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FOUNDATIONS

L-fuzzy rough approximation operators via three new types of *L*-fuzzy relations

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Abstract

Considering L being a frame with an order-reversing involution, three new types of L-fuzzy relations are introduced, which are called mediate, Euclidean and adjoint L-fuzzy relations, respectively. By means of these L-fuzzy relations, three types of L-fuzzy rough approximation operators are constructed and their connections with those three L-fuzzy relations are examined, respectively. An axiomatic approach is adopted to deal with L-fuzzy rough approximation operators. It is shown that each type of L-fuzzy rough approximation operators corresponding to mediate, Euclidean and adjoint L-fuzzy relations as well as their compositions can be characterized by single axioms.

Keywords Approximation operator · Rough set · Fuzzy rough set

1 Introduction

Since Pawlak (1982) proposed the concept of rough sets, rough set theory has recently received wide attention in both of the theoretical research and practical applications. There are usually two approaches for the development of this theory, i.e., the constructive approach and the axiomatic approach. In the constructive approach, upper and lower rough approximation operators are constructed from the primitive concepts, such as binary relations (Yao 1998a; Zhu 2007), neighborhood systems (Lin 1992; Mi et al. 2005; Yao 1998b) and coverings (Kryszkiewicz 1998; Zhu and Wang 2007). In contrast to the constructive approach, the axiomatic approach takes set-theoretic operators as primitive notions. Under some axioms on a pair of set-theoretic operators, there exists a binary relation such that the upper and lower rough approximation operators coincide with the set-theoretic operators (Lin 1992; Thiele 2000). More systematic axiomatic studies for classical rough sets were made by Yao (1996, 1998a) and so on.

With the development of fuzzy mathematics, many mathematical structures have been combined with fuzzy set theory, such as fuzzy convergence structures (Pang 2014, 2017a, b, 2018; Pang and Xiu 2018b), fuzzy convex structures (Pang and Shi 2017, 2018, 2019; Pang and Xiu 2018a, 2019; Pang and Zhao 2016; Pang et al. 2018; Xiu and Pang 2017, 2018a, b) and so on. Rough sets have also been generalized to the fuzzy case. In the framework of fuzzy rough set theory, various fuzzy generalizations of approximation operators, based on fuzzy binary relations, have been proposed and investigated, such as Liu (2006), Liu and Sai (2010), Mi and Zhang (2004), Mi et al. (2008), Pang et al. (2019), Morsi and Yakout (1998), Radzikowska and Kerre (2002), She and Wang (2009), Yao et al. (2019), Thiele (2001), Wu and Zhang (2004), Wu et al. (2013), Wu et al. (2015), Wu et al. (2016). In the above-mentioned works, researchers usually considered serial, reflexive, symmetric and transitive L-fuzzy relations. As we know, there are some other types of binary relations in the classical case, such as Euclidean relations (Yao 1998a), mediate relations (Zhu 2007) and (positive, negative) alliance relations (Zhu 2007). Furthermore, classical rough approximation operators corresponding to these types

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of binary relations have been investigated in the constructive and axiomatic approaches (Yao 1998a; Zhu 2007). This motivates us to consider fuzzy generalizations of these classical binary relations and their induced fuzzy rough approximation operators. From the viewpoint of fuzzy set theory, we will consider fuzzy counterparts of classical binary relations except serial, reflexive, symmetric and transitive relations and will construct fuzzy rough approximation operators based on the resulting fuzzy relations. As the first aim of this paper, we will adopt a frame L with an order-reversing involution "$'$" as the lattice background. Then we will propose three new types of L-fuzzy relations, including mediate, Euclidean and adjoint L-fuzzy relations. Moreover, we will explore the connections between these new types of L-fuzzy relations and their induced L-fuzzy rough approximation operators.

Using single axioms to characterize rough approximation operators is important in the study of crisp and fuzzy rough set theory. Following this idea, many researchers sought single axioms to describe classical and fuzzy rough approximation operators, see for example, Bao et al. (2018), Liu (2013), Wang (2018), Wu et al. (2015, 2016), Yang (2007). In these literatures, researchers usually considered single axioms to characterize L-fuzzy rough approximation operators corresponding to serial, reflexive, symmetric and transitive L-fuzzy relations as well as their compositions. Following the first aim of this paper, we have proposed mediate, Euclidean and adjoint L-fuzzy relations. So we will focus on axiomatic characterizations of L-fuzzy rough approximation operators corresponding to mediate, Euclidean and adjoint L-fuzzy relations in this paper. Concretely, as the second aim of this paper, we will provide single axioms to characterize upper and lower L-fuzzy rough approximation operators corresponding to mediate, Euclidean and adjoint L-fuzzy relations as well as their compositions.

This paper is organized as follows. In Sect. 2, we recall some necessary concepts and notations. In Sect. 3, we propose three new types of L-fuzzy relations and provide their characterizations by their induced upper and lower L-fuzzy rough approximation operators. In Sect. 4, we provide an axiomatic approach to L-fuzzy rough approximation operators corresponding to mediate, Euclidean and adjoint L-fuzzy relations. Further, we show that L-fuzzy rough approximation operators corresponding to three types of L-fuzzy relations as well as their compositions can be characterized by single axioms. In Sect. 5, we conclude the paper with a summary.

2 Preliminaries

Throughout this paper, let L denote a frame. That is a complete lattice, where finite meets is distributive over arbitrary joins, i.e.,

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i) \qquad \text{(ID)}$$

holds for all $a,\ b_i \in L\ (i \in I)$. Let 0 and 1 denote the smallest element and the biggest element in L, respectively. Further, L is equipped with an order-reversing involution "$'$", which means that $a'' = a$ and $a \leqslant b$ implies $b' \leqslant a'$.

The concept of L-fuzzy sets was first proposed by Goguen (1967) and it was considered as a generalization of the notion of Zadeh's fuzzy sets. In what follows, we first recall the definition of L-fuzzy sets.

Definition 2.1 (Goguen 1967) For a nonempty set U, a mapping $A : U \longrightarrow L$ is called an L-subset on U.

The family of all L-subsets on U will be denoted by $\mathcal{F}_L(U)$. Let $\leq$ denote the pointwise order on $\mathcal{F}_L(U)$, that is, for $A, B \in \mathcal{F}_L(U)$, $A \leq B$ means $A(x) \leq B(x)$. Then 0_U and 1_U defined by

$$0_U(x) = 0, \quad \forall x \in U,$$
$$1_U(x) = 1, \quad \forall x \in U,$$

are the smallest element and the largest element in $\mathcal{F}_L(U)$, respectively.

Given $A, B \in \mathcal{F}_L(U)$, $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$, we can define new L-fuzzy sets as follows:

$$(A \wedge B)(x) = A(x) \wedge B(x), \quad \forall x \in U,$$
$$(A \vee B)(x) = A(x) \vee B(x), \quad \forall x \in U,$$
$$\Big(\bigwedge_{i\in I} A_i\Big)(x) = \bigwedge_{i\in I} A_i(x), \quad \forall x \in U,$$
$$\Big(\bigvee_{i\in I} A_i\Big)(x) = \bigvee_{i\in I} A_i(x), \quad \forall x \in U.$$

For each $a \in L$ and each $x \in U$, let $\widehat{a}$ denote the constant L-subset and let a_x denote the L-subset which is defined by $a_x(y) = a$ if $y = x$, and $a_x(y) = 0$ if $y \neq x$. The characteristic function of each crisp set V is denoted by 1_V, and the Cartesian product of U and U is denoted by $U \times U$.

