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FOUNDATIONS

On novel hesitant fuzzy rough sets

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Abstract

This paper presents a novel framework for the study of hesitant fuzzy rough sets by integrating rough sets with hesitant fuzzy sets. Lower and upper approximations of hesitant fuzzy sets with respect to a hesitant fuzzy approximation space are first defined. Properties of hesitant fuzzy approximation operators are examined. Relationships between hesitant fuzzy approximation spaces and hesitant fuzzy topological spaces are then established. It is proved that the set of all lower approximation sets based on a hesitant fuzzy reflexive and transitive approximation space forms a hesitant fuzzy topology. And conversely, for a hesitant fuzzy rough topological space, there exists a hesitant fuzzy reflexive and transitive approximation space such that the topology in the hesitant fuzzy rough topological space is exactly the set of all lower approximation sets in the hesitant fuzzy reflexive and transitive approximation space. That is to say, there exists a one-to-one correspondence between the set of all hesitant fuzzy reflexive and transitive approximation spaces and the set of all hesitant fuzzy rough topological spaces.

Keywords Approximation operators · Hesitant fuzzy rough sets · Hesitant fuzzy sets · Hesitant fuzzy topological spaces · Rough sets

1 Introduction

Since fuzzy set was introduced by Zadeh (1965) as a mathematical way to represent and deal with vagueness in everyday life, several extensions have been developed, such as intuitionistic fuzzy sets (Atanassov 1986), interval-valued fuzzy sets (Gorzalczany 1987; Deschrijver and Kerre 2005), type-2 fuzzy sets (Dubois and Prade 1980; Miyamoto 2005) and type-*n* fuzzy sets (Dubois and Prade 1980). However, when defining the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have a margin of error (as in an intuitionistic fuzzy set or an interval-valued fuzzy set), or some possibility distribution (as in a type-2 fuzzy set) on the possible values, but because we have a set of possible values. To deal with such case, Torra and Narukawa (2009) and Torra (2010) introduced the concept of hesitant fuzzy (HF) sets. As another generalization of fuzzy sets, an HF set permits the membership of an element to a given set having several possible different values between 0 and 1, and is powerful to determine the membership degree especially when we have several different values on it. Since its appearance, HF set theory has become a fast-growing field of research in recent years (Xia and Xu 2011; Xu and Xia 2011; Chen et al. 2013; Farhadinia 2013; Liao and Xu 2013, 2015; Liu et al. 2016; Liang and Liu 2015).

Another important method used to deal with insufficient and incomplete information is the theory of rough sets originated by Pawlak (1982, 1991). The equivalence relation is a key notion in the Pawlak's rough set model. However, the requirement of an equivalence relation in Pawlak's rough set model seems to be a very restrictive condition that may limit the applications of this model. From both theoretic and practical needs, many researchers have generalized the notion of Pawlak's rough set model by replacing the equivalence relation with non-equivalence relations. More generally, rough set approximations have been extended to fuzzy environment in which the results are called rough fuzzy sets (Dubois and Prade 1990; Li and Zhang 2008; Thiele 2001) and fuzzy

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rough sets (Dubois and Prade 1990; Radzikowska and Kerre 2002; Yeung et al. 2005; Tiwari and Srivastava 2013). Meanwhile, by combining rough set theory with other uncertainty theories, lots of fruitful results have been achieved (Cornelis et al. 2003; Chakrabarty et al. 1998; Nanda and Majumda 1992; Jena and Ghosh 2002; Rizvi et al. 2002; Samanta and Mondal 2001; Sun et al. 2008; Zhang and Shu 2015; Yang et al. 2014; Zhang 2013; Deepak and John 2014; He and Xiong 2017; Zhan et al. 2017).

Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics but also in many real-life applications. An interesting research for rough approximation operators is to compare them with the topological structures. In fact, many authors studied topological structures of crisp rough sets (Chuchro 1994; Kondo 2006; Lashin et al. 2005; Zhu 2007; Wu et al. 2008). On the other hand, some authors discussed topological structures of rough sets in the fuzzy environment (see, e.g., Qin and Pei 2005; Qin et al. 2008; Wu 2011; Zhou et al. 2009; Wu and Zhou 2011; Zhang 2013). In Wu (2011), Wu presented a general framework for the study of T-fuzzy rough approximation operators determined by triangular norms in infinite universes of discourse, and proved that a pair of dual T-fuzzy rough approximation operators can induce a fuzzy topological space if and only if the fuzzy relation in the fuzzy approximation space is reflexive. Furthermore, under certain conditions, a fuzzy interior (respectively, a fuzzy closure) operator derived from a fuzzy topological space can be associated with a reflexive and T-transitive fuzzy relation such that the induced lower (respectively, upper) T-fuzzy rough approximation operator is exactly the fuzzy interior (respectively, the fuzzy closure) operator of the given topological space. Subsequently, Zhou et al. (2009), Wu and Zhou (2011)generalized these results to intuitionistic fuzzy rough sets and established the relationship between intuitionistic fuzzy rough approximations and intuitionistic fuzzy topologies. More recently, Zhang (2013) constructed interval type-2 rough fuzzy sets and interval type-2 fuzzy rough sets by integrating rough set theory with interval type-2 fuzzy set theory, and also examined their topological structures.

Rough set theory and HF set theory are two different tools to deal with uncertainty. Apparently there is no direct relationship between the two theories, however, a major step is taken by Yang et al. (2014). They introduced the concept of HF rough sets and proposed an axiomatic approach to the model. However, the disadvantage of this model is that the order on the HF power set for representing inclusion relation of two HF sets is not necessarily antisymmetric. That is, for any two HF sets $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{A} \sqsubseteq \mathbb{B}$ and $\mathbb{B} \sqsubseteq \mathbb{A}$, then the formula $\mathbb{A} = \mathbb{B}$ does not necessarily true. In classical set theory, it is well known that the antisymmetry is a critical condition for the equality relationship of ordinary sets. So is for HF sets. Only in this way can we better investigate the properties and operations of HF sets. Therefore, it is natural to ask, "Can we define an HF rough set model in which the order for characterizing inclusion relation of HF sets is antisymmetric?" If we can, it may also provide a theoretical basis for our study on topological structures of the HF rough sets. In the present paper, we are to develop a novel HF rough set model in order to find a positive answer to this question.

The remainder of this paper is organized as follows. In the next section, some basic notions and operations related to HF sets are introduced. In Sect. 3, a novel pair of lower and upper HF rough approximation operators are defined and their properties are examined. The connections between special HF relations and properties of HF rough approximation operators are also established. Section 4 introduces some basic notions and results about HF topological spaces. In Sect. 5, we further establish the relationship between HF approximation spaces and HF topological spaces. We then conclude the paper with a summary in Sect. 6.

2 Preliminaries

In this section, we briefly review some basic concepts related to HF sets. Throughout this paper, unless otherwise stated, U refers to a finite universe of discourse.

Definition 1 (Torra and Narukawa 2009; Torra 2010) Let U be a fixed set. An HF set $\mathbb{A}$ on U is in terms of a function $h_{\mathbb{A}}(x)$ that when applied to U returns a subset of [0, 1], that is,

$$\mathbb{A} = \{< x, h_{\mathbb{A}}(x) > | x \in U\},$$

where $h_{\mathbb{A}}(x)$ is a finite set of some different values in [0, 1], representing the possible membership degrees of the element $x \in U$ to $\mathbb{A}$.

For convenience, we call $h_{\mathbb{A}}(x)$ an HF element. The set of all HF sets on U, called the HF power set of U, is denoted by $HF(U)$. From Definition 1, we note that an HF set $\mathbb{A}$ can be viewed as a fuzzy set if there is only one element in $h_{\mathbb{A}}(x)$. In this situation, HF sets include fuzzy sets as a special case.

Example 1 If $U = \{x_1, x_2, x_3\}$ is the universe of discourse, $h_{\mathbb{A}}(x_1) = \{0.7, 0.4, 0.5\}$, $h_{\mathbb{A}}(x_2) = \{0.2, 0.4\}$ and $h_{\mathbb{A}}(x_3) = \{0.3, 0.1, 0.7, 0.6\}$ are the HF elements of $x_i (i = 1, 2, 3)$ to the set $\mathbb{A}$, respectively. Then $\mathbb{A}$ can be considered as an HF set, that is,

$$\mathbb{A} = \{ < x_1, \{0.7, 0.4, 0.5\} >, < x_2, \{0.2, 0.4\} >, < x_3, \{0.3, 0.1, 0.7, 0.6\} >\}.$$

Here, we introduce some special HF sets as follows (Torra 2010; Yang et al. 2014): for $\mathbb{A} \in HF(U)$,

1. $\mathbb{A}$ is referred to as an empty HF set if and only if $h_{\mathbb{A}}(x) = \{0\}$ for all $x \in U$. In this paper, the empty HF set is denoted by $\emptyset$.
2. $\mathbb{A}$ is referred to as the HF universe set if and only if $h_{\mathbb{A}}(x) = \{1\}$ for all $x \in U$. In this paper, the HF universe set is denoted by $\mathbb{U}$.
3. $\mathbb{A}$ is referred to as a constant HF set if and only if $h_{\mathbb{A}}(x) = \{a_1, a_2, \ldots, a_m\}$ for all $x \in U$, where $a_i \in [0, 1]$, $i = 1, 2, \ldots, m$, i.e., $h_{\mathbb{A}}(x) \in 2^{[0,1]}$. In this paper, the constant HF set is denoted by $\widehat{a_1, \ldots, a_m}$.

Meanwhile, for any $y \in U$ and $M \subseteq U$, two special HF sets 1_y and 1_M are, respectively, defined as follows: for $x \in U$,

$$h_{1_y}(x) = \begin{cases} \{1\}, & \text{if } x = y, \\ \{0\}, & \text{otherwise.} \end{cases} \quad h_{1_M}(x) = \begin{cases} \{1\}, & \text{if } x \in M, \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is noted that the number of values in different HF elements may be different. Suppose that $l(h_{\mathbb{A}}(x))$ stands for the number of values in $h_{\mathbb{A}}(x)$. To operate correctly, Xu and Xia (2011) gave the following assumptions:

(A1) All the elements in each HF element $h_{\mathbb{A}}(x)$ are arranged in increasing order, and then $h_{\mathbb{A}}^{\sigma(k)}(x)$ is referred to as the kth largest value in $h_{\mathbb{A}}(x)$.

(A2) If, for two HF elements $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$, $l(h_{\mathbb{A}}(x)) \neq l(h_{\mathbb{B}}(x))$, then $l = \max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$. To have a correct comparison, the two HF elements $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$ should have the same length l. If there are fewer elements in $h_{\mathbb{A}}(x)$ than in $h_{\mathbb{B}}(x)$, an extension of $h_{\mathbb{A}}(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_{\mathbb{B}}(x)$.

In Liao and Xu (2013), Liao et al. pointed out that the dimension of the derived HF element may increase as the addition or multiplicative operations are done when adopting the operations on HF elements defined by Torra and Narukawa (2009), Torra (2010) and Xia and Xu (2011), which may increase the amount of calculation drastically. Therefore, on the basis of the assumptions given by Xu and Xia (2011) and Liao and Xu (2013) developed some new methods to decrease the dimension of the derived HF element when operating the HF elements, which are slightly different from the ones introduced by Torra and Narukawa (2009), Torra (2010) and Xia and Xu (2011). The adjusted operational laws are defined as follows:

Definition 2 (Liao and Xu 2013) Let U be a nonempty and finite universe of discourse. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two HF sets, then, for any $x \in U$,

(1) the complement of $\mathbb{A}$, denoted by $\mathbb{A}^c$, is given by

$$h_{\mathbb{A}^c}(x) = \sim h_{\mathbb{A}}(x) = \{1 - h_{\mathbb{A}}^{\sigma(k)}(x) | k = 1, 2, \ldots, l\};$$

(2) the union of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cup \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cup\mathbb{B}}(x) = h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x) = \{h_{\mathbb{A}}^{\sigma(k)}(x) \vee h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, \ldots, l\};$$

(3) the intersection of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cap \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cap\mathbb{B}}(x) = h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x) = \{h_{\mathbb{A}}^{\sigma(k)}(x) \wedge h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, \ldots, l\};$$

(4) the ring sum of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxplus \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxplus\mathbb{B}}(x) = h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x) = \{h_{\mathbb{A}}^{\sigma(k)}(x) + h_{\mathbb{B}}^{\sigma(k)}(x) - h_{\mathbb{A}}^{\sigma(k)}(x) h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, \ldots, l\};$$

(5) the ring product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxtimes \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxtimes\mathbb{B}}(x) = h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x) = \{h_{\mathbb{A}}^{\sigma(k)}(x) h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, \ldots, l\},$$

where $h_{\mathbb{A}}^{\sigma(k)}(x)$ and $h_{\mathbb{B}}^{\sigma(k)}(x)$ are, respectively, the kth largest value in $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$, and $l = \max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$.

