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METHODOLOGIES AND APPLICATION

MCDM based on new membership and non-membership accuracy functions on trapezoidal-valued intuitionistic fuzzy numbers

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Abstract

Ranking of trapezoidal-valued intuitionistic fuzzy numbers (TVIFNs) plays an important role in multi-criteria decision making (MCDM) based on the TVIFNs. The main objective of this paper is to introduce new membership and non-membership accuracy functions on the classes of interval-valued intuitionistic fuzzy numbers (IVIFNs) and TVIFNs by which the orderings on IVIFNs and TVIFNs are done. This paper reveals the better part of the proposed accuracy functions than the existing or previous functions. Further, some operations on IVIFNs and TVIFNs are defined. Finally, a new method is proposed to solve the MCDM problem based on the multi-criteria trapezoidal-valued intuitionistic fuzzy index matrix and illustrated through numerical examples.

Keywords Interval-valued intuitionistic fuzzy number · Trapezoidal-valued intuitionistic fuzzy numbers · Multi-criteria decision making · Index matrix · Accuracy function

1 Introduction

Multi-criteria decision-making (MCDM) problem utilizes the accuracy functions to rank the alternatives (Wang and Wang 2008; Wang et al. 2008). Any decision-making problem involves three steps, specifically (1) gathering data from resource persons and designing the decision matrix, (2) aggregating the performance of each alternative with respect to each criteria and (3) ranking of alternatives in accordance with its aggregated performance.

Xu and Chen (2007) introduced the concept of score function and accuracy function for interval-valued intuitionistic fuzzy number (IVIFN). He has also proposed weighted arithmetic average operator and weighted geometric average operator to aggregate the performances of alternative with respect to criteria. Next, Ye (2009) has pointed out some drawbacks of Xu's work and has presented a novel accuracy function of IVIFN. Further, many authors like Nayagam (Nayagam et al. 2008, 2017, 2016a, b, c, 2018, 2011; Nayagam and Sivaraman 2011; Sahin 2015; Sivaraman et al. 2014; Bai 2013; Liu and Xia luo 2016; Garg 2016) have developed different concepts of score and accuracy functions.

The concepts of index matrix (IM), intuitionistic fuzzy index matrix (IFIM) and extended intuitionistic fuzzy index matrix (EIFIM) were introduced by Atanassov in 1987, and many binary operations and aggregation operations on IFIM and EIFIM have been studied in Atanassov (1987, 2010, 2013b) and Pap (1997, 2002). The inter-criteria decision making based on EIFIM is studied in Atanassov (2013a, 2014) and Atanassov et al. (2014).

The approach of this paper is coordinated as follows: Necessary basic definitions are briefly introduced in Sect. 2. In Sect. 3, a short review of ranking methods presented by some authors are given and new membership and non-membership accuracy functions on IVIFNs are introduced. Some properties of operations on IVIFNs are studied. In Sect. 4, new membership and non-membership accuracy functions on TVIFN are proposed by which the ordering on TVIFN is defined. In Sect. 5, trapezoidal-valued intuitionistic fuzzy index matrix (TVIFIM) is introduced and an algorithmic

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procedure is given to apply the proposed ranking method in MCDM based on multi-criteria TVIFIM. Finally, illustrative example is also studied to show its applicability. In Sect. 6, conclusions and future scope are given.

2 Preliminaries

A short review of preliminaries is given below.

Definition 2.0.1 (Atanassov 1986) An IFS A of a non-empty set X is defined as $A_1 = \{(x, \mu_{A_1}(x), \nu_{A_1}(x))/x \in X\}$ where $\mu_{A_1} : X \to [0, 1]$ and $\nu_{A_1} : X \to [0, 1]$ define the degree of membership $\mu_{A_1}(x)$ and degree of non-membership $\nu_{A_1}(x)$ of x in X to lie in A with $0 \le \mu_{A_1}(x) + \nu_{A_1}(x) \le 1, \forall x \in X$.

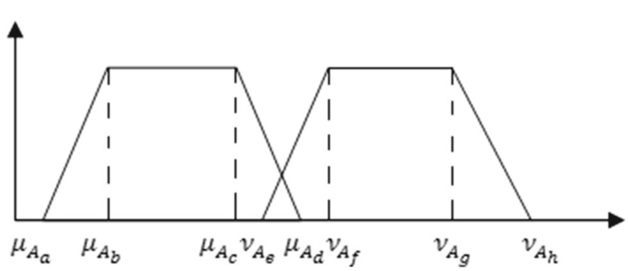
Definition 2.0.2 (Atanassov and Gargov 1989) An IVIFS on a non-empty set X is defined as $A_1 = \{(x, \mu_{A_1}(x), \nu_{A_1}(x) \in X\}$, where $\mu_{A_1}(x) = \left[\underline{\mu}_{A_1(x)}, \overline{\mu}_{A_1(x)}\right]$ and $\nu_{A_1(x)} = \left[\underline{\nu}_{A_1(x)}, \overline{\nu}_{A_1(x)}\right]$ are closed subintervals of [0, 1] which satisfy the condition $0 \le \overline{\mu}_{A_1(x)} + \overline{\nu}_{A_1(x)} \le 1$. The collection of all IVIFS on X is denoted by IVIFS(X). An IVIFS on singleton set is called IVIF number. The collection of all IVIF numbers is denoted by IVIFNs.

Definition 2.0.3 (Nayagam et al. 2008) Let $A = ([\mu_{A_a}, \mu_{A_b}, \mu_{A_c}, \mu_{A_d}],[\nu_{A_e}, \nu_{A_f}, \nu_{A_g}, \nu_{A_h}])$ (where $\nu_{A_e} \ge \mu_{A_c}$ and $\nu_{A_f} \ge \mu_{A_d}$ or $\nu_{A_g} \le \mu_{A_a}$ and $\nu_{A_h} \le \mu_{A_b}$) be a TVIFN. Then the degree of acceptance and degree of rejection functions are defined as

$$\mu_A(x) = \begin{cases} \dfrac{x - \mu_{A_a}}{\mu_{A_b} - \mu_{A_a}} & \mu_{A_a} \le x \le \mu_{A_b} \\ 1 & \mu_{A_b} \le x \le \mu_{A_c} \\ \dfrac{x - \mu_{A_d}}{\mu_{A_c} - \mu_{A_d}} & \mu_{A_c} \le x \le \mu_{A_d} \\ 0 & \text{otherwise} \end{cases};$$

$$\nu_A(x) = \begin{cases} \dfrac{x - \nu_{A_e}}{\nu_{A_f} - \nu_{A_e}} & \nu_{A_e} \le x \le \nu_{A_f} \\ 1 & \nu_{A_f} \le x \le \nu_{A_g} \\ \dfrac{x - \nu_{A_h}}{\nu_{A_g} - \nu_{A_h}} & \nu_{A_g} \le x \le \nu_{A_h} \\ 0 & \text{otherwise.} \end{cases}$$

The graphical representation of TVIFN is shown below



Definition 2.0.4 (Nayagam et al. 2008) The α cut $\mu_A^\alpha, \nu_A^\alpha$ of TVIFN A is defined as $\mu_A^\alpha = \{[\mu_{A_a} + \alpha(\mu_{A_b} - \mu_{A_a}), \mu_{A_d} + \alpha(\mu_{A_c} - \mu_{A_d})]\}$ and $\nu_A^\alpha = \{[\nu_{A_e} + \alpha(\nu_{A_f} - \nu_{A_e}), \nu_{A_h} + \alpha(\nu_{A_g} - \nu_{A_h})]\}$.

