \text{if } 0.45 < \alpha < 0.55 \\ 4.1-\alpha & \text{if } 0.55 \le \alpha \le 0.9, \end{cases}$$

then $\hat{a}_1, \hat{a}_2 \in \tilde{A}$.

Since $\hat{a} \in \tilde{A}$ is a gradual element in U, this gradual element $\hat{a}$ can also be regarded as a gradual set $\mathfrak{G}$ given by $\mathfrak{G}(\alpha) = \{\hat{a}(\alpha)\}$ that is a singleton set. According to (12), we can induce a fuzzy set $\tilde{A}^{\mathfrak{G}}$ in U with membership function given by

$$\xi_{\tilde{A}^{\mathfrak{G}}}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\mathfrak{G}(\alpha)}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\{\hat{a}(\alpha)\}}(x)$$
$$= \begin{cases} 0, & \text{if there is no } \alpha \in (0,1] \text{ satisfying } \hat{a}(\alpha) = x \\ \sup\limits_{\{\alpha \in (0,1] : \hat{a}(\alpha) = x\}} \alpha, & \text{otherwise.} \end{cases}$$

Therefore, the formal definition of membership function of element $\hat{a} \in \tilde{A}$ is proposed below.

Definition 3.6 Let $\tilde{A}$ be a fuzzy set in a topological space U. Given any $\hat{a} \in \tilde{A}$, the membership function of $\hat{a}$ is defined by

$$\xi_{\hat{a}}(x) = \begin{cases} 0, & \text{if there is no } \alpha \in (0,1] \text{ satisfying } \hat{a}(\alpha) = x \\ \sup\limits_{\{\alpha \in (0,1] : \hat{a}(\alpha) = x\}} \alpha, & \text{otherwise.} \end{cases} \tag{15}$$

Example 3.7 Example 3.5 says that $\hat{a}_1 \in \tilde{A}$. The membership function of $\hat{a}_1$ can be obtained from (15). In particular, for $x = 1.2$, the membership value of $\xi_{\hat{a}_1}(1.2)$ is given by

$$\xi_{\hat{a}_1}(1.2) = \sup_{\{\alpha \in (0,1] : \hat{a}_1(\alpha) = 1.2\}} \alpha = 0.2.$$

We also have

$$\xi_{\hat{a}_1}(1.45) = \sup_{\{\alpha \in (0,1] : \hat{a}_1(\alpha) = 1.45\}} \alpha = \sup[0.45, 0.55] = 0.55.$$

According to Definition 3.4, a fuzzy set $\tilde{A}$ in U can be regarded as a family consisting of elements (i.e., gradual elements) in $\tilde{A}$. Therefore, according to (13), this family $\tilde{A}$ of gradual elements can induce a gradual set $\mathfrak{G}$ given by

$$\mathfrak{G}(\alpha) = \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}\right\} \text{ for } \alpha \in I_{\tilde{A}}, \tag{16}$$

where $I_{\tilde{A}}$ is given in (7). According to (11), the fuzzy set $\tilde{A}$ can induce another gradual set $\mathfrak{G}_{\tilde{A}}$ given by $\mathfrak{G}_{\tilde{A}}(\alpha) = \tilde{A}_\alpha$ for $\alpha \in I_{\tilde{A}}$. On the other hand, according to (12), the gradual set $\mathfrak{G}$ in (16) can also induce another fuzzy set $\tilde{A}^{\mathfrak{G}}$ in U. In order to claim the consistency of Definition 3.4, we need to show $\mathfrak{G}(\alpha) = \mathfrak{G}_{\tilde{A}}(\alpha)$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}^{\mathfrak{G}} = \tilde{A}$, which will be presented below.

Proposition 3.8 *Let $\tilde{A}$ be a fuzzy set in a topological space U. Then, the following statements hold true.*

(i) *The gradual set $\mathfrak{G}$ induced by the family $\tilde{A}$ of gradual elements satisfies*

$$\mathfrak{G}(\alpha) = \tilde{A}_\alpha = \mathfrak{G}_{\tilde{A}}(\alpha) \text{ for each } \alpha \in I_{\tilde{A}},$$

where $I_{\tilde{A}}$ is given in (7).

(ii) *Let $\tilde{A}^{\mathfrak{G}}$ be a fuzzy set in U induced by the gradual set $\mathfrak{G}$ in part (i). Then, $\tilde{A}^{\mathfrak{G}} = \tilde{A}$.*

Proof To prove part (i), according to (16), we see that $\mathfrak{G}(\alpha) \subseteq \tilde{A}_\alpha$ for $\alpha \in I_{\tilde{A}}$. On the other hand, given any fixed $\alpha \in I_{\tilde{A}}$ and any $x \in \tilde{A}_\alpha$, we define a function $\hat{a}$ on $I_{\tilde{A}}$ by

$$\hat{a}(\beta) = \begin{cases} x, & \text{if } \beta = \alpha \\ y \text{ for some } y \in \tilde{A}_\beta, & \text{if } \beta \neq \alpha. \end{cases}$$

Then, it is clear to see that $\hat{a} \in \tilde{A}$. This shows that $\hat{a}(\alpha) = x \in \mathfrak{G}(\alpha)$, i.e., $\tilde{A}_\alpha \subseteq \mathfrak{G}(\alpha)$. Therefore, we obtain $\mathfrak{G}(\alpha) = \tilde{A}_\alpha$ for $\alpha \in I_{\tilde{A}}$.

To prove part (ii), from (12), the membership function of $\tilde{A}^{\mathfrak{G}}$ is given by

$$\xi_{\tilde{A}^{\mathfrak{G}}}(x) = \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\mathfrak{G}(\alpha)}(x).$$

Using Theorem 2.4, the membership function of $\tilde{A}$ can be expressed as

$$\xi_{\tilde{A}}(x) = \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x).$$

Since $\mathfrak{G}(\alpha) = \tilde{A}_\alpha$ for each $\alpha \in I_{\tilde{A}}$ by part (i), we obtain $\tilde{A}^{\mathfrak{G}} = \tilde{A}$ by referring to their membership functions. This completes the proof. □

Example 3.9 Continued from Example 3.5, using (16), we can induce a gradual set $\mathfrak{G}$ given by

$$\mathfrak{G}(\alpha) = \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}\right\} \text{ for } \alpha \in [0, 0.9], \text{ where } \hat{a}(\alpha) \in \tilde{A}_\alpha.$$

By referring to (14), it is clear to see that $\mathfrak{G}(\alpha) = \tilde{A}_\alpha$ for $\alpha \in [0, 0.9]$, which verifies Proposition 3.8 and says that the elements presented in Example 3.5 are well defined.

4 Complement set

Let $\tilde{A}$ be a fuzzy set in U with membership function $\xi_{\tilde{A}}$. Recall that the complement of $\tilde{A}$ is denoted by $\tilde{A}^c$ with membership function defined by

$$\xi_{\tilde{A}^c}(x) = 1 - \xi_{\tilde{A}}(x).$$

The strong α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_{\alpha+} = \left\{x \in U : \xi_{\tilde{A}}(x) > \alpha\right\}$$

for $\alpha \in [0, 1)$. We also recall that $\tilde{A}_{0+}$ is the support of $\tilde{A}$. It is clear that $\tilde{A}_{\alpha+} \subseteq \tilde{A}_\alpha$ for all $\alpha \in (0, 1]$. Then, for $\alpha \in (0, 1]$, the α-level set of $\tilde{A}^c$ is given by

$$\begin{aligned}\tilde{A}^c_\alpha &= \left\{x \in U : \xi_{\tilde{A}^c}(x) \geq \alpha\right\} = \left\{x \in U : 1 - \xi_{\tilde{A}}(x) \geq \alpha\right\}\\ &= \left\{x \in U : \xi_{\tilde{A}}(x) \leq 1 - \alpha\right\} = U \setminus \left\{x \in U : \xi_{\tilde{A}}(x) > 1 - \alpha\right\}\\ &= U \setminus \tilde{A}_{(1-\alpha)+} = \left[\tilde{A}_{(1-\alpha)+}\right]^c. \qquad (17)\end{aligned}$$

We need to remark that $(\tilde{A}_\alpha)^c$ means the complement set of the α-level set $\tilde{A}_\alpha$ of $\tilde{A}$, which is different from the α-level set $\tilde{A}^c_\alpha$ of $\tilde{A}^c$. Moreover, for $\alpha < \beta$, since $\tilde{A}$ and $\tilde{A}^c$ are fuzzy sets in U, the nestedness says that

$$\tilde{A}_{\beta+} \subseteq \tilde{A}_\beta \subseteq \tilde{A}_\alpha \text{ and } \tilde{A}^c_{\beta+} \subseteq \tilde{A}^c_\beta \subseteq \tilde{A}^c_\alpha,$$

which also says that

$$(\tilde{A}_{\beta+})^c \supseteq (\tilde{A}_\beta)^c \supseteq (\tilde{A}_\alpha)^c \text{ and } \tilde{A}^c_{\beta+} \subseteq \tilde{A}^c_\beta \subseteq \tilde{A}^c_\alpha. \qquad (18)$$

Next we shall define the complement of $\tilde{A}$ based on the gradual element.