Definition 2.2 (Goguen 1967) An L-subset R on $U \times U$ is called an L-fuzzy relation on U. $R(x, y)$ represents the degree of relation between x and y, where $(x, y) \in U \times U$.

By equipping additional conditions on L-fuzzy relations, several types of L-fuzzy relations are introduced, such as serial, reflexive, symmetric and transitive L-fuzzy relations. Here we presented the definition of reflexive L-fuzzy relations.

Definition 2.3 (Bělohlávek 2004) An L-fuzzy relation R on U is called reflexive if for each $x \in U$, $R(x, x) = 1$.

Based on fuzzy relations, where the lattice background is taken as the unit interval [0, 1], fuzzy rough approximation spaces are proposed and studied in many literatures (Mi and Zhang 2004; Mi et al. 2008; Wu and Zhang 2004). By means of L-fuzzy relations, where L is a more general lattice, such as a residuated lattice, L-fuzzy rough approximation spaces are proposed as follows:

Definition 2.4 (She and Wang 2009) Suppose that U is a nonempty set and R is an L-fuzzy relation on U. Then the pair (U, R) is called an L-fuzzy rough approximation space.

3 Constructions of *L*-fuzzy rough approximation operators

In this section, we will first introduce the concept of L-fuzzy rough sets by constructing upper and lower L-fuzzy rough approximation operators from an L-fuzzy relation and explore some of its basic properties. Then we will propose three new types of L-fuzzy relations and characterize them by their induced upper and lower L-fuzzy rough approximation operators.

Definition 3.1 Suppose that (U, R) is an L-fuzzy rough approximation space. Define $\overline{R}, \underline{R} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ as follows: $\forall A \in \mathcal{F}_L(U), \forall x \in U$,

$$\overline{R}(A)(x) = \bigvee_{y\in U} (R(x, y) \wedge A(y)),$$

$$\underline{R}(A)(x) = \bigwedge_{y\in U} (R(x, y)' \vee A(y)).$$

Then $\overline{R}$ and $\underline{R}$ are called the upper and the lower L-fuzzy rough approximation operators of (U, R), respectively, and the pair $(\overline{R}(A), \underline{R}(A))$ is called the L-fuzzy rough set of A with respect to (U, R).

Remark 3.2 Definition 3.1 can be viewed as generalizations of fuzzy rough sets in Wu and Zhang (2004) and Dubois and Prade (1990). Concretely,

(1) If $L = [0, 1]$ and $a' = 1 - a$ for each $a \in L$, then Definition 3.1 coincides with Definition 4 in Wu and Zhang (2004).
(2) If $L = [0, 1]$, $a' = \mathcal{N}(a)$ and R is a fuzzy similarity relation on U, then Definition 3.1 is exactly the same as the fuzzy rough sets in Dubois and Prade (1990).

Theorem 3.3 *Let (U, R) be an L-fuzzy rough approximation space. Then for each $A \in \mathcal{F}_L(U)$,*

(DFUL) $\overline{R}(A) = \underline{R}(A')'$,
(DFLU) $\underline{R}(A) = \overline{R}(A')'$.

Proof (DFUL) Take each $x \in U$. Then

$$\underline{R}(A')'(x) = \Big(\bigwedge_{y\in U} (R(x, y)' \vee A'(y)) \Big)' = \bigvee_{y\in U} (R(x, y) \wedge A(y)) = \overline{R}(A)(x).$$

This means $\overline{R}(A) = \underline{R}(A')'$.
(DFLU) can be proved similarly.

Now let us study the properties of upper and lower L-fuzzy rough approximation operators.

Theorem 3.4 *Let (U, R) be an L-fuzzy rough approximation space. Then for each $A \in \mathcal{F}_L(U)$, $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$ and $a \in L$,*

(FU1) $\overline{R}\left(\bigvee_{i\in I} A_i\right) = \bigvee_{i\in I} \overline{R}(A_i)$,
(FU2) $\overline{R}(\widehat{a} \wedge A) = \widehat{a} \wedge \overline{R}(A)$,
(FU3) $\overline{R}(\widehat{a}) \le \widehat{a}$,
(FL1) $\underline{R}\left(\bigwedge_{i\in I} A_i\right) = \bigwedge_{i\in I} \underline{R}(A_i)$,
(FL2) $\underline{R}(\widehat{a} \vee A) = \widehat{a} \vee \underline{R}(A)$,
(FL3) $\underline{R}(\widehat{a}) \ge \widehat{a}$.

Proof We first show that $\overline{R}$ satisfies (FU1)–(FU3). Indeed,
(FU1) Take each $x \in U$. Then

$$\begin{aligned}\overline{R}\Big(\bigvee_{i\in I} A_i\Big)(x) &= \bigvee_{y\in U} \Big(R(x, y) \wedge \bigvee_{i\in I} A_i(y)\Big)\\ &= \bigvee_{y\in U} \bigvee_{i\in I} (R(x, y) \wedge A_i(y)) \ \ \text{(by (ID))}\\ &= \bigvee_{i\in I} \bigvee_{y\in U} (R(x, y) \wedge A_i(y))\\ &= \bigvee_{i\in I} \overline{R}(A_i)(x).\end{aligned}$$

This shows $\overline{R}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \overline{R}(A_i)$.
(FU2) Take each $x \in U$. Then

$$\begin{aligned}\overline{R}(\widehat{a} \wedge A)(x) &= \bigvee_{y\in U} (R(x, y) \wedge (\widehat{a} \wedge A)(y))\\ &= \bigvee_{y\in U} (R(x, y) \wedge a \wedge A(y))\\ &= a \wedge \bigvee_{y\in U} (R(x, y) \wedge A(y)) \ \ \text{(by (ID))}\\ &= a \wedge \overline{R}(A)(x)\\ &= (\widehat{a} \wedge \overline{R}(A))(x).\end{aligned}$$

This shows $\overline{R}(\widehat{a} \wedge A) = \widehat{a} \wedge \overline{R}(A)$.

(FU3) Take each $x \in U$. Then

$$\begin{aligned}\overline{R}(\widehat{a})(x) &= \bigvee_{y\in U}(R(x,y)\wedge \widehat{a}(y))\\ &= \bigvee_{y\in U}(R(x,y)\wedge a)\\ &= a\wedge \bigvee_{y\in U} R(x,y) \le a = \widehat{a}(x).\end{aligned}$$

This implies $\overline{R}(\widehat{a}) \le \widehat{a}$.

By Theorem 3.3, it is easy to check that $\underline{R}$ satisfies (FL1)–(FL3).

Corollary 3.5 *Let (U, R) be an L-fuzzy rough approximation space. Then for each $A, B \in \mathcal{F}_L(U)$,*

(1) *$A \le B$ implies $\overline{R}(A) \le \overline{R}(B)$,*
(2) *$A \le B$ implies $\underline{R}(A) \le \underline{R}(B)$.*

Actually, serial, reflexive, symmetric and transitive L-fuzzy relations are usually discussed in the framework of fuzzy rough sets. In the following, we will propose three new types of L-fuzzy relations, including mediate, Euclidean and adjoint L-fuzzy relations, and examine their connections with L-fuzzy rough approximation operators.