Remark 2.0.5 Through out this paper, $A = ([\mu_{A_a}, \mu_{A_b}, \mu_{A_c}, \mu_{A_d}],[\nu_{A_e}, \nu_{A_f}, \nu_{A_g}, \nu_{A_h}])$ denotes the TVIFN with $\nu_{A_e} \ge \mu_{A_c}$ and $\nu_{A_f} \ge \mu_{A_d}$. The similar results for the TVIFN A with $\nu_{A_g} \le \mu_{A_a}$ and $\nu_{A_h} \le \mu_{A_b}$ can be proved analogously, and hence, they are left to the readers.

Definition 2.0.6 (Nayagam et al. 2011) Two TVIFNs, $A = \left(\left[\mu_{A_{a_1}}, \mu_{A_{b_1}}, \mu_{A_{c_1}}, \mu_{A_{d_1}}\right],\left[\nu_{A_{e_1}}, \nu_{A_{f_1}}, \nu_{A_{g_1}}, \nu_{A_{h_1}}\right]\right)$ and $B = \left(\left[\mu_{A_{a_2}}, \mu_{A_{b_2}}, \mu_{A_{c_2}}, \mu_{A_{d_2}}\right],\left[\nu_{A_{e_2}}, \nu_{A_{f_2}}, \nu_{A_{g_2}}, \nu_{A_{h_2}}\right]\right)$ are said to be comparable, $A \le_1 B$, if $\mu_{A_{a_1}} \le \mu_{A_{a_2}}$, $\mu_{A_{b_1}} \le \mu_{A_{b_2}}, \mu_{A_{c_1}} \le \mu_{A_{c_2}}, \mu_{A_{d_1}} \le \mu_{A_{d_2}}; \nu_{A_{e_1}} \ge \nu_{A_{e_2}}, \nu_{A_{f_1}} \ge \nu_{A_{f_2}}, \nu_{A_{g_1}} \ge \nu_{A_{g_2}}, \nu_{A_{h_1}} \ge \nu_{A_{h_2}}$.

Remark 2.0.7 If any one of the inequalities is strict $<$, then $A <_1 B$.

Definition 2.0.8 (Nayagam et al. 2008) Let $A = ([\mu_{A_a}, \mu_{A_b}, \mu_{A_c}, \mu_{A_d}],[\nu_{A_e}, \nu_{A_f}, \nu_{A_g}, \nu_{A_h}])$ be a TVIFN. If $\nu_{A_e} \ge \mu_{A_c}$ and $\nu_{A_f} \ge \mu_{A_d}$, then the intuitionistic fuzzy score of A is defined by (T, NTc), where T and NTc are the membership and the non-membership score of M which are given by $T = \dfrac{(1 + R - L)}{2}$ and $NT_c = \dfrac{(1 - NL + NR)}{2}$ with $R = \dfrac{\mu_{A_d}}{1 + \mu_{A_d} - \mu_{A_c}}$, $L = \dfrac{1 - \mu_{A_a}}{1 + \mu_{A_b} - \mu_{A_a}}$, $NL = \dfrac{\nu_{A_e}}{1 + \nu_{A_e} - \nu_{A_f}}$ and $NR = \dfrac{1 - \nu_{A_h}}{1 + \nu_{A_g} - \nu_{A_h}}$.

Definition 2.0.9 (Xu and Chen 2007) The score function S of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is given as $S(A) = (\mu_{A_a} + \mu_{A_b} - \nu_{A_c} - \nu_{A_d})/2$, where $S(A) \in [-1, 1]$.

Definition 2.0.10 (Xu and Chen 2007) The accuracy function H of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $H(A) = (\mu_{A_a} + \mu_{A_b} + \nu_{A_c} + \nu_{A_d})/2$, where $H(A) \in [0, 1]$.

Definition 2.0.11 (Ye 2009) A novel accuracy function M of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $M(A) = \mu_{A_a} + \mu_{A_b} - 1 + (\nu_{A_c} + \nu_{A_d})/2$, where $M(A) \in [-1, 1]$.

Definition 2.0.12 (Sahin 2015) An improved accuracy function K of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $K(A) = (\mu_{A_a} + \mu_{A_b}(1 - \mu_{A_a} - \nu_{A_c}) + \mu_{A_b} + \mu_{A_a}(1 - \mu_{A_b} - \nu_{A_d})/2$, where $K(A) \in [0, 1]$.

Definition 2.0.13 (Nayagam et al. 2011) An accuracy function L of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $L(A) = ((\mu_{A_a} + \mu_{A_b} - \nu_{A_d}(1 - \mu_{A_b}) - \nu_{A_c}(1 - \mu_{A_a}))/2$, where $L(A) \in [-1, 1]$.

Definition 2.0.14 (Nayagam and Sivaraman 2011) A general accuracy function LG of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $LG(A) = ((\mu_{A_a} + \mu_{A_b})(1-\delta) + \delta(2 - \nu_{A_c} - \nu_{A_d}))/2$, where $LG(A) \in [0, 1]$.

Definition 2.0.15 (Liu and Xia luo 2016) A new accuracy function $A(\alpha)$ of IVIFN $\alpha = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $A(\alpha) = (\mu_{A_a} + \delta_1(1 - \mu_{A_a} - \nu_{A_c}) + \mu_{A_b} + \delta_2(1 - \mu_{A_b} - \nu_{A_d}))/2$, where $A(\alpha) \in [-1, 1]$.

Definition 2.0.16 (Bai 2013) An improved score function I of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $I(A) = (\mu_{A_a} + \mu_{A_a}(1 - \mu_{A_a} - \nu_{A_c}) + \mu_{A_b} + \mu_{A_b}(1 - \mu_{A_b} - \nu_{A_d}))/2$, where $I(A) \in [0, 1]$.

Definition 2.0.17 (Garg 2016) A generalized improved score function GIS of IVIFN $A = ([\mu_{A_a}, \mu_{A_b}], [\nu_{A_c}, \nu_{A_d}])$ is expressed as $GIS(A) = \left(\frac{\mu_{A_a} + \mu_{A_b}}{2} + k_1\mu_{A_a}(1 - \mu_{A_a} - \nu_{A_c}) + k_2\mu_{A_b}(1 - \mu_{A_b} - \nu_{A_d})\right)$, where $GIS(A) \in [0, 1]$.

Definition 2.0.18 (Nayagam et al. 2017) Let $A = ([\mu_{A_a}, \mu_{A_b}, \mu_{A_c}, \mu_{A_d}], [\nu_{A_e}, \nu_{A_f}, \nu_{A_g}, \nu_{A_h}])$ be a trapezoidal intuitionistic fuzzy number. The score function L of A is defined as

$$\begin{aligned} L(A) = [&2(\mu_{A_a} + \mu_{A_b} + \mu_{A_c} + \mu_{A_d}) \\ &-2(\nu_{A_e} + \nu_{A_f} + \nu_{A_g} + \nu_{A_h}) \\ &+(\mu_{A_a} + \mu_{A_b})(\nu_{A_e} + \nu_{A_f}) \\ &+(\mu_{A_c} + \mu_{A_d})(\nu_{A_g} + \nu_{A_h})]/8. \end{aligned}$$

3 Ranking by new accuracy function

In this section, it is showed that the existing accuracy functions proposed by several authors do not give reliable information about alternatives. Therefore, it is necessary to pay attention to this issue and to study other measuring functions. New membership and non-membership accuracy functions for membership degree, non-membership degree by taking the unknown degree (upper hesitancy degree $(1 - a - c)$, lower hesitancy degree $(1 - b - d)$) of IVIFNs are introduced and analyzed by giving illustrative examples to show that the proposed new functions are more reliable in multi-criteria decision process.