Let $\tilde{A}$ be a fuzzy set in U with membership function $\xi_{\tilde{A}}$ and interval range $I_{\tilde{A}}$ given in (7). Inspired by (17), we consider the following family of gradual elements

$$\mathcal{A} = \left\{\hat{a} : \hat{a}(\alpha) \in \left[\tilde{A}_{(1-\alpha)+}\right]^c \text{ for all } \alpha \in I_{\tilde{A}}\right\}.$$

According to (13), this family can induce a gradual set $\mathfrak{G}^{\dagger c}$ in U defined on $I_{\tilde{A}}$ by

$$\begin{aligned}\mathfrak{G}^{\dagger c}(\alpha) &= \left\{\hat{a}(\alpha) : \hat{a} \in \mathcal{A}\right\}\\ &= \left\{\hat{a}(\alpha) : \hat{a}(\alpha) \in \left[\tilde{A}_{(1-\alpha)+}\right]^c \text{ for all } \alpha \in I_{\tilde{A}}\right\}.\end{aligned} \qquad (19)$$

According to (12), this gradual set $\mathfrak{G}^{\dagger c}$ can also induce a fuzzy set $\tilde{A}^{\dagger c}$ with membership function given by

$$\xi_{\tilde{A}^{\dagger c}}(x) = \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\mathfrak{G}^{\dagger c}(\alpha)}(x), \qquad (20)$$

where the fuzzy set $\tilde{A}^{\dagger c}$ is defined to be a new type of complement of $\tilde{A}$. We remark that the gradual element $\hat{a}$ in (19) is defined on the interval range $I_{\tilde{A}}$ of $\tilde{A}$, which contains 0. In other words, the gradual element $\hat{a}$ must have the assignment at 0. Therefore, the gradual element proposed by Dubois and Prade (2008) cannot be used in (19), since the assignment at 0 was not considered by Dubois and Prade (2008). Using the assignment at 0 can define the complement of $\tilde{A}$.

We are going to claim $\tilde{A}^{\dagger c} = \tilde{A}^c$, although $\tilde{A}^{\dagger c}$ is based on the gradual element and gradual set and $\tilde{A}^c$ is based on the membership function.

Theorem 4.1 *Let $\tilde{A}$ be a fuzzy set in a topological space U. Then, the gradual set $\mathfrak{G}^{\dagger c}$ induced by the family $\mathcal{A}$ of gradual elements satisfies*

$$\mathfrak{G}^{\dagger c}(\alpha) = \left[\tilde{A}_{(1-\alpha)+}\right]^c = \tilde{A}^c_\alpha \text{ for each } \alpha \in I_{\tilde{A}}.$$

Moreover, we have $\tilde{A}^{\dagger c} = \tilde{A}^c$.

Proof According to (19), we see that $\mathfrak{G}^{\dagger c}(\alpha) \subseteq [\tilde{A}_{(1-\alpha)+}]^c$ for $\alpha \in I_{\tilde{A}}$. On the other hand, given any fixed $\alpha \in I_{\tilde{A}}$ and any $x \in [\tilde{A}_{(1-\alpha)+}]^c$, we define a function $\hat{a}$ on $I_{\tilde{A}}$ by

$$\hat{a}(\beta) = \begin{cases} x, & \text{if } \beta = \alpha \\ y \text{ for some } y \in [\tilde{A}_{(1-\beta)+}]^c, & \text{if } \beta \neq \alpha. \end{cases}$$

Then, it is clear to see that $\hat{a} \in \mathcal{A}$. This shows that $\hat{a}(\alpha) = x \in \mathfrak{G}^{\dagger c}(\alpha)$, i.e., $[\tilde{A}_{(1-\alpha)+}]^c \subseteq \mathfrak{G}^{\dagger c}(\alpha)$. Therefore, we obtain $\mathfrak{G}^{\dagger c}(\alpha) = [\tilde{A}_{(1-\alpha)+}]^c$ for all $\alpha \in I_{\tilde{A}}$. Using (17), we obtain the following equalities

$$\mathfrak{G}^{\dagger c}(\alpha) = \left[\tilde{A}_{(1-\alpha)+}\right]^c = \tilde{A}^c_\alpha \text{ for each } \alpha \in I_{\tilde{A}}.$$

On the other hand, according to Theorem 2.4, the membership function of $\tilde{A}^c$ is given by

$$\xi_{\tilde{A}^c}(x) = \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}^c_\alpha}(x).$$

By referring to the membership function (20), since $\mathfrak{G}^{\dagger c}(\alpha) = \tilde{A}^c_\alpha$, it follows that $\tilde{A}^{\dagger c} = \tilde{A}^c$. This completes the proof. □

5 Intersection and union

Let $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ be fuzzy sets in a topological space U, and let $\alpha_i^* = \sup \mathcal{R}\left(\xi_{\tilde{A}^{(i)}}\right)$. From Proposition 2.2 and (7), we see that $\tilde{A}^{(i)}_\alpha \neq \emptyset$ for $\alpha \in I_i$, where I_i is given by

$$I_i = \begin{cases} [0, \alpha_i^*), & \text{if } \max \mathcal{R}(\xi_{\tilde{A}^{(i)}}) \text{ does not exist} \\ [0, \alpha_i^*], & \text{if } \max \mathcal{R}(\xi_{\tilde{A}^{(i)}}) \text{ exists.} \end{cases} \tag{21}$$

According to Definition 3.4, each $\hat{a}_i \in \tilde{A}^{(i)}$ is a gradual element in U for $i = 1, \ldots, n$. The gradual element $\hat{a}_i : I_i \to U$ is defined on I_i satisfying $\hat{a}_i(\alpha) \in \tilde{A}^{(i)}_\alpha \neq \emptyset$ for each $\alpha \in I_i$.

In the sequel, we assume that $I_i \equiv I$ are all identical for $i = 1, \ldots, n$. Then, $\tilde{A}^{(i)}_\alpha \neq \emptyset$ for all $\alpha \in I$ and $i = 1, \ldots, n$. In particular, if we assume that the fuzzy sets $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ are normal, then $I \equiv I_i = [0, 1]$ for all $i = 1, \ldots, n$.

Let $A_1, \ldots, A_n$ be (crisp) subsets of U. Recall that their intersection is given by

$$A_1 \cap \cdots \cap A_n = \{x \in U : x \in A_i \text{ for all } i = 1, \ldots, n\}.$$

Inspired by the above form, we are going to consider the intersection of $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ using the gradual elements. Now, we consider the following family

$$\left\{\hat{a} : \hat{a} \in \tilde{A}^{(i)} \text{ for all } i = 1, \ldots, n\right\}$$

that consists of all common gradual elements from $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$. Then, this family can induce a gradual set $\mathfrak{G}^\cap$ in U defined on I by

$$\mathfrak{G}^\cap(\alpha) = \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}^{(i)} \text{ for all } i = 1, \ldots, n\right\} \text{ for } \alpha \in I.$$

We can also define the union of $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ using the gradual elements. Recall that the union of (crisp) subsets $A_1, \ldots, A_n$ of U is given by

$$A_1 \cup \cdots \cup A_n = \{x \in U : x \in A_i \text{ for some } i = 1, \ldots, n\}.$$

Now, we consider the following family

$$\left\{\hat{a} : \hat{a} \in \tilde{A}^{(i)} \text{ for some } i = 1, \ldots, n\right\}$$

that consists of all gradual elements taken from some $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$. Then, this family can also induce a gradual set $\mathfrak{G}^\cup$ in U defined on I by

$$\mathfrak{G}^\cup(\alpha) = \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}^{(i)} \text{ for some } i = 1, \ldots, n\right\} \text{ for } \alpha \in I.$$

It is clear to see that

$$\hat{a}(\alpha) \in \mathfrak{G}^\cap(\alpha) \text{ implies } \hat{a}(\alpha) \in \tilde{A}^{(1)}_\alpha \cap \cdots \cap \tilde{A}^{(n)}_\alpha \text{ for } \alpha \in I \tag{22}$$

and

$$\hat{a}(\alpha) \in \mathfrak{G}^\cup(\alpha) \text{ implies } \hat{a}(\alpha) \in \tilde{A}^{(1)}_\alpha \cup \cdots \cup \tilde{A}^{(n)}_\alpha \text{ for } \alpha \in I. \tag{23}$$

Based on these two gradual sets $\mathfrak{G}^\cap$ and $\mathfrak{G}^\cup$, we can induce two fuzzy sets $\tilde{A}^\cap$ and $\tilde{A}^\cup$ in U with membership functions given by

$$\xi_{\tilde{A}^\cap}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\mathfrak{G}^\cap(\alpha)}(x) \text{ and } \xi_{\tilde{A}^\cup}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\mathfrak{G}^\cup(\alpha)}(x). \tag{24}$$

In this case, we define the intersection and union of $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ as follows:

$$\tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)} = \tilde{A}^\cap \text{ and } \tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)} = \tilde{A}^\cup,$$

where the membership functions of $\tilde{A}^\cap$ and $\tilde{A}^\cup$ are given in (24). This kind of intersection and union is based on the concept of gradual elements in fuzzy sets. We remark that the intersection and union of $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ are based on the interval ranges in (21) that contains 0. Therefore, the gradual element proposed by Dubois and Prade (2008) cannot be used in (21) to define the union and intersection, since the assignment at 0 was not considered by Dubois and Prade (2008).