Definition 3.6 An L-fuzzy relation R on U is called mediate if it satisfies

$$\forall x, y \in L,\ \ R(x, y) \le \bigvee_{z\in U}(R(x,z)\wedge R(z,y)).$$

Example 3.7 Suppose that $U = \{x, y, z\}$, $L = [0, 1]$ and $a' = 1 - a$ for each $a \in [0, 1]$. Then $R_m : U \times U \longrightarrow L$ defined by

R_m	x	y	z
x	0.3	0.1	0.5
y	0.2	0.4	0.4
z	0.6	0.4	0.6

is a mediate L-fuzzy relation on U.

Remark 3.8 (1) If R is a reflexive L-fuzzy relation on U, then it is mediate.

(2) If $L = \{0, 1\}$, then Definition 3.6 reduces to the crisp mediate binary relation in Zhu (2007).

Theorem 3.9 *Let (U, R) be an L-fuzzy rough approximation space. Then R is mediate if and only if one of the following conditions holds:*

(FU4) $\overline{R}(\overline{R}(A)) \ge \overline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(FL4) $\underline{R}(\underline{R}(A)) \le \underline{R}(A), \forall A \in \mathcal{F}_L(U)$.

Proof The equivalence of (FU4) and (FL4) follows from Theorem 3.3. Now we only need to show that

R is mediate $\Longleftrightarrow$ (FU4) holds.

($\Longrightarrow$) Assume that R is mediate. Take each $A \in \mathcal{F}_L(U)$ and $x \in U$. Then

$$\begin{aligned}\overline{R}(\overline{R}(A))(x) &= \bigvee_{y\in U}(R(x,y)\wedge \overline{R}(A)(y))\\ &= \bigvee_{y\in U}\Big(R(x,y)\wedge \bigvee_{z\in U}(R(y,z)\wedge A(z))\Big)\\ &= \bigvee_{y\in U}\bigvee_{z\in U}(R(x,y)\wedge R(y,z)\wedge A(z)) \ \ \text{(by (ID))}\\ &= \bigvee_{z\in U}\Big(\bigvee_{y\in U}(R(x,y)\wedge R(y,z))\wedge A(z)\Big) \ \ \text{(by (ID))}\\ &\ge \bigvee_{z\in U}(R(x,z)\wedge A(z))\\ &= \overline{R}(A)(x).\end{aligned}$$

($\Longleftarrow$) Assume that (FU4) holds. Take each $(x, y) \in U \times U$. Then

$$\begin{aligned}\bigvee_{z\in U}(R(x,z)\wedge R(z,y)) &= \bigvee_{z\in U}(R(x,z)\wedge \overline{R}(1_y)(z))\\ &= \overline{R}(\overline{R}(1_y))(x)\\ &\ge \overline{R}(1_y)(x) = R(x,y).\end{aligned}$$

This shows that R is mediate, as desired. □

Definition 3.10 An L-fuzzy relation R on U is called Euclidean if it satisfies

$$\forall x, y \in L,\ \ R(x, y)' \ge \bigvee_{z\in U}(R(x,z)\wedge R(z,y)').$$

Example 3.11 Suppose that $U = \{x, y, z\}$, $L = [0, 1]$ and $a' = 1 - a$ for each $a \in [0, 1]$. Then $R_e : U \times U \longrightarrow L$ defined by

R_e	x	y	z
x	0.2	0.3	0.6
y	0.5	0.4	0.3
z	0.5	0.4	0.6

is a Euclidean L-fuzzy relation on U.

Remark 3.12 When $L = \{0, 1\}$, Definition 3.10 can be translated as follows:

(NA) If $\forall x, y \in U$, $\exists z \in U$ such that $(x, z) \in R$ and $(z, y) \notin R$, then $(x, y) \notin R$.

This is exactly the definition of negative alliance relations in the sense of Zhu (see Definition 4 in Zhu 2007). As we all know, a Euclidean relation R is defined in the following way.

(E) If $\forall x, y \in U$, $\exists z \in U$ such that $(z, x) \in R$ and $(z, y) \in R$, then $(x, y) \in R$.

It is easy to verify that **(NA)** and **E** are equivalent. That is, negative alliance relations in the sense of Zhu (2007) are equivalent to Euclidean relations. So we define Euclidean L-fuzzy relations in Definition 3.10 by generalizing negative alliance relations to the fuzzy case.

In what follows, we will use the upper and lower L-fuzzy rough approximation operators corresponding to Euclidean L-fuzzy relations to characterize Euclidean L-fuzzy relations. To this end, we first present the following lemma.

Lemma 3.13 *Let $A \in \mathcal{F}_L(U)$. Then $A = \bigvee_{x\in U}(\widehat{A(x)} \wedge 1_x)$.*

Proof Take each $y \in U$. Then

$$\begin{aligned}
&\bigvee_{x\in U}(\widehat{A(x)} \wedge 1_x)(y)\\
&= \bigvee_{x=y}(\widehat{A(x)} \wedge 1_x)(y) \vee \bigvee_{x\neq y}(\widehat{A(x)} \wedge 1_x)(y)\\
&= \widehat{A(y)}(y) \vee 0\\
&= A(y),
\end{aligned}$$

as desired.

Theorem 3.14 *Let (U, R) be an L-fuzzy rough approximation space. Then R is Euclidean if and only if one of the following conditions holds:*

(FU5) $\overline{R}(\overline{R}(A)')' \geq \overline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(FL5) $\underline{R}(\underline{R}(A)')' \leq \underline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(ULE1) $\underline{R}(\overline{R}(A)) \geq \overline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(ULE2) $\overline{R}(\underline{R}(A)) \leq \underline{R}(A), \forall A \in \mathcal{F}_L(U)$.

Proof By Theorem 3.3, it is straightforward to verify that (FU5), (FL5), (ULE1) and (ULE2) are equivalent. Now we only need to show that

R is Euclidean $\Longleftrightarrow$ (ULE1) holds.

($\Longrightarrow$) For each $(x, y) \in U \times U$, it follows that

$$\begin{aligned}
\underline{R}(\overline{R}(1_x))(y) &= \bigwedge_{z\in U}(R(y,z)' \vee \overline{R}(1_x)(z))\\
&= \bigwedge_{z\in U}(R(y,z)' \vee R(z,x))\\
&= \Big(\bigvee_{z\in U}(R(y,z) \wedge R(z,x)')\Big)'\\
&\geq R(y,x) \quad \text{(by Definition 3.10)}\\
&= \overline{R}(1_x)(y).
\end{aligned}$$

Then we have

$$\begin{aligned}
\underline{R}(\overline{R}(A)) &= \underline{R}\Big(\overline{R}\Big(\bigvee_{x\in U}(\widehat{A(x)} \wedge 1_x)\Big)\Big) \quad \text{(by Lemma 3.13)}\\
&= \underline{R}\Big(\bigvee_{x\in U}\overline{R}(\widehat{A(x)} \wedge 1_x)\Big) \quad \text{(by Theorem 3.4)}\\
&\geq \bigvee_{x\in U}\underline{R}(\overline{R}(\widehat{A(x)} \wedge 1_x)) \quad \text{(by Corollary 3.5)}\\
&= \bigvee_{x\in U}\underline{R}(\widehat{A(x)} \wedge \overline{R}(1_x)) \quad \text{(by Theorem 3.4)}\\
&= \bigvee_{x\in U}\underline{R}(\widehat{A(x)}) \wedge \underline{R}(\overline{R}(1_x)) \quad \text{(by Theorem 3.4)}\\
&\geq \bigvee_{x\in U}(\widehat{A(x)} \wedge \overline{R}(1_x)) \quad \text{(by Theorem 3.4)}\\
&= \bigvee_{x\in U}\overline{R}(\widehat{A(x)} \wedge 1_x) \quad \text{(by Theorem 3.4)}\\
&= \overline{R}\Big(\bigvee_{x\in U}(\widehat{A(x)} \wedge 1_x)\Big) \quad \text{(by Theorem 3.4)}\\
&= \overline{R}(A). \quad \text{(by Lemma 3.13)}
\end{aligned}$$

This proves that (ULE1) holds.