Sahin (2015) and Bai (2013) have introduced the new improved accuracy function to rank IVIFNs, and both the authors claim that their method is far better than the existing methods, but unfortunately their methods also fail to rank in some places which is shown in Example 3.0.1.

Example 3.0.1 Illogicality of Sahin's (2015) and Bai's (2013) ranking methods: let $A = ([0, 0], [c_1, d_1])$ and $B = ([0, 0], [c_2, d_2])$ where $c_1 \geq c_2$ and $d_1 \geq d_2$ be two IVIFNs for two alternatives. Clearly $A \leq_1 B$. By applying Definitions 2.0.12 and 2.0.16, we obtain $K(A) = 0 = K(B)$, $I(A) = 0 = I(B)$, which is contradictory.

Garg (2016) has rectified the illogicality of the previous score function and the generalized improved score function to rank IVIFN. But his method also fails to rank in some cases which is shown in Example 3.0.2.

Example 3.0.2 Illogicality of Garg ranking method: let $A = ([0, 0], [0, 0])$ be any IVIFN. By applying Definition 2.0.17, we obtain $\mathrm{GIS}(A) = 0$ which is illogical.

3.1 New accuracy functions on IVIFN

In this subsection, new accuracy functions for membership degree, non-membership degree by taking the unknown degree (upper hesitancy degree $(1 - a - c)$, lower hesitancy degree $(1 - b - d)$) are defined and operations on IVIFNs are studied.

Definition 3.1.1 Let $A = ([a, b], [c, d])$ be an IVIFN, and the membership accuracy function NA_1 of an IVIFN based on the upper hesitancy degree and non-membership accuracy function NA_2 of an IVIFN based on the lower hesitancy degree are defined by membership accuracy function $= NA_1(A) = \frac{a + b + \delta(1 - a - c)}{2}$ and non-membership accuracy function $= NA_2(A) = \frac{c + d + \delta'(1 - b - d)}{2}$, where $\delta, \delta' \in [0, 1]$ with $\delta + \delta' \leq 1$ are optimistic and pessimistic parameters which depends on the individuals level of confidence.

Remark 3.1.2 Let $A = ([a, b], [c, d])$ be an IVIFN. When $\delta = 1$ and $\delta' = 0$ (for an optimist), $NA_1(A) = (1 + b - c)/2$, $NA_2(A) = (c + d)/2$. When $\delta = 0$ and $\delta' = 1$ (for a pessimist), $NA_1(A) = (a + b)/2$, $NA_2(A) = (1 - b + c)/2$. When $\delta = 1/2$ and $\delta' = 1/2$, $NA_1(A) = (1 + a + 2b - c)/4$, $NA_2(A) = (1 - b + 2c + d)/4$.

Remark 3.1.3 Let $A = a$ be any fuzzy number defined on singleton set. By considering $A = ([a, a], [1 - a, 1 - a])$ in the form of IVIFN, we have $NA_1(A) = a$ and $NA_2(A) = 1 - a$. Since there is no hesitation on fuzzy numbers, membership and non-membership accuracy functions on fuzzy numbers are independent of δ and δ'. It is also noted that $NA_1(A) + NA_2(A) = 1$ for fuzzy numbers.
The intuitionistic fuzzy value (IFV) $A = (a, c)$ with $a + c \leq 1$ defined on singleton set is considered as IVIFN of the form $A = ([a, a], [c, c])$ and the proof of the following is immediate from the definition.

Theorem 3.1.4 *Let $A = (a, c)$ be any intuitionistic fuzzy value (IFV). Then* $NA_1(A) = a + \frac{\delta(1-a-c)}{2}$ *and* $NA_2(A) = c + \frac{\delta'(1-a-c)}{2}$.

The interval-valued fuzzy number (IVFN) $A = [a, b]$ defined on singleton set is in the interval form $A = ([a, b], [1-b, 1-a])$ which is not an IVIFN since $b+1-a > 1$ except at $a = b$. Hence $NA_1(A)$ and $NA_2(A)$ for an IVFN are not applicable except at $a = b$.

Remark 3.1.5 The proofs of the following results which favor our intuition are immediate applications of the definition, and hence, proofs are omitted.
if $A = ([1, 1], [0, 0])$, $NA_1(A) = 1$, $NA_2(A) = 0$
if $A = ([0, 0], [1, 1])$, $NA_1(A) = 0$, $NA_2(A) = 1$
if $A = ([0, 0], [0, 0])$, $NA_1(A) = \frac{\delta}{2}$, $NA_2(A) = \frac{\delta'}{2}$.

Theorem 3.1.6 *Let $A = ([a_1, b_1], [c_1, d_1])$ and $B = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs. Let $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$.*
Then $\lambda_1 A + \lambda_2 B = ([\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 b_1 + \lambda_2 b_2], [\lambda_1 c_1 + \lambda_2 c_2, \lambda_1 d_1 + \lambda_2 d_2])$.

Remark 3.1.7 Let $A = ([a_1, b_1], [c_1, d_1])$ and $B = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs. By definition, we know that $b_1 + d_1 \leq 1$ and $b_2 + d_2 \leq 1$. Hence $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_1 d_1 + \lambda_2 d_2 = \lambda_1(b_1+d_1) + \lambda_2(b_2+d_2) \leq \lambda_1 + \lambda_2 \leq 1$, and hence, the operation is well defined and the resultant is again an IVIFN. So the following theorem is meaningful and the proof is immediate by the definition.

Theorem 3.1.8 *Let $A = ([a_1, b_1], [c_1, d_1])$ and $B = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs. Then*

1. $NA_1(\lambda_1 A + \lambda_2 B) = \lambda_1 NA_1(A) + \lambda_2 NA_1(B)$
2. $NA_2(\lambda_1 A + \lambda_2 B) = \lambda_1 NA_2(A) + \lambda_2 NA_2(B)$

Theorem 3.1.9 *For any two IVIFNs, A, B if $A \leq_1 B$, i.e., $a_1 \leq a_2, b_1 \leq b_2, c_1 \geq c_2, d_1 \geq d_2$, then*

1. $NA_1(A) \leq NA_1(B)$,
2. $NA_2(A) \geq NA_2(B)$.