Example 5.1 Let $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ be two normal fuzzy sets with membership functions given by

$$\xi_{\tilde{A}^{(1)}}(x) = \begin{cases} x - 1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } 2 < x < 3 \\ 4 - x & \text{if } 3 \leq x \leq 4 \\ 0 & \text{otherwise,} \end{cases}$$
$$\text{and } \xi_{\tilde{A}^{(2)}}(x) = \begin{cases} x - 2 & \text{if } 2 \leq x \leq 3 \\ 1 & \text{if } 3 < x < 4 \\ 5 - x & \text{if } 4 \leq x \leq 5 \\ 0 & \text{otherwise,} \end{cases}$$

Then, the interval ranges of $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ are identical to $I = [0, 1]$. We also see that the α-level sets of $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ are bounded closed intervals given by

$$\tilde{A}^{(1)}_\alpha = [1 + \alpha, 4 - \alpha] \text{ and } \tilde{A}^{(2)}_\alpha = [2 + \alpha, 5 - \alpha].$$

The gradual element $\hat{a} \in \tilde{A}^{(1)}$ must satisfy $\hat{a}(\alpha) \in \tilde{A}^{(1)}_\alpha$ for each $\alpha \in [0, 1]$. For example, if we take

$$\hat{a}^{(1)}_1(\alpha) = 1 + \alpha \text{ or } \hat{a}^{(1)}_2(\alpha) = 4 - \alpha \text{ for } \alpha \in [0, 1],$$

then $\hat{a}^{(1)}_1, \hat{a}^{(1)}_2 \in \tilde{A}^{(1)}$. Also, if we take

$$\hat{a}^{(2)}_1(\alpha) = 2 + \alpha \text{ or } \hat{a}^{(2)}_2(\alpha) = 5 - \alpha \text{ for } \alpha \in [0, 1],$$

then $\hat{a}^{(2)}_1, \hat{a}^{(2)}_2 \in \tilde{A}^{(2)}$. Now we consider two gradual sets given by

$$\mathfrak{G}^{\cap}(\alpha) = \left\{\hat{a}(\alpha) : \hat{a}\ in \tilde{A}^{(1)} \text{ and } \hat{a} \in \tilde{A}^{(2)}\right\} \text{ for } \alpha \in [0, 1]$$

and

$$\mathfrak{G}^{\cup}(\alpha) = \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}^{(1)} \text{ or } \hat{a} \in \tilde{A}^{(2)}\right\} \text{ for } \alpha \in [0, 1].$$

Then, it is clear to see that

$$\left\{\hat{a}^{(1)}_2(\alpha), \hat{a}^{(2)}_1(\alpha)\right\} \subset \mathfrak{G}^{\cap}(\alpha) \text{ for } \alpha \in [0, 1]$$

and

$$\left\{\hat{a}^{(1)}_1(\alpha), \hat{a}^{(1)}_2(\alpha), \hat{a}^{(2)}_1(\alpha), \hat{a}^{(2)}_2(\alpha)\right\} \subset \mathfrak{G}^{\cup}(\alpha) \text{ for } \alpha \in [0, 1].$$

In general, for $\alpha \in [0, 1]$, we have

$$\begin{aligned}
\mathfrak{G}^{\cap}(\alpha) &= \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}^{(1)} \text{ and } \hat{a} \in \tilde{A}^{(2)}\right\} \\
&= \{\hat{a}(\alpha) : 1 + \alpha \leq \hat{a}(\alpha) \leq 4 - \alpha \\
&\qquad \text{and } 2 + \alpha \leq \hat{a}(\alpha) \leq 5 - \alpha\} \\
&= \{\hat{a}(\alpha) : 2 + \alpha \leq \hat{a}(\alpha) \leq 4 - \alpha\} \\
&= [2 + \alpha, 4 - \alpha] = \tilde{A}^{(1)}_\alpha \cap \tilde{A}^{(2)}_\alpha
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{G}^{\cup}(\alpha) &= \left\{\hat{a}(\alpha) : \hat{a} \in \tilde{A}^{(1)} \text{ or } \hat{a} \in \tilde{A}^{(2)}\right\} \\
&= \{\hat{a}(\alpha) : 1 + \alpha \leq \hat{a}(\alpha) \leq 4 - \alpha \\
&\qquad \text{or } 2 + \alpha \leq \hat{a}(\alpha) \leq 5 - \alpha\} \\
&= \{\hat{a}(\alpha) : 1 + \alpha \leq \hat{a}(\alpha) \leq 5 - \alpha\} \\
&= [1 + \alpha, 5 - \alpha] = \tilde{A}^{(1)}_\alpha \cup \tilde{A}^{(2)}_\alpha.
\end{aligned}$$

According to (24), the membership functions of $\tilde{A}^{(1)} \cap \tilde{A}^{(2)}$ and $\tilde{A}^{(1)} \cup \tilde{A}^{(2)}$ are given by

$$\begin{aligned}
\xi_{\tilde{A}^{(1)} \cap \tilde{A}^{(2)}}(x) &= \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{[2+\alpha, 4-\alpha]}(x) \text{ and } \xi_{\tilde{A}^{(1)} \cup \tilde{A}^{(2)}}(x) \\
&= \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{[1+\alpha, 5-\alpha]}(x).
\end{aligned}$$

The α-level sets of $\tilde{A}^{(1)} \cap \tilde{A}^{(2)}$ and $\tilde{A}^{(1)} \cup \tilde{A}^{(2)}$ will be investigated in the subsequent discussion.

Proposition 5.2 *Let $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ be fuzzy sets in a topological space U such that $I_i \equiv I$ for all $i = 1, \ldots, n$. Then, the following statements hold true.*

(i) *We have*

$$\mathfrak{G}^{\cup}(\alpha) = \tilde{A}^{(1)}_\alpha \cup \cdots \cup \tilde{A}^{(n)}_\alpha \text{ for all } \alpha \in I$$

and

$$\mathfrak{G}^{\cup}(\beta) \subseteq \mathfrak{G}^{\cup}(\alpha) \text{ for } \beta > \alpha.$$

(ii) *Suppose that $\tilde{A}^{(1)}_\alpha \cap \cdots \cap \tilde{A}^{(n)}_\alpha \neq \emptyset$ for all $\alpha \in I$. Then, we have*

$$\mathfrak{G}^{\cap}(\alpha) = \tilde{A}^{(1)}_\alpha \cap \cdots \cap \tilde{A}^{(n)}_\alpha \text{ for all } \alpha \in I$$

and

$$\mathfrak{G}^{\cap}(\beta) \subseteq \mathfrak{G}^{\cap}(\alpha) \text{ for } \beta > \alpha.$$

Proof To prove part (i), for $\alpha \in I$, the inclusion

$$\mathfrak{G}^{\cup}(\alpha) \subseteq \tilde{A}^{(1)}_\alpha \cup \cdots \cup \tilde{A}^{(n)}_\alpha$$

follows from (23). Now, given any fixed $\alpha \in I$ and any $x \in \tilde{A}^{(1)}_\alpha \cup \cdots \cup \tilde{A}^{(n)}_\alpha$, we have $x \in \tilde{A}^{(i)}_\alpha$ for some $i = 1, \ldots, n$. Then, we define a function $\hat{a}$ on I by

$$\hat{a}(\beta) = \begin{cases} x, & \text{if } \beta = \alpha \\ y \text{ for some } y \in \tilde{A}^{(i)}_\beta, & \text{if } \beta \neq \alpha. \end{cases}$$

Then, $\hat{a} \in \tilde{A}^{(i)}$. Therefore, we obtain $x = \hat{a}(\alpha) \in \tilde{A}^{(i)}_\alpha$ with $\hat{a} \in \tilde{A}^{(i)}$ for some $i = 1, \ldots, n$. This says that $x = \hat{a}(\alpha) \in \mathfrak{G}^{\cup}(\alpha)$. Therefore, we obtain the inclusion

$$\tilde{A}^{(1)}_\alpha \cup \cdots \cup \tilde{A}^{(n)}_\alpha \subseteq \mathfrak{G}^{\cup}(\alpha),$$

for all $\alpha \in I$. Using the nestedness of the α-level sets $\tilde{A}^{(i)}_\alpha$ for $i = 1, \ldots, n$, it is clear to see that the gradual set $\mathfrak{G}^{\cup}$ is also nested in the sense of $\mathfrak{G}^{\cup}(\beta) \subseteq \mathfrak{G}^{\cup}(\alpha)$ for $\beta > \alpha$.

