Soft Computing (2019) 23:12529–12545
https://doi.org/10.1007/s00500-019-03803-z

METHODOLOGIES AND APPLICATION

A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis

İrfan Deli[1]

Published online: 4 February 2019

Abstract

The aim of this paper is to investigate the multiple attribute decision-making (MADM) problems where both the attribute value and attribute weight of alternatives are single-valued trapezoidal neutrosophic numbers (SVTN-numbers). Ranking of SVTN-numbers are always a necessary step in solving the MADM problems under SVTN environment, and the literature review reckoned the existence of six to seven ranking methods. After all the existing ranking methods of SVTN-numbers are examined, we firstly define the concept of centroid point and examine several useful properties of the developed concept. Then, we develop hamming ranking value and Euclidean ranking value of SVTN-numbers to compare the SVTN-numbers. Furthermore, based on the proposed ranking values, we develop a novel defuzzification method to MADM with linguistic information and give a real example deal with manufacturing company to illustrate the feasibility and effectiveness of the developed approach. Finally, we present some examples to compare the proposed method with the existing ranking results and the results verified through comparative analysis.

Keywords Neutrosophic sets · Trapezoidal neutrosophic numbers · Defuzzification · Centroid point · Hamming ranking value · Euclidean ranking value · Decision making

1 Introduction

The multiple attribute decision making (MADM) is an important part of decision making to choose the most desirable alternative from a set of alternatives according to the decision information provided by a decision maker. In MADM, since most of the decision information about attribute values is not known precisely, Zadeh (1965) has proposed the fuzzy set theory to model the uncertainty and vagueness that is nature of human thinking in the process of decision making. By added non-membership degree to fuzzy sets, Atanassov (1999) initially proposed intuitionistic fuzzy set theory as an extension of the concept of a fuzzy set. It is worth noting that the above decision-making information is based on fuzzy and intuitionistic fuzzy sets. To provide more flexibility to present imprecise data and uncertainty, Smarandache (1998) defined neutrosophic sets, which independently contain truth-membership degree, indeterminacy-membership degree and falsity-membership degree. Because of their advantages, the theories have been studied by many authors such as: on fuzzy sets (Kahraman and Otay 2019a; Rao and Shankar 2011; Yager and Zadeh 1992; Ying 2000; Zadeh 2005), on intuitionistic fuzzy sets (Liu and Li 2017; Wan et al. 2017; Wei 2010; Zhang 2017) and on neutrosophic sets in Broumi et al. (2014a, b), Kahraman and Otay (2019b), Peng et al. (2014, 2016) and Ye (2018).

However, due to uncertainty and complexity in some real situations, we often only can use a single-valued neutrosophic number that is a special neutrosophic set on the real number R. Thus, it seems to be suitable for employing in the real number R to estimate both the attribute value and attribute weight of alternatives in MADM problems. Hereby, single-valued trapezoidal neutrosophic numbers (SVTN-numbers) were defined by Deli and Şubaş (2017b) and Ye (2017, 2015). Currently, many researchers pay attention to the methods and related properties of SVTN-numbers. For example, Basset et al. (2018a) proposed a decision-making method by using Analytic Hierarchy Process for SVTN-numbers. Liu and Zhang (2018) introduced some operators on SVTN-numbers based on Maclaurin symmetric mean. Broumi et al. (2016b)

Communicated by V. Loia.

✉ İrfan Deli
irfandeli@kilis.edu.tr

[1] Kilis 7 Aralık University, 79000 Kilis, Turkey

developed an approach for solving shortest path problem in a network where edge weights are characterized by SVTN-numbers. Then, same authors presented a new method for solving shortest path problems in a network in which the edges length are characterized by SVTN-numbers in Broumi et al. (2016a). Biswas et al. (2016a, b) defined some operators and applied them to decision-making problems including medical representative selection problem. Deli and Şubaş (2017a) defined the values and ambiguities of the SVTN-numbers under cut sets. Liang et al. (2017a) presented new operations, an improved comparison method, and two aggregation operators to improve the methods in SVTN-numbers. Since the traditional arithmetic mean operator or geometric mean operator only deals with independent criteria, Liang et al. (2017b) introduced different operators by using entropy-weighted method. Basset et al. (2018b) developed a novel algorithm for the group decision-making problem with triangular neutrosophic additive reciprocal preference relations.