($\Longleftarrow$) For each $(x, y) \in U \times U$, put $A = 1_y$. Then it follows from (ULE1) that

$$\begin{aligned}
&\bigvee_{z\in U}(R(x,z) \wedge R(z,y)')\\
&= \bigvee_{z\in U}(R(x,z) \wedge \overline{R}(1_y)'(z))\\
&= \overline{R}(\overline{R}(1_y)')(x)\\
&= \underline{R}(\overline{R}(1_y))'(x)\\
&\leq \overline{R}(1_y)'(x)\\
&= R(x,y)'.
\end{aligned}$$

This shows that R is Euclidean.

In Wu and Zhang (2004), Wu and Zhang proposed a fuzzy counterpart of Euclidean binary relations and gave a counterexample (Example 1 in Wu and Zhang 2004) to show that Euclidean fuzzy relations cannot be characterized by (ULE1) or (ULE2). Here, we defined Euclidean L-fuzzy relations in Definition 3.10 by generalizing an equivalent form of Euclidean binary relations. Then we show that Euclidean L-fuzzy relations can be characterized by (ULE1) or (ULE2). This result provided a reasonable generalization of the results in the classical case and gave an answer to the problem with respect to Euclidean fuzzy relations in Wu and Zhang (2004).

Definition 3.15 An L-fuzzy relation R on U is called adjoint if it satisfies

$$\forall x, y \in L,\ R(x, y)' \geq \bigwedge_{z\in U} \bigvee_{w\neq y} (R(x, z)' \vee R(z, w)).$$

Example 3.16 Suppose that $U = \{x, y, z\}$, $L = [0, 1]$ and $a' = 1 - a$ for each $a \in [0, 1]$. Then $R_a : U \times U \longrightarrow L$ defined by

R_a	x	y	z
x	0.2	0.4	0.3
y	0.1	0.3	0.4
z	0.4	0.6	0.5

is an adjoint L-fuzzy relation on U.

In order to characterize adjoint L-fuzzy relations, we first present the following lemma.

Lemma 3.17 *Let $A \in \mathcal{F}_L(U)$. Then $A = \bigwedge_{x\in U}(\widehat{A(x)} \vee 1_{U-x})$.*

Proof Take each $y \in U$. Then

$$\begin{aligned}
&\bigwedge_{x\in U} (\widehat{A(x)} \vee 1_{U-x})(y)\\
&= \bigwedge_{x=y} (\widehat{A(x)} \vee 1_{U-x})(y) \wedge \bigwedge_{x\neq y} (\widehat{A(x)} \vee 1_{U-x})(y)\\
&= \widehat{A(y)}(y) \wedge 1\\
&= A(y),
\end{aligned}$$

as desired.

Theorem 3.18 *Let (U, R) be an L-fuzzy rough approximation space. Then R is adjoint if and only if one of the following conditions holds:*

(FU6) $\overline{R}(\overline{R}(A')') \geq \overline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(FL6) $\underline{R}(\underline{R}(A')') \leq \underline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(ULA1) $\overline{R}(\underline{R}(A)) \geq \overline{R}(A), \forall A \in \mathcal{F}_L(U)$,
(ULA2) $\underline{R}(\overline{R}(A)) \leq \underline{R}(A), \forall A \in \mathcal{F}_L(U)$.

Proof The equivalence of (FU6), (FL6), (ULA1) and (ULA2) follows immediately from Theorem 3.3. Next we only need to show that

R is adjoint $\Longleftrightarrow$ (ULA2) holds.

($\Longrightarrow$) For each $(x, y) \in U \times U$, it follows that

$$\begin{aligned}
\underline{R}(1_{U-y})(x) &= R(x, y)'\\
&\geq \bigwedge_{z\in U} \bigvee_{w\neq y} (R(x, z)' \vee R(z, w)) \quad \text{(by Definition 3.15)}\\
&= \bigwedge_{z\in U} \Big(R(x, z)' \vee \bigvee_{w\neq y} R(z, w)\Big)\\
&= \bigwedge_{z\in U} \Big(R(x, z)' \vee \bigvee_{w\in U} (R(z, w) \wedge 1_{U-y}(w))\Big)\\
&= \bigwedge_{z\in U} (R(x, z)' \vee \overline{R}(1_{U-y})(z))\\
&= \underline{R}(\overline{R}(1_{U-y}))(x).
\end{aligned}$$

This means $\underline{R}(1_{U-y}) \geq \underline{R}(\overline{R}(1_{U-y}))$. Then we have

$$\begin{aligned}
\underline{R}(\overline{R}(A)) &= \underline{R}\Big(\overline{R}\Big(\bigwedge_{y\in U} (\widehat{A(y)} \vee 1_{U-y})\Big)\Big) \quad \text{(by Lemma 3.17)}\\
&\leq \underline{R}\Big(\bigwedge_{y\in U} \overline{R}(\widehat{A(y)} \vee 1_{U-y})\Big) \quad \text{(by Corollary 3.5)}\\
&= \bigwedge_{y\in U} \underline{R}(\overline{R}(\widehat{A(y)} \vee 1_{U-y})) \quad \text{(by Theorem 3.4)}\\
&= \bigwedge_{y\in U} \underline{R}(\overline{R}(\widehat{A(y)}) \vee \overline{R}(1_{U-y})) \quad \text{(by Theorem 3.4)}\\
&= \bigwedge_{y\in U} (\overline{R}(\widehat{A(y)}) \vee \underline{R}(\overline{R}(1_{U-y}))) \quad \text{(by Theorem 3.4)}\\
&\leq \bigwedge_{y\in U} (\widehat{A(y)} \vee \underline{R}(\overline{R}(1_{U-y}))) \quad \text{(by Theorem 3.4)}\\
&\leq \bigwedge_{y\in U} (\widehat{A(y)} \vee \underline{R}(1_{U-y}))\\
&= \bigwedge_{y\in U} \underline{R}(\widehat{A(y)} \vee 1_{U-y}) \quad \text{(by Theorem 3.4)}\\
&= \underline{R}\Big(\bigwedge_{y\in U} (\widehat{A(y)} \vee 1_{U-y})\Big) \quad \text{(by Theorem 3.4)}\\
&= \underline{R}(A). \quad \text{(by Lemma 3.17)}
\end{aligned}$$

This proves that (ULA2) holds.

