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METHODOLOGIES AND APPLICATION

A novel distance-based multiple attribute decision-making with hesitant fuzzy sets

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Abstract

Up to now, various types of distance measures have been developed and investigated in-depth for hesitant fuzzy sets (HFSs). The analytical study of the existing distance measures for HFSs shows that they have still some limitations. In an attempt to overcome the limitations, this study develops a class of Hausdorff-based distances to measure the distance among HFSs which are not restricted to the same length of their hesitant fuzzy elements (HFEs) and of course the arranging order of values in the HFEs. Furthermore, these HFS distance measures do satisfy all well-known and essential axioms, specially, the triangle inequality property. Eventually, we present some examples to illustrate the efficiency of the new developed HFS distance measures together with a comparative analysis with other existing ones.

Keywords Hesitant fuzzy set · Distance measure · Multiple criteria decision-making

1 Introduction

Cognitive information has been recently paid attention as a focal topic related to the decision-making literature. For instance, Farhadinia and Xu (2017) introduced ordered weighted hesitant fuzzy sets in order to characterize cognitive information. Meng et al. (2016) proposed linguistic interval hesitant fuzzy sets for deriving cognitive information by emphasizing on the application of decision-making process. Zhao et al. (2016) implemented the concept of dual hesitant fuzzy preferences for extracting cognitive information. Moreover, Liu and Tang (2016) indicated that interval neutrosophic uncertain linguistic variables are able to be used in handling the uncertainty in the cognitive processes. The other contributions focus on the subject of linguistic variables in decision-making context are those presented by Dong et al. (2015), Wu et al. (2018) and Li et al. (2018).

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However, there exist recently a growing number of studies that are focused on the distance and interchangeably on the similarity measures for hesitant fuzzy sets (HFSs) (Farhadinia 2013b, 2014a, b, 2016; Farhadinia and Herrera-Viedma 2018; Farhadinia and Xu 2017, 2019; Li et al. 2015a, b; Tang et al. 2018), and of course for some extensions of HFSs (Farhadinia and Herrera-Viedma 2018; Peng et al. 2013; Rodriguez et al. 2016; Xu 2012). On the basis of the fact that a distance measure can be transformed to a similarity measure and vice versa (Farhadinia 2013a), we here only deal with the distance measures for HFSs.

Needless to say that the distance measures are fundamentally important in various fields such as decision-making, market prediction, pattern recognition and distance-based consensus in multiple criteria decision-making (MCDM) (Cabrerizo et al. 2017; Moral et al. 2018).

The first attempt was made for extending the theory of information measures for HFSs by investigating a connection between distance measures and similarity measures (Xu and Xia 2011). Then, different kinds of distance measures for HFSs including hesitant ordered weighted distance measures, the generalized hesitant fuzzy weighted distance measure, the generalized hesitant fuzzy ordered weighted distance measure and the generalized form of hesitant fuzzy synergetic weighted distance measure (Peng et al. 2013) were introduced and used in developing the methods dealing with MCDM (Zhou and Li 2012). In the sequel works, the topic of

HFS distance measures was enhanced by taking the concept of hesitancy degree of HFS into account (Li et al. 2015a, b; Zeng et al. 2016; Zhang and Xu 2015). This concept is used actually to describe the hesitancy feature of the decision makers in a decision-making procedure. Subsequently, a class of HFS distance measures was constructed without judging about the decision makers' risk preference, and furthermore, the involved HFSs do not need to be re-considered by adding any element into the corresponding smaller HFEs (Hu et al. 2016). However, this attempt is continued and developed by relaxing the two assumptions of equalizing the length of involved HFEs and of arranging increasingly or decreasingly the elements of the HFEs (Tang et al. 2018).

Although the existing distance measures for HFSs may well offer some advantages, but they are still subject to a number of limitations: they should be re-ordered beforehand by the help of arranging in an increasing or decreasing order together with making equal length of involved HFEs. Often, the process of making HFEs with equal length is performed by adding several artificial elements in that HFE with shorter length. In addition to the mentioned shortcomings, some of distance measures for HFSs do not satisfy the conventional axiom, known as, triangle inequality property. The above-mentioned analysis provides extra motivation to develop further the study of HFS distance measures.

The present paper is organized as follows: Firstly, a brief overview of HFS is given in Sect. 2, and we then present a through discussion on the existing HFS distance measures by emphasizing on their shortcomings. Section 3 is devoted to introducing a variety of novel distance measures for HFSs on the basis of the Hausdorff metric concept. In Sect. 4, we apply the proposed HFS distance measures to a MCDM problem in order to demonstrate the applicability of the proposed measures. Finally, the conclusion is drawn in Sect. 5.

2 Discussion on the existing distance measures for HFSs

In this section, we first provide a brief overview of the concept of hesitant fuzzy set (HFS) (Torra 2010) that usually plays a basic role in the case where there exist some difficulties in determining the membership for an element to a set.

Definition 2.1 (Torra 2010) In the case where X stands for the reference set, we define a hesitant fuzzy set (HFS) on X in terms of a function that when it is applied to X, it returns a subset of [0, 1].

In the light of Torra's (2010) HFS definition, Xia and Xu (2011) represented the following mathematical form of HFS for a better understanding:

$$A = \{\langle x, h_A(x)\rangle : x \in X\},$$

where $h_A(x)$ indicates all possible membership degrees of $x \in X$ belonging to the set A. Moreover, the set $h_A(x)$ is called hesitant fuzzy element (HFE) of A.

Before investigation of the existing distance measures for HFSs, let us present a general form of HFE arithmetic operations (Xu 2012) in which the number of elements included in HFEs is not considered to be the same in advance.

Let $h = \{h^{\delta(i)} \mid i = 1, \ldots, l_h\}$, $h_1 = \{h_1^{\delta(i)} \mid i = 1, \ldots, l_{h_1}\}$ and $h_2 = \{h_2^{\delta(i)} \mid i = 1, \ldots, l_{h_2}\}$ be three HFEs arranged in an increasing order of their elements where $\delta : (1, 2, \ldots, l_{h_*}) \longrightarrow (1, 2, \ldots, l_{h_*})$ indicates the permutation operator. Then, it is defined that

$$h_1 \oplus h_2 = \bigcup_{h_1^{\delta(i)} \in h_1, h_2^{\delta(j)} \in h_2} \left\{h_1^{\delta(i)} + h_2^{\delta(j)} - h_1^{(\delta(i))} h_2^{(\delta(j))}\right\};$$

$$h_1 \otimes h_2 = \bigcup_{h_1^{\delta(i)} \in h_1, h_2^{\delta(j)} \in h_2} \left\{h_1^{\delta(i)} h_2^{\delta(j)}\right\};$$

$$\lambda h = \bigcup_{h^{\delta(i)} \in h} \left\{1 - (1 - h^{\delta(i)})^{\lambda}\right\},\ \lambda > 0;$$

$$h^{\lambda} = \bigcup_{h^{\delta(i)} \in h} \left\{(h^{\delta(i)})^{\lambda}\right\},\ \lambda > 0.$$

Although up to now most studies on the HFS distance measures have focused on the unification of the length of HFSs in computational cases including (i) in the *pessimistic* case where the shortest value is repeated until the length of HFEs is the same; (ii) in the *optimistic* case in which the largest value is repeated, and (iii) in the case where the convex combination of maximum and minimum values is taken into account, but such a unification process is not necessary here for the reasons to be discussed later.

It is clear that the distance and the similarity measures are very useful tools in distinguishing the difference between two objects. By the current portion, we basically intended to analyse a number of existing distance measures for HFSs from different aspects. In the first attempt in this regard, Xu and Xia (2011) proposed a class of distance measures for HFSs including the Euclidean, the Hamming and the generalized hesitant normalized distances. As a representation, the generalized hesitant normalized distance for HFSs is given as:

$$d_{gXX}(A, B) = \left[\frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k) - h_B^{\delta(t)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \tag{1}$$

in which $l_k = \max\{l_{h_A}(x_k), l_{h_B}(x_k)\}$ for $k = 1, 2, \ldots, N$.