Proof Let $A = ([a_1, b_1], [c_1, d_1])$ and $B = ([a_2, b_2], [c_2, d_2])$ be two comparable IVIFNs such that $A \leq_1 B$. By definition, $NA_1(A) = \frac{a_1 + b_1 + \delta(1 - a_1 - c_1)}{2}$ and $NA_1(B) = \frac{a_2 + b_2 + \delta(1 - a_2 - c_2)}{2}$. Now $2(NA_1(A) - NA_1(B)) = a_1 + b_1 + \delta(1 - a_1 - c_1) - (a_2 + b_2 + \delta(1 - a_2 - c_2)) = ([(a_1 - a_2)(1 - \delta) + (b_1 - b_2)] + \delta[(c_2 - c_1)])$. Since $A \leq_1 B, a_1 \leq a_2, b_1 \leq b_2$ and $c_1 \geq c_2$ and $\delta \in [0, 1]$, $2(NA_1(A) - NA_1(B)) \leq 0$. Hence $NA_1(A) \leq NA_1(B)$. Similarly $NA_2(A) \geq NA_2(B)$. □

Definition 3.1.10 Let A and B be two IVIFNs. Then the ranking principle is defined as follows: If $NA_1(A) < NA_1(B)$, then $A < B$. If $NA_1(A) = NA_1(B)$ and if $NA_2(A) > NA_2(B)$, then $A < B$.

Remark 3.1.11 The class C of all comparable IVIFNs are completely ranked by the definition of ranking principle $<$.

Proof Let $A = ([a_1, b_1], [c_1, d_1])$ and $B = ([a_2, b_2], [c_2, d_2]) \in C$ such that $A \neq B$. Let us assume that $A <_1 B$. If any one of the inequalities $a_1 \leq a_2, b_1 \leq b_2, c_1 \geq c_2$ is a strict inequality, by the above theorem, $(NA_1(A) - NA_1(B)) < 0$, and hence, $NA_1(A) < NA_1(B)$. So $A < B$. If $d_2 < d_1$ and $a_1 = a_2, b_1 = b_2, c_1 = c_2$, then $NA_1(A) = NA_1(B)$ and $NA_2(A) > NA_2(B)$. So $A < B$. Hence the above definition of ranking principle $<$ is a total order on the class of comparable IVIFNs. □

3.2 Significance of the proposed method

In this subsection, the proposed definition of ranking is applied through numerical example to show the validity and significance of the proposed method.

Example 3.2.1 In Example 3.0.1, $A = ([0, 0], [c_1, d_1])$ and $B = ([0, 0], [c_2, d_2])$, where $c_1 \geq c_2$ and $d_1 \geq d_2$ be two IVIFNs for two alternatives. It is clear that $A \leq_1 B$. By Definition 3.1.1, we obtain $NA_1(A) \leq NA_1(B)$, for all δ and $NA_2(A) \geq NA_2(B)$, for all δ'.

Example 3.2.2 In Example 3.0.2, $A = ([0, 0], [0, 0])$. By Definition 3.1.1, we obtain $NA_1(A) = \frac{\delta}{2}$, $NA_2(A) = \frac{\delta'}{2}$ which supports our expectation.

Table 1 shows the drawbacks of existing methods and the efficiency of proposed accuracy function. In Table 1, first column shows that the ranking methods presented by many authors, the second column illustrates the illogicality of various ranking methods in comparing arbitrary IVIFNs, and finally, the third column shows the significance of our proposed accuracy function. When most of the existing methods give anti-intuitive results, the proposed method gives better result. In many methods, the ranking depends on IVIFNs only and not on the expert's degree of optimism and degree of pessimism. But the proposed method considers expert's degree of optimism on measuring membership score and degree of pessimism on measuring non-membership score to have optimistic membership and non-membership scores for ranking.

4 New accuracy functions on TVIFN

In this section, the new membership and non-membership accuracy functions on TVIFNs by extending membership

Table 1 Significance of the proposed method

Existing methods	Anti-intuitive results of existing methods	Proposed method, NA_1 and NA_2
Xu and Chen (2007)	$A=([0.1,0.3],[0.4,0.5]),\ B=([0.2,0.2],[0.2,0.7])$	$NA_1(A)=0.2+\delta(0.25)$
$S(A)=(\mu_{A_a}+\mu_{A_b}-\nu_{A_c}-\nu_{A_d})/2$	$S(A)=-0.25=S(B)$	$\leq NA_1(B)=0.2+\delta(0.3)$
	Hence we go for accuracy function	$\Rightarrow A\leq B$
$H(A)=(\mu_{A_a}+\mu_{A_b}+\nu_{A_c}+\nu_{A_d})/2$	$H(A)=0.65=H(B)\Rightarrow A=B$	
Ye (2009)	$A=([0.1,0.3],[0.4,0.5]),\ B=([0.2,0.2],[0.2,0.7])$	$NA_1(A)=0.2+\delta(0.25)$
$M(A)=\mu_{A_a}+\mu_{A_b}-1+$	$M(A)=-0.15=M(B)$	$\leq NA_1(B)=0.2+\delta(0.3)$
$(\nu_{A_c}+\nu_{A_d})/2$		$\Rightarrow A\leq B$
Nayagam and Sivaraman (2011)	$A=([0.1,0.3],[0.4,0.5]),\ B=([0.2,0.2],[0.2,0.7])$	$NA_1(A)=0.2+\delta(0.25)$
$LG(A)=((\mu_{A_a}+\mu_{A_b})(1-\delta)$	$LG(A)=(0.2)(1-\delta)+\delta(0.55)=LG(B)$	$\leq NA_1(B)=0.2+\delta(0.3)$
$+\delta(2-\nu_{A_c}-\nu_{A_d}))/2$		$\Rightarrow A\leq B$
Sahin (2015)	$A=([0,0],[c_1,d_1]),\ B=([0,0],[c_2,d_2])$	$NA_1(A)=\delta(1-c_1)$
$K(A)=(\mu_{A_a}+\mu_{A_b}(1-\mu_{A_a}-\nu_{A_c})+$	$K(A)=0=K(B)$	$< NA_1(B)=\delta(1-c_2)$
$\mu_{A_b}+\mu_{A_a}(1-\mu_{A_b}-\nu_{A_d})/2$	when $c_1>c_2$ and $d_1>d_2\Rightarrow A<B$	When $c_1>c_2\Rightarrow A<B$
Bai (2013)	$A=([0,0],[c_1,d_1]),\ B=([0,0],[c_2,d_2])$	$NA_1(A)=\delta(1-c_1)$
$I(A)=(\mu_{A_a}+\mu_{A_a}(1-\mu_{A_a}-\nu_{A_c})+$	$I(A)=0=I(B)$	$< NA_1(B)=\delta(1-c_2)$
$\mu_{A_b}+\mu_{A_b}(1-\mu_{A_b}-\nu_{A_d}))/2$	when $c_1>c_2$ and $d_1>d_2\Rightarrow A<B$	When $c_1>c_2\Rightarrow A<B$
Garg (2016)	$A=([0,0],[0,0]), GIS(A)=0$	$NA_1(A)=\frac{\delta}{2}$
$GIS(A)=\left(\frac{\mu_{A_a}+\mu_{A_b}}{2}+k_1\mu_{A_a}\right.$		
$\left.(1-\mu_{A_a}-\nu_{A_c})+k_2\mu_{A_b}(1-\mu_{A_b}-\nu_{A_d})\right)$		
Sivaraman et al. (2014)	$A=([0,0.4],[0.2,0.3]),$	$NA_1(A)=0.2+\delta(0.4)$
$L(A)=(\mu_{A_a}+\mu_{A_b}-\nu A_c-\nu A_d+$	$B=\left(\left[0.1,\frac{\sqrt{0.37}+0.1}{2}\right],\left[0.3,\frac{\sqrt{0.37}-0.1}{2}\right]\right)$	$< NA_1(B)=0.23+\delta(0.3)$
$\mu_{A_a}\nu_{A_c}+\mu_{A_b}\nu_{A_d})/2$	$L(A)=0.01=L(B)$	$\Rightarrow A<B$
$LG(A)=(-\mu_{A_a}-\mu_{A_b}+\nu A_c+$	$LG(A)=0.11=LG(B)$	
$\nu A_d+\mu_{A_a}\nu_{A_c}+\mu_{A_b}\nu_{A_d})/2$	$P(A)=-0.09=P(B)$	
$P(A)=(\mu_{A_a}-\mu_{A_b}-\nu A_c+$	$IP(A)=0.31\geq IP(B)=0.13$	
$\nu A_d+\mu_{A_a}\nu_{A_c}+\mu_{A_b}\nu_{A_d})/2$		
$IP(A)=(-\mu_{A_a}+\mu_{A_b}-\nu A_c+$		
$\nu A_d-\mu_{A_a}\nu_{A_c}+\mu_{A_b}\nu_{A_d})/2$		