($\Longleftarrow$) For each $(x, y) \in U \times U$, it follows from (ULA2) that

$$\begin{aligned}
&\bigwedge_{z\in U} \bigvee_{w\neq y} (R(x, z)' \vee R(z, w))\\
&= \bigwedge_{z\in U} \Big(R(x, z)' \vee \bigvee_{w\neq y} R(z, w)\Big)\\
&= \bigwedge_{z\in U} \Big(R(x, z)' \vee \bigvee_{w\in U} (R(z, w) \wedge 1_{U-y}(w))\Big)\\
&= \bigwedge_{z\in U} (R(x, z)' \vee \overline{R}(1_{U-y})(z))\\
&= \underline{R}(\overline{R}(1_{U-y}))(x)\\
&\leq \underline{R}(1_{U-y})(x)\\
&= R(x, y)'.
\end{aligned}$$

This shows that R is adjoint.

In this section, we actually applied a constructive approach to L-fuzzy rough approximation operators (i.e., L-fuzzy rough sets) corresponding to mediate, Euclidean and adjoint L-fuzzy relations. We further provided some equivalent descriptions of three new types of L-fuzzy relations. In the next section, we will explore L-fuzzy rough approximation operators corresponding to mediate, Euclidean and adjoint L-fuzzy relations in an axiomatic approach.

4 Axiomatic characterizations of *L*-fuzzy rough approximation operators by single axioms

In this section, we will provide an axiomatic approach to L-fuzzy rough approximation operators and will use single axioms to characterize each kind of L-fuzzy rough approximation operators corresponding to mediate, Euclidean, adjoint L-fuzzy relations as well as their compositions.

Definition 4.1 Suppose that $\mathcal{U}, \mathcal{L}\colon \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are two mappings. They are called dual L-fuzzy operators if they satisfy the following conditions:

(DFUL) $\mathcal{U}(A) = \mathcal{L}(A')', \forall A \in \mathcal{F}_L(U)$,
(DFLU) $\mathcal{L}(A) = \mathcal{U}(A')', \forall A \in \mathcal{F}_L(U)$.

Now let us give an axiomatic characterization of dual L-fuzzy operators by single axioms.

Theorem 4.2 *Let $\mathcal{U}, \mathcal{L}\colon \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if $\mathcal{U}$ satisfies* (GFU1) *or $\mathcal{L}$ satisfies* (GFL1)*:*
$\forall a \in L, \forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,

$$\text{(GFU1)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i),$$
$$\text{(GFL1)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I} \mathcal{L}(A_i).$$

Proof Since $\mathcal{U}$ and $\mathcal{L}$ are dual L-fuzzy operators, we know (GFU1) and (GFL1) are equivalent. Now we only need to show that there exists a unique L-fuzzy relation on U such that $\overline{R} = \mathcal{U}$ if and only if $\mathcal{U}$ satisfies (GFU1).

($\Longrightarrow$) If there exists an L-fuzzy relation on U such that $\overline{R} = \mathcal{U}$, then it follows from Theorem 3.4 that $\mathcal{U}$ satisfies (FU1) and (FU2). This implies that

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \mathcal{U}\Big(\bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i).$$

Thus, $\mathcal{U}$ satisfies (GFU1).

($\Longleftarrow$) Suppose that $\mathcal{U}$ satisfies (GFU1). It is easy to see that $\mathcal{U}$ satisfies (FU1) and (FU2). Define $R : U \times U \longrightarrow L$ by

$$\forall (x, y) \in U \times U,\ R(x, y) = \mathcal{U}(1_y)(x).$$

Then we have

$$\begin{aligned}
\overline{R}(A)(x) &= \bigvee_{y\in W} (R(x, y) \wedge A(y)) \\
&= \bigvee_{y\in W} (\mathcal{U}(1_y)(x) \wedge A(y)) \\
&= \bigvee_{y\in W} (\mathcal{U}(1_y) \wedge \widehat{A(y)})(x) \\
&= \bigvee_{y\in W} \mathcal{U}(\widehat{A(y)} \wedge 1_y)(x) \quad \text{(by (FU2))} \\
&= \mathcal{U}\Big(\bigvee_{y\in W} (\widehat{A(y)} \wedge 1_y)\Big)(x) \quad \text{(by (FU1))} \\
&= \mathcal{U}(A)(x). \quad \text{(by Lemma 3.13)}
\end{aligned}$$

This means $\overline{R} = \mathcal{U}$. The existence of R is proved. Suppose that R^* is another L-fuzzy relation satisfying $\overline{R^*} = \mathcal{U}$. Then for each $(x, y) \in U \times U$,

$$R^*(x, y) = \overline{R^*}(1_y)(x) = \mathcal{U}(1_y)(x) = R(x, y).$$

That is, $R^* = R$, which shows the uniqueness.

For L-fuzzy operators, we can also give another axiomatic characterization in the following theorem.

Theorem 4.3 *Let $\mathcal{U}, \mathcal{L}\colon \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if $\mathcal{U}$ satisfies* (GFU2) *or $\mathcal{L}$ satisfies* (GFL2)*:*
$\forall A, B \in \mathcal{F}_L(U)$,

$$\text{(GFU2)} \bigvee_{x\in U} (A(x) \wedge \mathcal{U}(B)(x)) = \bigvee_{x\in U} \bigvee_{y\in U} (B(x) \wedge \mathcal{U}(1_x)(y) \wedge A(y)),$$
$$\text{(GFL2)} \bigwedge_{x\in U} (A(x) \vee \mathcal{L}(B)(x)) = \bigwedge_{x\in U} \bigwedge_{y\in U} (B(x) \vee \mathcal{L}(1_{U-x})(y) \vee A(y)).$$

Proof Since $\mathcal{U}$ and $\mathcal{L}$ are dual L-fuzzy operators, we know (GFU2) and (GFL2) are equivalent. Then by Theorem 4.2, it suffices to show that

(GFU1)$\Longleftrightarrow$(GFU2).

($\Longrightarrow$) Suppose that (GFU1) holds. Then (FU1) and (FU2) hold. Take each $A, B \in \mathcal{F}_L(U)$. Then

$$\begin{aligned}
&\bigvee_{x\in U} (A(x) \wedge \mathcal{U}(B)(x)) \\
&= \bigvee_{x\in U} \Big(A(x) \wedge \mathcal{U}\Big(\bigvee_{y\in U} (\widehat{B(y)} \wedge 1_y)\Big)(x)\Big) \quad \text{(by Lemma 3.13)} \\
&= \bigvee_{x\in U} \Big(A(x) \wedge \bigvee_{y\in U} \mathcal{U}(\widehat{B(y)} \wedge 1_y)(x)\Big) \quad \text{(by (FU1))} \\
&= \bigvee_{x\in U} \Big(A(x) \wedge \bigvee_{y\in U} (\widehat{B(y)} \wedge \mathcal{U}(1_y))(x)\Big) \quad \text{(by (FU2))} \\
&= \bigvee_{x\in U} \bigvee_{y\in U} (A(x) \wedge B(y) \wedge \mathcal{U}(1_y)(x)) \quad \text{(by (ID))} \\
&= \bigvee_{y\in U} \bigvee_{x\in U} (B(y) \wedge \mathcal{U}(1_y)(x) \wedge A(x)) \\
&= \bigvee_{x\in U} \bigvee_{y\in U} (B(x) \wedge \mathcal{U}(1_x)(y) \wedge A(y)).
\end{aligned}$$

This shows (GFU2) holds.