In addition to the above distance measure, Xu and Xia (2011) introduced the following generalized hesitant weighted distance measure:

$$d_{wgXX}(A,B)=\left[\sum_{k=1}^{N}\omega_k\left(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \quad (2)$$

where ω_k indicates the weight of $x_k \in X$ for $k = 1, 2, \ldots, N$ with the property $\sum_{k=1}^{N}\omega_k = 1$.

Needless to say that the above-mentioned distance measure d_{gXX} given by (1) is indeed the mean value of distances between all elements of the HFSs A and B. Taking this fact into account, some other HFS distance measures were proposed by Zhou and Li (2012) which are much like that introduced by Xu and Xia (2011), and they are only different in the coefficients.

Zhou and Li (2012) defined the type-2 generalized hesitant distance measure between the HFSs A and B in the form of

$$d_{gZL}(A,B)=\left[\sum_{k=1}^{N}\left(\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \quad (3)$$

for $\lambda > 0$.

In the sequel, Peng et al. (2013) extended this line of research by introducing a HFS generalized hesitant fuzzy synergetic weighted distance measure as

$$d_{ghfswP}(A,B)=\left[\frac{\sum_{k=1}^{N}\omega_k\left(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|^{\lambda}\right)w_{\rho(k)}}{\sum_{k=1}^{N}\omega_k w_{\rho(k)}}\right]^{\frac{1}{\lambda}}, \quad (4)$$

where $\rho : (1, 2, \ldots, N) \longrightarrow (1, 2, \ldots, N)$ indicates a permutation operator such that $(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|)$ stands for the $\rho(k)$-th largest of the individual distances $(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|)$ for $k = 1, 2, \ldots, N$. Moreover, the positive relative weighting vector of individual distances $(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|)$, which is denoted by ω_k, satisfies the property of normality in the sense that $\sum_{k=1}^{N}\omega_k = 1$.

A critical shortcoming in calculating the above-mentioned distance measures for HFEs is that the number of included elements may be different. This may lead to a problem, specially in the case where we are going to calculate $d(h_A, h_B)$ and $d(h_A, h_C)$, and $l_{AB} = \max\{l_{h_A}, l_{h_B}\} \neq l_{AC} = \max\{l_{h_A}, l_{h_C}\}$.

To eliminate this shortcoming, Li et al. (2015a) extended initially the set of HFEs $\{h_{A_1}, h_{A_2}, \ldots, h_{A_t}\}$ including, respectively, $\{l_{h_{A_1}}, l_{h_{A_2}}, \ldots, l_{h_{A_t}}\}$ numbers of elements as the points in the same space with the dimension of $l = \max\{l_{h_{A_1}}, l_{h_{A_2}}, \ldots, l_{h_{A_t}}\}$. Then, they proposed the generalized hesitant weighted distance measure in the form of

$$d_{gwL}(A_1,A_2)=\left[\sum_{k=1}^{N}\left(\frac{\omega_k}{l_k}\sum_{t=1}^{l_k}|h_{A_1}^{\delta(t)}(x_k)-h_{A_2}^{\delta(t)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \quad (5)$$

where $l_k = \max\{l_{h_{A_1}}, l_{h_{A_2}}, \ldots, l_{h_{A_t}}\}$ stands for the number of elements in the set of HFEs $\{h_{A_1}(x_k), h_{A_2}(x_k), \ldots, h_{A_t}(x_k)\}$.

Recently, Li et al. (2015a) and Zeng et al. (2016) showed that the above-mentioned distance measures only consider the difference between elements in HFEs and not the difference between the numbers of values in HFEs. In this regard, they demonstrated that the above-mentioned distance measures do not obey the triangle inequality axiom (see Property 3 in Theorem 3.2 below), and therefore tried to overcome this drawback by appending the concept of hesitancy of HFEs to the definition of distance measures.

Li et al. (2015a) and Zeng et al. (2016) defined the generalized hesitant weighted distance as follows:

$$d_{wgLZ}(A,B)=\frac{1}{2}\left[\sum_{k=1}^{N}\omega_k\left(\frac{1}{l_k}\sum_{t=1}^{l_k}|h_A^{\delta(t)}(x_k)-h_B^{\delta(t)}(x_k)|^{\lambda}\right)+|\Xi(h_A(x_k))-\Xi(h_B(x_k))|^{\lambda}\right]^{\frac{1}{\lambda}}, \quad (6)$$

where ω_k indicates the weight of $x_k \in X$ for $k = 1, 2, \ldots, N$ with the property $\sum_{k=1}^{N}\omega_k = 1$, and moreover, $\Xi(h_A(x_k)) = 1 - \frac{1}{l_{h_A}(x_k)}$ such that $l_k = \max\{l_{h_A}(x_k), l_{h_B}(x_k)\}$.

There exists still a problem that the calculation of the above-mentioned distance measures of HFSs is mainly based on the extension of HFEs uniformly. Thus, the reasonability of the original HFEs will be mainly influenced by expanding them in terms of a number of artificial values.

Tang et al. (2018) relaxed the property given by Farhadinia (2013a) and Zhou and Li (2012) stating that

$$\text{if} \quad A \prec B \prec C \quad \text{then} \quad d(A,B) \leq d(A,C) \quad \text{and} \quad d(B,C) \leq d(A,C), \quad (7)$$

where $A \prec B$ means that $h_A(x_k) \prec h_B(x_k)$, that is, $h_A^{\delta(t)}(x_k) \leq h_B^{\delta(s)}(x_k)$ for each $x_k \in X$, and then, they replaced that property with the *conditional reflexivity* property in terms of

$$d(A,B)=0 \quad \text{if and only if}$$
$$h_A(x_k)=h_B(x_k) \quad \text{and} \quad l_{h_A}(x_k)=l_{h_B}(x_k)=1 \tag{8}$$

for the singular-value HFEs $h_A(x_k)$ and $h_B(x_k)$ for $k = 1, 2, \ldots, N$.

Later, Tang et al. (2018) defined the generalized hesitant weighted distance measure between HFSs A and B in the form of

$$d_{wgT}(A,B)=\left[\sum_{k=1}^{N}\omega_k\left(\frac{1}{l_{h_A}(x_k)l_{h_B}(x_k)}\sum_{t=1}^{l_{h_A}(x_k)}\times\sum_{s=1}^{l_{h_B}(x_k)}|h_A^{\delta(t)}(x_k)-h_B^{\delta(s)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \tag{9}$$

where ω_k indicates the weight of $x_k \in X$ for $k = 1, 2, \ldots, N$ with the property $\sum_{k=1}^{N}\omega_k = 1$. Moreover, $l_{h_A}(x_k)$ and $l_{h_B}(x_k)$ stand for the number of values in the HFEs $h_A(x_k)$ and $h_B(x_k)$, respectively.

From Tang et al.'s (2018) conditional reflexivity property given by (8), one can easily deduce that this property holds true whenever the HFSs A and B are reduced to two equivalent fuzzy sets, and it does not return any result to the general form of HFSs.

In addition to the above-mentioned distance measures, Hu et al. (2016) presented the following generalized hesitant distance measure:

$$d_{wgH}(A,B)=\frac{1}{2}\left[\sum_{k=1}^{N}\omega_k\left(\frac{1}{l_{h_A}(x_k)}\sum_{t=1}^{l_{h_A}(x_k)}\min_{s=1,2,\ldots,l_{h_B}(x_k)}|h_A^{\delta(t)}(x_k)-h_B^{\delta(s)}(x_k)|^{\lambda}+\frac{1}{l_{h_B}(x_k)}\sum_{s=1}^{l_{h_B}(x_k)}\min_{t=1,2,\ldots,l_{h_A}(x_k)}|h_A^{\delta(t)}(x_k)-h_B^{\delta(s)}(x_k)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \tag{10}$$

where ω_k indicates the weight of $x_k \in X$ for $k = 1, 2, \ldots, N$ with the property $\sum_{k=1}^{N}\omega_k = 1$, moreover, $l_{h_A}(x_k)$ and $l_{h_B}(x_k)$ stand for the number of values in the HFEs $h_A(x_k)$ and $h_B(x_k)$, respectively.