and non-membership accuracy functions on IVIFNs by considering upper hesitancy degree and lower hesitancy degree of TVIFN are defined and operations on TVIFNs are studied by which (ave, ave) column aggregation operator for trapezoidal-valued intuitionistic fuzzy index matrix is defined in the next section.

Definition 4.0.1 Let $A = ([a, b, c, d], [e, f, g, h])$ be a TVIFN. The new membership accuracy function NA_1 and non-membership accuracy function NA_2 of a TVIFN based on the upper and lower hesitancy degrees are defined as

New membership accuracy function

$$NA_1(A)=\frac{a+b+c+d+\delta(2-(a+b+e+f))}{4}$$

New non-membership accuracy function

$$NA_2(A)=\frac{e+f+g+h+\delta^{'}(2-(c+d+g+h))}{4},$$

where $\delta, \delta^{'} \in [0, 1]$ with $\delta+\delta^{'} \leq 1$ are parameters depending on the individual optimistic and pessimistic intention.

Definition 4.0.2 Let A and B be two TVIFNs. Then the ranking principle is defined as follows: If $NA_1(A) < NA_1(B)$, then $A < B$. If $NA_1(A) = NA_1(B)$ and if $NA_2(A) > NA_2(B)$, then $A < B$.

The above definition is applied in the following examples to study how it works.

Example 4.0.3 Let $A = ([0.15, 0.2, 0.25, 0.3], [0.4, 0.45, 0.5, 0.7])$ and $B = ([0.2, 0.25, 0.3, 0.35], [0.36, 0.4, 0.45, 0.6])$ be two TVIFNs for two alternatives. Clearly $A \leq_1 B$. By applying Definition 4.0.1, we obtain $NA_1(A) = 0.225+\delta(0.2)$, $NA_1(B) = 0.275+\delta(0.198) \Rightarrow NA_1(A) \leq NA_1(B)$, for all δ.

Example 4.0.4 Let $A = ([0, 0, 0, 0], [0, 0, g_1, h_1])$ and $B = ([0, 0, 0, 0], [0, 0, g_2, h_2])$, where $g_1 \geq g_2$ and $h_1 \geq h_2$ be two TVIFNs for two alternatives. Clearly $A \leq_1 B$. By applying Definition 4.0.2, we obtain $NA_1(A) = NA_1(B)$, for all δ. But $A \neq B$ which is illogical.

But, we get $NA_2(A) = (g_1 + h_1 + \delta^{'}(2 - (g_1 + h_1)))/4$, $NA_2(B) = (g_2 + h_2 + \delta^{'}(2 - (g_2 + h_2)))/4$. So, we have $NA_2(A) - NA_2(B) = g_1 - g_2 + h_1 - h_2 - \delta^{'}(g_1 - g_2) - \delta^{'}(h_1 - h_2) = (1 - \delta^{'})(g_1 - g_2 + h_1 - h_2) \geq 0$. Hence $NA_2(A) \geq NA_2(B)$, which favors our intuition.

Remark 4.0.5 When $\delta = 1$ and $\delta^{'} = 0$, $NA_1(A) = (2 + c + d - (e + f))/4$, $NA_2(A) = (e + f + g + h)/4$. When $\delta = 0$ and $\delta^{'} = 1$, $NA_1(A) = (a + b + c + d)/4$, $NA_2(A) = (2 - (c + d) + e + f)/4$. When $\delta = 1/2$ and $\delta^{'} = 1/2$, $NA_1(A) = (2 + a + b + 2(c + d) - (e + f))/8$, $NA_2(A) = (2 - (c + d) + 2(e + f) + g + h)/8$.

Remark 4.0.6 Let $A = a$ be any fuzzy value defined on singleton set. By considering $A = [a, a, a, a], [1 - a, 1 - a, 1 - a, 1 - a]$ in the form of TVIFN, we have $NA_1(A) = a$ and $NA_2(A) = 1 - a$. Since there is no hesitation on fuzzy numbers, membership and non-membership accuracy functions on fuzzy numbers are independent of δ and $\delta^{'}$. It is also noted that $NA_1(A) + NA_2(A) = 1$ for fuzzy numbers on singleton.

The IFN $A = (a, c)$ defined on a singleton set is considered as TVIFN of the form $A = ([a, a, a, a], [c, c, c, c])$, and the proof of the following theorem is immediate from the definition.

Theorem 4.0.7 *Let $A = (a, c)$ be any IFN. Then* $NA_1(A) = a + \dfrac{\delta(1 - a - c)}{2}$ *and* $NA_2(A) = c + \dfrac{\delta^{'}(1 - a - c)}{2}$.

The interval fuzzy number $A = [a, b]$ with $b \leq \frac{1}{2}$ or $a \geq \frac{1}{2}$ defined on a singleton set is a TVIFN in the form $A = [a, a, a, a], [1 - b, 1 - b, 1 - a, 1 - a]$, and hence, the proof of the following theorem is immediate from the definition.

Theorem 4.0.8 *Let $A = [a, b]$ be any IVFN. Then* $NA_1(A) = \dfrac{a + b + \delta(b - a)}{2}$ *and* $NA_2(A) = 1 - \left[\dfrac{a + b + \delta^{'}(b - a)}{2}\right]$.

The proof of the following result is an immediate application of the definition which are omitted.

Theorem 4.0.9 *When $A = 1 = ([1, 1, 1, 1], [0, 0, 0, 0])$, then the new accuracy function $NA_1(A) = 1$, $NA_2(A) = 0$, When $A = 0 = ([0, 0, 0, 0], [1, 1, 1, 1])$, then the new accuracy function*
$NA_1(A) = 0$, $NA_2(A) = 1$,

When $A = [0, 0] = ([0, 0, 0, 0], [0, 0, 0, 0])$, then the new accuracy function
$NA_1(A) = \dfrac{\delta}{2}$, $NA_2(A) = \dfrac{\delta^{'}}{2}$, *which supports our intuition.*

Definition 4.0.10 Let $A = ([a_1, b_1, c_1, d_1], [e_1, f_1, g_1, h_1])$ and $B = ([a_2, b_2, c_2, d_2], [e_2, f_2, g_2, h_2])$ be two TVIFNs. Then

1. $A + B = ([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2], [e_1 + e_2, f_1 + f_2, g_1 + g_2, h_1 + h_2])$.
2. $A * B = ([a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2], [e_1 e_2, f_1 f_2, g_1 g_2, h_1 h_2])$.
3. $\lambda A = ([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1], [\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1])$, $\lambda > 0$.