Remark 1 From the above assumptions, we see that multiple occurrences of the greatest element or the smallest element in an HF element are permitted. For example, $h = \{0.2, 0.4, 0.5, 0.5\}$ and $h^c = \{0.5, 0.5, 0.6, 0.8\}$ can be seen as HF elements. However, multiple occurrences of any elements except the greatest element or the smallest element in an HF element are not permitted. For example, $h = \{0.2, 0.4, 0.4, 0.5\}$ cannot be viewed as an HF element, but $h = \{0.2, 0.4, 0.5\}$ is an HF element.

Example 2 Let $h_{\mathbb{A}}(x) = \{0.3, 0.5, 0.8, 0.9\}$ and $h_{\mathbb{B}}(x) = \{0.4, 0.6, 0.7\}$ be two HF elements. According to Definition 2, we have

$$h_{\mathbb{A}\Cup\mathbb{B}}(x) = h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x)$$
$$= \{h_{\mathbb{A}}^{\sigma(k)}(x) \vee h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, 3, 4\}$$
$$= \{0.4 \vee 0.3, 0.5 \vee 0.6, 0.8 \vee 0.7, 0.9 \vee 0.7\}$$
$$= \{0.4, 0.6, 0.8, 0.9\},$$
$$h_{\mathbb{A}\Cap\mathbb{B}}(x) = h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x)$$
$$= \{h_{\mathbb{A}}^{\sigma(k)}(x) \wedge h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, 3, 4\}$$
$$= \{0.4 \wedge 0.3, 0.5 \wedge 0.6, 0.8 \wedge 0.7, 0.9 \wedge 0.7\}$$
$$= \{0.3, 0.5, 0.7, 0.7\},$$
$$h_{\mathbb{A}\boxplus\mathbb{B}}(x) = h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x)$$
$$= \{h_{\mathbb{A}}^{\sigma(k)}(x) + h_{\mathbb{B}}^{\sigma(k)}(x) - h_{\mathbb{A}}^{\sigma(k)}(x) h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, 3, 4\}$$
$$= \{0.4 + 0.3 - 0.4 \times 0.3, 0.5 + 0.6 - 0.5 \times 0.6, 0.8 + 0.7 - 0.8 \times 0.7, 0.9 + 0.7 - 0.9 \times 0.7\}$$
$$= \{0.58, 0.8, 0.94, 0.97\},$$
$$h_{\mathbb{A}\boxtimes\mathbb{B}}(x) = h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x)$$
$$= \{h_{\mathbb{A}}^{\sigma(k)}(x) h_{\mathbb{B}}^{\sigma(k)}(x) | k = 1, 2, 3, 4\}$$
$$= \{0.4 \times 0.3, 0.5 \times 0.6, 0.8 \times 0.7, 0.9 \times 0.7\}$$
$$= \{0.12, 0.3, 0.56, 0.63\}.$$

It is noted that the following theorem is valid for the new operations developed by Liao and Xu (2013).

Theorem 1 (Liao and Xu 2013) *For two HF sets $\mathbb{A}$ and $\mathbb{B}$, the followings are valid:*

(1) $(\mathbb{A} \Cup \mathbb{B})^c = \mathbb{A}^c \Cap \mathbb{B}^c$, $\sim (h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \barwedge (\sim h_{\mathbb{B}}(x))$,
(2) $(\mathbb{A} \Cap \mathbb{B})^c = \mathbb{A}^c \Cup \mathbb{B}^c$, $\sim (h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \veebar (\sim h_{\mathbb{B}}(x))$,
(3) $(\mathbb{A} \boxplus \mathbb{B})^c = \mathbb{A}^c \boxtimes \mathbb{B}^c$, $\sim (h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \otimes (\sim h_{\mathbb{B}}(x))$,
(4) $(\mathbb{A} \boxtimes \mathbb{B})^c = \mathbb{A}^c \boxplus \mathbb{B}^c$, $\sim (h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \oplus (\sim h_{\mathbb{B}}(x))$.

In order to construct a novel HF rough set and further investigate its topological structures, we introduce the concept of HF subsets which is different from the one given by Yang et al. (2014).

Definition 3 Let U be a nonempty and finite universe of discourse. For $\mathbb{A}, \mathbb{B} \in HF(U)$, $\mathbb{A}$ is said to be an HF subset of $\mathbb{B}$, if $h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x)$ holds for all $x \in U$, i.e.,

$$h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x) \Leftrightarrow h_{\mathbb{A}}^{\sigma(k)}(x) \leq h_{\mathbb{B}}^{\sigma(k)}(x), k = 1, 2, \ldots, l.$$

We denote it by $\mathbb{A} \sqsubseteq \mathbb{B}$.

Obviously, we can easily verify the following conclusions: for $\mathbb{A}, \mathbb{B}, \mathbb{C} \in HF(U)$,

(1) $\mathbb{A} \sqsubseteq \mathbb{A}$,
(2) $\mathbb{A} \sqsubseteq \mathbb{B}, \mathbb{B} \sqsubseteq \mathbb{C} \Longrightarrow \mathbb{A} \sqsubseteq \mathbb{C}$,
(3) $\mathbb{A} \sqsubseteq \mathbb{B}, \mathbb{B} \sqsubseteq \mathbb{A} \Longleftrightarrow \mathbb{A} = \mathbb{B}$.
That is, the notation $\sqsubseteq$ is reflexive, transitive and anti-symmetric on $HF(U)$.

Example 3 Let $U = \{x_1, x_2\}$. Suppose that $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ are four HF sets on U defined as follows:

$$\mathbb{A} = \{< x_1, \{0.3, 0.4, 0.5\} >, < x_2, \{0.4, 0.6\} >\},$$
$$\mathbb{B} = \{< x_1, \{0.5, 0.6, 0.7\} >, < x_2, \{0.5, 0.6, 0.7\} >\},$$
$$\mathbb{C} = \{< x_1, \{0.7, 0.8, 0.8\} >, < x_2, \{0.6, 0.9\} >\},$$
$$\mathbb{D} = \{< x_1, \{0.7, 0.8\} >, < x_2, \{0.6, 0.9, 0.9\} >\}.$$

Then, by Definition 3, we have $\mathbb{A} \sqsubseteq \mathbb{B}, \mathbb{B} \sqsubseteq \mathbb{C}, \mathbb{A} \sqsubseteq \mathbb{C}$ and $\mathbb{C} = \mathbb{D}$.

3 Construction of hesitant fuzzy rough approximation operators

In this section, we first introduce the HF relation presented by Yang et al. (2014).

Definition 4 Suppose that U is a nonempty and finite universe of discourse. An HF relation $\mathbb{R}$ on U is an HF subset of $U \times U$, namely, $\mathbb{R}$ is given by

$$\mathbb{R} = \{< (x, y), h_{\mathbb{R}}(x, y) > | (x, y) \in U \times U\},$$

where $h_{\mathbb{R}}(x, y)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the relationship between x and y.

Yang et al. (2014) also presented several special HF relations as follows.

Definition 5 Let $\mathbb{R}$ be an HF relation on U. Then

(1) $\mathbb{R}$ is said to be serial if for any $x \in U$ there exists a $y \in U$ such that $h_{\mathbb{R}}(x, y) = \{1\}$;
(2) $\mathbb{R}$ is said to be reflexive if $h_{\mathbb{R}}(x, x) = \{1\}$ for all $x \in U$;
(3) $\mathbb{R}$ is said to be symmetric if $h_{\mathbb{R}}(x, y) = h_{\mathbb{R}}(y, x)$ for all $(x, y) \in U \times U$;
(4) $\mathbb{R}$ is said to be transitive if $h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{R}}(y, z) \preceq h_{\mathbb{R}}(x, z)$ for all $(x, z) \in U \times U$.
Alternatively, $\mathbb{R}$ is transitive if the following condition is satisfied:

$$h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge h_{\mathbb{R}}^{\sigma(k)}(y, z) \leq h_{\mathbb{R}}^{\sigma(k)}(x, z), k = 1, 2, \ldots, l,$$

where $l = \max\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{R}}(y, z)), l(h_{\mathbb{R}}(x, z))\}$.

Next, we define novel HF rough approximation operators induced from an HF approximation space.

Definition 6 Let U be a nonempty and finite universe of discourse and $\mathbb{R}$ an HF relation on U, then the pair $(U, \mathbb{R})$ is called an HF approximation space. For any $\mathbb{A} \in HF(U)$, the lower and upper approximations of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two HF sets and are, respectively, defined as follows:

$$\underline{\mathbb{R}}(\mathbb{A}) = \{< x, h_{\underline{\mathbb{R}}(\mathbb{A})}(x) > | x \in U\}, \tag{1}$$
$$\overline{\mathbb{R}}(\mathbb{A}) = \{< x, h_{\overline{\mathbb{R}}(\mathbb{A})}(x) > | x \in U\}, \tag{2}$$

where

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \barwedge_{y \in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}}(y)\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \veebar_{y \in U}\{h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{A}}(y)\}.$$

The pair $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ is called the HF rough set of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, and $\underline{\mathbb{R}}, \overline{\mathbb{R}} : HF(U) \to HF(U)$ are referred to as lower and upper HF rough approximation operators, respectively.

Clearly, we can observe that

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \left\{ \bigwedge_{y \in U} h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y) | k = 1, 2, \ldots, l_x \right\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \left\{ \bigvee_{y \in U} h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)}(y) | k = 1, 2, \ldots, l_x \right\},$$

where $l_x = \max \max_{y \in U}\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{A}}(y))\}$.

Remark 2 By using an HF relation, Yang et al. (2014) introduced the concept of HF rough sets. However, HF subset based on the HF rough sets proposed by them is not necessarily antisymmetric. For example, let $\mathbb{F} = \{0.2, 0.3, 0.8\}$ and $\mathbb{G} = \{0.2, 0.4, 0.6, 0.8\}$. According to the definition of the HF subset in Yang et al. (2014), we have $\mathbb{F} \sqsubseteq \mathbb{G}$ and $\mathbb{G} \sqsubseteq \mathbb{F}$. However, it is obvious that $\mathbb{F} \neq \mathbb{G}$. These situations do not occur in classical set theory.

We point out that the HF rough set in Definition 6 is different from the one given by Yang et al. (2014) because of different operational laws on HF elements. Since lower and upper approximations $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ of the HF set $\mathbb{A}$ are still HF sets, the advantage of the novel HF rough set is that HF subset based on the model is antisymmetric. Thus, it provides theoretical basis for our study of its topological structures in the next sections.