Furthermore, Hu et al. (2016) considered a weak form of properties given in Theorem 3.2 below, that is, they mainly considered the property given by (7) instead of the triangle inequality axiom in Theorem 3.2.

Now we are in a position to describe a formula that Farhadinia (2013a) provided on how the above-mentioned distance measures for HFSs can be transformed to HFS similarity measures.

Theorem 2.2 (Farhadinia 2013a) *Assume that* $\Gamma : [0,1] \to [0,1]$ *is a strictly monotone decreasing real function, and let d be a distance between HFSs. Then, for any HFSs A and B on X*

$$S_d(A,B)=\frac{\Gamma(d(A,B))-\Gamma(1)}{\Gamma(0)-\Gamma(1)},$$

defines a HFS similarity measure based on the corresponding distance d.

By the way, if we keep the aforesaid analysis in mind, then it will be obvious that the distance measure for HFSs needs to be debated further. In the next section, we are going to propose a number of distance measures for HFSs that avoid the aforementioned drawbacks.

3 Novel distance measures for HFSs

Let us first review the common interpretation of distance measure between a single point b and a set of points A. Whenever we are going to calculate the distance measure θ of b from A, the Euclidean distance $\|.\|$ is usually employed to show how b is near to the points of A and it is denoted by $\theta(b,A)=\min_{a\in A}\|a-b\|$.

If we suppose $A=\{a_1,a_2,\ldots,a_{|A|}\}$ and $B=\{b_1,b_2, v, b_{|B|}\}$, then many different ways exist that define the *directed distance measure* between A and B as follows (see Dubuisson and Jain (1994)):

$$\Theta_{1H}(B,A)=\min_{b\in B}\theta(b,A)=\min_{b\in B}\min_{a\in A}\|a-b\|; \tag{11}$$
$$\Theta_{2H}(B,A)=\max_{b\in B}\theta(b,A)=\max_{b\in B}\min_{a\in A}\|a-b\|; \tag{12}$$
$$\Theta_{3H}(B,A)=\frac{1}{|B|}\sum_{b\in B}\theta(b,A)=\frac{1}{|B|}\sum_{b\in B}\min_{a\in A}\|a-b\|; \tag{13}$$
$$\Theta_{4H}(B,A)={}^{\gamma}\kappa^{th}_{b\in B}\theta(b,A)={}^{\gamma}\kappa^{th}_{b\in B}\min_{a\in A}\|a-b\|, \tag{14}$$

where ${}^{\gamma}\kappa^{th}_{b\in B}$ indicates the κ^{th} largest value in the set B with the property $\frac{\kappa}{|B|}=\%\gamma$.

For instance, ${}^{50}\kappa^{th}_{b\in B}$ indicates the median of the distance measures $\theta(b,A)=\min_{a\in A}\|a-b\|$ for any $b\in B$.

Now, it is interesting to note that there are many different ways of combining the direct distance measures $\Theta_{rH}(B,A)$ and $\Theta_{rH}(A,B)$ for $(r=1,2,3,4)$ to obtain a class of undirected distance measures as follows:

$$\Phi_1(\Theta_{rH}(A,B),\Theta_{rH}(B,A))=\min\{\Theta_{rH}(A,B),\Theta_{rH}(B,A)\}; \tag{15}$$
$$\Phi_2(\Theta_{rH}(A,B),\Theta_{rH}(B,A))=\max\{\Theta_{rH}(A,B),\Theta_{rH}(B,A)\}; \tag{16}$$

$$\Phi_3(\Theta_{rH}(A,B),\Theta_{rH}(B,A)) = \frac{1}{2}(\Theta_{rH}(A,B)+\Theta_{rH}(B,A)); \quad (17)$$

$$\Phi_4(\Theta_{rH}(A,B),\Theta_{rH}(B,A)) = \frac{1}{|A|+|B|}(|A|\Theta_{rH}(A,B)+|B|\Theta_{rH}(B,A)). \quad (18)$$

For instance, if we take Θ_{2H} given by (12) and Φ_2 given by (16) into account, then we result in

$$\Phi_2(\Theta_{2H}(A,B),\Theta_{2H}(B,A)) = \max\{\Theta_{2H}(A,B),\Theta_{2H}(B,A)\} = \max\left\{\max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{a\in A}\|a-b\|\right\}, \quad (19)$$

which is known as the *Hausdorff distance measure*.

The latter formula of Hausdorff distance measure is interpreted by the help of the two terms $\Theta_{2H}(A,B)$ and $\Theta_{2H}(B,A)$ where each of them is, respectively, called as the *direct* Hausdorff distance measure from the set A (respectively, B) to the set B (respectively, A). For instance, with the term $\Theta_{2H}(A,B)$ at hand, we can identify the farthest point $a\in A$ from any point $b\in B$, and measure the distance value of the point $a\in A$ from its nearest neighbour in the set B. Indeed, $\Theta_{2H}(A,B)$ ranks each point $a\in A$ on the basis of its distance measure to the nearest point of B, and consequently introduces the largest ranked point as the distance measure. For more explanation, let $\Theta_{2H}(A,B)=\alpha$ which means that any point of A is within the distance α from some point of B. Then, some point of A has the distance value α from the nearest point of B. By this explanation, we can say that the Hausdorff distance measure $\Phi_2(\Theta_{2H}(A,B),\Theta_{2H}(B,A))$ which is shown hereafter by the notation $\Phi_2(A,B)$ in brief, is the maximum value of the two terms $\Theta_{2H}(A,B)$ and $\Theta_{2H}(B,A)$ and further measures the distance of the point in A being the farthest from any point of B.

Now, we are in a position to introduce a new class of distance measures for HFSs by taking the above class of Hausdorff distance measures into consideration. Let $h_A=\{h_A^{\delta(i)}\mid i=1,\ldots,l_{h_A}\}$ and $h_B=\{h_B^{\delta(i)}\mid i=1,\ldots,l_{h_B}\}$ be two HFEs arranged in an increasing order, where $\delta:(1,2,\ldots,l_{h_i})\longrightarrow(1,2,\ldots,l_{h_i})$ for $i=A,B$ indicates the permutation operator.

Needless to say that the assumption of increasing order of elements in a HFE is not essential in the proposed setting of Hausdorff distance measures through this contribution, and therefore, it cannot be seen as a shortcoming of the theory.

We define

$$\Phi_1(\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)) = \min\{\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)\}; \quad (20)$$

$$\Phi_2(\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)) = \max\{\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)\}; \quad (21)$$

$$\Phi_3(\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)) = \frac{1}{2}(\Theta_{rH}(h_A,h_B)+\Theta_{rH}(h_B,h_A)); \quad (22)$$

$$\Phi_4(\Theta_{rH}(h_A,h_B),\Theta_{rH}(h_B,h_A)) = \frac{1}{l_{h_A}+l_{h_B}}(l_{h_A}\Theta_{rH}(h_A,h_B)+l_{h_B}\Theta_{rH}(h_B,h_A)). \quad (23)$$

where

$$\Theta_{1H}(h_B,h_A) = \min_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)},h_A^{\delta(i)}) = \min_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|; \quad (24)$$

$$\Theta_{2H}(h_B,h_A) = \max_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)},h_A^{\delta(i)}) = \max_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|; \quad (25)$$

$$\Theta_{3H}(h_B,h_A) = \frac{1}{l_{h_B}}\sum_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)},h_A^{\delta(i)}) = \frac{1}{l_{h_B}}\sum_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|; \quad (26)$$

$$\Theta_{4H}(h_B,h_A) = {}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)},h_A^{\delta(i)}) = {}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|, \quad (27)$$

such that ${}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}$ indicates the κ^{th} largest value in the HFE h_B with the property $\frac{\kappa}{l_{h_B}}=\%\gamma$.