Remark 4.0.11 Let $A = ([a_1, b_1, c_1, d_1], [e_1, f_1, g_1, h_1])$ and $B = ([a_2, b_2, c_2, d_2], [e_2, f_2, g_2, h_2])$ be two TVIFNs. By remark 2.0.5, we know that $c_1 \leq e_1$ and $d_1 \leq f_1$. Hence $c_1 + c_2 \leq e_1 + e_2$ and $d_1 + d_2 \leq f_1 + f_2$. Hence $A + B$ is a TVIFN, and hence, the operation $+$ is well defined. Similarly we can prove for (2) and (3).

The proof of the following results is an immediate application of the definition which are omitted.

Theorem 4.0.12 *Let $A = ([a_1, b_1, c_1, d_1], [e_1, f_1, g_1, h_1])$ and $B = ([a_2, b_2, c_2, d_2], [e_2, f_2, g_2, h_2])$ be two TVIFVs. Then*

1. $$NA_1(A + B) = \frac{a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2 + \delta(2 - (a_1 + a_2 + b_1 + b_2 + e_1 + e_2 + f_1 + f_2))}{4}.$$
$$= NA_1(A) + NA_1(B) - \frac{\delta}{2}$$
2. $$NA_2(A + B) = \frac{e_1 + e_2 + f_1 + f_2 + g_1 + g_2 + h_1 + h_2 + \delta^{'}(2 - (c_1 + c_2 + d_1 + d_2 + g_1 + g_2 + h_1 + h_2))}{4}.$$
$$= NA_2(A) + NA_2(B) - \frac{\delta'}{2}$$
3. $$NA_1(A * B) = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 + \delta(2 - (a_1 a_2 + b_1 b_2 + e_1 e_2 + f_1 f_2))}{4}.$$

4. $NA_2(A*B) = \dfrac{e_1e_2 + f_1f_2 + g_1g_2 + h_1h_2 + \delta^{'}(2 - (c_1c_2 + d_1d_2 + g_1g_2 + h_1h_2))}{4}$.

5. $$NA_1(\lambda A) = \frac{\lambda a_1 + \lambda b_1 + \lambda c_1 + \lambda d_1 + \delta(2 - (\lambda a_1 + \lambda b_1 + \lambda e_1 + \lambda f_1))}{4} = \lambda[NA_1(A)] + \frac{\delta}{2}(1-\lambda), \quad \forall \lambda > 0.$$

6. $$NA_2(\lambda A) = \frac{\lambda e_1 + \lambda f_1 + \lambda g_1 + \lambda h_1 + \delta^{'}(2 - (\lambda c_1 + \lambda d_1 + \lambda g_1 + \lambda h_1))}{4} = \lambda[NA_2(A)] + \frac{\delta'}{2}(1-\lambda), \quad \forall \lambda > 0.$$

Theorem 4.0.13 *For any two TVIFNs, A, B if $A \le_1 B$. Then $(i).NA_1(A) \le NA_1(B)$, $(ii).NA_2(A) \ge NA_2(B)$.*

Proof Let $A = ([a_1, b_1, c_1, d_1], [e_1, f_1, g_1, h_1])$ and $B = ([a_2, b_2, c_2, d_2], [e_2, f_2, g_2, h_2])$ be two comparable TVIFNs such that $A \le_1 B$.
By definition,

$$NA_1(A) = \frac{(a_1+b_1+c_1+d_1 + \delta(2 - (a_1 + b_1 + e_1 + f_1))}{4}$$

and

$$NA_1(B) = \frac{(a_2 + b_2+c_2+d_2+\delta(2 - (a_2 + b_2 + e_2 + f_2))}{4}.$$

Now $4(NA_1(A) - NA_1(B)) = a_1 + b_1 + c_1 + d_1 + \delta(2 - (a_1+b_1+e_1+f_1))) - (a_2+b_2+c_2+d_2+\delta(2-(a_2+b_2+e_2+f_2))) = ([(a_1 - a_2) + (b_1 - b_2)](1-\delta) + (c_1 - c_2) + (d_1 - d_2) + \delta[(e_2 - e_1) + (f_2 - f_1)])$. Since $A \le_1 B, a_1 \le a_2, b_1 \le b_2, c_1 \le c_2, d_1 \le d_2$ and $e_1 \ge e_2, f_1 \ge f_2, g_1 \ge g_2, h_1 \ge h_2$ and $\delta \in [0, 1], 4(NA_1(A) - NA_1(B)) \le 0$. Hence $NA_1(A) \le NA_1(B)$. Similarly $NA_2(A) \ge NA_2(B)$. □

4.1 Significance of the proposed method

In this subsection, the proposed definition of ranking is applied through numerical example to show the validity and significance of the proposed method.

In the literature, the study of TVIFNs is limited and in infant stage. The ranking of IVIFNs as a particular case of TVIFNs is rich in the literature. The ranking of TVIFNs is studied in Nayagam et al. (2016a, 2017, 2008). The importance of our proposed method over existing methods is explained with example. In many methods, the ranking depends only on TVIFNs and not on the expert's degree of optimism and degree of pessimism. But the proposed method includes expert's degree of optimism on measuring membership score and the degree of pessimism on measuring non-membership score to have optimistic membership and non-membership scores for ranking.

4.1.1 Comparison between our proposed accuracy function with a ranking of intuitionistic fuzzy numbers in Nayagam et al. (2008)

Our proposed method is compared with ranking of intuitionistic fuzzy numbers in Nayagam et al. (2008).

Let $A = (a_1, c_1)$ and $B = (a_2, c_2)$ be two IFNs. By applying definition in (2.0.8), the intuitionistic fuzzy scores of A and B are $T(A) = (a_1, 1 - c_1)$ and $T(B) = (a_2, 1 - c_2)$. If $A \le B$, that is, $a_1 \le a_2$ and $c_1 \ge c_2$, then intuitionistic fuzzy score method cannot rank. But by applying our proposed method, $NA_1(A) \le NA_1(B)$ for every δ, and hence, we obtain $A \le B$.

4.1.2 Comparison between our proposed accuracy function with a complete of incomplete trapezoidal information in Nayagam et al. (2016a)

Our proposed method is compared with a complete ranking of incomplete trapezoidal information in Nayagam et al. (2016a).

Let $A = [(0.3, 0.3, 0.4, 0.4), (0.6, 0.6, 0.6, 0.6)]$ and B=[(0.256608438, 0.256608438, 0.42783361, 0.42783361), (0.506608438, 0.506608438, 0.67783361, 0.67783361)] be two TVIFNs. $L(A) = -0.04 = L(B), LG(A) = 0.46 = LG(B), P_1(A) = -0.8 = P_1(B) \Rightarrow A = B, P_2(A) = 0.02, P_2(B) = -0.08$. It is noted that the evaluations of the TVIFN are equal when $L(A)$, $LG(A)$, $P_1(A)$ and $P_2(A)$ is used. It is found that $P_2(A) \ne P_2(B)$ only in P_2, and hence, $A \le B$ which is much laborious. But we can apply the proposed method $NA_2(A) = 0.6$ and $NA_2(B) = 0.592221024 - \delta'(0.05283361)$, for every $\delta' \in [0, 1] \Rightarrow A < B$.