Example 4 Let $(U, \mathbb{R})$ be an HF approximation space, where $U = \{x_1, x_2, x_3\}$ and $\mathbb{R}$ is an HF relation on U defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \{0.4\} & \{0.4, 0.6\} & \{0.3, 0.5, 0.7\} \\ \{0.4, 0.7, 0.8\} & \{0.5\} & \{0.1, 0.4, 0.7\} \\ \{0.2, 0.4, 0.5\} & \{0.3, 0.4, 0.6\} & \{0.5, 0.8\} \end{pmatrix} \end{array}$$

If an HF set

$$\mathbb{A} = \{ < x_1, \{0.3, 0.4, 0.6\} >, < x_2, \{0.5, 0.7\} >, < x_3, \{0.2, 0.4, 0.8\} >\},$$

then by Definition 6, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x_1) &= \barwedge_{y \in U}\{h_{\mathbb{R}^c}(x_1, y) \veebar h_{\mathbb{A}}(y)\} \\ &= (\{0.6, 0.6, 0.6\} \veebar \{0.3, 0.4, 0.6\}) \\ &\quad \barwedge (\{0.4, 0.4, 0.6\} \veebar \{0.5, 0.7, 0.7\}) \\ &\quad \barwedge (\{0.3, 0.5, 0.7\} \veebar \{0.2, 0.4, 0.8\}) \\ &= \{0.6, 0.6, 0.6\} \barwedge \{0.5, 0.7, 0.7\} \barwedge \{0.3, 0.5, 0.8\} \\ &= \{0.3, 0.5, 0.6\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x_2) &= \{0.3, 0.4, 0.6\}, \\ h_{\underline{\mathbb{R}}(\mathbb{A})}(x_3) &= \{0.2, 0.4, 0.7\}; \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_1) &= \{0.4, 0.6, 0.7\}, \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_2) &= \{0.5, 0.5, 0.7\}, \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_3) &= \{0.3, 0.4, 0.8\}. \end{aligned}$$

Hence, we can conclude that

$$\underline{\mathbb{R}}(\mathbb{A}) = \{ < x_1, \{0.3, 0.5, 0.6\} >, < x_2, \{0.3, 0.4, 0.6\} >, < x_3, \{0.2, 0.4, 0.7\} >\},$$
$$\overline{\mathbb{R}}(\mathbb{A}) = \{ < x_1, \{0.4, 0.6, 0.7\} >, < x_2, \{0.5, 0.5, 0.7\} >, < x_3, \{0.3, 0.4, 0.8\} >\}.$$

On the other hand, note that

$$\mathbb{A}^c = \{ < x_1, \{0.4, 0.6, 0.7\} >, < x_2, \{0.3, 0.3, 0.5\} >, < x_3, \{0.2, 0.6, 0.8\} >\},$$

then

$$\overline{\mathbb{R}}(\mathbb{A}^c) = \{ < x_1, \{0.4, 0.5, 0.7\} >, < x_2, \{0.4, 0.6, 0.7\} >, < x_3, \{0.3, 0.6, 0.8\} >\}.$$

In general, the conclusions $\overline{\mathbb{R}}(\mathbb{A}^c) = (\underline{\mathbb{R}}(\mathbb{A}))^c$ and $\underline{\mathbb{R}}(\mathbb{A}^c) = (\overline{\mathbb{R}}(\mathbb{A}))^c$ hold, but $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$ cannot hold.

Theorem 2 *Let $(U, \mathbb{R})$ be an HF approximation space. Then the lower and upper HF rough approximation operators induced from $(U, \mathbb{R})$ satisfy the following properties: for all $\mathbb{A}, \mathbb{B} \in HF(U)$, and all $a_i \in [0, 1]$, $i = 1, 2, \ldots, m$,*

(HFL1) $\underline{\mathbb{R}}(\mathbb{A}^c) = (\overline{\mathbb{R}}(\mathbb{A}))^c$.
(HFU1) $\overline{\mathbb{R}}(\mathbb{A}^c) = (\underline{\mathbb{R}}(\mathbb{A}))^c$.
(HFL2) $\mathbb{A} \sqsubseteq \mathbb{B} \Rightarrow \underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}}(\mathbb{B})$.
(HFU2) $\mathbb{A} \sqsubseteq \mathbb{B} \Rightarrow \overline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{B})$.
(HFL3) $\underline{\mathbb{R}}(\mathbb{A} \Cap \mathbb{B}) = \underline{\mathbb{R}}(\mathbb{A}) \Cap \underline{\mathbb{R}}(\mathbb{B})$.
(HFU3) $\overline{\mathbb{R}}(\mathbb{A} \Cup \mathbb{B}) = \overline{\mathbb{R}}(\mathbb{A}) \Cup \overline{\mathbb{R}}(\mathbb{B})$.
(HFL4) $\underline{\mathbb{R}}(\mathbb{A} \Cup \mathbb{B}) \sqsupseteq \underline{\mathbb{R}}(\mathbb{A}) \Cup \underline{\mathbb{R}}(\mathbb{B})$.
(HFU4) $\overline{\mathbb{R}}(\mathbb{A} \Cap \mathbb{B}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A}) \Cap \overline{\mathbb{R}}(\mathbb{B})$.
(HFL5) $\underline{\mathbb{R}}(\mathbb{A} \Cup \widehat{a_1, \ldots, a_m}) = \underline{\mathbb{R}}(\mathbb{A}) \Cup \widehat{a_1, \ldots, a_m}$.
(HFU5) $\overline{\mathbb{R}}(\mathbb{A} \Cap \widehat{a_1, \ldots, a_m}) = \overline{\mathbb{R}}(\mathbb{A}) \Cap \widehat{a_1, \ldots, a_m}$.
(HFL6) $\underline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$.
(HFU6) $\overline{\mathbb{R}}(\emptyset) = \emptyset$.

Proof We only investigate the case of the lower approximation $\underline{R}$.

(HFL1) By Definitions 6, 2 and Theorem 1, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A}^c)}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}^c}(y)\} \\ &= \barwedge_{y\in U}\{(\sim h_{\mathbb{R}}(x, y)) \veebar (\sim h_{\mathbb{A}}(y))\} \\ &= \barwedge_{y\in U}\{\sim (h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{A}}(y))\} \\ &=\sim (\veebar_{y\in U}\{h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{A}}(y)\}) = h_{(\overline{\mathbb{R}}(\mathbb{A}))^c}(x). \end{aligned}$$

Hence, $\underline{\mathbb{R}}(\mathbb{A}^c) = (\overline{\mathbb{R}}(\mathbb{A}))^c$.

(HFL2) Since $\mathbb{A} \sqsubseteq \mathbb{B}$, by Definition 3, we have $h_{\mathbb{A}}^{\sigma(k)}(y) \leq h_{\mathbb{B}}^{\sigma(k)}(y)$ for all $y \in U$. Then, $\forall x \in U$,

$$\bigwedge_{y\in U}(h_{\mathbb{A}}^{\sigma(k)}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)}(x, y)) \leq \bigwedge_{y\in U}(h_{\mathbb{B}}^{\sigma(k)}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)}(x, y)), 1 \leq k \leq l_x.$$

Consequently, for each $x \in U$, $h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \preceq h_{\underline{\mathbb{R}}(\mathbb{B})}(x)$. Hence, $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}}(\mathbb{B})$.

(HFL3) For any $x \in U$, by Eq. (1), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A}\Cap\mathbb{B})}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}\Cap\mathbb{B}}(y)\} \\ &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar (h_{\mathbb{A}}(y) \barwedge h_{\mathbb{B}}(y))\} \\ &= \left\{\bigwedge_{y\in U}(h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee (h_{\mathbb{A}}^{\sigma(k)}(y) \wedge h_{\mathbb{B}}^{\sigma(k)}(y)))|k = 1, 2, \ldots, l_x\right\} \\ &= \left\{\bigwedge_{y\in U}(h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y)) \wedge \bigwedge_{y\in U}(h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{B}}^{\sigma(k)}(y))|k = 1, 2, \ldots, l_x\right\} \\ &= h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \barwedge h_{\underline{\mathbb{R}}(\mathbb{B})}(x) = h_{\underline{\mathbb{R}}(\mathbb{A})\Cap\underline{\mathbb{R}}(\mathbb{B})}(x), \end{aligned}$$

where $l_x = \max \max_{y\in U}\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{A}}(y)), l(h_{\mathbb{B}}(y))\}$.

Hence, (HFL3) holds.

(HFL4) It follows immediately from (HFL2).

(HFL5) For any $x \in U$, by Eq. (1), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A}\Cup\widehat{a_1,\ldots,a_m})}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{(\mathbb{A}\Cup\widehat{a_1,\ldots,a_m})}(y)\} \\ &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar (h_{\mathbb{A}}(y) \veebar \{a_1, a_2, \ldots, a_m\})\} \\ &= \left\{\bigwedge_{y\in U}(h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee (h_{\mathbb{A}}^{\sigma(k)}(y) \vee a_{1,\ldots,m}^{\sigma(k)}))|k = 1, 2, \ldots, l_x\right\} \\ &= \left\{\bigwedge_{y\in U}(h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y)) \vee a_{1,\ldots,m}^{\sigma(k)}|k = 1, 2, \ldots, l_x\right\} \\ &= h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \veebar h_{\widehat{a_1,\ldots,a_m}}(x) = h_{\underline{\mathbb{R}}(\mathbb{A})\Cup\widehat{a_1,\ldots,a_m}}(x), \end{aligned}$$

where $l_x = \max \max_{y\in U}\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{A}}(y)), l(\widehat{a_1, \ldots, a_m})\}$, and $a_{1,\ldots,m}^{\sigma(k)}$ is the kth largest value in $\widehat{a_1, \ldots, a_m}$. Hence, (HFL5) holds.

(HFL6) It follows immediately from Eq. (1). □

Properties (HFL1) and (HFU1) show that the HF rough approximation operators $\overline{\mathbb{R}}$ and $\underline{\mathbb{R}}$ are dual with each other. Properties with the same number may be also considered as dual properties.

Theorem 3 below states that an HF relation can be represented by the HF rough approximation operators.

Theorem 3 *Let $\mathbb{R}$ be an HF relation on U, then, for any $(x, y) \in U \times U$, and $M \subseteq U$,*

(1) $h_{\underline{\mathbb{R}}(1_M)}(x) = \barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y)$.
(2) $h_{\overline{\mathbb{R}}(1_M)}(x) = \veebar_{y\in M} h_{\mathbb{R}}(x, y)$.
(3) $h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(x) = h_{\mathbb{R}^c}(x, y)$.
(4) $h_{\overline{\mathbb{R}}(1_y)}(x) = h_{\mathbb{R}}(x, y)$.

Proof (1) For any $x \in U$, according to Eq. (1), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(1_M)}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{1_M}(y)\} \\ &= \{1\} \barwedge (\barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y)) = \barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y). \end{aligned}$$

(2) It follows immediately from the result (1) and the duality.
(3) For any $x \in U$, by Eq. (1), we conclude

$$\begin{aligned} h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(x) &= \barwedge_{z\in U}\{h_{\mathbb{R}^c}(x, z) \veebar h_{1_{U-\{y\}}}(z)\} \\ &= h_{\mathbb{R}^c}(x, y) \barwedge \{1\} = h_{\mathbb{R}^c}(x, y). \end{aligned}$$

(4) It follows immediately from the result (3) and the duality. □

The following Theorems 4 and 5 show that an HF relation with some special properties, such as serializability, reflexivity, symmetry, and transitivity, can be, respectively, characterized by the essential properties of the lower and upper HF rough approximation operators.

Theorem 4 *Let $\mathbb{R}$ be an HF relation on U. Suppose that $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ are the lower and upper HF rough approximation operators given in Definition 6, then $\mathbb{R}$ is serial iff one of the following properties holds:*

(HFL0) $\underline{\mathbb{R}}(\emptyset) = \emptyset$.
(HFU0) $\overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$.
(HFLU0) $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A}), \forall \mathbb{A} \in HF(U)$.
(HFL0)′ $\underline{\mathbb{R}}(\widehat{a_1, \ldots, a_m}) = \widehat{a_1, \ldots, a_m}, \forall a_i \in [0, 1], 1 \leq i \leq m$.
(HFU0)′ $\overline{\mathbb{R}}(\widehat{a_1, \ldots, a_m}) = \widehat{a_1, \ldots, a_m}, \forall a_i \in [0, 1], 1 \leq i \leq m$.

Proof First, we can deduce from the dual properties of $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ that (HFL0) and (HFU0) are equivalent. Similarly, (HFL0)′ and (HFU0)′ are also equivalent.