As performed in Farhadinia (2013a, b); Farhadinia and Xu (2017); Farhadinia and Herrera-Viedma (2018); Xu and Xia (2011), we can easily extend the above results of HFEs to that for HFSs by the following rules:

$$\Phi_1(A,B) = \sum_{k=1}^{N}\Phi_1(\Theta_{rH}(h_A(x_k),h_B(x_k)), \Theta_{rH}(h_B(x_k),h_A(x_k))) = \sum_{k=1}^{N}\min\{\Theta_{rH}(h_A(x_k),h_B(x_k)), \Theta_{rH}(h_B(x_k),h_A(x_k))\}; \quad (28)$$

$$\Phi_2(A,B)=\sum_{k=1}^{N}\Phi_2(\Theta_{rH}(h_A(x_k),h_B(x_k)),\Theta_{rH}(h_B(x_k),h_A(x_k)))=\sum_{k=1}^{N}\max\{\Theta_{rH}(h_A(x_k),h_B(x_k)),\Theta_{rH}(h_B(x_k),h_A(x_k))\};\quad(29)$$

$$\Phi_3(A,B)=\sum_{k=1}^{N}\Phi_3(\Theta_{rH}(h_A(x_k),h_B(x_k)),\Theta_{rH}(h_B(x_k),h_A(x_k)))=\sum_{k=1}^{N}\frac{1}{2}(\Theta_{rH}(h_A(x_k),h_B(x_k))+\Theta_{rH}(h_B(x_k),h_A(x_k));\quad(30)$$

$$\Phi_4(A,B)=\sum_{k=1}^{N}\Phi_4(\Theta_{rH}(h_A(x_k),h_B(x_k)),\Theta_{rH}(h_B(x_k),h_A(x_k)))=\sum_{k=1}^{N}\frac{1}{l_{h_A}+l_{h_B}}(l_{h_A}\Theta_{rH}(h_A(x_k),h_B(x_k))+l_{h_B}\Theta_{rH}(h_B(x_k),h_A(x_k)),\quad(31)$$

where

$$\Theta_{1H}(h_B(x_k),h_A(x_k))=\min_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)}(x_k),h_A^{\delta(i)}(x_k))=\min_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}(x_k)-h_B^{\delta(j)}(x_k)\|;\quad(32)$$

$$\Theta_{2H}(h_B(x_k),h_A(x_k))=\max_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)}(x_k),h_A^{\delta(i)}(x_k))=\max_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}(x_k)-h_B^{\delta(j)}(x_k)\|;\quad(33)$$

$$\Theta_{3H}(h_B(x_k),h_A(x_k))=\frac{1}{l_{h_B}}\sum_{h_B^{\delta(j)}\in h_B}\theta(h_B^{\delta(j)}(x_k),h_A^{\delta(i)}(x_k))=\frac{1}{l_{h_B}}\sum_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}(x_k)-h_B^{\delta(j)}(x_k)\|;\quad(34)$$

$$\Theta_{4H}(h_B(x_k),h_A(x_k))={}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}\,\theta(h_B^{\delta(j)}(x_k),h_A^{\delta(i)}(x_k))={}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}\min_{h_A^{\delta(i)}\in h_A}\|h_A^{\delta(i)}(x_k)-h_B^{\delta(j)}(x_k)\|.\quad(35)$$

In the latter formula, ${}^{\gamma}\kappa^{th}_{h_B^{\delta(j)}\in h_B}$ indicates the κ^{th} largest value in the HFE h_B with the property $\frac{\kappa}{l_{h_B}}=\%\gamma$.

We should emphasize here that in defining the above-proposed Hausdorff distance measures, it is required that all the combinations of possible values existing in the two HFEs h_A and h_B be handled, and moreover, the restriction of length equality of HFEs is relaxed in calculating the proposed distance measures.

Remark 3.1 It needs to be mentioned that only the Hausdorff distance measure Φ_2 given by (19) is a metric, and the other combinations do not satisfy the triangle inequality property 3 below:

Theorem 3.2 *Suppose that* $A=\{\langle x,h_A(x)\rangle : x\in X\}=\{\langle x,\{h_A^{\delta(i)}(x)\,|\,i=1,\ldots,l_{h_A}\}\rangle : x\in X\}$, $B=\{\langle x,h_B(x)\rangle : x\in X\}=\{\langle x,\{h_B^{\delta(j)}(x)\mid j=1,\ldots,l_{h_B}\}\rangle : x\in X\}$ *and* $C=\{\langle x,h_C(x)\rangle : x\in X\}=\{\langle x,\{h_C^{\delta(k)}(x)\mid k=1,\ldots,l_{h_C}\}\rangle : x\in X\}$ *are three HFSs, and* Φ_2 *is that defined by* (29). *Then,* Φ_2 *is a distance measure for HFSs and satisfies*

1. $0\le\Phi_2(A,B)\le 1$ *(Boundedness property)*
2. $\Phi_2(A,B)=\Phi_2(B,A)$ *(Symmetric property)*
3. $\Phi_2(A,C)\le\Phi_2(A,B)+\Phi_2(B,C)$ *(Triangle inequality property)*
4. $\Phi_2(A,B)=0$ *if and only if* $A=B$. *(Conditional reflexivity property)*

Proof From the definition of HFS distance measure Φ_2 given by (29), it can be easily seen that the proofs of the properties 1 and 2 are clear and we do not give them here.

Property 3. First of all, suppose that

$$\|h_A^{\delta(i)}-h_C^{\delta(k_0)}\|=\min_{h_C^{\delta(k)}\in h_C}\|h_A^{\delta(i)}-h_C^{\delta(k)}\|,\quad\text{for}\quad k_0\in\{1,2,\ldots,l_{h_C}\},\quad(36)$$

$$\|h_C^{\delta(k_0)}-h_B^{\delta(j_0)}\|=\min_{h_B^{\delta(j)}\in h_B}\|h_C^{\delta(k_0)}-h_B^{\delta(j)}\|,\quad\text{for}\quad j_0\in\{1,2,\ldots,l_{h_B}\}.\quad(37)$$

On the other hand, we get that

$$\min_{h_B^{\delta(j)}\in h_B}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|\le\|h_A^{\delta(i)}-h_B^{\delta(j_0)}\|\le\|h_A^{\delta(i)}-h_C^{\delta(k_0)}\|+\|h_C^{\delta(k_0)}-h_B^{\delta(j_0)}\|.$$

Now, by the use of relations (36) and (37) together with the above inequality, we conclude that

$$\min_{h_B^{\delta(j)}\in h_B}\|h_A^{\delta(i)}-h_B^{\delta(j)}\|\le\min_{h_C^{\delta(k)}\in h_C}\|h_A^{\delta(i)}-h_C^{\delta(k)}\|+\min_{h_B^{\delta(j)}\in h_B}\|h_C^{\delta(k_0)}-h_B^{\delta(j)}\|\le\min_{h_C^{\delta(k)}\in h_C}\|h_A^{\delta(i)}-h_C^{\delta(k)}\|+\max_{h_C^{\delta(k)}\in h_C}\min_{h_B^{\delta(j)}\in h_B}\|h_C^{\delta(k)}-h_B^{\delta(j)}\|$$

$$\leq \max_{h_A^{\delta(i)} \in h_A} \min_{h_C^{\delta(k)} \in h_C} \|h_A^{\delta(i)} - h_C^{\delta(k)}\| + \max_{h_C^{\delta(k)} \in h_C} \min_{h_B^{\delta(j)} \in h_B} \|h_C^{\delta(k)} - h_B^{\delta(j)}\|$$
$$\leq \max\{\Theta_{2H}(h_A, h_C), \Theta_{2H}(h_C, h_A)\} + \max\{\Theta_{2H}(h_C, h_B), \Theta_{2H}(h_B, h_C)\}$$
$$= \Phi_2(A, C) + \Phi_2(C, B), \tag{38}$$

for all $h_A^{\delta(i)} \in h_A$.