Das and Guha (2013) said that "However, after analyzing the aforementioned ranking procedures it has been observed that, for some cases, they fail to calculate the ranking results correctly. Furthermore, many of them produce different ranking outcomes for the same problem. Under these circumstances, the decision maker may not be able to carry out the comparison and recognition properly. This creates problem in practical applications". In order to overcome these problems of the existing methods, a new method for ranking has been proposed based on centroid point of intuitionistic fuzzy numbers in Das and Guha (2013). Then, in Das and Guha (2016), a new ranking method of intuitionistic fuzzy numbers was developed by utilizing the concept of centroid point. Varghese and Kuriakose (2012) defined a formula for finding the centroid of an intuitionistic fuzzy number. Arun et al. (2016) developed a new approach to rank intuitionistic fuzzy numbers based on centroid point of intuitionistic fuzzy numbers. Hajek and Olej (2014) gave an intuitionistic fuzzy inference system of Takagi-Sugeno type which is defuzzification methods. As far as we know, however, there has been a few researches on methods of SVTN-numbers.

Therefore, the primary goal of this paper is to define the defuzzification methods of SVTN-numbers and hereby develop a simplified and an effective method for computing defuzzification methods of SVTN-numbers. The rest of this paper is organized as follows. In Sect. 2, we briefly introduced some concepts of SVTN-numbers. In Sect. 3, we presented the existing ranking methods with examples. In Sect. 4, we defined the concept of centroid point and examined several useful properties of the developed concept. Then, we proposed the hamming ranking value and Euclidean ranking value of SVTN-numbers to compare the SVTN-numbers. In Sect. 5, we developed a novel defuzzification method to MADM with linguistic information and gave a real example deal with manufacturing company to illustrate the feasibility and effectiveness of the developed approach. Finally, we presented some examples to compare the proposed method with the existing ranking results, and the results were verified through comparative analysis.

2 Preliminary

Definition 2.1 (Wang et al. 2010) Let X be a universe. A single-valued neutrosophic set A in X is characterized by a truth-membership function $T_A : X \to [0, 1]$, an indeterminacy-membership function $I_A : X \to [0, 1]$ and a falsity-membership function $F_A : X \to [0, 1]$ as:

$$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in X\}.$$

Definition 2.2 (Şubaş 2015) Let $w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \in [0, 1]$ be any real number. A single-valued neutrosophic number $\tilde{a} = \langle([a_1, b_1, c_1, d_1)], w_{\tilde{a}}), ([a_2, b_2, c_2, d_2], u_{\tilde{a}}), ([a_3, b_3, c_3, d_3], y_{\tilde{a}})\rangle$ is a special single-valued neutrosophic set on the set of real numbers R, whose truth-membership function $\mu_{\tilde{a}} : R \to [0, w_{\tilde{a}}]$, an indeterminacy-membership function $\nu_{\tilde{a}} : R \to [u_{\tilde{a}}, 1]$ and a falsity-membership function $\lambda_{\tilde{a}} : R \to [y_{\tilde{a}}, 1]$ as given by;

$$\mu_{\tilde{a}}(x) = \begin{cases} f_{\mu l}(x) & (a_1 \le x < b_1) \\ w_{\tilde{a}} & (b_1 \le x < c_1) \\ f_{\mu r}(x) & (c_1 \le x \le d_1) \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} f_{\nu l}(x) & (a_2 \le x < b_2) \\ u_{\tilde{a}} & (b_2 \le x < c_2) \\ f_{\nu r}(x) & (c_2 \le x \le d_2) \\ 1 & \text{otherwise} \end{cases}$$

and

$$\lambda_{\tilde{a}}(x) = \begin{cases} f_{\lambda l}(x) & (a_3 \le x < b_3) \\ y_{\tilde{a}} & (b_3 \le x < c_3) \\ f_{\lambda r}(x) & (c_3 \le x \le d_3) \\ 1 & \text{otherwise} \end{cases}$$