Second, we are to prove that $\mathbb{R}$ is serial if and only if (HFU0) holds.

Suppose that $\mathbb{R}$ is serial. For any $x \in U$, by the definition, there exists a $z \in U$ such that $h_{\mathbb{R}}(x, z) = \{1\}$. Hence, by Eq. (2), we have

$$\begin{aligned} h_{\overline{\mathbb{R}}(\mathbb{U})}(x) &= \left\{ \bigvee_{y \in U} (h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge h_{\mathbb{U}}^{\sigma(k)}(y)) | k=1, 2, \ldots, l_x \right\} \\ &= \left\{ \bigvee_{y \in U} (h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge 1) | k = 1, 2, \ldots, l_x \right\} \\ &= \left\{ h_{\mathbb{R}}^{\sigma(k)}(x, z) \vee \left(\bigvee_{y \neq z} h_{\mathbb{R}}^{\sigma(k)}(x, y) \right) | k = 1, 2, \ldots, l_x \right\} \\ &= \{1\} = h_{\mathbb{U}}(x). \end{aligned}$$

Thus, $\overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$, that is, (HFU0) holds.

Conversely, assume that (HFU0) holds, that is, $h_{\overline{\mathbb{R}}(\mathbb{U})}(x) = \{1\}$ for all $x \in U$. If $\mathbb{R}$ is not serial, then there exists an $x_0 \in U$ such that $h_{\mathbb{R}}(x_0, y) \neq \{1\}$ for all $y \in U$. Since $h_{\mathbb{U}}(y) = \{1\}$ for all $y \in U$, we have $h_{\mathbb{R}}(x_0, y) \overline{\wedge} h_{\mathbb{U}}(y) = h_{\mathbb{R}}(x_0, y) \neq \{1\}$ for all $y \in U$, that is, $h_{\overline{\mathbb{R}}(\mathbb{U})}(x_0) \neq \{1\}$, which contradicts the assumption.

Third, we are to prove that $\mathbb{R}$ is serial if and only if (HFLU0) holds.

Suppose that $\mathbb{R}$ is serial. For any $x \in U$, by the definition, there exists a $z \in U$ such that $h_{\mathbb{R}}(x, z) = \{1\}$. Hence $h_{\mathbb{R}^c}(x, z) = \{0\}$. By Eq. (1), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x) &= \overline{\wedge}_{y \in U} \{h_{\mathbb{R}^c}(x, y) \underline{\vee} h_{\mathbb{A}}(y)\} \\ &= \left\{ \bigwedge_{y \in U} (h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y)) | k = 1, 2, \ldots, l_x \right\} \\ &= \Bigg\{ (h_{\mathbb{R}^c}^{\sigma(k)}(x, z) \vee h_{\mathbb{A}}^{\sigma(k)}(z)) \\ &\quad \wedge \left(\bigwedge_{y \neq z} (h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y)) \right) | k = 1, 2, \ldots, l_x \Bigg\} \\ &= \Bigg\{ h_{\mathbb{A}}^{\sigma(k)}(z) \wedge \\ &\quad \left(\bigwedge_{y \neq z} (h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)}(y)) \right) | k = 1, 2, \ldots, l_x \Bigg\} \\ &\preceq \{h_{\mathbb{A}}^{\sigma(k)}(z) | k = 1, 2, \ldots, l_x\} = h_{\mathbb{A}}(z). \end{aligned}$$

On the other hand, according to Eq. (2), we conclude

$$\begin{aligned} h_{\overline{\mathbb{R}}(\mathbb{A})}(x) &= \underline{\vee}_{y \in U} \{h_{\mathbb{R}}(x, y) \overline{\wedge} h_{\mathbb{A}}(y)\} \\ &= \left\{ \bigvee_{y \in U} (h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)}(y)) | k = 1, 2, \ldots, l_x \right\} \\ &= \Bigg\{ (h_{\mathbb{R}}^{\sigma(k)}(x, z) \\ &\quad \wedge h_{\mathbb{A}}^{\sigma(k)}(z)) \vee \Bigg(\bigvee_{y \neq z} (h_{\mathbb{R}}^{\sigma(k)}(x, y) \\ &\quad \wedge h_{\mathbb{A}}^{\sigma(k)}(y)) \Bigg) | k = 1, 2, \ldots, l_x \Bigg\} \\ &= \Bigg\{ h_{\mathbb{A}}^{\sigma(k)}(z) \vee \Bigg(\bigvee_{y \neq z} (h_{\mathbb{R}}^{\sigma(k)}(x, y) \\ &\quad \wedge h_{\mathbb{A}}^{\sigma(k)}(y)) \Bigg) | k = 1, 2, \cdots, l_x \Bigg\} \\ &\succeq \{h_{\mathbb{A}}^{\sigma(k)}(z) | k = 1, 2, \ldots, l_x\} = h_{\mathbb{A}}(z). \end{aligned}$$

Consequently, $h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \preceq h_{\overline{\mathbb{R}}(\mathbb{A})}(x)$. Thus, we have proved that $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$, i.e., (HFLU0) holds.

Conversely, assume that (HFLU0) holds. For any $\mathbb{A} \in HF(U)$, we have $h_{\underline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)}(x) \leq h_{\overline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)}(x)$ for all $x \in U$. Hence $h_{\underline{\mathbb{R}}(\emptyset)}^{\sigma(k)}(x) \leq h_{\overline{\mathbb{R}}(\emptyset)}^{\sigma(k)}(x)$ for all $x \in U$. On the other hand, by Eqs. (1) and (2), we obtain

$$\begin{aligned} h_{\underline{\mathbb{R}}(\emptyset)}(x) &= \overline{\wedge}_{y \in U} h_{\mathbb{R}^c}(x, y) \\ &= \left\{ \bigwedge_{y \in U} h_{\mathbb{R}^c}^{\sigma(k)}(x, y) | k = 1, 2, \ldots, l_x \right\} \end{aligned}$$

and $h_{\overline{\mathbb{R}}(\emptyset)}(x)=\{0\}$. Then, for any $x\in U$, there exists a $y\in U$ such that $h^{\sigma(k)}_{\mathbb{R}^c}(x,y)=0$. It follows that $h_{\mathbb{R}}(x,y)=\{1\}$. Thus, $\mathbb{R}$ is serial.

Finally, we are to prove that $\mathbb{R}$ is serial if and only if (HFL0)$'$ holds.

Assume that $\mathbb{R}$ is serial. For any $a_i\in[0,1], i=1,2,\dots,m$, and $x\in U$, by the definition, there exists a $z\in U$ such that $h_{\mathbb{R}}(x,z)=\{1\}$. Then, by Eq. (1), we have

$$\begin{aligned}
&h_{\underline{\mathbb{R}}(\widehat{a_1,\dots,a_m})}(x)=\overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,h_{\widehat{a_1,\dots,a_m}}(y)\}\\
&=\Bigg\{(h^{\sigma(k)}_{\mathbb{R}^c}(x,z)\vee a^{\sigma(k)}_{1,\dots,m})\\
&\quad\wedge\left(\bigwedge_{y\neq z}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee a^{\sigma(k)}_{1,\dots,m})\right)|k=1,2,\dots,l_x\Bigg\}\\
&=\Bigg\{a^{\sigma(k)}_{1,\dots,m}\wedge\Bigg(\bigwedge_{y\neq z}h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\\
&\quad\vee a^{\sigma(k)}_{1,\dots,m}\Bigg)|k=1,2,\dots,l_x\Bigg\}\\
&=\{a^{\sigma(k)}_{1,\dots,m}|k=1,2,\dots,l_x\}=h_{\widehat{a_1,\dots,a_m}}(x).
\end{aligned}$$

Thus, we have proved that (HFL0)$'$ holds.

Conversely, assume that (HFL0)$'$ holds. Let $x\in U$, for any $a_i\in[0,1], i=1,2,\dots,m$, by (HFL0)$'$, we have

$$\begin{aligned}
&h_{\underline{\mathbb{R}}(\widehat{a_1,\dots,a_m})}(x)=\overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,h_{\widehat{a_1,\dots,a_m}}(y)\}\\
&=\left\{\bigwedge_{y\in U}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee a^{\sigma(k)}_{1,\dots,m})|k=1,2,\dots,l_x\right\}\\
&=\left\{\left(\bigwedge_{y\in U}h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\right)\vee a^{\sigma(k)}_{1,\dots,m}|k=1,2,\dots,l_x\right\}\\
&=\{a^{\sigma(k)}_{1,\dots,m}|k=1,2,\dots,l_x\}.
\end{aligned}$$

Hence $\bigwedge_{y\in U}h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\le a^{\sigma(k)}_{1,\dots,m}$. By taking $a^{\sigma(k)}_{1,\dots,m}=0$, we see that there must exist a $y\in U$ such that $h_{\mathbb{R}}(x,y)=\{1\}$. Thus, $\mathbb{R}$ is serial. □

Remark 3 Theorem 4 is different from Theorem 6(1) in Yang et al. (2014). Since absorption laws do not hold for the operations on HF elements in Yang et al. (2014), we cannot draw the conclusion that "$\mathbb{R}$ is serial $\Longleftrightarrow$(HFL0)$'$ $\Longleftrightarrow$ (HFU0)$'$".

Theorem 5 *Let (U,R) be an HF approximation space, and $\underline{R}$ and $\overline{R}$ the HF approximation operators induced from $(U,\mathbb{R})$. Then, $\forall\mathbb{A}\in HF(U)$, $(x,y)\in U\times U$,*

$$\begin{aligned}
&(1)\ \mathbb{R}\text{is reflexive} \Longleftrightarrow \text{(HFLR) } \underline{\mathbb{R}}(\mathbb{A})\sqsubseteq\mathbb{A}.\\
&\qquad\qquad\Longleftrightarrow \text{(FHUR)}\mathbb{A}\sqsubseteq\overline{\mathbb{R}}(\mathbb{A}).\\
&(2)\mathbb{R}\text{ is symmetric} \Longleftrightarrow \text{(HFLS) } h_{\underline{\mathbb{R}}(1_{U-\{x\}})}(y)=h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(x),\\
&\qquad\qquad\Longleftrightarrow \text{(HFUS) } h_{\overline{\mathbb{R}}(1_x)}(y)=h_{\overline{\mathbb{R}}(1_y)}(x),\\
&(3)\ \mathbb{R}\text{ is transitive} \Longleftrightarrow \text{(HFLT) } \underline{\mathbb{R}}(\mathbb{A})\sqsubseteq\underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A})).\\
&\qquad\qquad\Longleftrightarrow \text{(HFUT) } \overline{\mathbb{R}}(\overline{\mathbb{R}}(\mathbb{A}))\sqsubseteq\overline{\mathbb{R}}(\mathbb{A}).
\end{aligned}$$

Proof (1) By the dual properties of the HF rough approximation operators, it is only to prove that $\mathbb{R}$ is reflexive if and only if (HFLR) holds.

Assume that $\mathbb{R}$ is reflexive. For any $\mathbb{A}\in HF(U)$ and $x\in U$, by the reflexivity of $\mathbb{R}$, we have $h_{\mathbb{R}}(x,x)=\{1\}$. Hence, $h_{\mathbb{R}^c}(x,x)=\{0\}$. Then, by Eq. (1), we conclude

$$\begin{aligned}
&h_{\underline{\mathbb{R}}(\mathbb{A})}(x)=\overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,h_{\mathbb{A}}(y)\}\\
&=\left\{\bigwedge_{y\in U}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee h^{\sigma(k)}_{\mathbb{A}}(y))|k=1,2,\dots,l_x\right\}\\
&=\Bigg\{(h^{\sigma(k)}_{\mathbb{R}^c}(x,x)\vee h^{\sigma(k)}_{\mathbb{A}}(x))\\
&\quad\wedge\left(\bigwedge_{y\neq x}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee h^{\sigma(k)}_{\mathbb{A}}(y))\right)|k=1,2,\dots,l_x\Bigg\}\\
&=\Bigg\{h^{\sigma(k)}_{\mathbb{A}}(x)\wedge\Bigg(\bigwedge_{y\neq x}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\\
&\quad\vee h^{\sigma(k)}_{\mathbb{A}}(y))\Bigg)|k=1,2,\dots,l_x\Bigg\}\\
&\preceq\{h^{\sigma(k)}_{\mathbb{A}}(x)|k=1,2,\dots,l_x\}=h_{\mathbb{A}}(x).
\end{aligned}$$

Thus, we have proved that (HFLR) holds.