If we now consider definition (33) and inequality (38), we then find that

$$\Theta_{2H}(h_A, h_B) = \max_{h_A^{\delta(i)} \in h_A} \min_{h_B^{\delta(j)} \in h_B} \|h_A^{\delta(i)} - h_B^{\delta(j)}\| \leq \Phi_2(A, C) + \Phi_2(C, B).$$

By the same procedure described above, it is deduced that

$$\Theta_{2H}(h_B, h_A) = \max_{h_B^{\delta(j)} \in h_B} \min_{h_A^{\delta(i)} \in h_A} \|h_A^{\delta(i)} - h_B^{\delta(j)}\| \leq \Phi_2(A, C) + \Phi_2(C, B),$$

and thus

$$\Phi_2(A, B) = \max\{\Theta_{2H}(h_A, h_B), \Theta_{2H}(h_B, h_A)\} \leq \Phi_2(A, C) + \Phi_2(C, B).$$

Property 4. Follows from definition (29), it is easily concluded that

$$\Phi_2(A, B) = \sum_{k=1}^{N} \Phi_2(\Theta_{2H}(h_A(x_k), h_B(x_k)), \Theta_{2H}(h_B(x_k), h_A(x_k))) = \sum_{k=1}^{N} \max\{\Theta_{2H}(h_A(x_k), h_B(x_k)), \Theta_{2H}(h_B(x_k), h_A(x_k))\} = 0 \tag{39}$$

if and only of

$$\Theta_{2H}(h_A(x_k), h_B(x_k)) = 0, \tag{40}$$
$$\Theta_{2H}(h_B(x_k), h_A(x_k)) = 0. \tag{41}$$

Now, from equality (40), we find that

$$\Theta_{2H}(h_A(x_k), h_B(x_k)) = \max_{h_B^{\delta(j)} \in h_B} \theta(h_B^{\delta(j)}(x_k), h_A^{\delta(i)}(x_k))$$
$$= \max_{h_B^{\delta(j)} \in h_B} \min_{h_A^{\delta(i)} \in h_A} \|h_A^{\delta(i)}(x_k) - h_B^{\delta(j)}(x_k)\| = 0,$$

which means that

$$\|h_A^{\delta(i)}(x_k) - h_B^{\delta(j)}(x_k)\| = 0,$$

for any $h_B^{\delta(j)} \in h_B$ and $h_A^{\delta(i)} \in h_A$. This says that $h_A^{\delta(i)}(x_k) = h_B^{\delta(j)}(x_k)$ for any $x_k \in X$, that is, $A = B$.

The proof of equality (41) is similar. □

In the case where ω_k indicates the weight of $x_k \in X$ for $k = 1, 2, \ldots, N$ with the property $\sum_{k=1}^{N} \omega_k = 1$, the following weighted Hausdorff distance measures for HFSs can be proposed as:

$$\Phi_1^{\omega}(A, B) = \sum_{k=1}^{N} \omega_k \Phi_1(\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))) = \sum_{k=1}^{N} \omega_k \min\{\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))\}; \tag{42}$$

$$\Phi_2^{\omega}(A, B) = \sum_{k=1}^{N} \omega_k \Phi_2(\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))) = \sum_{k=1}^{N} \omega_k \max\{\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))\}; \tag{43}$$

$$\Phi_3^{\omega}(A, B) = \sum_{k=1}^{N} \omega_k \Phi_3(\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))) = \sum_{k=1}^{N} \omega_k \frac{1}{2}(\Theta_{rH}(h_A(x_k), h_B(x_k)) + \Theta_{rH}(h_B(x_k), h_A(x_k)); \tag{44}$$

$$\Phi_4^{\omega}(A, B) = \sum_{k=1}^{N} \omega_k \Phi_4(\Theta_{rH}(h_A(x_k), h_B(x_k)), \Theta_{rH}(h_B(x_k), h_A(x_k))) = \sum_{k=1}^{N} \omega_k \left(\frac{1}{l_{h_A} + l_{h_B}} (l_{h_A} \Theta_{rH}(h_A(x_k), h_B(x_k)) + l_{h_B} \Theta_{rH}(h_B(x_k), h_A(x_k)) \right). \tag{45}$$

If the reference set X and moreover the continuous weights of elements are taken into account, then the continuous weighted Hausdorff distance measures for HFSs are given as follows:

$$\Phi_1^{\omega}(A,B) = \int_a^b \omega(x)\Phi_1(\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x)))dx = \int_a^b \omega(x)\min\{\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x))\}dx; \quad (46)$$

$$\Phi_2^{\omega}(A,B) = \int_a^b \omega(x)\Phi_2(\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x)))dx = \int_a^b \omega(x)\max\{\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x))\}dx; \quad (47)$$

$$\Phi_3^{\omega}(A,B) = \int_a^b \omega(x)\Phi_3(\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x)))dx = \int_a^b \omega(x)\frac{1}{2}(\Theta_{rH}(h_A(x),h_B(x)) + \Theta_{rH}(h_B(x),h_A(x))dx; \quad (48)$$

$$\Phi_4^{\omega}(A,B) = \int_a^b \omega(x)\Phi_4(\Theta_{rH}(h_A(x),h_B(x)), \Theta_{rH}(h_B(x),h_A(x)))dx = \int_a^b \omega(x)(\frac{1}{l_{h_A}+l_{h_B}}(l_{h_A} \Theta_{rH}(h_A(x),h_B(x)) + l_{h_B}\Theta_{rH}(h_B(x),h_A(x)))dx, \quad (49)$$

where any weight of element $x \in X = [a, b]$ is denoted by $\omega(x)$ such that $0 \le \omega(x) \le 1$ and $\int_a^b \omega(x)dx = 1$.

4 Applications of the proposed distance measures

In this section, three portions are provided to demonstrate the applicability and validity of the proposed distance measures for HFSs. The first portion relating to the evaluation of energy policy is taken from Tang et al. (2018) and Xu and Xia (2011). The second portion is adapted from Hu et al. (2018) where they employed the TOPSIS technique to conduct a hesitant fuzzy MCDM approach. The third portion deals with developing an approach for multiple criteria group decision-making in order to evaluate the classification modes of the China's college entrance examination.

4.1 Appropriate energy policy based on the distance measures

Here, we intend to demonstrate the application of suggested distance measures for HFSs by the use of two numerical examples that were discussed previously by Tang et al. (2018).

Example 4.1 [Adopted from Tang et al. (2018) and Xu and Xia (2011)] These days, energy roles as an indispensable parameter in developing the socio-economic scenarios. This implies that selecting the most appropriate energy policy is so important. It is assumed that we are going to invest five energy projects (alternatives) $A_i (i = 1, 2, 3, 4, 5)$ with respect to four criteria as C_1: technological; C_2: environmental; C_3: sociopolitical; and C_4: economic. The criteria weight vector is supposed to be in the form of $\omega = (0.15, 0.3, 0.2, 0.35)$. In the case where the decision makers return anonymously their evaluation of each alternative under the four criteria, we then face to the sets of evaluated values in the form of HFS as given in Table 1. Let the ideal solution be as the special HFS $A^* = \{1\}$. This set is seen as the reference set for comparing all alternatives. In this regard, the best alternative is that with the least value of distance measure from the ideal solution $A^* = \{1\}$.