($\Longleftarrow$) Take each $x \in U$. Then

$$\begin{aligned}
&\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big)(x) \\
&= \bigvee_{y\in U} \Big(1_x(y) \wedge \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big)(y)\Big) \\
&= \bigvee_{y\in U} \bigvee_{z\in U} \Big(\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big)(y) \wedge \mathcal{U}(1_y)(z) \wedge 1_x(z)\Big) \quad \text{(by (GFU2))} \\
&= a \wedge \bigvee_{i\in I} \bigvee_{y\in U} \bigvee_{z\in U} (A_i(y) \wedge \mathcal{U}(1_y)(z) \wedge 1_x(z)) \quad \text{(by (ID))} \\
&= a \wedge \bigvee_{i\in I} \bigvee_{y\in U} (1_x(y) \wedge \mathcal{U}(A_i)(y)) \quad \text{(by (GFU2))} \\
&= a \wedge \bigvee_{i\in I} \mathcal{U}(A_i)(x) \\
&= \Big(\widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i)\Big)(x).
\end{aligned}$$

This proves that $\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i)$.

According to Theorems 3.9 and 4.2, we can characterize the upper (resp. lower) L-fuzzy rough approximation operators generated by mediate L-fuzzy relations by the axioms (GFU1) and (FU4) (resp. (GFL1) and (FL4)). In the following theorem, we will replace the axioms (GFU1) and (FU4) (resp. (GFL1) and (FL4)) by a single axiom (MFU) (resp. (MFL)).

Theorem 4.4 *Let $\mathcal{U}, \mathcal{L}$: $\mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique mediate L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if $\mathcal{U}$ satisfies* (MFU) *or $\mathcal{L}$ satisfies* (MFL)*:*

$\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,

$$\text{(MFU)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} (\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i))),$$

$$\text{(MFL)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I} (\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i))).$$

Proof Since $\mathcal{U}$ and $\mathcal{L}$ are dual, it is easy to verify that (MFU) and (MFL) are equivalent. Now we only need to show that there is a unique mediate L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ if and only if (MFU) holds.

($\Longrightarrow$) If there is a mediate L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$, then it follows from Theorem 3.9 that $\mathcal{U}(\mathcal{U}(A_i)) \geq \mathcal{U}(A_i)$ for each $i \in I$. Then by Theorem 4.2, we have

$$\begin{aligned}
\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) &= \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i) \\
&= \widehat{a} \wedge \bigvee_{i\in I} (\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i))).
\end{aligned}$$

($\Longleftarrow$) For each $A \in \mathcal{F}_L(U)$, put $a = 1$, $I = \{1\}$ and $A_1 = A$. Then it follows that

$$\mathcal{U}(A) = \mathcal{U}(A) \wedge \mathcal{U}(\mathcal{U}(A)).$$

This means that $\mathcal{U}(\mathcal{U}(A)) \geq \mathcal{U}(A)$ for each $A \in \mathcal{F}_L(U)$. Then it follows from (MFU) that

$$\begin{aligned}
\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) &= \widehat{a} \wedge \bigvee_{i\in I} (\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i))) \\
&= \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i).
\end{aligned}$$

By Theorem 4.2, there exists a unique L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$. Further, we have

$$\overline{R}(\overline{R}(A)) = \mathcal{U}(\mathcal{U}(A)) \geq \mathcal{U}(A) = \overline{R}(A).$$

This implies that R is mediate.

According to Theorems 3.14 and 4.2, we observe that some axioms can be used to characterize the upper and lower L-fuzzy rough approximation operators with respect to Euclidean L-fuzzy relations. For example, (GFU1) and (FU5) can be used to characterize the upper and lower L-fuzzy rough approximation operators generated by Euclidean L-fuzzy relations. Actually, these axioms can be replaced by single axioms.

Theorem 4.5 *Let $\mathcal{U}, \mathcal{L}$: $\mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique Euclidean L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if one of the following conditions holds:*

$\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,

$$\text{(EFU1)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)')'),$$

$$\text{(EFU2)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{L}(\mathcal{U}(A_i))),$$

$$\text{(EFL1)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)')'),$$

$$\text{(EFL2)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{U}(\mathcal{L}(A_i))).$$

Proof The equivalence of (EFU1), (EFU2), (EFL1) and (EFL2) follows immediately from the duality of $\mathcal{U}$ and $\mathcal{L}$. Now we only need to show that there is a unique Euclidean L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$ if and only if (EFU2) holds.

($\Longrightarrow$) If there is a Euclidean L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$, then it follows from Theorem 3.14 that $\mathcal{U}(A_i) \leq \mathcal{L}(\mathcal{U}(A_i))$ for each $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$. Then by Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i)$$
$$= \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{L}(\mathcal{U}(A_i))).$$

This shows (EFU2) holds.

($\Longleftarrow$) For each $A \in \mathcal{F}_L(U)$, put $a = 1$, $I = \{1\}$ and $A_1 = A$. Then it follows from (EFU2) that

$$\mathcal{U}(A) = \mathcal{U}(A) \wedge \mathcal{L}(\mathcal{U}(A)).$$

This means that $\mathcal{U}(A) \leq \mathcal{L}(\mathcal{U}(A))$ for each $A \in \mathcal{F}_L(U)$. Then for each $A_i \in \mathcal{F}_L(U)$, it follows that $\mathcal{U}(A_i) \leq \mathcal{L}(\mathcal{U}(A_i))$. Thus, we obtain

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{L}(\mathcal{U}(A_i)))$$
$$= \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i).$$

By Theorem 4.2, there exists a unique L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$. Then it follows that

$$\underline{R}(\overline{R}(A)) = \mathcal{L}(\mathcal{U}(A)) \geq \mathcal{U}(A) = \overline{R}(A).$$

By Theorem 3.14, we know R is Euclidean.

According to Theorems 3.18 and 4.2, we can examine that the upper (resp. lower) L-fuzzy rough approximation operators generated by adjoint L-fuzzy relations can be characterized by some axioms, such as (FYU6) and (GFU1) (resp. (FYL5) and (GFU2)). In the following theorem, we will use single axioms to characterize L-fuzzy rough approximation operators corresponding to adjoint L-fuzzy relations.

Theorem 4.6 *Let* $\mathcal{U}, \mathcal{L}: \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *be a pair of dual* L*-fuzzy operators. Then there exists a unique adjoint* L*-fuzzy relation* R *on* U *such that* $\overline{R} = \mathcal{U}$ *and* $\underline{R} = \mathcal{L}$ *if and only if one of the following conditions holds:*
$\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,

$$\text{(AFU1)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i')')),$$

$$\text{(AFU2)}\ \mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{L}(A_i))),$$

$$\text{(AFL1)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i')')),$$

$$\text{(AFL2)}\ \mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{U}(A_i))).$$

Proof The equivalence of (AFU1), (AFU2), (AFL1) and (AFL2) follows immediately from the duality of $\mathcal{U}$ and $\mathcal{L}$. Now we only need to show that there is a unique adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$ if and only if (AFU2) holds.

($\Longrightarrow$) If there is an adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$, then it follows from Theorem 3.18 that $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{L}(A_i))$ for each $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$. Then by Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i)$$
$$= \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{L}(A_i))).$$

This shows (AFU2) holds.