Conversely, assume that (HFLR) holds. For any $x\in U$, by taking $\mathbb{A}=1_{U-\{x\}}$ in (HFLR), we obtain $h^{\sigma(k)}_{\underline{\mathbb{R}}(1_{U-\{x\}})}(x)\le h^{\sigma(k)}_{1_{U-\{x\}}}(x)=0$. That is, $h_{\underline{\mathbb{R}}(1_{U-\{x\}})}(x)=\{0\}$. On the other hand, by Eq. (1), we have

$$\begin{aligned}
&h_{\underline{\mathbb{R}}(1_{U-\{x\}})}(x)=\overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,h_{1_{U-\{x\}}}(y)\}\\
&=\left\{\bigwedge_{y\in U}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee h^{\sigma(k)}_{1_{U-\{x\}}}(y))|k=1,2,\dots,l_x\right\}\\
&=\Bigg\{(h^{\sigma(k)}_{\mathbb{R}^c}(x,x)\vee h^{\sigma(k)}_{1_{U-\{x\}}}(x))\\
&\quad\wedge\left(\bigwedge_{y\neq x}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee h^{\sigma(k)}_{1_{U-\{x\}}}(y))\right)|k=1,2,\dots,l_x\Bigg\}\\
&=\Bigg\{h^{\sigma(k)}_{\mathbb{R}^c}(x,x)\\
&\quad\wedge\left(\bigwedge_{y\neq x}(h^{\sigma(k)}_{\mathbb{R}^c}(x,y)\vee 1)\right)|k=1,2,\dots,l_x\Bigg\}
\end{aligned}$$

$$= \{h_{\mathbb{R}^c}^{\sigma(k)}(x, x) | k = 1, 2, \ldots, l_x\}$$
$$= h_{\mathbb{R}^c}(x, x) = \{0\}.$$

Consequently, $h_{\mathbb{R}}(x, x) = \{1\}$. Thus, $\mathbb{R}$ is reflexive.

(2) It follows immediately from Theorem 3.
(3) By the dualities of the HF rough approximation operators, it is easy to verify that (HFLT) and (HFUT) are equivalent. We are only to prove that the transitivity of $\mathbb{R}$ is equivalent to (HFLT).

Suppose that $\mathbb{R}$ is transitive. For any $\mathbb{A} \in HF(U)$ and $x \in U$, by Eq. (1), we have

$$h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A}))}(x) = \bar{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\underline{\mathbb{R}}(\mathbb{A})}(y)\}$$
$$= \left\{ \bigwedge_{y\in U} (h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee \left(\bigwedge_{z\in U} (h_{\mathbb{R}^c}^{\sigma(k)}(y, z) \vee h_{\mathbb{A}}^{\sigma(k)}(z))\right) \right) | k = 1, 2, \ldots, l_x \right\}$$
$$= \left\{ \bigwedge_{y\in U} \bigwedge_{z\in U} (h_{\mathbb{R}^c}^{\sigma(k)}(x, y) \vee h_{\mathbb{R}^c}^{\sigma(k)}(y, z) \vee h_{\mathbb{A}}^{\sigma(k)}(z)) | k = 1, 2, \ldots, l_x \right\}$$
$$= \left\{ \bigwedge_{z\in U} \bigwedge_{y\in U} ((1 - h_{\mathbb{R}}^{\sigma(k)}(x, y)) \vee (1 - h_{\mathbb{R}}^{\sigma(k)}(y, z))) \vee h_{\mathbb{A}}^{\sigma(k)}(z) | k = 1, 2, \ldots, l_x \right\}$$
$$= \left\{ \bigwedge_{z\in U} \left(\bigwedge_{y\in U} (1 - (h_{\mathbb{R}}^{\sigma(k)}(x, y) \wedge h_{\mathbb{R}}^{\sigma(k)}(y, z))) \right) \vee h_{\mathbb{A}}^{\sigma(k)}(z) | k = 1, 2, \ldots, l_x \right\}$$
$$\succeq \left\{ \bigwedge_{z\in U} (h_{\mathbb{R}^c}^{\sigma(k)}(x, z) \vee h_{\mathbb{A}}^{\sigma(k)}(z)) | k = 1, 2, \ldots, l_x \right\}$$
$$= h_{\underline{\mathbb{R}}(\mathbb{A})}(x).$$

Thus, we have proved that (HFLT) holds.

Conversely, assume that (HFLT) holds. For any $x, y \in U$, by taking $\mathbb{A} = 1_{U-\{y\}}$ in (HFLT), we observe that $h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}(1_{U-\{y\}}))}(x) \succeq h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(x)$. On the other hand, by Eq. (1) and Theorem 3, we obtain

$$h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}(1_{U-\{y\}}))}(x) = \bar{\wedge}_{z\in U}\{h_{\mathbb{R}^c}(x, z) \veebar h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(z)\}$$
$$= \bar{\wedge}_{z\in U}\{h_{\mathbb{R}^c}(x, z) \veebar h_{\mathbb{R}^c}(z, y)\},$$

and $h_{\underline{\mathbb{R}}(1_{U-\{y\}})}(x) = h_{\mathbb{R}^c}(x, y)$. Consequently,

$$\bigwedge_{z\in U} (h_{\mathbb{R}^c}^{\sigma(k)}(x, z) \vee h_{\mathbb{R}^c}^{\sigma(k)}(z, y)) \geq h_{\mathbb{R}^c}^{\sigma(k)}(x, y).$$

Hence $h_{\mathbb{R}^c}^{\sigma(k)}(x, z) \vee h_{\mathbb{R}^c}^{\sigma(k)}(z, y) \geq h_{\mathbb{R}^c}^{\sigma(k)}(x, y)$ for all $z \in U$. It follows that $h_{\mathbb{R}}^{\sigma(k)}(x, z) \wedge h_{\mathbb{R}}^{\sigma(k)}(z, y) \leq h_{\mathbb{R}}^{\sigma(k)}(x, y)$ for all $k \in \{1, 2, \ldots, l\}$. Thus, we conclude that $\mathbb{R}$ is transitive. □

Combining (1) and (3) in Theorem 5, we can easily obtain the following corollary.

Corollary 1 *If $\mathbb{R}$ is a reflexive and transitive HF relation on U, then*

(HFLRT) $\underline{\mathbb{R}}(\mathbb{A}) = \underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A})), \forall \mathbb{A} \in HF(U)$.
(HFURT) $\overline{\mathbb{R}}(\overline{\mathbb{R}}(\mathbb{A})) = \overline{\mathbb{R}}(\mathbb{A}), \forall \mathbb{A} \in HF(U)$.

4 Hesitant fuzzy topological spaces

In this section, we introduce some basic concepts related to HF topological spaces in the sense of Lowen (1976).

Definition 7 An HF topology in the sense of Lowen on a nonempty set U is a family τ of HF sets on U satisfying the following conditions:

(T_1) $\widehat{a_1, \ldots, a_m} \in \tau$ for all $a_i \in [0, 1], i = 1, 2, \ldots, m$.
(T_2) $\mathbb{A} \Cap \mathbb{B} \in \tau$ for any $\mathbb{A}, \mathbb{B} \in \tau$.
(T_3) $\Cup_{i\in I} \mathbb{A}_i \in \tau$ for any $\mathbb{A}_i \in \tau, i \in I$, where I is an index set.

The pair (U, τ) is called an HF topological space and each HF set $\mathbb{A}$ in τ is referred to as an HF open set in (U, τ). The complement of an HF open set in the HF topological space (U, τ) is called an HF closed set in (U, τ).

It is noted that if the condition (T_1) in Definition 7 is replaced by $\emptyset, \mathbb{U} \in \tau$, then τ is an HF topology in the sense of Chang (1968). It is evident that an HF topology in the sense of Lowen must be an HF topology in the sense of Chang. Throughout this paper, we always consider the HF topology in the sense of Lowen.

Example 5 Let $U = \{x_1, x_2, x_3\}$ and $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ four HF sets on U defined as follows:

$$\mathbb{A} = \{ < x_1, \{0.4, 0.5, 0.6, 0.7\} >, < x_2, \{0.4, 0.6\} >, < x_3, \{0.5, 0.7\} > \},$$
$$\mathbb{B} = \{ < x_1, \{0.6, 0.7, 0.8\} >, < x_2, \{0.3, 0.4, 0.6\} >, < x_3, \{0.4, 0.5\} > \},$$
$$\mathbb{C} = \{ < x_1, \{0.4, 0.5, 0.6, 0.7\} >, < x_2, \{0.3, 0.4, 0.6\} >, < x_3, \{0.4, 0.5\} > \},$$
$$\mathbb{D} = \{ < x_1, \{0.6, 0.7, 0.8\} >, < x_2, \{0.4, 0.6\} >, < x_3, \{0.5, 0.7\} > \}.$$

Then, the family $\tau = \{\emptyset, \mathbb{U}, \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}\}$ is an HF topology on U.

Now we define HF interior and closure operators in an HF topological space.

Definition 8 Let (U, τ) be an HF topological space. For any $\mathbb{A} \in HF(U)$, the HF interior and HF closure of $\mathbb{A}$ are, respectively, defined as follows:
$int(\mathbb{A}) = \Cup\{\mathbb{G}|\mathbb{G} \in \tau \text{ and } \mathbb{G} \sqsubseteq \mathbb{A}\}$,
$cl(\mathbb{A}) = \Cap\{\mathbb{K}|\mathbb{K}^c \in \tau \text{ and } \mathbb{A} \sqsubseteq \mathbb{K}\}$,
and int *and* $cl : HF(U) \longrightarrow HF(U)$ are, respectively, called the HF interior operator and the HF closure operator of τ.

Example 6 Reconsider Example 5. Let $\mathbb{E}$ be another HF set on U defined as follows:

$$\mathbb{E} = \{ < x_1, \{0.7, 0.8, 0.9\} >, < x_2, \{0.5, 0.6, 0.8\} >, < x_3, \{0.7, 0.8\} > \}.$$

By Definition 8, it can be calculated that

$$int(\mathbb{E}) = \mathbb{D} = \{ < x_1, \{0.6, 0.7, 0.8\} >, < x_2, \{0.4, 0.6\} >, < x_3, \{0.5, 0.7\} > \},$$

and $cl(\mathbb{E}) = \mathbb{U}$.

Theorem 6 *Let (U, τ) be an HF topological space. For any $\mathbb{A} \in HF(U)$, then*

(1) *$\mathbb{A}$ is an HF open set in (U, τ) iff $int(\mathbb{A}) = \mathbb{A}$.*
(2) *$\mathbb{A}$ is an HF closed set in (U, τ) iff $cl(\mathbb{A}) = \mathbb{A}$.*

Proof It is straightforward from Definition 8. □

Theorem 7 *Let (U, τ) be an HF topological space. Then the following properties hold: for any $\mathbb{A}, \mathbb{B} \in HF(U)$ and $a_i \in [0, 1], i = 1, 2, \ldots, m,$*

(Int0) $(int(\mathbb{A}))^c = cl(\mathbb{A}^c)$, (Cl0) $(cl(\mathbb{A}))^c = int(\mathbb{A}^c)$.
(Int1) $int(\widehat{a_1, \cdots, a_m}) = \widehat{a_1, \cdots, a_m}$, (Cl1) $cl(\widehat{a_1, \cdots, a_m}) = \widehat{a_1, \cdots, a_m}$.
(Int2) $int(\mathbb{A}) \sqsubseteq \mathbb{A}$, (Cl2) $\mathbb{A} \sqsubseteq cl(\mathbb{A})$.
(Int3) $int(int(\mathbb{A})) = int(\mathbb{A})$, (Cl3) $cl(cl(\mathbb{A})) = cl(\mathbb{A})$.
(Int4) $int(\mathbb{A} \Cap \mathbb{B}) = int(\mathbb{A}) \Cap int(\mathbb{B})$, (Cl4) $cl(\mathbb{A} \Cup \mathbb{B}) = cl(\mathbb{A}) \Cup cl(\mathbb{B})$.