We employ the weighted distance measures proposed in this contribution as given by Eqs. (42)–(45) and then calculate the hesitant weighted distance measure of each alternative and the ideal solution $A^* = \{1\}$ together with the ranking order of alternatives which are all represented in Table 2. Besides that, the results of Tang et al.'s (2018) and Xu and Xia's (2011) distance measures are given in Table 2.

Example 4.2 By keeping all the data structures used in Example 4.1, we only update the ideal solution as $A^{\odot} = \{\langle C_1, \{0.7, 0.6\}\rangle, \langle C_2, \{0.9, 0.6\}\rangle, \langle C3, \{0.9, 0.8, 0.7\}, \langle C4, \{0.9, 0.8, 0.6\}\rangle\}$.

Once again, by the use of the proposed distance measures, we can obtain the results being shown in Table 3. Also, the results of applying the two classes of distance measures proposed by Tang et al. (2018) and Xu and Xia (2011) to this example are shown in Table 3.

By taking a look at the results presented in Tables 2 and 3, it can be observed that although the ranking orders are not exactly coincided, they often propose that the most appropriate energy policy is A_5. The results of Tang et al.'s (2018) and Xu and Xia's (2011) distance measures depend highly on the variation of parameter λ, that is, they return A_5 as the most appropriate energy policy for $\lambda \in [1, 3]$, and A_3 for $\lambda \in (3, \infty]$.

It is worthwhile to state that the extra parameter λ does not play any role in describing the proposed distance measures for HFSs.

Moreover, what should be noted here is that Tang et al.'s (2018) and Xu and Xia's (2011) distance measures only provide the decision makers with more choices of λ, and do not explain how we can be aware of λ values. But, the proposed

Table 1 Hesitant fuzzy decision matrix

Alternative\criteria	C_1	C_2	C_3	C_4
A_1	$\{0.5, 0.4, 0.3\}$	$\{0.9, 0.8, 0.7, 0.1\}$	$\{0.5, 0.4, 0.2\}$	$\{0.9, 0.6, 0.5, 0.3\}$
A_2	$\{0.5, 0.3\}$	$\{0.9, 0.7, 0.6, 0.5, 0.2\}$	$\{0.8, 0.6, 0.5, 0.1\}$	$\{0.7, 0.3, 0.4\}$
A_3	$\{0.7, 0.6\}$	$\{0.9, 0.6\}$	$\{0.7, 0.5, 0.3\}$	$\{0.6, 0.4\}$
A_4	$\{0.8, 0.7, 0.4, 0.3\}$	$\{0.7, 0.4, 0.2\}$	$\{0.8, 0.1\}$	$\{0.9, 0.8, 0.6\}$
A_5	$\{0.9, 0.7, 0.6, 0.3, 0.1\}$	$\{0.8, 0.7, 0.6, 0.4\}$	$\{0.9, 0.8, 0.7\}$	$\{0.9, 0.7, 0.6, 0.3\}$

Table 2 The distance values of the five energy projects

Distance	A_1	A_2	A_3	A_4	A_5	Ranking order
d_{wgXX} (Xu and Xia 2011)						
$\lambda = 1$	0.4799	0.5027	0.4025	0.4292	0.3558	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 2$	0.5378	0.5451	0.4366	0.5052	0.4129	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 6$	0.6599	0.6476	0.5156	0.6704	0.5699	$A_3 > A_5 > A_2 > A_1 > A_4$
$\lambda = 10$	0.7213	0.7046	0.5607	0.7373	0.6537	$A_3 > A_5 > A_2 > A_1 > A_4$
d_{wgT} (Tang et al. 2018)						
$\lambda = 1$	0.4779	0.5027	0.4025	0.4292	0.3558	$A_5 > A_3 > A_4 > A_1 > A_2$
$\lambda = 2$	0.5378	0.5451	0.4366	0.5052	0.4129	$A_5 > A_3 > A_4 > A_1 > A_2$
$\lambda = 6$	0.6599	0.6476	0.5156	0.6704	0.5699	$A_3 > A_5 > A_2 > A_1 > A_4$
$\lambda = 10$	0.7213	0.7047	0.5603	0.7374	0.6537	$A_3 > A_5 > A_2 > A_1 > A_4$
Proposed measure						
Φ_1^{ω} (with $r = 1$)	0.2400	0.2500	0.2750	0.1950	0.1300	$A_5 > A_4 > A_1 > A_2 > A_3$
Φ_2^{ω} (with $r = 2$)	0.7800	0.7350	0.5300	0.6650	0.6200	$A_3 > A_5 > A_4 > A_2 > A_1$
Φ_3^{ω} (with $r = 3$)	0.3590	0.3505	0.4363	0.3121	0.2391	$A_5 > A_4 > A_2 > A_1 > A_3$
Φ_4^{ω} (with $r = 4$)	0.4002	0.4800	0.3633	0.3204	0.2885	$A_5 > A_4 > A_3 > A_1 > A_2$

Table 3 The distance values of the five energy projects

Distance	A_1	A_2	A_3	A_4	A_5	Ranking order
d_{wgXX} (Xu and Xia 2011)						
$\lambda = 1$	0.2342	0.2335	0.1650	0.1996	0.0980	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 2$	0.2850	0.2773	0.2306	0.2949	0.1580	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 6$	0.3771	0.3842	0.3105	0.4744	0.2872	$A_3 > A_5 > A_2 > A_1 > A_4$
$\lambda = 10$	0.4154	0.4475	0.3395	0.5446	0.3530	$A_3 > A_5 > A_2 > A_1 > A_4$
d_{wgT} (Tang et al. 2018)						
$\lambda = 1$	0.2971	0.2886	0.2058	0.2583	0.1940	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 2$	0.3577	0.3495	0.2697	0.3449	0.2543	$A_5 > A_3 > A_4 > A_2 > A_1$
$\lambda = 6$	0.5079	0.4934	0.3775	0.5263	0.3901	$A_3 > A_5 > A_2 > A_1 > A_4$
$\lambda = 10$	0.5897	0.5687	0.4318	0.6014	0.4512	$A_3 > A_5 > A_2 > A_1 > A_4$
Proposed measure						
Φ_1^{ω} (with $r = 1$)	0.0550	0.0500	0.0001	0.0300	0.0001	$A_5 > A_3 > A_4 > A_2 > A_1$
Φ_2^{ω} (with $r = 2$)	0.1100	0.1200	0.1450	0.0950	0.0600	$A_5 > A_4 > A_1 > A_2 > A_3$
Φ_3^{ω} (with $r = 3$)	0.5625	0.4725	0.3525	0.2850	0.2250	$A_5 > A_4 > A_3 > A_2 > A_1$
Φ_4^{ω} (with $r = 4$)	0.0695	0.0538	0.0380	0.0390	0.0150	$A_5 > A_3 > A_4 > A_2 > A_1$

Table 4 Hesitant fuzzy decision matrix

Alternative\criteria	C_1	C_2	C_3	C_4
A_1	{0.7, 0.4, 0.2}	{0.7, 0.5, 0.2, 0.1}	{0.8, 0.7, 0.5, 0.3, 0.2}	{0.6, 0.4, 0.1}
A_2	{0.7, 0.6, 0.4}	{0.6, 0.4, 0.2, 0.1}	{0.9, 0.8, 0.6, 0.4, 0.3}	{0.4, 0.2, 0.1}
A_3	{0.6, 0.3, 0.2}	{0.9, 0.5, 0.4, 0.3}	{0.8, 0.7, 0.6, 0.4, 0.2}	{0.8, 0.4, 0.3}
A_4	{0.5, 0.3, 0.2}	{0.7, 0.5, 0.3, 0.2}	{0.9, 0.8, 0.7, 0.6, 0.4}	{0.7, 0.2, 0.1}

distance measures do not depend on any parameter like λ in Tang et al. (2018) and Xu and Xia' (2011) measures.