($\Longleftarrow$) For each $A \in \mathcal{F}_L(U)$, put $a = 1$, $I = \{1\}$ and $A_1 = A$. Then it follows from (AFU2) that

$$\mathcal{U}(A) = \mathcal{U}(A) \wedge \mathcal{U}(\mathcal{L}(A)).$$

This means that $\mathcal{U}(A) \leq \mathcal{U}(\mathcal{L}(A))$ for each $A \in \mathcal{F}_L(U)$. Then for each $A_i \in \mathcal{F}_L(U)$, it follows that $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{L}(A_i))$. Thus, we obtain

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{L}(A_i)))$$
$$= \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i).$$

By Theorem 4.2, there exists a unique L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$. Then it follows that

$$\overline{R}(\underline{R}(A)) = \mathcal{U}(\mathcal{L}(A)) \geq \mathcal{U}(A) = \overline{R}(A).$$

By Theorem 3.18, we know R is adjoint.

In order to show the reasonability of Theorems 4.4–4.6, we will show each type of L-fuzzy relations really exists. Here, we provide some concrete examples.

Example 4.7 (1) R_m in Example 3.7 is mediate but not Euclidean or adjoint.
(2) R_e in Example 3.11 is Euclidean but not mediate or adjoint.
(3) R_a in Example 3.16 is adjoint but not mediate or Euclidean.

In the sequel, we will investigate axiomatic characterizations of L-fuzzy rough approximation operators with respect to compositions of mediate, Euclidean and adjoint L-fuzzy relations.

Theorem 4.8 *Let $\mathcal{U}, \mathcal{L}$: $\mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique mediate and Euclidean L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if one of the following conditions holds: $\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,*

(MEFU1) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{U}(A_i)')'),$$

(MEFU2) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} (\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{L}(\mathcal{U}(A_i))),$$

(MEFL1) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{L}(\mathcal{L}(A_i)')'),$$

(MEFL2) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{U}(\mathcal{L}(A_i))).$$

Proof It is enough to show that there is a unique mediate and Euclidean L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$ if and only if (MEFU2) holds.

($\Longrightarrow$) If there is a mediate and Euclidean L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$, then it follows from Theorems 3.9 and 3.14 that $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{U}(A_i))$ and $\mathcal{U}(A_i) \leq \mathcal{L}(\mathcal{U}(A_i))$ for each $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$. By Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{L}(\mathcal{U}(A_i))).$$

This shows (MEFU2) holds.

($\Longleftarrow$) For each $A \in \mathcal{F}_L(U)$, put $a = 1$, $I = \{1\}$ and $A_1 = A$. Then it follows from (MEFU2) that

$$\mathcal{U}(A) = \mathcal{U}(A) \wedge \mathcal{U}(\mathcal{U}(A)) \wedge \mathcal{L}(\mathcal{U}(A)).$$

This implies that $\mathcal{U}(\mathcal{U}(A)) \geq \mathcal{U}(A)$. Then it follows from (MEFU2) that

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{L}(\mathcal{U}(A_i))).$$

By Theorem 4.5, we know there exists a unique Euclidean L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$. Also, we have

$$\overline{R}(\overline{R}(A)) = \mathcal{U}(\mathcal{U}(\mathcal{A})) \geq \mathcal{U}(A) = \overline{R}(A).$$

Thus, R is also mediate.

Theorem 4.9 *Let $\mathcal{U}, \mathcal{L}$: $\mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique mediate and adjoint L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if one of the following conditions holds: $\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,*

(MAFU1) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{U}(A_i')')),$$

(MAFU2) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in' I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{L}(A_i))),$$

(MAFL1) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{L}(\mathcal{L}(A_i')')),$$

(MAFL2) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{L}(\mathcal{U}(A_i))).$$

Proof It is enough to show that there is a unique mediate and adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$ if and only if (MAFU2) holds.

($\Longrightarrow$) If there is a mediate and adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$, then it follows from Theorems 3.9 and 3.18 that $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{U}(A_i))$ and $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{L}(A_i))$ for each $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$. By Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)=\widehat{a}\wedge\bigvee_{i\in I}\mathcal{U}(A_i)$$
$$=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{U}(\mathcal{U}(A_i))\wedge\mathcal{U}(\mathcal{L}(A_i))).$$

This shows (MAFU2) holds.

($\Longleftarrow$) For each $A\in\mathcal{F}_L(U)$, put $a=1$, $I=\{1\}$ and $A_1=A$. Then it follows from (MAFU2) that

$$\mathcal{U}(A)=\mathcal{U}(A)\wedge\mathcal{U}(\mathcal{U}(\mathcal{A}))\wedge\mathcal{U}(\mathcal{L}(A)).$$

This implies that $\mathcal{U}(\mathcal{U}(\mathcal{A}))\geq\mathcal{U}(A)$. Then it follows from (MAFU2) that

$$\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{U}(\mathcal{L}(A_i))).$$

By Theorem 4.6, we know there exists a unique adjoint L-fuzzy relation R on U such that $\mathcal{U}=\overline{R}$ and $\mathcal{L}=\underline{R}$. Also, we have

$$\overline{R}(\overline{R}(A))=\mathcal{U}(\mathcal{U}(\mathcal{A}))\geq\mathcal{U}(A)=\overline{R}(A).$$

Then it follows from Theorem 3.9 that R is mediate.

Theorem 4.10 *Let $\mathcal{U},\mathcal{L}$: $\mathcal{F}_L(U)\longrightarrow\mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique Euclidean and adjoint L-fuzzy relation R on U such that $\overline{R}=\mathcal{U}$ and $\underline{R}=\mathcal{L}$ if and only if one of the following conditions holds: $\forall a\in L$, $\forall\{A_i\}_{i\in I}\subseteq\mathcal{F}_L(U)$,*

(EAFU1) $\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)$
$=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{U}(\mathcal{U}(A_i')'\wedge\mathcal{U}(\mathcal{U}(A_i)')'),$

(EAFU2) $\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)$
$=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{U}(\mathcal{L}(A_i))\wedge\mathcal{L}(\mathcal{U}(A_i))),$

(EAFL1) $\mathcal{L}\Big(\widehat{a}\vee\bigwedge_{i\in I}A_i\Big)$
$=\widehat{a}\vee\bigwedge_{i\in I}(\mathcal{L}(A_i)\vee\mathcal{L}(\mathcal{L}(A_i')')\vee\mathcal{L}(\mathcal{L}(A_i)')'),$

(EAFL2) $\mathcal{L}\Big(\widehat{a}\vee\bigwedge_{i\in I}A_i\Big)$
$=\widehat{a}\vee\bigwedge_{i\in I}(\mathcal{L}(A_i)\vee\mathcal{L}(\mathcal{U}(A_i))\vee\mathcal{U}(\mathcal{L}(A_i))).$

Proof We only need to show that there is a unique Euclidean and adjoint L-fuzzy relation R such that $\mathcal{U}=\overline{R}$ and $\mathcal{L}=\underline{R}$ if and only if (EAFU2) holds.

($\Longrightarrow$) If there is a Euclidean and adjoint L-fuzzy relation R such that $\mathcal{U}=\overline{R}$ and $\mathcal{L}=\underline{R}$, then it follows from Theorems 3.14 and 3.18 that $\mathcal{U}(A_i)\leq\mathcal{U}(\mathcal{L}(A_i))$ and $\mathcal{U}(A_i)\leq\mathcal{L}(\mathcal{U}(A_i))$ for each $\{A_i\}_{i\in I}\subseteq\mathcal{F}_L(U)$. By Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)=\widehat{a}\wedge\bigvee_{i\in I}\mathcal{U}(A_i)$$
$$=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{U}(\mathcal{L}(A_i))\wedge\mathcal{L}(\mathcal{U}(A_i))).$$

This shows (EAFU2) holds.