Proof It is straightforward from Definition 8 and Theorem 6. □

Properties (Int0) and (Cl0) state that the HF interior operator and the HF closure operator of τ are dual with each other. Moreover, it is easy to observe that properties (Int4) and (Cl4) imply, respectively, the following properties (Int4)′ and (Cl4)′:

(Int4)′ $\mathbb{A} \sqsubseteq \mathbb{B} \Longrightarrow int(\mathbb{A}) \sqsubseteq int(\mathbb{B})$,
(Cl4)′ $\mathbb{A} \sqsubseteq \mathbb{B} \Longrightarrow cl(\mathbb{A}) \sqsubseteq cl(\mathbb{B})$.

The following theorem shows that an HF operator satisfying properties (Int1)–(Int4) (respectively, properties (Cl1)–(Cl4)) is the HF interior operator (respectively, the HF closure operator) of some IF topology.

Theorem 8 (1) *If an HF operator $int : HF(U) \longrightarrow HF(U)$ satisfies properties* (Int1)–(Int4)*, then there exists an HF topology τ_{int} on U such that $int_{\tau_{int}} = int$.*
(2) *If an HF operator $cl : HF(U) \longrightarrow HF(U)$ satisfies properties* (Cl1)–(Cl4)*, then there exists an HF topology τ_{cl} on U such that $cl_{\tau_{cl}} = cl$.*

Proof (1) Define $\tau_{int} = \{\mathbb{A} \in HF(U)|int(\mathbb{A}) = \mathbb{A}\}$. We are to prove that τ_{int} is an HF topology on U.

(T_1) For any $a_i \in [0, 1], i = 1, 2, \ldots, m$, by (Int1), we have $\widehat{a_1, \cdots, a_m} \in \tau_{int}$.
(T_2) For any $\mathbb{A}, \mathbb{B} \in \tau_{int}$, that is, $int(\mathbb{A}) = \mathbb{A}$ and $int(\mathbb{B}) = \mathbb{B}$. By (Int4), we have $int(\mathbb{A} \Cap \mathbb{B}) = int(\mathbb{A}) \Cap int(\mathbb{B}) = \mathbb{A} \Cap \mathbb{B}$. Thus $\mathbb{A} \Cap \mathbb{B} \in \tau_{int}$.
(T_3) Suppose that $\mathbb{A}_i \in \tau_{int}, i \in I, I$ is any index set. Since $int(\mathbb{A}_i) = \mathbb{A}_i$, for all $i \in I$, by (Int2), we have $int(\Cup_{i \in I} \mathbb{A}_i) \sqsubseteq \Cup_{i \in I} \mathbb{A}_i$.

On the other hand, obviously, $\Cup_{i \in I} int(\mathbb{A}_i) \sqsupseteq int(\mathbb{A}_i)$ for all $i \in I$, then, by (Int4)′ and (Int3), we obtain $int(\Cup_{i \in I} int(\mathbb{A}_i)) \sqsupseteq int(int(\mathbb{A}_i)) = int(\mathbb{A}_i)$ for all $i \in I$. Hence $int(\Cup_{i \in I} int(\mathbb{A}_i)) \sqsupseteq \Cup_{i \in I} int(\mathbb{A}_i)$. Moreover, by the assumption, we have $int(\Cup_{i \in I} \mathbb{A}_i) \sqsupseteq \Cup_{i \in I} \mathbb{A}_i$ for all $i \in I$. Consequently, $int(\Cup_{i \in I} \mathbb{A}_i) = \Cup_{i \in I} \mathbb{A}_i$. Therefore, $\Cup_{i \in I} \mathbb{A}_i \in \tau_{int}$.

Thus, we have proved that τ_{int} is an HF topology on U. Obviously, $int_{\tau_{int}} = int$.

(2) By defining $\tau_{cl} = \{\mathbb{A} \in HF(U) | cl(\mathbb{A}^c) = \mathbb{A}^c\}$, it is similar to the proof of (1). □

Theorem 9 (1) *Let* $int : HF(U) \longrightarrow HF(U)$ *be an HF operator satisfying properties* (Int1)–(Int4). *Define* $\tau'_{int} = \{int(\mathbb{A}) | \mathbb{A} \in HF(U)\}$, *then* $\tau'_{int} = \tau_{int}$.
(2) *Let* $cl : HF(U) \longrightarrow HF(U)$ *be an HF operator satisfying properties* (Cl1)–(Cl4). *Define* $\tau'_{cl} = \{(cl(\mathbb{A}))^c | \mathbb{A} \in HF(U)\}$,

then $\tau'_{cl} = \tau_{cl}$.

Proof (1) It is evident that $\tau_{int} = \{\mathbb{A} \in HF(U) | int(\mathbb{A}) = \mathbb{A}\} \subseteq \tau'_{int}$. On the other hand, for any $\mathbb{A} \in HF(U)$, by (Int3) we have $int(int(\mathbb{A})) = int(\mathbb{A})$. Thus $int(\mathbb{A}) \in \tau_{int}$. Hence, $\tau'_{int} \subseteq \tau_{int}$. Consequently, $\tau'_{int} = \tau_{int}$.
(2) It is similar to the proof of (1). □

Theorem 10 *Let* $int : HF(U) \longrightarrow HF(U)$ *be an HF operator satisfying properties* (Int1)–(Int4) *and* $cl : HF(U) \longrightarrow HF(U)$ *satisfying properties* (Cl1)–(Cl4). *If* (Int0) *and* (Cl0) *hold, then* $\tau'_{int} = \tau_{int} = \tau'_{cl} = \tau_{cl}$.

Proof According to Theorem 9, we are only to prove that $\tau'_{int} = \tau'_{cl}$.

In fact, by (Int0) and (Cl0), we have

$$\begin{aligned}\tau'_{int} &= \{int(\mathbb{A}) | \mathbb{A} \in HF(U)\} = \{(cl(\mathbb{A}^c))^c | \mathbb{A} \in HF(U)\} \\ &= \{(cl(\mathbb{A}))^c | \mathbb{A}^c \in HF(U)\} = \{(cl(\mathbb{A}))^c | \mathbb{A} \in HF(U)\} \\ &= \tau'_{cl}.\end{aligned}$$

□

5 Relationships between HF approximation spaces and HF topological spaces

In this section, we generalize the HF rough set theory in the framework of HF topological spaces and discuss relationships between HF rough approximation spaces and HF topological spaces.

5.1 From HF approximation spaces to HF topological spaces

In this subsection, we assume that U is a nonempty and finite universe of discourse, $\mathbb{R}$ an HF relation on U, and $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ the HF rough approximation operators in Definition 6.

Denote

$$\tau_{\mathbb{R}} = \{\mathbb{A} \in HF(U) | \underline{\mathbb{R}}(\mathbb{A}) = \mathbb{A}\}. \quad (3)$$

Proposition 1 *Let* I *be an index set, and* $\mathbb{A}_i \in HF(U)$ *for all* $i \in I$. *If* $\mathbb{R}$ *is an HF reflexive and transitive relation on* U, *then* $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) = \Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)$.

Proof On one hand, by the reflexivity of $\mathbb{R}$ and Theorem 5, we have $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) \sqsubseteq \Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)$. On the other hand, since $\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i) \sqsupseteq \underline{\mathbb{R}}(\mathbb{A}_i)$, in terms of (HFL2) in Theorem 2, we have $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) \sqsupseteq \underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A}_i))$. Since $\mathbb{R}$ is an HF reflexive and transitive relation on U, by Corollary 1, we have $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) \sqsupseteq \underline{\mathbb{R}}(\mathbb{A}_i)$. Hence $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) \sqsupseteq \Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)$. Consequently, $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) = \Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)$. □

Theorem 11 below shows that an HF reflexive and transitive relation on U can induce an HF topology on U.

Theorem 11 *If* $\mathbb{R}$ *is an HF reflexive and transitive relation on* U, *then* $\tau_{\mathbb{R}}$ *defined in Eq.* (3) *is an HF topology on* U.

Proof (T_1) For any $a_i \in [0, 1], i = 1, 2, \ldots, m$, since an HF reflexive relation must be serial, by Theorem 4, we have $\underline{\mathbb{R}}(\widehat{a_1, \cdots, a_m}) = \widehat{a_1, \ldots, a_m}$. Thus, $\widehat{a_1, \ldots, a_m} \in \tau_{\mathbb{R}}$.

(T_2) For any $\mathbb{A}, \mathbb{B} \in \tau_{\mathbb{R}}$, that is, $\underline{\mathbb{R}}(\mathbb{A}) = \mathbb{A}$ and $\underline{\mathbb{R}}(\mathbb{B}) = \mathbb{B}$. Then, by Theorem 2, we have $\underline{\mathbb{R}}(\mathbb{A} \Cap \mathbb{B}) = \underline{\mathbb{R}}(\mathbb{A}) \Cap \underline{\mathbb{R}}(\mathbb{B}) = \mathbb{A} \Cap \mathbb{B}$. Thus, $\mathbb{A} \Cap \mathbb{B} \in \tau_{\mathbb{R}}$.

(T_3) Suppose that $\mathbb{A}_i \in \tau_{\mathbb{R}}$ for all $i \in I$, where I is any index set. Obviously, $\underline{\mathbb{R}}(\mathbb{A}_i) = \mathbb{A}_i$ for all $i \in I$. Since $\mathbb{R}$ is reflexive and transitive, by virtue of Proposition 1, we have $\underline{\mathbb{R}}(\Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)) = \Cup_{i \in I} \underline{\mathbb{R}}(\mathbb{A}_i)$. Thus, we can conclude that $\underline{\mathbb{R}}(\Cup_{i \in I} \mathbb{A}_i) = \Cup_{i \in I} \mathbb{A}_i$, which means that $\Cup_{i \in I} \mathbb{A}_i \in \tau_{\mathbb{R}}$.

Therefore, $\tau_{\mathbb{R}}$ is an HF topology on U. □

Example 7 Let $(U, \mathbb{R})$ be an HF approximation space, where $U = \{x_1, x_2, x_3\}$ and $\mathbb{R}$ is defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \{0.2, 0.5\} & \{0.4, 0.6, 0.7\} & \{0.3, 0.5, 0.8\} \\ \{0.3, 0.6, 0.9\} & \{1\} & \{0.3, 0.6\} \\ \{0.2, 0.4, 0.7\} & \{0.5, 0.8\} & \{0.4\} \end{pmatrix} \end{array}$$

By Definition 6, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\emptyset)}(x_1) &= \overline{\wedge}_{y \in U} \{h_{\mathbb{R}^c}(x_1, y) \veebar h_{\emptyset}(y)\} \\ &= \{0.5, 0.5, 0.8\} \barwedge \{0.3, 0.4, 0.6\} \barwedge \{0.2, 0.5, 0.7\} \\ &= \{0.2, 0.4, 0.6\}, \\ h_{\underline{\mathbb{R}}(\emptyset)}(x_2) &= \overline{\wedge}_{y \in U} \{h_{\mathbb{R}^c}(x_2, y) \veebar h_{\emptyset}(y)\} \\ &= \{0.1, 0.4, 0.7\} \barwedge \{0\} \barwedge \{0.4, 0.4, 0.7\} = \{0\}, \\ h_{\underline{\mathbb{R}}(\emptyset)}(x_3) &= \overline{\wedge}_{y \in U} \{h_{\mathbb{R}^c}(x_3, y) \veebar h_{\emptyset}(y)\} \\ &= \{0.3, 0.6, 0.8\} \barwedge \{0.2, 0.2, 0.5\} \barwedge \{0.6\} \\ &= \{0.2, 0.2, 0.5\}. \end{aligned}$$

Thus

$$\begin{aligned} \underline{\mathbb{R}}(\emptyset) = \{ &< x_1, \{0.2, 0.4, 0.6\} >, < x_2, \{0\} >, \\ &< x_3, \{0.2, 0.2, 0.5\} >\} \neq \emptyset. \end{aligned}$$

Hence, $\emptyset \notin \tau_{\mathbb{R}}$. That is, $\tau_{\mathbb{R}}$ does not form an HF topology. Obviously, $\mathbb{R}$ is not reflexive.