4.2 Appropriate supplier based on the distance measures

On the basis of the relationship between distance measure and similarity measure for HFSs (Farhadinia 2013a), we discuss here about the novelty of the proposed distance measures from applied intelligence perspective. In this regard, we adopt an experimentation which is discussed thoroughly by Hu et al. (2018). In Hu et al. (2018), a class of similarity measures for HFSs are further investigated from their performance. Hu et al. (2018) employed the TOPSIS technique to conduct a hesitant fuzzy MCDM method where the alternatives set contains $\{A_1, A_2, \ldots, A_m\}$ and the set of criteria includes $\{C_1, C_2, \ldots, C_n\}$ with the weight vector of $(w_1, w_2, \ldots, w_n)$ satisfying $w_j \geq 0$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. It is needless to say that here, the decision maker evaluation is in the form of hesitant fuzzy decision matrix $D = [h_{ij}]_{m\times n} = [\{h_{ij}^{\delta(1)}, \ldots, h_{ij}^{\delta(l_{ij})}\}]_{m\times n}$. The latter-mentioned MCDM process is described by the following steps:

Step 1. Given the weights w_j $(j = 1, 2, \ldots, n)$, the hesitant fuzzy positive ideal solution (PIS) and negative ideal solution (NIS) can be calculated by the following manner:

$$h^+ = (h_1^+, h_2^+, \ldots, h_n^+), \tag{50}$$

$$h^- = (h_1^-, h_2^-, \ldots, h_n^-), \tag{51}$$

where $h_j^+ = \max_i \max_{1\leq k\leq l_{ij}} \{h_{ij}^{\delta(k)}\}$ and $h_j^- = \min_i \min_{1\leq k\leq l_{ij}} \{h_{ij}^{\delta(k)}\}$ for any benefit criterion C_j, and $h_j^+ = \min_i \min_{1\leq k\leq l_{ij}} \{h_{ij}^{\delta(k)}\}$ and $h_j^- = \max_i \max_{1\leq k\leq l_{ij}} \{h_{ij}^{\delta(k)}\}$ for any cost criterion C_j.

Step 2. The weighted distance measures Φ_{ir}^{ω} and Φ_{ir}^{ω} $(r = 1, 2, 3, 4)$ of alternative A_i from the PIS and NIS are computed by

$$\Phi_{ir}^+ = \sum_{j=1}^{n} w_j \Phi_r^{\omega}(h_{ij}, h_j^+), \tag{52}$$

$$\Phi_{ir}^- = \sum_{j=1}^{n} w_j \Phi_r^{\omega}(h_{ij}, h_j^-), \tag{53}$$

for any $i = 1, 2, \ldots, m$ and $r = 1, 2, 3, 4$.

Step 3. Compute the relative closeness of each alternative A_i $(i = 1, 2, \ldots, m)$ to the ideal solution as the following:

$$RC_i = \frac{\Phi_{ir}^+}{\Phi_{ir}^+ + \Phi_{ir}^-}, \quad i = 1, 2, \ldots, m, \ r = 1, 2, 3, 4. \tag{54}$$

Step 4. Taking the above relative closeness into account, the alternative can be ranked in decreasing order, that is, $A_i > A_j$ whenever $RC_i < RC_j$.

Here, it should be emphasized that the ranking of alternatives according to Hu et al.'s (2018) relative closeness, which is based on the similarity measure, is determined by $A_i < A_j$ whenever $RC_i < RC_j$.

Illustrative example 4.3 (Hu et al. 2018) An automotive company is going to select the most appropriate supplier which plays a key role in its manufacturing process. According to the initial evaluation, the company team select four suppliers A_i $(i = 1, 2, 3, 4)$ as candidate alternatives in order to further evaluation. Moreover, in order to have a more accurately evaluation of the suppliers, the company team considers the most important criteria as C_1 (product quality), C_2 (relationship closeness), C_3 (delivery performance) and C_4 (price). Needless to say that the criteria C_1, C_2, and C_3 are benefit; meanwhile, the criterion C_4 is cost. In such a situation, employing experts with different backgrounds helps to perform the evaluation more reliable and accurately. Regarding such a viewpoint on this issue, we are able to reflect the information provided by the experts in the form of HFSs which are presented as the decision matrix given in Table 4.

Since the main intention here is to compare the results of the proposed distance measures for HFSs with that of presented in Hu et al. (2018), once again, we restate Hu et

Table 5 The relative closeness and ranking of the alternatives

Measure	A_1	A_2	A_3	A_4	Ranking order
S_{ghs_1} (Hu et al. 2018)	0.4881	0.5500	0.4756	0.5118	$A_2 > A_4 > A_1 > A_3$
S_{ghs_2} (Hu et al. 2018)	0.4854	0.5388	0.4801	0.5159	$A_2 > A_4 > A_1 > A_3$
S_{ghs_5} (Hu et al. 2018)	0.4774	0.5153	0.4897	0.5161	$A_4 > A_2 > A_3 > A_1$
$S_{ghs_{10}}$ (Hu et al. 2018)	0.4698	0.5047	0.4984	0.5111	$A_4 > A_2 > A_3 > A_1$
S_{ghhs_1} (Hu et al. 2018)	0.4881	0.5500	0.4756	0.5118	$A_2 > A_4 > A_1 > A_3$
S_{ghhs_2} (Hu et al. 2018)	0.4854	0.5388	0.4801	0.5159	$A_2 > A_4 > A_1 > A_3$
S_{ghhs_5} (Hu et al. 2018)	0.4774	0.5143	0.4897	0.5161	$A_4 > A_2 > A_3 > A_1$
$S_{ghhs_{10}}$ (Hu et al. 2018)	0.4699	0.5047	0.4984	0.5111	$A_4 > A_2 > A_3 > A_1$
S_1 (Hu et al. 2018)	0.4881	0.5500	0.4756	0.5118	$A_2 > A_4 > A_1 > A_3$
S_2 (Hu et al. 2018)	0.4754	0.5466	0.4598	0.5026	$A_2 > A_4 > A_1 > A_3$
S_3 (Hu et al. 2018)	0.4684	0.5327	0.4524	0.4910	$A_2 > A_4 > A_1 > A_3$
Φ_1^{ω} (with $r = 1$)	0.3796	0.2111	0.6871	0.5869	$A_2 > A_1 > A_4 > A_3$
Φ_2^{ω} (with $r = 2$)	0.5536	0.4902	0.6019	0.4793	$A_4 > A_2 > A_1 > A_3$
Φ_3^{ω} (with $r = 3$)	0.8083	0.6381	0.6917	0.5958	$A_4 > A_2 > A_3 > A_1$
Φ_4^{ω} (with $r = 4$)	0.4375	0.2473	0.6056	0.5051	$A_2 > A_1 > A_4 > A_3$

Table 6 Hesitant fuzzy decision matrix

Alternative\criteria	C_1	C_2	C_3	C_4
A_1	{0.6, 0.4, 0.3}	{0.8, 0.6}	{0.5, 0.4}	{0.7, 0.6}
A_2	{0.7, 0.5}	{0.5, 0.3}	{0.9, 0.7}	{0.6, 0.5}
A_3	{0.7, 0.6}	{0.9, 0.8, 0.7}	{0.6, 0.5}	{0.7, 0.6}
A_4	{0.5, 0.4}	{0.9, 0.8}	{0.5, 0.4}	{0.9, 0.8, 0.6}
A_5	{0.8, 0.6}	{0.5, 0.4}	{0.6, 0.4, 0.3}	{0.8, 0.7}

al.'s (2018) evaluation of the weight, the hesitant fuzzy PIS and the hesitant fuzzy NIS as follows:

$$w = (w_1, w_2, w_3, w_4) = (0.2394, 0.2483, 0.1756, 0.3367),$$
$$h^+ = (h_1^+, h_2^+, h_3^+, h_4^+) = (0.7, 0.9, 0.9, 0.1),$$
$$h^- = (h_1^-, h_2^-, h_3^-, h_4^-) = (0.2, 0.1, 0.2, 0.8).$$

Now, by performing the steps of the above-mentioned TOPSIS method, the results are those given in the four last rows of Table 5 together with the results of similarity-based techniques being adopted from Table 8 in Hu et al. (2018).