To prove part (ii), for $\alpha \in I$, the inclusion

$$\mathfrak{G}^{\cap}(\alpha) \subseteq \tilde{A}^{(1)}_\alpha \cap \cdots \cap \tilde{A}^{(n)}_\alpha$$

follows from (22). Given any fixed $\alpha \in I$ and any $x \in \tilde{A}^{(1)}_\alpha \cap \cdots \cap \tilde{A}^{(n)}_\alpha \neq \emptyset$, we have $x \in \tilde{A}^{(i)}_\alpha$ for all $i = 1, \ldots, n$. Then, we define a function $\hat{a}$ on I by

$$\hat{a}(\beta) = \begin{cases} x, & \text{if } \beta = \alpha \\ y \text{ for some } y \in \tilde{A}^{(1)}_\beta \cap \cdots \cap \tilde{A}^{(n)}_\beta \neq \emptyset, & \text{if } \beta \neq \alpha. \end{cases}$$

Then, $\hat{a} \in \tilde{A}^{(i)}$ for all $i = 1, \ldots, n$. This says that $x = \hat{a}(\alpha) \in \mathfrak{G}^{\cap}(\alpha)$. Therefore, we obtain the inclusion

$$\tilde{A}_{\alpha}^{(1)} \cap \cdots \cap \tilde{A}_{\alpha}^{(n)} \subseteq \mathfrak{G}^{\cap}(\alpha)$$

for all $\alpha \in I$. This shows the desired equality. The nestedness of the gradual set $\mathfrak{G}^{\cap}$ can be similarly realized, and the proof is complete. □

Remark 5.3 Since we assume that $I_i \equiv I$ for all $i = 1, \ldots, n$, by referring to (21) and Remark 3.3, we see that the interval ranges of membership functions $\tilde{A}^{\cap}$ and $\tilde{A}^{\cup}$ are given by

$$I_{\tilde{A}^{\cap}} = I_{\tilde{A}^{\cup}} = I.$$

Using Proposition 2.2, we have $\tilde{A}_{\alpha}^{\cap} \neq \emptyset$ for all $\alpha \in I$ and $\tilde{A}_{\alpha}^{\cap} = \emptyset$ for all $\alpha \notin I$. We also have $\tilde{A}_{\alpha}^{\cup} \neq \emptyset$ for all $\alpha \in I$ and $\tilde{A}_{\alpha}^{\cup} = \emptyset$ for all $\alpha \notin I$.

Now we want to study the α-level sets of $\tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)}$ and $\tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)}$. Let $\tilde{A}$ be a fuzzy set in U with $I_{\tilde{A}}$ given in (7). Given any $\alpha \in I_{\tilde{A}}$ with $\alpha > 0$ and any increasing convergent sequence $\{\alpha_m\}_{m=1}^{\infty}$ in $I_{\tilde{A}}$ with $\alpha_m > 0$ for all m and $\alpha_m \uparrow \alpha$, the basic property of fuzzy set says that

$$\bigcap_{m=1}^{\infty} \tilde{A}_{\alpha_m} = \tilde{A}_{\alpha}, \tag{25}$$

which will be used for the subsequent discussion. We need two useful lemmas.

Lemma 5.4 (Royden 1968) *Let U be a topological space, and let K be a compact subset of U. Let f be a real-valued function defined on U. If f is upper semi-continuous, then f assumes its maximum on a compact subset of U; that is, the supremum is attained in the following sense*

$$\sup_{x \in K} f(x) = \max_{x \in K} f(x).$$

Lemma 5.5 *Let S be a subset of $(0, 1]$. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in a universal set U with membership functions $\xi_{\tilde{A}}$ and $\xi_{\tilde{B}}$, respectively, satisfying $\tilde{A}_{\alpha} = \tilde{B}_{\alpha}$ for all $\alpha \in S$.*

(i) *Suppose that $\mathcal{R}(\xi_{\tilde{A}}) \setminus \{0\} \subseteq S$ and $\mathcal{R}(\xi_{\tilde{B}}) \setminus \{0\} \subseteq S$. Then, $\xi_{\tilde{A}}(x) = 0$ if and only if $\xi_{\tilde{B}}(x) = 0$, i.e., $\xi_{\tilde{A}}(x) > 0$ if and only if $\xi_{\tilde{B}}(x) > 0$.*
(ii) *We consider the following set*

$$U^S = \left\{x \in U : \xi_{\tilde{A}}(x), \xi_{\tilde{B}}(x) \in S \text{ and } \xi_{\tilde{A}}(x) > 0, \xi_{\tilde{B}}(x) > 0\right\}.$$

Then, $\xi_{\tilde{A}}(x) = \xi_{\tilde{B}}(x)$ for any $x \in U^S$. Moreover, if $\mathcal{R}(\xi_{\tilde{A}}) \setminus \{0\} \subseteq S$ and $\mathcal{R}(\xi_{\tilde{B}}) \setminus \{0\} \subseteq S$, then

$$U^S = \left\{x \in U : \xi_{\tilde{A}}(x) > 0\right\} = \left\{x \in U : \xi_{\tilde{B}}(x) > 0\right\} \tag{26}$$

and $\xi_{\tilde{A}} = \xi_{\tilde{B}}$ with $\mathcal{R}(\xi_{\tilde{A}}) = \mathcal{R}(\xi_{\tilde{B}})$.

Proof To prove part (i), if $\xi_{\tilde{A}}(x) = 0$, then $x \notin \tilde{A}_{\alpha}$ for all $\alpha > 0$, which implies $x \notin \tilde{B}_{\alpha}$ for all $\alpha \in \mathcal{R}(\xi_{\tilde{B}}) \setminus \{0\} \subseteq S$, since $\tilde{A}_{\alpha} = \tilde{B}_{\alpha}$ for all $\alpha \in S$. From Theorem 2.4, we have

$$\xi_{\tilde{B}}(x) = \sup_{\alpha \in \mathcal{R}(\xi_{\tilde{B}})} \alpha \cdot \chi_{\tilde{B}_{\alpha}}(x) = \sup_{\mathcal{R}(\xi_{\tilde{B}}) \setminus \{0\}} \alpha \cdot \chi_{\tilde{B}_{\alpha}}(x) = 0.$$

We can similarly prove that $\xi_{\tilde{B}}(x) = 0$ implies $\xi_{\tilde{A}}(x) = 0$.

To prove part (ii), for $x \in U^S$, let $0 < \alpha = \xi_{\tilde{A}}(x)$ and $0 < \beta = \xi_{\tilde{B}}(x)$ with $\alpha, \beta \in S$, i.e., $x \in \tilde{A}_{\alpha}$ and $x \in \tilde{B}_{\beta}$. We consider the following two cases.

- For $x \in \tilde{A}_{\alpha}$, since $\tilde{A}_{\alpha} = \tilde{B}_{\alpha}$, i.e., $x \in \tilde{B}_{\alpha}$, we have $\xi_{\tilde{B}}(x) \geq \alpha$, i.e., $\beta \geq \alpha$.
- For $x \in \tilde{B}_{\beta}$, since $\tilde{A}_{\beta} = \tilde{B}_{\beta}$, i.e., $x \in \tilde{A}_{\beta}$, we also have $\xi_{\tilde{A}}(x) \geq \beta$, i.e., $\alpha \geq \beta$.

Therefore, we obtain $\alpha = \beta$. This shows that $\xi_{\tilde{A}}(x) = \xi_{\tilde{B}}(x)$ for any $x \in U^S$. Using part (i), we can obtain (26) and conclude that $\xi_{\tilde{A}} = \xi_{\tilde{B}}$ with $\mathcal{R}(\xi_{\tilde{A}}) = \mathcal{R}(\xi_{\tilde{B}})$. This completes the proof. □

We say that two fuzzy sets $\tilde{A}$ and $\tilde{B}$ in U are identical, written by $\tilde{A} = \tilde{B}$, if and only if $\xi_{\tilde{A}} = \xi_{\tilde{B}}$, i.e., $\xi_{\tilde{A}}(x) = \xi_{\tilde{B}}(x)$ for all $x \in U$.