From Example 7, we observe that if an HF relation $\mathbb{R}$ is not reflexive, then $\tau_{\mathbb{R}}$ defined by Eq. (3) may not be an HF topology.

Theorem 12 below shows that an HF reflexive and transitive approximation space can generate an HF topological space such that the family of all lower approximations of HF sets with respect to the HF approximation space forms the HF topology.

Theorem 12 *If $\mathbb{R}$ is an HF reflexive and transitive relation on U, then $\{\underline{\mathbb{R}}(\mathbb{A}) | \mathbb{A} \in HF(U)\}$ is an HF topology on U.*

Proof Obviously, $\tau_{\mathbb{R}} \subseteq \{\underline{\mathbb{R}}(\mathbb{A}) | \mathbb{A} \in HF(U)\}$. On the other hand, since $\mathbb{R}$ is an HF reflexive and transitive relation on U, by Corollary 1, we have $\underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A})) = \underline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in HF(U)$, which means that $\underline{\mathbb{R}}(\mathbb{A}) \in \tau_{\mathbb{R}}$ for all $\mathbb{A} \in HF(U)$. Hence $\{\underline{\mathbb{R}}(\mathbb{A}) | \mathbb{A} \in HF(U)\} \subseteq \tau_{\mathbb{R}}$. Consequently, $\{\underline{\mathbb{R}}(\mathbb{A}) | \mathbb{A} \in HF(U)\} = \tau_{\mathbb{R}}$. Thus, by Theorem 11, we conclude that $\{\underline{\mathbb{R}}(\mathbb{A}) |\ \mathbb{A} \in HF(U)\}$ is an HF topology on U. □

Theorem 13 *Let $(U, \tau_{\mathbb{R}})$ be the HF topological space induced from an HF reflexive and transitive approximation space $(U, \mathbb{R})$, i.e., $\tau_{\mathbb{R}} = \{\underline{\mathbb{R}}(\mathbb{A}) | \mathbb{A} \in HF(U)\}$. Then, for any $\mathbb{A} \in HF(U)$,*

$$(1)\underline{\mathbb{R}}(\mathbb{A}) = int_{\tau_{\mathbb{R}}}(\mathbb{A}) = \Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\},$$

$$\begin{aligned}(2)\overline{\mathbb{R}}(\mathbb{A}) &= cl_{\tau_{\mathbb{R}}}(\mathbb{A}) \\ &= \Cap\{(\underline{\mathbb{R}}(\mathbb{B}))^c | (\underline{\mathbb{R}}(\mathbb{B}))^c \sqsupseteq \mathbb{A}, \mathbb{B} \in HF(U)\} \\ &= \Cap\{\overline{\mathbb{R}}(\mathbb{B}) | \overline{\mathbb{R}}(\mathbb{B}) \sqsupseteq \mathbb{A}, \mathbb{B} \in HF(U)\}.\end{aligned}$$

Proof (1) Since $\mathbb{R}$ is reflexive, by Theorem 5, we have $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \mathbb{A}$. It follows that $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\}$. On the other hand, from $\Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\} \sqsubseteq \mathbb{A}$, we see that $\underline{\mathbb{R}}(\Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\}) \sqsubseteq \underline{\mathbb{R}}(\mathbb{A})$. Furthermore, by Proposition 1, we obtain $\Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\} \sqsubseteq \underline{\mathbb{R}}(\mathbb{A})$. Thus, we conclude that $\Cup\{\underline{\mathbb{R}}(\mathbb{B}) | \underline{\mathbb{R}}(\mathbb{B}) \sqsubseteq \mathbb{A}, \mathbb{B} \in HF(U)\} = \underline{\mathbb{R}}(\mathbb{A})$.

(2) It follows immediately from the result (1) and the duality of $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$. □

Theorem 13 states that the lower and upper HF rough approximation operators induced from an HF reflexive and transitive approximation space are, respectively, the interior and closure operators of an HF topological space. Theorem 14 below shows that an HF reflexive and transitive relation can also be represented by its producing HF topology.

Theorem 14 *Let $(U, \mathbb{R})$ be an HF reflexive and transitive approximation space and $(U, \tau_{\mathbb{R}})$ the HF topological space induced by $(U, \mathbb{R})$. Then*

$$h_{\mathbb{R}}(x, y) = \overline{\wedge}_{B \in (y)_{\tau_{\mathbb{R}}}} h_{\mathbb{B}}(x),$$

$$where\ (y)_{\tau_{\mathbb{R}}} = \{\mathbb{B} \in HF(U) | \mathbb{B}^c \in \tau_{\mathbb{R}}, h_{\mathbb{B}}(y) = \{1\}\}.$$

Proof For any $x, y \in U$, by Theorem 13, it is clear that $\overline{\mathbb{R}}(1_y) = cl_{\tau_{\mathbb{R}}}(1_y)$. And, by Theorem 3, it can be seen that $h_{\mathbb{R}}(x, y) = h_{\overline{\mathbb{R}}(1_y)}(x)$. On the other hand, since $cl_{\tau_{\mathbb{R}}}(1_y) = \Cap\{\mathbb{B} \in HF(U) | \mathbb{B}^c \in \tau_{\mathbb{R}}\ and\ 1_y \sqsubseteq \mathbb{B}\}$, we have

$$\begin{aligned}h_{cl_{\tau_{\mathbb{R}}}(1_y)}(x) &= \overline{\wedge}\{h_{\mathbb{B}}(x) | \mathbb{B}^c \in \tau_{\mathbb{R}},\ h_{1_y}(x) \preceq h_{\mathbb{B}}(x)\} \\ &= \overline{\wedge}\{h_{\mathbb{B}}(x) | \mathbb{B}^c \in \tau_{\mathbb{R}}, h_{\mathbb{B}}(y) = \{1\}\} \\ &= \overline{\wedge}_{B \in (y)_{\tau_{\mathbb{R}}}} h_{\mathbb{B}}(x).\end{aligned}$$

Consequently, we conclude that $h_{\mathbb{R}}(x, y) = \overline{\wedge}_{B \in (y)_{\tau_{\mathbb{R}}}} h_{\mathbb{B}}(x)$. □

5.2 From HF topological spaces to HF approximation spaces

As seen from Sect. 5.1, an HF reflexive and transitive approximation space can yield an HF topological space such that its HF interior and closure operators are, respectively, the lower and upper approximation operators of the given HF approximation space. In this subsection, we consider the reverse problem, that is, under which conditions can an HF topological space be associated with an HF approximation space producing the same HF topological space? Theorem 15 below answers this question.

Theorem 15 *Let (U, τ) be an HF topological space and $int, cl : HF(U) \longrightarrow HF(U)$ its HF interior operator and HF closure operator, respectively. Then there exists an HF reflexive and transitive relation $\mathbb{R}_\tau$ on U such that $\underline{\mathbb{R}_\tau}(\mathbb{A}) = int(\mathbb{A})$ and $\overline{\mathbb{R}_\tau}(\mathbb{A}) = cl(\mathbb{A})$ for all $\mathbb{A} \in HF(U)$ iff int satisfies axioms (I2) and (I3), or equivalently, cl satisfies axioms (C2) and (C3): for all $\mathbb{A}, \mathbb{B} \in HF(U)$, $\widehat{a_1, \ldots, a_m} \in 2^{[0,1]}$,*

(I2) $int(\mathbb{A} \Cup \widehat{a_1, \cdots, a_m}) = int(\mathbb{A}) \Cup \widehat{a_1, \ldots, a_m}$.
(I3) $int(\mathbb{A} \Cap \mathbb{B}) = int(\mathbb{A}) \Cap int(\mathbb{B})$.
(C2) $cl(\mathbb{A} \Cap \widehat{a_1, \ldots, a_m}) = cl(\mathbb{A}) \Cap \widehat{a_1, \ldots, a_m}$.
(C3) $cl(\mathbb{A} \Cup \mathbb{B}) = cl(\mathbb{A}) \Cup cl(\mathbb{B})$.

Proof "$\Longrightarrow$" Suppose that there exists an HF reflexive and transitive relation $\mathbb{R}_\tau$ on U such that $\underline{\mathbb{R}_\tau}(\mathbb{A}) = int(\mathbb{A})$ and $\overline{\mathbb{R}_\tau}(\mathbb{A}) = cl(\mathbb{A})$ for all $\mathbb{A} \in HF(U)$. By Theorem 2, it can be easily observed that the conditions (I2), (I3), (C2) and (C3) hold.

"$\Longleftarrow$" Assume that the operator cl satisfies axioms (C2) and (C3). Then we define an HF relation $\mathbb{R}_\tau = \{< (x, y),$

$h_{\mathbb{R}_\tau}(x, y) > |(x, y) \in U \times U\}$ on U by cl as follows:

$$h_{\mathbb{R}_\tau}(x, y) = h_{cl(1_y)}(x), \ (x, y) \in U \times U. \tag{4}$$

Moreover, we can prove that for any $\mathbb{A} \in HF(U)$,
$\mathbb{A} = \Cup_{y \in U}(1_y \Cap \widehat{h_\mathbb{A}(y)})$.
In fact, for any $x \in U$, we have

$$\begin{aligned} h_{\Cup_{y \in U}(1_y \Cap \widehat{h_\mathbb{A}(y)})}(x) &= \veebar_{y \in U} h_{(1_y \Cap \widehat{h_\mathbb{A}(y)})}(x) \\ &= \veebar_{y \in U}(h_{1_y}(x) \barwedge h_{\widehat{h_\mathbb{A}(y)}}(x)) \\ &= (\{1\} \barwedge h_\mathbb{A}(x)) \veebar \{0\} = h_\mathbb{A}(x). \end{aligned}$$

Thus, $\mathbb{A} = \Cup_{y \in U}(1_y \Cap \widehat{h_\mathbb{A}(y)})$.

For any $\mathbb{A} \in HF(U)$ and $x \in U$, by Eq. (2), (C2) and (C3), we have

$$\begin{aligned} h_{\overline{\mathbb{R}_\tau}(\mathbb{A})}(x) &= \veebar_{y \in U}\{h_{\mathbb{R}_\tau}(x, y) \barwedge h_\mathbb{A}(y)\} \\ &= \veebar_{y \in U}\{h_{cl(1_y)}(x) \barwedge h_{\widehat{h_\mathbb{A}(y)}}(x)\} \\ &= \veebar_{y \in U}\{h_{cl(1_y) \Cap \widehat{h_\mathbb{A}(y)}}(x)\} \\ &= \veebar_{y \in U}\{h_{cl(1_y \Cap \widehat{h_\mathbb{A}(y)})}(x)\} \\ &= h_{\Cup_{y \in U} cl(1_y \Cap \widehat{h_\mathbb{A}(y)})}(x) \\ &= h_{cl(\Cup_{y \in U}(1_y \Cap \widehat{h_\mathbb{A}(y)}))}(x) \\ &= h_{cl(\mathbb{A})}(x). \end{aligned}$$

That is, $cl(\mathbb{A}) = \overline{\mathbb{R}_\tau}(\mathbb{A})$. Since cl and int are dual with each other and notice that $cl(\mathbb{A}) = \overline{\mathbb{R}_\tau}(\mathbb{A})$, we obtain $int(\mathbb{A}) = \underline{\mathbb{R}_\tau}(\mathbb{A})$. Furthermore, in terms of (Int2) of Theorem 7, we have $\underline{\mathbb{R}_\tau}(\mathbb{A}) \sqsubseteq \mathbb{A}$. Then, by Theorem 5, we see that $\mathbb{R}_\tau$ is reflexive. Moreover, by (Int4)′ of Theorem 7 again, we have $\underline{\mathbb{R}_\tau}(\underline{\mathbb{R}_\tau}(\mathbb{A})) \sqsubseteq \underline{\mathbb{R}_\tau}(\mathbb{A})$. Meanwhile, by (Int3) of Theorem 7, we conclude that $\underline{\mathbb{R}_\tau}(\underline{\mathbb{R}_\tau}(\mathbb{A})) = \underline{\mathbb{R}_\tau}(\mathbb{A})$. Consequently, $\underline{\mathbb{R}_\tau}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}_\tau}(\underline{\mathbb{R}_\tau}(\mathbb{A}))$. Therefore, by Theorem 5, we see that $\mathbb{R}_\tau$ is transitive. Thus, we have proved that the HF relation $\mathbb{R}_\tau$ is reflexive and transitive. □

Theorem 15 provides the sufficient and necessary conditions that an HF interior (closure, respectively) operator in an HF topological space can be associated with an HF reflexive and transitive relation such that the induced lower (upper, respectively) HF rough approximation operator is just the HF interior (closure, respectively) operator.