4.1.3 Comparison between our proposed accuracy function with ranking of incomplete trapezoidal information Nayagam et al. (2017)

Our proposed method is compared with ranking of incomplete trapezoidal information in Nayagam et al. (2017).

Let $A = [(0.3, 0.3, 0.4, 0.4), (0.6, 0.6, 0.6, 0.6)]$ and B=[(0.256608438, 0.256608438, 0.42783361, 0.42783361), (0.506608438, 0.506608438, 0.67783361, 0.67783361)] be two TVIFNs. $L(A) = -0.04 = L(B), LG(A) = 0.46 = LG(B) \Rightarrow A = B, P(A) = 0.16, P(B) = 0.21$. It is noted that the evaluations of the TVIFN are equal when $L(A), LG(A)$ are used. It is found that $P(A) \neq P(B)$ only in $P(A)$, and hence, $A \leq B$ which is much laborious. But we can apply the proposed method $NA_2(A) = 0.6$ and $NA_2(B) = 0.592221024 - \delta'(0.05283361)$, for every $\delta' \in [0, 1] \Rightarrow A < B$.

5 Application of the proposed accuracy function in multi-criteria decision-making problem using index matrix

The concept of index matrix (IM) was introduced by Atanassov (1987). Let I be a fixed set of indices and R be the set of all real numbers. Let $K = \{k_1, k_2, \ldots, k_m\}, L = \{l_1, l_2, \ldots, l_n\} \subset I$. The general form of IM with real numbers $R - IM$ is given as

$[K, L, \{a_{k_i,l_j}\}] =$

	$l_1 \cdots$	$l_j \cdots$	l_n
k_1	$a_{k_1,l_1} \cdots$	$a_{k_1,l_j} \cdots$	a_{k_1,l_n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_i	$a_{k_i,l_1} \cdots$	$a_{k_i,l_j} \cdots$	a_{k_i,l_n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_m	$a_{k_m,l_1} \cdots$	$a_{k_m,l_j} \cdots$	a_{k_m,l_n}

where for $(1 \leq i \leq m$ and $1 \leq j \leq n) : a_{k_i,l_j} \in R$. In the above index matrix, if $a_{k_i,l_j} \in [0, 1]$, then it is called $(0, 1)-$IM.

Further, $(0, 1)-$ IM was extended to intuitionistic fuzzy index matrix (IFIM) by Atanassov (2010). Let $K = \{k_1, k_2, \ldots, k_m\}, L = \{l_1, l_2, \ldots, l_n\} \subset I$. The general form of IFIM is given by $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}] =$

	$l_1 \cdots$	$l_j \cdots$	l_n
k_1	$\langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle \cdots$	$\langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle \cdots$	$\langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_i	$\langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle \cdots$	$\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle \cdots$	$\langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_m	$\langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle \cdots$	$\langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle \cdots$	$\langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle$

where, for every $1 \leq i \leq m$ and $1 \leq j \leq n, 0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1.\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ is an intuitionistic fuzzy pair.

Now, we extend IFIM to trapezoidal intuitionistic fuzzy index matrix (TVIFIM) as follows.

Definition 5.0.1 Let I be the fixed set of indices. Let $K = \{k_1, k_2, \ldots, k_m\}, L = \{l_1, l_2, \ldots, l_n\} \subset I$. The general form of TVIFM is given by $[K, L, \{\langle (a, b, c, d)_{k_i,l_j}, (e, f, g, h)_{k_i,l_j} \rangle\}] =$

	$l_1 \cdots$	$l_j \cdots$	l_n
k_1	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_1,l_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_1,l_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_1,l_n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_i	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_i,l_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_i,l_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_i,l_n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k_m	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_m,l_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_m,l_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{k_m,l_n}$

where, for every $1 \leq i \leq m, 1 \leq j \leq n, \langle (a, b, c, d), (e, f, g, h) \rangle_{k_i,l_j}$ is a trapezoidal intuitionistic fuzzy number.

Using IFIM, Deyan Marrov, Vassia Atanssova and Atanssova introduced an inter-criteria multi-criteria decision making based on IFIM in Atanassov et al. (2014). Now by using TVIFIM defined above, we introduce multi-criteria decision making based on TVIFIM as follows.

Definition 5.0.2 Let I be a fixed set of indices and R be the set of all real numbers. Let $G = \{G_1, G_2, \cdots, G_m\}, H = \{H_1, H_2, \cdots, H_n\} \subset I$. The general form of multi-criteria decision making based on TVIFIM A_{TVIFIM} is given as

	$H_1 \cdots$	$H_j \cdots$	H_n
G_1	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_1,H_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_1,H_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_1,H_n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
G_i	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_i,H_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_i,H_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_i,H_n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
G_m	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_m,H_1} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_m,H_j} \cdots$	⟨(a,b,c,d),(e,f,g,h)⟩$_{G_m,H_n}$

where for every $p, q (1 \leq p \leq m, 1 \leq q \leq n)$, G_p is the object being evaluated and H_q is the criterion taking part in the evaluation and $\langle (a, b, c, d), (e, f, g, h) \rangle_{G_p, H_q}$ is a TVIFIM that is comparable by the ranking principle $<$ defined in Definition 4.0.2

There are 18 aggregation operators introduced in Atanassov (2013b) in which we consider (ave, ave) column aggregation for given IFIM $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$. Let $l_0 \notin L$ be a fixed index. The (ave, ave) column aggregation for given IFIM $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$ is defined by

$$\sigma_{\max}([K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], l_0) = \begin{array}{c|c} & l_0 \\ \hline k_1 & \left\langle \frac{1}{n} \sum_{j=1}^{n} \mu_{k_1,l_j}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_1,l_j} \right\rangle \\ \vdots & \vdots \\ k_i & \left\langle \frac{1}{n} \sum_{j=1}^{n} \mu_{k_i,l_j}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_i,l_j} \right\rangle \\ \vdots & \vdots \\ k_m & \left\langle \frac{1}{n} \sum_{j=1}^{n} \mu_{k_m,l_j}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_m,l_j} \right\rangle \end{array}$$

Now we introduce (ave, ave) column aggregation for a given multi-criteria TVIFIM (A_{TVIFIM}) using 4.0.10.