($\Longleftarrow$) For each $A\in\mathcal{F}_L(U)$, put $a=1$, $I=\{1\}$ and $A_1=A$. Then it follows from (EAFU2) that

$$\mathcal{U}(A)=\mathcal{U}(A)\wedge\mathcal{U}(\mathcal{L}(A))\wedge\mathcal{L}(\mathcal{U}(A)).$$

Thus, we have $\mathcal{U}(\mathcal{L}(A))\geq\mathcal{U}(A)$. Then it follows from (EAFU2) that

$$\mathcal{U}\Big(\widehat{a}\wedge\bigvee_{i\in I}A_i\Big)=\widehat{a}\wedge\bigvee_{i\in I}(\mathcal{U}(A_i)\wedge\mathcal{L}(\mathcal{U}(A_i))).$$

By Theorem 4.5, there exists a unique Euclidean L-fuzzy relation R on U such that $\mathcal{U}=\overline{R}$ and $\mathcal{L}=\underline{R}$. Further, we have

$$\overline{R}(\underline{R}(A))=\mathcal{U}(\mathcal{L}(A))\geq\mathcal{U}(A)=\overline{R}(A).$$

This implies that R is also adjoint.

In order to show the reasonability of Theorems 4.8–4.10, it is necessary to give some concrete examples of each kind of composite L-fuzzy relations by two types of L-fuzzy relations.

Example 4.11 Suppose that $U=\{x,y,z\}$, $L=[0,1]$ and $a'=1-a$ for each $a\in[0,1]$. Then

(1) $R_{\mathrm{me}}:U\times U\longrightarrow L$ defined by

R_{me}	x	y	z
x	0.2	0.6	0.3
y	0.5	0.6	0.4
z	0.4	0.6	0.4

is mediate and Euclidean but not adjoint.

(2) $R_{\mathrm{ma}}:U\times U\longrightarrow L$ defined by

R_{ma}	x	y	z
x	0.3	0.1	0.4
y	0.2	0.4	0.4
z	0.6	0.5	0.6

is mediate and adjoint but not Euclidean.
(3) $R_{ea} : U \times U \longrightarrow L$ defined by

R_{ea}	x	y	z
x	0.3	0.4	0.2
y	0.4	0.1	0.5
z	0.1	0.3	0.4

is Euclidean and adjoint but not mediate.

Theorem 4.12 *Let $\mathcal{U}, \mathcal{L}$: $\mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be a pair of dual L-fuzzy operators. Then there exists a unique mediate, Euclidean and adjoint L-fuzzy relation R on U such that $\overline{R} = \mathcal{U}$ and $\underline{R} = \mathcal{L}$ if and only if one of the following conditions holds:*
$\forall a \in L$, $\forall \{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$,

(MEAFU1) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{U}(A_i'))' \wedge \mathcal{U}(\mathcal{U}(A_i)')'),$$

(MEAFU2) $$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{L}(A_i)) \wedge \mathcal{L}(\mathcal{U}(A_i))),$$

(MEAFL1) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{L}(\mathcal{L}(A_i')') \vee \mathcal{L}(\mathcal{L}(A_i)')'),$$

(MEAFL2) $$\mathcal{L}\Big(\widehat{a} \vee \bigwedge_{i\in I} A_i\Big) = \widehat{a} \vee \bigwedge_{i\in I}(\mathcal{L}(A_i) \vee \mathcal{L}(\mathcal{L}(A_i)) \vee \mathcal{L}(\mathcal{U}(A_i)) \vee \mathcal{U}(\mathcal{L}(A_i))).$$

Proof We only need to show that there is a unique mediate, Euclidean and adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$ if and only if (MEAFU2) holds.

($\Longrightarrow$) If there is a mediate, Euclidean and adjoint L-fuzzy relation R such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$, then it follows from Theorems 3.9, 3.14 and 3.18 that $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{U}(A_i))$, $\mathcal{U}(A_i) \leq \mathcal{U}(\mathcal{L}(A_i))$ and $\mathcal{U}(A_i) \leq \mathcal{L}(\mathcal{U}(A_i))$ for each $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$. By Theorem 4.2, we have

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I} \mathcal{U}(A_i) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{U}(A_i)) \wedge \mathcal{U}(\mathcal{L}(A_i)) \wedge \mathcal{L}(\mathcal{U}(A_i))).$$

This shows (MEAFU2) holds.

($\Longleftarrow$) For each $A \in \mathcal{F}_L(U)$, put $a = 1$, $I = \{1\}$ and $A_1 = A$. Then it follows from (MEAFU2) that

$$\mathcal{U}(A) = \mathcal{U}(A) \wedge \mathcal{U}(\mathcal{U}(A)) \wedge \mathcal{U}(\mathcal{L}(A)) \wedge \mathcal{L}(\mathcal{U}(A)).$$

Thus, we have $\mathcal{U}(\mathcal{U}(A)) \geq \mathcal{U}(A)$. Then it follows from (MEAFU2) that

$$\mathcal{U}\Big(\widehat{a} \wedge \bigvee_{i\in I} A_i\Big) = \widehat{a} \wedge \bigvee_{i\in I}(\mathcal{U}(A_i) \wedge \mathcal{U}(\mathcal{L}(A_i)) \wedge \mathcal{L}(\mathcal{U}(A_i))).$$

By Theorem 4.10, there exists a unique Euclidean and adjoint L-fuzzy relation R on U such that $\mathcal{U} = \overline{R}$ and $\mathcal{L} = \underline{R}$. Further, we have

$$\overline{R}(\overline{R}(A)) = \mathcal{U}(\mathcal{U}(A)) \geq \mathcal{U}(A) = \overline{R}(A).$$

This implies that R is also mediate, as desired.

In the final, we provide a concrete example which satisfies all the mediate, Euclidean and adjoint conditions.

Example 4.13 Suppose that $U = \{x, y, z\}$, $L = [0, 1]$ and $a' = 1 - a$ for each $a \in [0, 1]$. Then $R_{mea} : U \times U \longrightarrow L$ defined by

R_{mea}	x	y	z
x	0.2	0.3	0.3
y	0.2	0.4	0.4
z	0.4	0.3	0.4

is a mediate, Euclidean and adjoint L-fuzzy relation on U.

5 Conclusions

In this paper, we first introduced three new types of L-fuzzy relations, including mediate, Euclidean and adjoint L-fuzzy relations and characterized them by their induced upper and lower L-fuzzy rough approximation operators. Secondly, we provided single axioms for axiomatic characterizations of L-fuzzy rough approximation corresponding to mediate, Euclidean and adjoint L-fuzzy relations as well as their compositions. Following the constructive and axiomatic approaches, we presented the above-mentioned results in this paper, which can be considered as important parts of fuzzy rough set theory. In the future, we will further consider the following problems:

- Using single axioms to characterize L-fuzzy rough approximation operators corresponding to compositions of serial, reflexive, symmetric, transitive, mediate, Euclidean and adjoint L-fuzzy relations.
- Generalizing the lattice background from a frame to a residuated lattice. In particular, a frame equipped with an order-reversing involution will be generalized to a regular residuated lattice.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this paper.

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