Definition 9 Let (U, τ) be an HF topological space and int and $cl : HF(U) \longrightarrow HF(U)$ the induced HF interior operator and HF closure operator, respectively. If int satisfies the conditions (I2) and (I3), or equivalently, cl obeys the conditions (C2) and (C3), then we call (U, τ) an HF rough topological space.

Let $\mathscr{R}$ be the set of all HF reflexive and transitive relations on U and $\mathscr{T}$ the set of all HF rough topological spaces.

Theorem 16 (1) *If $\mathbb{R} \in \mathscr{R}$, $\tau_\mathbb{R}$ is defined by Eq. (3) and $\mathbb{R}_{\tau_\mathbb{R}}$ by Eq. (4), then $\mathbb{R}_{\tau_\mathbb{R}} = \mathbb{R}$.*
(2) *If $\tau \in \mathscr{T}$, $\mathbb{R}_\tau$ is defined by Eq. (4) and $\tau_{\mathbb{R}_\tau}$ by Eq. (3), then $\tau_{\mathbb{R}_\tau} = \tau$.*

Proof (1) Since $\mathbb{R}$ is an HF reflexive and transitive relation on U, by Theorem 13, we obtain $\underline{\mathbb{R}} = int_{\tau_\mathbb{R}}$ and $\overline{\mathbb{R}} = cl_{\tau_\mathbb{R}}$. According to Eq. (4) and Theorem 3, we have $h_{\mathbb{R}_{\tau_\mathbb{R}}}(x, y) = h_{cl_{\tau_\mathbb{R}}(1_y)}(x) = h_{\overline{\mathbb{R}}(1_y)}(x) = h_\mathbb{R}(x, y), \forall x, y \in U$.
Thus, $\mathbb{R}_{\tau_\mathbb{R}} = \mathbb{R}$.
(2) By Eq. (3) and Theorem 15, we have $\tau_{\mathbb{R}_\tau} = \{\mathbb{A} \in HF(U) | \underline{\mathbb{R}_\tau}(\mathbb{A}) = \mathbb{A}\} = \{\mathbb{A} \in HF(U) | int(\mathbb{A}) = \mathbb{A}\} = \tau$. □

Theorem 17 *There exists a one-to-one correspondence between $\mathscr{R}$ and $\mathscr{T}$.*

Proof Define a mapping $f : \mathscr{R} \longrightarrow \mathscr{T}$ as follows:

$$f(\mathbb{R}) = \tau_\mathbb{R}, \ \mathbb{R} \in \mathscr{R}.$$

On the other hand, define a mapping $g : \mathscr{T} \longrightarrow \mathscr{R}$ as follows:

$$g(\tau) = \mathbb{R}_\tau, \ \tau \in \mathscr{T}.$$

Then, by Theorem 16, it is easy to verify that both f and g are one-to-one correspondences between $\mathscr{R}$ and $\mathscr{T}$. □

Theorem 17 shows that there exists a one-to-one correspondence between the set of all HF reflexive and transitive approximation spaces and the set of all HF rough topological spaces such that the lower and upper HF rough approximation operators are, respectively, the HF interior and closure operators.

6 Conclusion

Yang et al. (2014) introduced an HF rough set model in which the order for characterizing inclusion relation of HF sets is not necessarily antisymmetric. That is, for any two HF sets $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{A} \sqsubseteq \mathbb{B}$ and $\mathbb{B} \sqsubseteq \mathbb{A}$, then the formula $\mathbb{A} = \mathbb{B}$ does not necessarily true. This can further keep us from investigating better the properties of HF rough sets. So in this case, we have first introduced a new order on the HF power set for representing the inclusion relation of two HF subsets. The new order is antisymmetric which means that the HF power set with this order is a partial ordered set. So it can be used to characterize equality relation of two HF sets and it also provides the theoretical basis for the study of HF rough sets.

In this paper, we have defined lower and upper approximations of HF sets with respect to an HF approximation space by employing the new operations on HF power set. That is, a novel HF rough set model has been developed by us to improve Yang et al.'s one in Yang et al. (2014). We have also examined some essential properties of the HF rough approximation operators. We have further explored the topological structures of HF rough sets. We have proved that a pair of dual HF rough approximation operators can induce an HF topological space in the sense of Lowen if and only if the HF relation is reflexive and transitive. And there exists a one-to-one correspondence between the set of all HF reflexive and transitive approximation spaces and the set of all HF topological spaces such that the lower and upper HF rough approximation operators are, respectively, the HF interior and closure operators. We believe that the model offered in this paper will facilitate further research in uncertain decision making under the HF environment.

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Compliance with ethical standards

Conflict of interest The author declares that there is no conflict of interests.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Atanassov K (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96
Chakrabarty K, Gedeon T, Koczy L (1998) Intuitionistic fuzzy rough set. In: Proceedings of fourth joint conference on information sciences (JCIS), Durham, NC, pp 211–214
Chang CL (1968) Fuzzy topological spaces. J Math Anal Appl 24:182–189
Chen N, Xu ZS, Xia MM (2013) Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Appl Math Model 37:2197–2211
Chuchro M (1994) On rough sets in topological Boolean algebras. In: Ziarko W (ed) Rough sets, fuzzy sets and knowledge discovery. Springer, Berlin, pp 157–160
Cornelis C, Cock MD, Kerre EE (2003) Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge. Expert Syst Appl 20:260–270
Deepak D, John SJ (2014) Hesitant fuzzy rough sets through hesitant fuzzy relations. Ann Fuzzy Math Inform 8:33–46
Deschrijver G, Kerre EE (2005) Implicators based on binary aggregation operators in interval-valued fuzzy set theory. Fuzzy Sets Syst 153:229–248
Dubois D, Prade H (1980) Fuzzy sets and systems: theory and applications. Academic Press, New York
Dubois D, Prade H (1990) Rough fuzzy sets and fuzzy rough sets. Int J Gen Syst 17:191–209
Farhadinia B (2013) Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Inf Sci 240:129–144
Gorzalczany MB (1987) A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst 21:1–17
He YP, Xiong LL (2017) Generalized interval-valued intuitionistic fuzzy soft rough set and its application. J Comput Anal Appl 23:1070–1088
Jena SP, Ghosh SK (2002) Intuitionistic fuzzy rough sets. Notes Intuitionistic Fuzzy Sets 8:1–18
Kondo M (2006) On the structure of generalized rough sets. Inf Sci 176:586–600
Lashin EF, Kozae AM, Khadra AAA et al (2005) Rough set theory for topological spaces. Int J Approx Reason 40:35–43
Li TJ, Zhang WX (2008) Rough fuzzy approximations on two universes of discourse. Inf Sci 178:892–906
Liang DC, Liu D (2015) A novel risk decision-making based on decision-theoretic rough sets under hesitant fuzzy information. IEEE Trans Fuzzy Syst 23:237–247
Liao HC, Xu ZS (2013) A VIKOR-based method for hesitant fuzzy multi-criteria decision making. Fuzzy Optim Decis Mak 2:373–392
Liao HC, Xu ZS (2015) Extended hesitant fuzzy hybrid weighted aggregation operators and their application in decision making. Soft Comput 19:2551–2564
Liu HF, Xu ZS, Liao HC (2016) The multiplicative consistency index of hesitant fuzzy preference relation. IEEE Trans Fuzzy Syst 24:82–93
Lowen R (1976) Fuzzy topological spaces and fuzzy compactness. J Math Anal Appl 56:621–633
Miyamoto S (2005) Remarks on basics of fuzzy sets and fuzzy multisets. Fuzzy Sets Syst 156:427–431
Nanda S, Majumda S (1992) Fuzzy rough sets. Fuzzy Sets Syst 45:157–160
Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11:145–172
Pawlak Z (1991) Rough sets-theoretical aspects to reasoning about data. Kluwer Academic Publisher, Boston
Qin KY, Pei Z (2005) On the topological properties of fuzzy rough sets. Fuzzy Sets Syst 151:601–613
Qin KY, Yang JL, Pei Z (2008) Generalized rough sets based on reflexive and transitive relations. Inf Sci 178:4138–4141
Radzikowska AM, Kerre EE (2002) A comparative study of fuzzy rough sets. Fuzzy Sets Syst 126:137–155
Rizvi S, Naqvi HJ, Nadeem D (2002) Rough intuitionistic fuzzy set. In: Proceedings of the sixth joint conference on information sciences (JCIS), Durham, NC, pp 101–104
Samanta SK, Mondal TK (2001) Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets. J Fuzzy Math 9:561–582
Sun BZ, Gong ZT, Chen DG (2008) Fuzzy rough set theory for the interval-valued fuzzy information systems. Inf Sci 178:2794–2815
Thiele H (2001) On aximatic characterisation of fuzzy approximation operators: II. The rough fuzzy set based case. In: Proceeding of the 31st IEEE international symposium on multiple-valued logic, pp 330–335
Tiwari SP, Srivastava AK (2013) Fuzzy rough sets, fuzzy preorders and fuzzy topologies. Fuzzy Sets Syst 210:63–68
Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25:529–539
Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: The 18th IEEE international conference on fuzzy systems, Jeju Island, Korea, pp 1378–1382

Wu WZ (2011) On some mathematical structures of T-fuzzy rough set algebras in infinite universes of discourse. Fundam Inform 108:337–369

Wu WZ, Zhou L (2011) On intuitionistic fuzzy topologies based on intuitionistic fuzzy reflexive and transitive relations. Soft Comput 15:1183–1194

Wu QE, Wang T, Huang YX et al (2008) Topology theory on rough sets. IEEE Trans Syst Man Cybern Part B Cybern 38:68–77

Xia MM, Xu ZS (2011) Hesitant fuzzy information aggregation in decision making. Int J Approx Reason 52:395–407

Xu ZS, Xia MM (2011) Distance and similarity measures for hesitant fuzzy sets. Inf Sci 181:2128–2138

Yang XB, Song XN, Qi YS et al (2014) Constructive and axiomatic approaches to hesitant fuzzy rough set. Soft Comput 18:1067–1077

Yeung DS, Chen DG, Tsang ECC et al (2005) On the generalization of fuzzy rough sets. IEEE Trans Fuzzy Syst 13:343–361

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

Zhan JM, Liu Q, Herawan T (2017) A novel soft rough set: soft rough hemirings and its multicriteria group decision making. Appl Soft Comput 54:393–402

Zhang ZM (2013) On characterization of generalized interval type-2 fuzzy rough sets. Inf Sci 219:124–150

Zhang HD, Shu L (2015) Generalized interval-valued fuzzy rough set and its application in decision making. Int J Fuzzy Syst 17:279–291

Zhou L, Wu WZ, Zhang WX (2009) On intuitionistic fuzzy rough sets and their topological structures. Int J Gener Syst 38:589–616

Zhu W (2007) Topological approaches to covering rough sets. Inf Sci 177:1499–1508