Definition 5.0.3 Let $H_0 \notin G$ be a fixed index. The (ave, ave) column aggregation for a given TVIFIM (A_{TVIFIM}) is defined by

$$\sigma_{\max}(A_{\text{TVIFIM}}, H_0) = \begin{array}{c|c} & H_0 \\ \hline G_1 & \left\langle \frac{1}{n} \sum_{j=1}^{n} (a, b, c, d)_{k_1,l_j}, \frac{1}{n} \sum_{j=1}^{n} (e, f, g, h)_{k_1,l_j} \right\rangle \\ \vdots & \vdots \\ G_i & \left\langle \frac{1}{n} \sum_{j=1}^{n} (a, b, c, d)_{k_i,l_j}, \frac{1}{n} \sum_{j=1}^{n} (e, f, g, h)_{k_i,l_j} \right\rangle \\ \vdots & \vdots \\ G_m & \left\langle \frac{1}{n} \sum_{j=1}^{n} (a, b, c, d)_{k_m,l_j}, \frac{1}{n} \sum_{j=1}^{n} (e, f, g, h)_{k_m,l_j} \right\rangle \end{array}$$

5.1 Algorithm for multi-criteria TVIFIM

The algorithmic procedure for the proposed method for the multi-criteria TVIFIM (A_{TVIFIM}) can be summarized as follows:

1. Obtain (ave, ave) column aggregation for a given multi-criteria TVIFIM (A_{TVIFIM}) which gives the aggregated trapezoidal score for G_i for every $1 \leq i \leq m$.
2. Compute the score value of $\sigma_{\max}(A_{\text{TVIFIM}}, H_0)$ corresponding to G_i by using Definition 2.0.18 for $i = 1, 2, \ldots, m$. Let a_i be the number of A_{TVIFIM} whose order is not changed by their score values, and we take $\delta = \max\limits_{1 \leq i \leq m} \left\{ \frac{a_i}{n} \right\}$ and $\delta' = \min\limits_{1 \leq i \leq m} \left\{ \frac{a_i}{n} \right\}$.
3. Compute the membership accuracy values $NA_1(G_i, H_0)$ $(i = 1, 2, \ldots, m)$ and $NA_2(G_p, H_0)$ for which NA_1 $(G_p, H_0) = NA_1(G_i, H_0)$, $(1 \leq p, i \leq n)$ using Definition 4.0.1.
4. Rank the alternatives $G_i (i = 1, 2, \ldots, m)$ using Definition 4.0.2.

5.2 Illustrative example

Now a numerical illustration of the algorithm for multi-criteria TVIFIM is given.

Example 5.2.1 There is a panel with four possible objects to invest the money: (1). G_1 is a cement firm; (2). G_2 is a computer firm; (3). G_3 is an alternating current firm; and (4). G_4 is a chemical firm. The investment company must take a decision according to the following three criteria: (1). H_1 is the sensitivity analysis; (2). H_2 is the cost benefit analysis; and (3). H_3 is the credit analysis. Four possible objects $G_p (1 \leq p \leq 4)$ are evaluated under the above three criteria $H_q (1 \leq q \leq 3)$ using the TVIFNs by the panel which is the multi-criteria TVIFIM (A_{TVIFIM}) given in Table 2 from which the best object is chosen.

1. By applying step 1 of the above algorithm, we obtain (ave, ave) column aggregation for the given multi-criteria TVIFIM (A_{TVIFIM}) with a fixed index $H_0 \notin G$ as follows:

$$\sigma_{\max}(A_{\text{TVIFIM}}, H_0) = \begin{array}{c|c} & H_0 \\ \hline G_1 & \langle (0.1500, 0.2000, 0.2500, 0.3000), (0.3367, 0.4000, 0.4500, 0.5167) \rangle \\ G_2 & \langle (0.1033, 0.1833, 0.2333, 0.2833), (0.3333, 0.3833, 0.4333, 0.4833) \rangle \\ G_3 & \langle (0.1667, 0.2333, 0.2833, 0.3333), (0.3833, 0.4333, 0.5000, 0.5667) \rangle \\ G_4 & \langle (0.1333, 0.2000, 0.2667, 0.3167), (0.3833, 0.4500, 0.5167, 0.5833) \rangle \end{array}$$

2. Compute the score value of A_{TVIFIM} by using Definition 2.0.18 for $i = 1, 2, \ldots, m$. Let a_i be the number of A_{TVIFIM} whose order is not changed by their score values and we take $\delta = \max_{1 \leq i \leq m} \left\{ \frac{a_i}{n} \right\} = 1$ and $\delta' = \min_{1 \leq i \leq m} \left\{ \frac{a_i}{n} \right\} = 0$.
3. We obtain $NA_1(G_i, H_0)$, $(i = 1, 2, 3, 4)$ as follows: $NA_1(G_1, H_0) = 0.4533$, $NA_1(G_2, H_0) = 0.4500$, $NA_1(G_3, H_0) = 0.4500$, $NA_1(G_4, H_0) = 0.4378$. Now by using NA_1, we have $G_4 < G_3 < G_1 <$ and $G_4 < G_2 < G_1$. But the new membership accuracy func-

Table 2 A_{TVIFIM}

	H_1	H_2	H_3
G_1	$\langle(0.1,0.15,0.2,0.25),(0.3,0.35,0.4,0.45)\rangle_{G_1,H_1}$	$\langle(0.15,0.2,0.25,0.3),(0.31,0.4,0.45,0.5)\rangle_{G_1,H_2}$	$\langle(0.2,0.25,0.3,0.35),(0.4,0.45,0.5,0.6)\rangle_{G_1,H_3}$
G_2	$\langle(0.01,0.1,0.15,0.2),(0.25,0.3,0.35,0.4)\rangle_{G_2,H_1}$	$\langle(0.1,0.15,0.2,0.25),(0.3,0.35,0.4,0.45)\rangle_{G_2,H_2}$	$\langle(0.2,0.3,0.35,0.4), (0.45,0.5,0.55,0.6)\rangle_{G_2,H_3}$
G_3	$\langle(0.15,0.2,0.25,0.3),(0.35,0.4,0.45,0.5)\rangle_{G_3,H_1}$	$\langle(0.1,0.2,0.25,0.3),(0.35,0.4,0.5,0.6)\rangle_{G_3,H_2}$	$\langle(0.25,0.3,0.35,0.4),(0.45,0.5,0.55,0.6)\rangle_{G_3,H_3}$
G_4	$\langle(0.1,0.15,0.2,0.25),(0.3,0.4,0.5,0.6)\rangle_{G_4,H_1}$	$\langle(0.1,0.2,0.25,0.3),(0.4,0.45,0.5,0.55)\rangle_{G_4,H_2}$	$\langle(0.2,0.25,0.35,0.4),(0.45,0.5,0.55,0.6)\rangle_{G_4,H_3}$

tion NA_1 fails to rank G_2, G_3 , *i.e.*, $NA_1(G_2, H_0) = NA_1(G_3, H_0)$, and hence, it is necessary to go for new non-membership accuracy function. We obtain $NA_2(G_2, H_0) = 0.4083, NA_2(G_3, H_0) = 0.4708$. Hence $G_2 > G_3$

4. Therefore, we get $G_4 < G_3 < G_2 < G_1$. Hence G_1 is the most desirable object from the given A_{TVIFIM}.

6 Conclusions and future scope

In this paper, we have proposed new accuracy functions for IVIFNs and TVIFNs, which can be used to rank IVIFNs and TVIFNs more accurately than the existing accuracy functions. Further, we have introduced TVIFIMs and an algorithmic procedure is given to apply the proposed method in multi-criteria decision making based on multi-criteria TVIFIM. Finally, illustrative example is also given to show its applicability. In near future, the proposed accuracy functions on TVIFNs can be extended to any nonlinear intuitionistic fuzzy numbers, and hence, MCDM problems involving nonlinear intuitionistic fuzzy index matrix can be solved.

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Compliance with ethical standards

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