Theorem 5.6 *Let $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ be fuzzy sets in a topological space U such that $I_i \equiv I$ for all $i = 1, \ldots, n$, that $\tilde{A}_{\alpha}^{(1)} \cap \cdots \cap \tilde{A}_{\alpha}^{(n)} \neq \emptyset$ for all $\alpha \in I$ and that the maximum $\max I$ exists. Then, the α-level set of $\tilde{A}^{\cap} = \tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)}$ is given by*

$$\begin{aligned}\tilde{A}_{\alpha}^{\cap} &= \left(\tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)}\right)_{\alpha} = \{x \in U : \xi_{\tilde{A}^{\cap}}(x) \geq \alpha\} \\ &= \mathfrak{G}^{\cap}(\alpha) = \tilde{A}_{\alpha}^{(1)} \cap \cdots \cap \tilde{A}_{\alpha}^{(n)}\end{aligned} \tag{27}$$

for every $\alpha \in I$ with $\alpha > 0$, and

$$\begin{aligned}\tilde{A}_{0+}^{\cap} &= \left(\tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)}\right)_{0+} = \bigcup_{\{\alpha \in I : \alpha > 0\}} \left(\tilde{A}^{(1)} \cap \cdots \cap \tilde{A}^{(n)}\right)_{\alpha} \\ &= \bigcup_{\{\alpha \in I : \alpha > 0\}} \mathfrak{G}^{\cap}(\alpha) = \bigcup_{\{\alpha \in I : \alpha > 0\}} \tilde{A}_{\alpha}^{(1)} \cap \cdots \cap \tilde{A}_{\alpha}^{(n)}.\end{aligned} \tag{28}$$

Proof Given any $\alpha \in I$ with $\alpha > 0$ and any increasing convergent sequence $\{\alpha_m\}_{m=1}^{\infty}$ in I with $\alpha_m > 0$ for all m and $\alpha_m \uparrow \alpha$, using (25), we have

$$\bigcap_{m=1}^{\infty}\left(\tilde{A}^{(1)}_{\alpha_m}\cap\cdots\cap\tilde{A}^{(n)}_{\alpha_m}\right)=\left(\bigcap_{m=1}^{\infty}\tilde{A}^{(1)}_{\alpha_m}\right)\cap\cdots\cap\left(\bigcap_{m=1}^{\infty}\tilde{A}^{(n)}_{\alpha_m}\right)$$
$$=\tilde{A}^{(1)}_{\alpha}\cap\cdots\cap\tilde{A}^{(n)}_{\alpha}. \tag{29}$$

From part (ii) of Proposition 5.2, we see that (29) is satisfied if and only if the following equality is satisfied

$$\bigcap_{m=1}^{\infty}\mathfrak{G}^{\cap}(\alpha_m)=\mathfrak{G}^{\cap}(\alpha). \tag{30}$$

We also have

$$\mathfrak{G}^{\cap}(\beta)\subseteq\mathfrak{G}^{\cap}(\alpha)\text{ for }\alpha,\beta\in I\text{ with }\beta>\alpha. \tag{31}$$

Given any fixed $x\in U$, we define the following set

$$F_{\rho}=\left\{\alpha\in I:\xi_{\tilde{A}^{\cap}}(x)=\alpha\cdot\chi_{\mathfrak{G}^{\cap}(\alpha)}(x)\geq\rho\right\}.$$

Since $I_i\equiv I$ for all $i=1,\ldots,n$, from Remark 5.3 and (21), it follows that $I_{\tilde{A}^{\cap}}=I$ is an interval. Since we assume that the maximum $\max I$ exists, Proposition 2.2 says that $I_{\tilde{A}^{\cap}}=I=[0,\alpha^*]$ for some $\alpha^*\in(0,1]$. We are going to claim that the set F_ρ is closed for each $\rho\in\mathbb{R}$. Given any $\rho\in I$ with $\rho>0$, for each $\alpha\in\text{cl}(F_\rho)$, there exists a sequence $\{\alpha_m\}_{m=1}^{\infty}$ in F_ρ such that $\alpha_m\to\alpha$. Therefore, we have $\alpha_m\in I$ with $\alpha_m\geq\rho>0$ and $x\in\mathfrak{G}^{\cap}(\alpha_m)$ for all m, which also says that $\alpha>0$, since

$$\alpha=\lim_{m\to\infty}\alpha_m\geq\rho>0.$$

Therefore, there exists a subsequence $\{\alpha_{m_k}\}_{k=1}^{\infty}$ of $\{\alpha_m\}_{m=1}^{\infty}$ such that $\alpha_{m_k}\uparrow\alpha$ or $\alpha_{m_k}\downarrow\alpha$ as $k\to\infty$.

- Suppose that $\alpha_{m_k}\downarrow\alpha$. Then, $\alpha_{m_k}\geq\alpha$ for all k. This says that $x\in\mathfrak{G}^{\cap}(\alpha_{m_k})\subseteq\mathfrak{G}^{\cap}(\alpha)$ by (31).
- Suppose that $\alpha_{m_k}\uparrow\alpha$. Since $\alpha_{m_k}\in F_\rho$ for each k, it follows that $\alpha\in\text{cl}(F_\rho)$. Since $F_\rho\subseteq I$, i.e., $\text{cl}(F_\rho)\subseteq\text{cl}(I)=I=[0,\alpha^*]$, it says that $\alpha\in I$. Since $x\in\mathfrak{G}^{\cap}(\alpha_{m_k})$ for all k, using (30), it follows that $x\in\mathfrak{G}^{\cap}(\alpha)$.

Therefore, we conclude that $x\in\mathfrak{G}^{\cap}(\alpha)$ for both cases. This also says that $\alpha\cdot\chi_{\mathfrak{G}^{\cap}(\alpha)}(x)\geq\rho$, i.e., $\alpha\in F_\rho$. Therefore, we obtain the inclusion $\text{cl}(F_\rho)\subseteq F_\rho$, which means that F_ρ is closed for each $\rho\in I$ with $\rho>0$. We are going to claim that for any fixed $x\in U$, the function $\eta_x(\alpha)=\alpha\cdot\chi_{\mathfrak{G}^{\cap}(\alpha)}(x)$ is upper semi-continuous on I. It is equivalent to show that the set F_ρ is closed for each $\rho\in\mathbb{R}$. We have shown that F_ρ is closed for each $\rho\in I$ with $\rho>0$. If $\rho\notin I$, then the empty set $F_\rho=\emptyset$ is closed. If $\rho=0$, then $F_\rho=I=[0,\alpha^*]$ is also a closed set. This shows that the function $\eta_x=\alpha\cdot\chi_{\mathfrak{G}^{\cap}(\alpha)}(x)$ is indeed upper semi-continuous on I.

Since $\tilde{A}^{\cap}_{\alpha}\neq\emptyset$ for all $\alpha\in I$ by Remark 5.3, given any fixed $0<\alpha\in I$, suppose that $x\in\tilde{A}^{\cap}_{\alpha}$. Now we also assume that $x\notin\mathfrak{G}^{\cap}(\beta)$ for all $\beta\in I$ with $\alpha\leq\beta$. We want to lead to a contradiction. Under this assumption, we see that $\beta\cdot\chi_{\mathfrak{G}^{\cap}(\beta)}(x)<\alpha$ for all $\beta\in I$. Since $\eta_x(\beta)=\beta\cdot\chi_{\mathfrak{G}^{\cap}(\beta)}(x)$ is upper semi-continuous on the compact set $I=[0,\alpha^*]$, the supremum of function η_x is achieved by Lemma 5.4. This says that

$$\xi_{\tilde{A}^{\cap}}(x)=\sup_{\beta\in I}\eta_x(\beta)=\sup_{\beta\in I}\beta\cdot\chi_{\mathfrak{G}^{\cap}(\beta)}(x)$$
$$=\max_{\beta\in I}\beta\cdot\chi_{\mathfrak{G}^{\cap}(\beta)}(x)=\beta^*\cdot\chi_{\mathfrak{G}(\beta^*)}(x)<\alpha$$

for some $\beta^*\in I$, which violates $x\in\tilde{A}^{\cap}_{\alpha}$. Therefore, there exists $\beta_0\in I$ with $\beta_0\geq\alpha$ such that $x\in\mathfrak{G}^{\cap}(\beta_0)\subseteq\mathfrak{G}^{\cap}(\alpha)$ by part (ii) of Proposition 5.2, which shows the inclusion $\tilde{A}^{\cap}_{\alpha}\subseteq\mathfrak{G}^{\cap}(\alpha)$. The following inclusion is obvious

$$\mathfrak{G}^{\cap}(\alpha)\subseteq\left\{x\in\mathbb{R}:\sup_{\beta\in I}\beta\cdot\chi_{\mathfrak{G}^{\cap}(\beta)}(x)\geq\alpha\right\}$$
$$=\{x\in\mathbb{R}:\xi_{\tilde{A}^{\cap}}(x)\geq\alpha\}=\tilde{A}^{\cap}_{\alpha}.$$

Therefore, we obtain the desired equalities (27) and (28). This completes the proof. □