As can be seen from the first row to eleventh row of Table 5, the most optimal alternative changes from A_2 to A_4 as the value of parameter λ increases. This phenomena can be observed at the four last rows of Table 5 where the proposed distance measures return such a result, but without needing to consider an extra parameter like λ.

4.3 Classification modes of China's college entrance examination

The college entrance examination system is indeed a very vital education system which is able to select the most qualified talents of higher education institutions. In recent years, such a system has undergone dozens of reforms and adjustments. By the way, in this part of the paper, we will investigate a MCDM problem in order to evaluate the classification modes of the China's college entrance examination in which the information is carried by hesitant fuzzy sets. Taking the idea of dependent aggregation into consideration, we employ the dependent hesitant fuzzy ordered weighted averaging (DHFOWA) operator whose weight coefficients depend only on the aggregated hesitant fuzzy arguments. Such a consideration allows us to relieve the influence of unfair hesitant fuzzy arguments on the aggregated results by the use of assigning low weights to those "false" and moreover "biased" ones and subsequently we apply them for developing a multiple attribute group decision-making-based technique to evaluate the classification modes of the China's college entrance examination.

Following Hua's (2017) consideration, we suppose five classification modes of the China's college entrance examination entitled as A_i ($i = 1, 2, 3, 4, 5$) which are evaluated by some experts using four criteria of college entrance examination including: the technical advancement C_1, the fairness C_2, the effectiveness C_3 and the society degree of recognizing C_4. Needless to say that the decision makers are requested here for assessing the five alternatives under the four criteria

Table 7 Hua's (2017) algorithm outcomes with respect to different distance measures

Distance	A_1	A_2	A_3	A_4	A_5	Ranking order
Xu and Xia's (2011) distance: d_{wgXX} (2)	0.1966	0.1379	0.2261	0.2296	0.2097	$A_4 > A_3 > A_5 > A_1 > A_2$
Zhou and Li's (2012) distance: d_{gZL} (3)	0.1966	0.1379	0.2261	0.2296	0.2097	$A_4 > A_3 > A_5 > A_1 > A_2$
Peng et al.'s (2013) distance: d_{ghfswP} (4)	0.1966	0.1379	0.2261	0.2296	0.2097	$A_4 > A_3 > A_5 > A_1 > A_2$
Li et al.'s (2015a) distance: d_{gwL} (5)	0.1966	0.1379	0.2261	0.2296	0.2097	$A_4 > A_3 > A_5 > A_1 > A_2$
Zeng et al.'s (2016) distance: d_{wgLZ} (6)	0.1966	0.1379	0.2261	0.2296	0.2097	$A_4 > A_3 > A_5 > A_1 > A_2$
Tang et al.'s (2018) distance: d_{wgT}(9)	0.331	0.276	0.402	0.457	0.378	$A_4 > A_3 > A_5 > A_1 > A_2$
Hu et al.'s (2016) distance:d_{wgH} (10)	0.1968	0.1373	0.2268	0.2301	0.2091	$A_4 > A_3 > A_5 > A_1 > A_2$
Hua's (2017) distance: d_{wgT} (9)	0.331	0.276	0.402	0.457	0.378	$A_4 > A_3 > A_5 > A_1 > A_2$
Proposed distance measure						
Φ_1^{ω} (with $r = 1$)	0.2472	0.0337	0.0337	0.3708	0.3146	$A_4 > A_5 > A_1 > A_3 > A_2$
Φ_2^{ω} (with $r = 2$)	0.2804	0.0280	0.0280	0.4019	0.2617	$A_4 > A_5 > A_1 > A_3 > A_2$
Φ_3^{ω} (with $r = 3$)	0.2650	0.0298	0.0298	0.3975	0.2779	$A_4 > A_5 > A_1 > A_3 > A_2$
Φ_4^{ω} (with $r = 4$)	0.2516	0.0325	0.0325	0.3796	0.3037	$A_4 > A_5 > A_1 > A_3 > A_2$

in the anonymous case, and this leads to present the decision matrix $H = [h_{ij}]_{5\times4}$ in the form of HFEs given in Table 6.

Hua (2017) developed a practical technique in the form of the following algorithm to deal with a MCDM problem concentrating on the evaluation of classification modes of the China's college entrance examination. There, the information describing the criterion weights is indeed completely unknown, and the criteria values are in the form of hesitant fuzzy information.

Hua's (2017) algorithm of MCDM problem:

Step 1. For any alternative A_i, we employ the DHFOWA operator

$$\mathfrak{h}_i = \text{DHFOWA}(h_{i1}, h_{i2}, \ldots, h_{in}) = \bigoplus_{j=1}^{n} w_j h_{ij} = \left\{ 1 - \prod_{j=1}^{n} (1 - h_{ij}^{\delta(k)})^{\frac{S_d(s(h_{ij}), s_i)}{\sum_{j=1}^{n} S_d(s(h_{ij}), s_i)}} \mid k = 1, \ldots, l_{h_{ij}} \right\} \tag{55}$$

to get the overall preference values. Here, the notation S_d denotes the degree of similarity measure based on the distance measure d given by

$$S_d(s(h_{ij}), s_i) = 1 - \frac{d(s(h_{ij}), s_i)}{\sum_{j=1}^{n} d(s(h_{ij}), s_i)}, \tag{56}$$

and $s(h_{ij})$ stands for the score function of the HFE $h_{ij} = \{h_{ij}^{\delta(k)} \mid k = 1, \ldots, l_{h_{ij}}\}$ given by

$$s(h_{ij}) = \frac{1}{l_{h_{ij}}} \sum_{k=1}^{l_{h_{ij}}} h_{ij}^{\delta(k)}, \quad \text{for} \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \tag{57}$$

and moreover,

$$s_i = \frac{1}{n} \sum_{j=1}^{n} s(h_{ij}), \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{58}$$

Step 2. Now, by considering

$$s(\mathfrak{h}_i) = \frac{1}{l_{\mathfrak{h}_i}} \sum_{k=1}^{l_{\mathfrak{h}_i}} \mathfrak{h}_i^{\delta(k)} \tag{59}$$

known as the score function of overall hesitant fuzzy preference values $\mathfrak{h}_i$ $(i = 1, 2, \ldots, m)$, we are able to determine the priorities of alternatives A_i $(i = 1, 2, \ldots, m)$.

Now, we are in a position to apply the above-mentioned algorithm to evaluate the classification modes of the China's college entrance examination. In the process of performing the algorithm, we implement the different distance measures proposed in this contribution to calculate the degree of similarity measure described in Step 1 by Eq. (56).

Here it needs to emphasize that Hua's (2017) distance measure being employed in Step 1 is indeed Tang et al.'s (2018) generalized hesitant weighted distance measure given in this contribution by (9).

From Table 7, it is evident that the best and worst classification modes of the China's college entrance examination which are deduced from various distance-based MCDM processes are the same, and only the medium classification modes may differ slightly.

5 Conclusions

In this contribution, we were interested to review a variety of existing distance measures for HFSs. Besides the critical review of the existing distance measures, we introduced a new class of HFS distance measures by emphasizing on these points that there is no need to make equal the lengths of HFEs as well as altering the arranging order of their values. It is interesting to note that not only the proposed Hausdorff-based distance measures for HFSs satisfy mainly the triangle inequality property, but also they inherit the other well-known properties. At the end of this contribution, we provided some numerical examples to illustrate the efficiency of the new developed distance measures for HFSs together with a comparative analysis with other existing distance measures. For the future work, we will explore the behaviour of the other HFS information measures for handling multiple criteria decision-making.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by the authors.

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