Example 5.7 Continued from Example 5.1, Theorem 5.6 says that the α-level sets of $\tilde{A}^{(1)}\cap\tilde{A}^{(2)}$ are given by

$$\left(\tilde{A}^{(1)}\cap\tilde{A}^{(2)}\right)_{\alpha}=\mathfrak{G}^{\cap}(\alpha)=\tilde{A}^{(1)}_{\alpha}\cap\tilde{A}^{(2)}_{\alpha}=[2+\alpha,4-\alpha].$$

Theorem 5.8 *Let $\tilde{A}^{(1)},\ldots,\tilde{A}^{(n)}$ be fuzzy sets in a topological space U such that $I_i\equiv I$ for all $i=1,\ldots,n$ and that the maximum $\max I$ exists. Suppose that any one of the following conditions is satisfied.*

(a) *For any fixed $x\in U$, the function $\eta_x(\alpha)=\alpha\cdot\chi_{\mathfrak{G}^{\cup}(\alpha)}(x)$ is upper semi-continuous on I.*
(b) *Given any $\alpha\in I$ with $\alpha>0$ and any increasing convergent sequence $\{\alpha_m\}_{m=1}^{\infty}$ in I with $\alpha_m>0$ for all m and $\alpha_m\uparrow\alpha$, the following inclusion is satisfied*

$$\bigcap_{m=1}^{\infty}\left(\tilde{A}^{(1)}_{\alpha_m}\cup\cdots\cup\tilde{A}^{(n)}_{\alpha_m}\right)\subseteq\tilde{A}^{(1)}_{\alpha}\cup\cdots\cup\tilde{A}^{(n)}_{\alpha}. \tag{32}$$

(c) *Given any increasing sequence $\{\alpha_m\}_{m=1}^{\infty}$ in I with $\alpha_m>0$ for all m, the following inclusion is satisfied*

$$\bigcap_{m=1}^{\infty}\left(\tilde{A}^{(1)}_{\alpha_m}\cup\cdots\cup\tilde{A}^{(n)}_{\alpha_m}\right)\subseteq\left(\bigcap_{m=1}^{\infty}\tilde{A}^{(1)}_{\alpha_m}\right)\cup\cdots\cup\left(\bigcap_{m=1}^{\infty}\tilde{A}^{(n)}_{\alpha_m}\right).$$

Then, the α-level set of $\tilde{A}^{\cup} = \tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)}$ is given by

$$\tilde{A}^{\cup}_{\alpha} = \left(\tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)}\right)_{\alpha} = \{x \in U : \xi_{\tilde{A}^{\cup}}(x) \geq \alpha\} = \mathfrak{G}^{\cup}(\alpha) = \tilde{A}^{(1)}_{\alpha} \cup \cdots \cup \tilde{A}^{(n)}_{\alpha} \quad (33)$$

for every $\alpha \in I$ with $\alpha > 0$, and

$$\tilde{A}^{\cup}_{0+} = \left(\tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)}\right)_{0+} = \bigcup_{\{\alpha \in I : \alpha > 0\}} \left(\tilde{A}^{(1)} \cup \cdots \cup \tilde{A}^{(n)}\right)_{\alpha} = \bigcup_{\{\alpha \in I : \alpha > 0\}} \mathfrak{G}^{\cup}(\alpha) = \bigcup_{\{\alpha \in I : \alpha > 0\}} \tilde{A}^{(1)}_{\alpha} \cup \cdots \cup \tilde{A}^{(n)}_{\alpha}. \quad (34)$$

Proof Suppose that condition (a) is satisfied. Since $\tilde{A}^{\cup}_{\alpha} \neq \emptyset$ for all $\alpha \in I$ by Remark 5.3, the arguments in the proof of Theorem 5.6 are still valid to obtain the desired equalities (33) and (34) by considering the upper semi-continuity of η_x.

Suppose that condition (b) is satisfied. From part (i) of Proposition 5.2, we see that the inclusion (32) is satisfied if and only if the following inclusion is satisfied

$$\bigcap_{m=1}^{\infty} \mathfrak{G}^{\cup}(\alpha_m) \subseteq \mathfrak{G}^{\cup}(\alpha). \quad (35)$$

We also have $\mathfrak{G}^{\cup}(\beta) \subseteq \mathfrak{G}^{\cup}(\alpha)$ for $\alpha, \beta \in I$. From the proof of Theorem 5.6, when the equality (30) is replaced by the following inclusion

$$\bigcap_{m=1}^{\infty} \mathfrak{G}^{\cup}(\alpha_m) \subseteq \mathfrak{G}^{\cup}(\alpha), \quad (36)$$

the same results can still be obtained. Therefore, by changing the role of (30) as (36) and using the arguments in the proof of Theorem 5.6, we can also show that the function $\eta_x(\alpha) = \alpha \cdot \chi_{\mathfrak{G}^{\cup}(\alpha)}(x)$ is upper semi-continuous on I. Therefore, using condition (a), the desired results can be obtained.

Suppose that condition (c) is satisfied. Then, from (25), it is clear to see that condition (b) is satisfied. Therefore, we also have the desired results, and the proof is complete.

Example 5.9 Continued from Example 5.1, we are going to apply Theorem 5.8 to obtain the α-level sets of $\tilde{A}^{(1)} \cup \tilde{A}^{(2)}$. We shall check that the inclusion (32) in condition (b) will be satisfied. For $\alpha_m \uparrow \alpha$, since $\tilde{A}^{(1)}_{\alpha} \cup \tilde{A}^{(2)}_{\alpha} = [1+\alpha, 5-\alpha]$, we need to claim the inclusion

$$\bigcap_{m=1}^{\infty} [1+\alpha_m, 5-\alpha_m] \subseteq [1+\alpha, 5-\alpha].$$

Given x satisfying $1+\alpha_m \leq x \leq 5-\alpha_m$ for all $m = 1, 2, \ldots$, by taking $m \to \infty$, we obtain $1+\alpha \leq x \leq 5-\alpha$, which proves the desired inclusion. Therefore, Theorem 5.8 says that

$$\left(\tilde{A}^{(1)} \cup \tilde{A}^{(2)}\right)_{\alpha} = \mathfrak{G}^{\cup}(\alpha) = \tilde{A}^{(1)}_{\alpha} \cup \tilde{A}^{(2)}_{\alpha} = [1+\alpha, 5-\alpha].$$

By referring to (32), we remark that the following inclusion

$$\left(\bigcap_{m=1}^{\infty} \tilde{A}^{(1)}_{\alpha_m}\right) \cup \cdots \cup \left(\bigcap_{m=1}^{\infty} \tilde{A}^{(n)}_{\alpha_m}\right) \subseteq \bigcap_{m=1}^{\infty} \left(\tilde{A}^{(1)}_{\alpha_m} \cup \cdots \cup \tilde{A}^{(n)}_{\alpha_m}\right).$$

is satisfied automatically, which is the reversed direction of inclusion.

6 Associativity

Now we are going to study the associativity. Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy sets in a topological space U with the interval ranges $I_{\tilde{A}}$, $I_{\tilde{B}}$ and I_C, respectively. In order to claim the following equalities

$$\left(\tilde{A} \cup \tilde{B}\right) \cup \tilde{C} = \tilde{A} \cup \left(\tilde{B} \cup \tilde{C}\right) = \tilde{A} \cup \tilde{B} \cup \tilde{C},$$

some sufficient conditions are needed.

Proposition 6.1 *Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy sets in a topological space U such that $I_{\tilde{A}} = I_{\tilde{B}} = I_C \equiv I$ are all identical and that the maximum* max I *exists. Suppose that any one of the following conditions is satisfied.*

- *For any fixed $x \in U$, the functions $\eta^{(1)}_x(\alpha) = \alpha \cdot \chi_{\tilde{A}_{\alpha} \cup \tilde{B}_{\alpha}}(x)$ and $\eta^{(2)}_x(\alpha) = \alpha \cdot \chi_{\tilde{B}_{\alpha} \cup \tilde{C}_{\alpha}}(x)$ are upper semi-continuous on I.*
- *Given any $\alpha \in I$ with $\alpha > 0$ and any increasing convergent sequence $\{\alpha_m\}_{m=1}^{\infty}$ in I with $\alpha_m > 0$ for all m and $\alpha_m \uparrow \alpha$, the following inclusions*

$$\bigcap_{m=1}^{\infty} \left(\tilde{A}_{\alpha_m} \cup \tilde{B}_{\alpha_m}\right) \subseteq \tilde{A}_{\alpha} \cup \tilde{B}_{\alpha}$$

$$\text{and} \bigcap_{m=1}^{\infty} \left(\tilde{B}_{\alpha_m} \cup \tilde{C}_{\alpha_m}\right) \subseteq \tilde{B}_{\alpha} \cup \tilde{C}_{\alpha}$$

are satisfied

- *Given any increasing sequence $\{\alpha_m\}_{m=1}^{\infty}$ in I with $\alpha_m > 0$ for all m, the following inclusions*

$$\bigcap_{m=1}^{\infty} \left(\tilde{A}_{\alpha_m} \cup \tilde{B}_{\alpha_m}\right) \subseteq \left(\bigcap_{m=1}^{\infty} \tilde{A}_{\alpha_m}\right) \cup \left(\bigcap_{m=1}^{\infty} \tilde{B}_{\alpha_m}\right)$$

and

$$\bigcap_{m=1}^{\infty}\left(\tilde{B}_{\alpha_m}\cup\tilde{C}_{\alpha_m}\right)\subseteq\left(\bigcap_{m=1}^{\infty}\tilde{B}_{\alpha_m}\right)\cup\left(\bigcap_{m=1}^{\infty}\tilde{C}_{\alpha_m}\right)$$

are satisfied. Then, we have

$$\left(\tilde{A}\cup\tilde{B}\right)\cup\tilde{C}=\tilde{A}\cup\left(\tilde{B}\cup\tilde{C}\right)=\tilde{A}\cup\tilde{B}\cup\tilde{C}.$$

Proof Let $\tilde{D}^{(1)}=\tilde{A}\cup\tilde{B}$ and $\tilde{D}^{(2)}=\tilde{B}\cup\tilde{C}$. Considering the function $\eta_x^{(1)}$ and using Theorem 5.8, we have

$$\tilde{D}_\alpha^{(1)}=\tilde{A}_\alpha\cup\tilde{B}_\alpha \text{ for all } \alpha\in I \text{ with } \alpha>0. \tag{37}$$

Similarly, considering the function $\eta_x^{(2)}$, we also have

$$\tilde{D}_\alpha^{(2)}=\tilde{B}_\alpha\cup\tilde{C}_\alpha \text{ for all } \alpha\in I \text{ with } \alpha>0. \tag{38}$$

Let $\tilde{E}^{(1)}=\tilde{D}^{(1)}\cup\tilde{C}$ and $\tilde{E}^{(2)}=\tilde{A}\cup\tilde{D}^{(2)}$. Then, the membership functions of $\tilde{E}^{(1)}$ and $\tilde{E}^{(2)}$ are given by

$$\begin{aligned}&\xi_{\tilde{E}^{(1)}}(x)\\&\quad=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{D}_\alpha^{(1)}\cup\tilde{C}_\alpha}(x)\text{(using (24) and part(i) of Proposition 5.2)}\\&\quad=\sup_{\alpha\in I}\alpha\cdot\chi_{(\tilde{A}_\alpha\cup\tilde{B}_\alpha)\cup\tilde{C}_\alpha}(x)\text{(using (37))}\\&\quad=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cup\tilde{B}_\alpha\cup\tilde{C}_\alpha}(x)\end{aligned}$$

and

$$\begin{aligned}&\xi_{\tilde{E}^{(2)}}(x)\\&\quad=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cup\tilde{D}_\alpha^{(2)}}(x)=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cup(\tilde{B}_\alpha\cup\tilde{C}_\alpha)}(x)\text{(using (38))}\\&\quad=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cup\tilde{B}_\alpha\cup\tilde{C}_\alpha}(x).\end{aligned}$$

Let $\tilde{E}\equiv\tilde{A}\cup\tilde{B}\cup\tilde{C}$. Using (24) and part (i) of Proposition 5.2, we see that the membership function of $\tilde{E}$ is given by

$$\xi_{\tilde{E}}(x)=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cup\tilde{B}_\alpha\cup\tilde{C}_\alpha}(x).$$

Therefore, we obtain $\tilde{E}^{(1)}=\tilde{E}^{(2)}=\tilde{E}$. This completes the proof. □

Example 6.2 Continued from Example 5.1, let $\tilde{A}^{(3)}$ be another fuzzy set with membership function given by

$$\xi_{\tilde{A}^{(3)}}(x)=\begin{cases}x-3 & \text{if } 3\le x\le 4\\ 1 & \text{if } 4<x<5\\ 6-x & \text{if } 5\le x\le 6\\ 0 & \text{otherwise,}\end{cases}$$

The α-level sets of $\tilde{A}^{(3)}$ are given by $\tilde{A}_\alpha^{(3)}=[3+\alpha,6-\alpha]$. The second condition in Proposition 6.1 is satisfied, which can be realized from the similar argument of Example 5.9. Therefore, Proposition 6.1 says that

$$\begin{aligned}\left(\tilde{A}^{(1)}\cup\tilde{A}^{(2)}\right)\cup\tilde{A}^{(3)}&=\tilde{A}^{(1)}\cup\left(\tilde{A}^{(2)}\cup\tilde{A}^{(3)}\right)\\&=\tilde{A}^{(1)}\cup\tilde{A}^{(2)}\cup\tilde{A}^{(3)}.\end{aligned}$$

The membership function is given by

$$\xi_{\tilde{A}^{(1)}\cup\tilde{A}^{(2)}\cup\tilde{A}^{(3)}}(x)=\sup_{\alpha\in[0,1]}\alpha\cdot\chi_{[1+\alpha,6-\alpha]}(x).$$

For guaranteeing the following equalities

$$\left(\tilde{A}\cap\tilde{B}\right)\cap\tilde{C}=\tilde{A}\cap\left(\tilde{B}\cap\tilde{C}\right)=\tilde{A}\cap\tilde{B}\cap\tilde{C},$$

we do not need any extra sufficient conditions.

Proposition 6.3 *Let $\tilde{A},\tilde{B},\tilde{C}$ be fuzzy sets in a topological space U such that $I_{\tilde{A}}=I_{\tilde{B}}=I_C\equiv I$ are all identical, that $\tilde{A}_\alpha\cap\tilde{B}_\alpha\cap\tilde{C}_\alpha\neq\emptyset$ for all $\alpha\in I$ and that the maximum* $\max I$ *exists. Then, we have*

$$\left(\tilde{A}\cap\tilde{B}\right)\cap\tilde{C}=\tilde{A}\cap\left(\tilde{B}\cap\tilde{C}\right)=\tilde{A}\cap\tilde{B}\cap\tilde{C}. \tag{39}$$

Proof Let $\tilde{D}^{(1)}=\tilde{A}\cap\tilde{B}$ and $\tilde{D}^{(2)}=\tilde{B}\cap\tilde{C}$. Using Theorem 5.6, we have

$$\tilde{D}_\alpha^{(1)}=\tilde{A}_\alpha\cap\tilde{B}_\alpha \text{ and } \tilde{D}_\alpha^{(2)}=\tilde{B}_\alpha\cap\tilde{C}_\alpha \text{ for all } \alpha\in I \text{ with } \alpha>0.$$

Let $\tilde{E}^{(1)}=\tilde{D}^{(1)}\cap\tilde{C}$ and $\tilde{E}^{(2)}=\tilde{A}\cap\tilde{D}^{(2)}$. Then, the membership functions of $\tilde{E}^{(1)}$ and $\tilde{E}^{(2)}$ are given by

$$\begin{aligned}\xi_{\tilde{E}^{(1)}}(x)&=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{D}_\alpha^{(1)}\cap\tilde{C}_\alpha}(x)=\sup_{\alpha\in I}\alpha\cdot\chi_{(\tilde{A}_\alpha\cap\tilde{B}_\alpha)\cap\tilde{C}_\alpha}(x)\\&=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cap\tilde{B}_\alpha\cap\tilde{C}_\alpha}(x)\end{aligned}$$

and

$$\begin{aligned}\xi_{\tilde{E}^{(2)}}(x)&=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cap\tilde{D}_\alpha^{(2)}}(x)=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cap(\tilde{B}_\alpha\cap\tilde{C}_\alpha)}(x)\\&=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cap\tilde{B}_\alpha\cap\tilde{C}_\alpha}(x).\end{aligned}$$

Let $\tilde{E}\equiv\tilde{A}\cap\tilde{B}\cap\tilde{C}$. Using (24) and part (ii) of Proposition 5.2, we see that the membership function of $\tilde{E}$ is given by

$$\xi_{\tilde{E}}(x)=\sup_{\alpha\in I}\alpha\cdot\chi_{\tilde{A}_\alpha\cap\tilde{B}_\alpha\cap\tilde{C}_\alpha}(x).$$

Therefore, we obtain $\tilde{E}^{(1)}=\tilde{E}^{(2)}=\tilde{E}$. This completes the proof. □

Propositions 6.1 and 6.3 can be inductively extended by considering the fuzzy sets $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(n)}$ in U. We also remark that if $\tilde{A}, \tilde{B}, \tilde{C}$ are assumed to be normal fuzzy sets, then

$$I_{\tilde{A}} = I_{\tilde{B}} = I_C = [0, 1] = I$$

and the maximum $\max I = 1$ exists. In this case, the associativity in (39) holds true. This also says that the sufficient conditions in Proposition 6.3 are not so strong.

Now we want to consider the mixed set operations by including the parentheses. For example, we consider the following expression

$$\tilde{A} \equiv \left(\left(\tilde{A}^{(1)} \cap \tilde{A}^{(2)} \cap \tilde{A}^{(3)}\right) \cup \left(\tilde{A}^{(4)} \cap \tilde{A}^{(5)}\right)\right) \cap \left(\tilde{A}^{(6)} \cup \tilde{A}^{(7)} \cup \tilde{A}^{(8)} \cup \tilde{A}^{(9)}\right).$$

We also assume that $I_i \equiv I$ are all identical for $i = 1, \ldots, 9$. Let $\tilde{B}^{(1)} \equiv \tilde{A}^{(1)} \cap \tilde{A}^{(2)} \cap \tilde{A}^{(3)}$, $\tilde{B}^{(2)} \equiv \tilde{A}^{(4)} \cap \tilde{A}^{(5)}$, $\tilde{B}^{(3)} \equiv \tilde{A}^{(6)} \cup \tilde{A}^{(7)} \cup \tilde{A}^{(8)} \cup \tilde{A}^{(9)}$ and $\tilde{B}^{(4)} = \tilde{B}^{(1)} \cup \tilde{B}^{(2)}$. Now we perform the following operations.

- Using (24) and part (ii) of Proposition 5.2, the membership function of $\tilde{B}^{(1)}$ is given by

  $$\xi_{\tilde{B}^{(1)}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\tilde{A}_\alpha^{(1)} \cap \tilde{A}_\alpha^{(2)} \cap \tilde{A}_\alpha^{(3)}}(x).$$

  Suppose that the conditions in Theorem 5.6 are satisfied. Then,

  $$\tilde{B}_\alpha^{(1)} = \tilde{A}_\alpha^{(1)} \cap \tilde{A}_\alpha^{(2)} \cap \tilde{A}_\alpha^{(3)} \text{ for all } \alpha \in I \text{ with } \alpha > 0.$$

- Using (24) and part (ii) of Proposition 5.2, the membership function of $\tilde{B}^{(2)}$ is given by

  $$\xi_{\tilde{B}^{(2)}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\tilde{A}_\alpha^{(4)} \cap \tilde{A}_\alpha^{(5)}}(x).$$

  Suppose that the conditions in Theorem 5.6 are satisfied. Then,

  $$\tilde{B}_\alpha^{(2)} = \tilde{A}_\alpha^{(4)} \cap \tilde{A}_\alpha^{(5)} \text{ for all } \alpha \in I \text{ with } \alpha > 0.$$

- Using (24) and part (i) of Proposition 5.2, the membership function of $\tilde{B}^{(3)}$ is given by

  $$\xi_{\tilde{B}^{(3)}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\tilde{A}_\alpha^{(6)} \cup \tilde{A}_\alpha^{(7)} \cup \tilde{A}_\alpha^{8} \cup \tilde{A}_\alpha^{9}}(x).$$

  Suppose that the conditions in Theorem 5.8 are satisfied. Then,

  $$\tilde{B}_\alpha^{(3)} = \tilde{A}_\alpha^{(6)} \cup \tilde{A}_\alpha^{(7)} \cup \tilde{A}_\alpha^{(8)} \cup \tilde{A}_\alpha^{(9)} \text{ for all } \alpha \in I \text{ with } \alpha > 0.$$

- Using (24) and part (i) of Proposition 5.2, the membership function of $\tilde{B}^{(4)}$ is given by

  $$\xi_{\tilde{B}^{(4)}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\tilde{B}_\alpha^{(1)} \cup \tilde{B}_\alpha^{(2)}}(x).$$

  Suppose that the conditions regarding $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$ in Theorem 5.8 are satisfied. Then,

  $$\tilde{B}_\alpha^{(4)} = \tilde{B}_\alpha^{(1)} \cup \tilde{B}_\alpha^{(2)} \text{ for all } \alpha \in I \text{ with } \alpha > 0.$$

- Finally, using (24) and part (ii) of Proposition 5.2, the membership function of $\tilde{A}$ is given by

  $$\xi_{\tilde{A}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\tilde{B}_\alpha^{(4)} \cap \tilde{B}_\alpha^{(5)}}(x).$$

  Suppose that the conditions regarding $\tilde{B}^{(4)}$ and $\tilde{B}^{(5)}$ in Theorem 5.6 are satisfied. Then,

  $$\tilde{A}_\alpha = \tilde{B}_\alpha^{(4)} \cap \tilde{B}_\alpha^{(5)} \text{ for all } \alpha \in I \text{ with } \alpha > 0.$$

  For $\alpha \in I$ with $\alpha > 0$, we define

  $$\mathfrak{G}_\alpha \equiv \left(\left(\tilde{A}_\alpha^{(1)} \cap \tilde{A}_\alpha^{(2)} \cap \tilde{A}_\alpha^{(3)}\right) \cup \left(\tilde{A}_\alpha^{(4)} \cap \tilde{A}_\alpha^{(5)}\right)\right) \cap \left(\tilde{A}_\alpha^{(6)} \cup \tilde{A}_\alpha^{(7)} \cup \tilde{A}_\alpha^{(8)} \cup \tilde{A}_\alpha^{(9)}\right).$$

  Then, the membership function of $\tilde{A}$ is given by

  $$\xi_{\tilde{A}}(x) = \sup_{\alpha \in I} \alpha \cdot \chi_{\mathfrak{G}_\alpha}(x).$$

Now we consider the following expression

$$\tilde{A}^\circ \equiv \left(\left(\tilde{A}^{(1)} \cap \tilde{A}^{(2)} \cap \tilde{A}^{(3)}\right) \cup \left(\tilde{A}^{(4)} \cap \tilde{A}^{(5)}\right)\right) \cap \left(\tilde{A}^{(6)} \cup \left(\tilde{A}^{(7)} \cup \tilde{A}^{(8)}\right) \cup \tilde{A}^{(9)}\right).$$

Then, $\tilde{A} \neq \tilde{A}^\circ$ in general. However, if the similar conditions regarding $\tilde{A}^{(6)}, \tilde{A}^{(7)}, \tilde{A}^{(8)}, \tilde{A}^{(9)}$ in Proposition 6.1 are satisfied, then $\tilde{A} = \tilde{A}^\circ$.

Compliance with ethical standards

Conflict of interests The author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

References

Boukezzoula R, Galichet S, Foulloy L, Elmasry M (2014) Extended gradual interval arithmetic and its application to gradual weighted averages. Fuzzy Sets Syst 257:67–84

Dubois D, Prade H (1985) A review of fuzzy set aggregation connectives. Inf Sci 36:85–121

Dubois D, Prade H (1988) Possibility theory. Springer, New York

Dubois D, Prade H (2008) Gradual elements in a fuzzy set. Soft Comput 12:165–175

Dubois D, Prade H (2012) Gradualness, uncertainty and bipolarity: making sense of fuzzy sets. Fuzzy Sets Syst 192:3–24

Fortin J, Dubois D, Prade H (2008) Gradual numbers and their application to fuzzy interval analysis. IEEE Trans Fuzzy Syst 16:388–402

Fullér R, Keresztfalvi T (1990) On generalization of Nguyen's theorem. Fuzzy Sets Syst 41:371–374

Goetschel RH (1997) Representations with fuzzy darts. Fuzzy Sets Syst 89:77–105

Herencia J, Lamata M (1999) A total order for the graded numbers used in decision problems. Int J Uncertain Fuzziness Knowl-Based Syst 7:267–276

Klement EP (1982) Operations on fuzzy sets - an axiomatic approach. Fuzzy Sets Syst 27:221–232

Klir GJ, Yuan B (1995) Fuzzy sets and fuzzy logic: theory and applications. Prentice-Hall, New York

Negoita CV, Ralescu DA (1975) Representation theorems for fuzzy concepts. Kybernetes 4:169–174

Nguyen HT (1978) a note on the extension principle for fuzzy sets. J Math Anal Appl 64:369–380

Ralescu DA (1992) A generalization of the representation theorem. Fuzzy Sets Syst 51:309–311

Royden HL (1968) Real Anal, 2nd edn. Macmillan, New York

Sanchez D, Delgado M, Vila MA, Chamorro-Martinez J (2012) On a non-nested level-based representation of fuzziness. Fuzzy Sets Syst 92:159–175

Tan SK, Wang PZ, Lee ES (1993) Fuzzy set operations based on the theory of falling shadows. J Math Anal Appl 174:242–255

Weber S (1983) A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. Fuzzy Sets Syst 11:115–134

Wu HC (2010) Generalized extension principle. Fuzzy Optim Decis Mak 9:31–68

Yager RR (1980) On a general class of fuzzy connectives. Fuzzy Sets Syst 4:235–242

Yager RR (1991) Non-monotonic set theoretic operations. Fuzzy Sets Syst 42:173–190