respectively, where the functions $f_{\mu l} : [a_1, b_1] \to [0, w_{\tilde{a}}]$, $f_{\nu r} : [c_2, d_2] \to [u_{\tilde{a}}, 1]$ $f_{\lambda r} : [c_3, d_3] \to [y_{\tilde{a}}, 1]$ are continuous and nondecreasing, and satisfy the conditions: $f_{\mu l}(a_1) = 0$, $f_{\mu l}(b_1) = w_{\tilde{a}}$, $f_{\nu r}(c_2) = u_{\tilde{a}}$, $f_{\nu r}(d_2) = 1$, $f_{\lambda r}(c_3) = y_{\tilde{a}}$, and $f_{\lambda r}(d_3) = 1$; the functions $f_{\mu r} : [c_1, d_1] \to [0, w_{\tilde{a}}]$, $f_{\nu l} : [a_2, b_2] \to [u_{\tilde{a}}, 1]$ and $f_{\lambda l} : [a_3, b_3] \to [y_{\tilde{a}}, 1]$ are continuous and nonincreasing, and satisfy the conditions: $f_{\mu r}(c_1) = w_{\tilde{a}}, f_{\mu r}(d_1) = 0$, $f_{\nu l}(a_2) = 1$, $f_{\nu l}(b_2) = u_{\tilde{a}}$, $f_{\lambda l}(a_3) = 1$ and $f_{\lambda l}(b_3) = y_{\tilde{a}}$. $[b_1, c_1], a_1$ and d_1 are called the mean interval and the lower and upper limits of the general neutrosophic number $\tilde{a}$ for the truth-membership function, respectively. $[b_2, c_2]$, a_2 and d_2 are called the mean interval and the lower and upper limits of the general neutrosophic number $\tilde{a}$ for the indeterminacy-membership function, respectively. $[b_3, c_3]$, a_3 and d_3 are called the mean interval and the lower and upper limits of the general neutrosophic number $\tilde{a}$ for the falsity-membership function, respectively. $w_{\tilde{a}}$, $u_{\tilde{a}}$ and $y_{\tilde{a}}$ are called the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree, respectively.

Note that a single-valued neutrosophic number $\tilde{a} = \langle([a_1, b_1, c_1, d_1)], w_{\tilde{a}}), ([a_2, b_2, c_2, d_2], u_{\tilde{a}}), ([a_3, b_3, c_3, d_3], y_{\tilde{a}})\rangle$ for $w_{\tilde{a}} = 1, u_{\tilde{a}} = 0, y_{\tilde{a}} = 0$ defined in Ye (2017) as: $\tilde{a} = \langle[a_1, b_1, c_1, d_1], [a_2, b_2, c_2, d_2], [a_3, b_3, c_3, d_3]\rangle$

Definition 2.3 (Deli and Şubaş 2017a) An SVTN-number $\tilde{a} = \langle(a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle$ on R is characterized by a truth-membership function $T_{\tilde{a}} : R \to [0, 1]$, an indeterminacy-membership function $I_{\tilde{a}} : R \to [0, 1]$ and a falsity-membership function $F_{\tilde{a}} : R \to [0, 1]$ are defined as:

$$< x, (T_{\tilde{a}}(x), I_{\tilde{a}}(x), F_{\tilde{a}}(x)) > = \begin{cases} < x, (\frac{(x-a)w_{\tilde{a}}}{b-a}, \frac{b-x+u_{\tilde{a}}(x-a)}{b-a}, \frac{b-x+y_{\tilde{a}}(x-a)}{b-a}) >, & a \le x < b \\ < x, (w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}) >, & b \le x \le c \\ < x, (\frac{(d-x)w_{\tilde{a}}}{d-c}, \frac{x-c+u_{\tilde{a}}(d-x)}{d-c}, \frac{x-c+y_{\tilde{a}}(d-x)}{d-c}) >, & c < x \le d \\ < x, (0, 1, 1) >, & \text{otherwise.} \end{cases}$$

Note that the set of all SVTN-number-numbers on R will be denoted by $\triangle$.

For any $\tilde{a} = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle$ and $\tilde{b} = \langle(a_2, b_2, c_2, d_2); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}\rangle$;

1. $\tilde{a} + \tilde{b} = \langle(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); w_{\tilde{a}} \wedge w_{\tilde{b}}, u_{\tilde{a}} \vee u_{\tilde{b}}, y_{\tilde{a}} \vee y_{\tilde{b}}\rangle$
2. $\tilde{a} - \tilde{b} = \langle(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2); w_{\tilde{a}} \wedge w_{\tilde{b}}, u_{\tilde{a}} \vee u_{\tilde{b}}, y_{\tilde{b}} \vee y_{\tilde{b}}\rangle$
3. $\tilde{a}\tilde{b} = \begin{cases} \langle(a_1a_2, b_1b_2, c_1c_2, d_1d_2; w_{\tilde{a}} \wedge w_{\tilde{b}}, u_{\tilde{a}} \vee u_{\tilde{b}}, y_{\tilde{a}} \vee y_{\tilde{b}}\rangle & (d_1 > 0, d_2 > 0) \\ \langle(a_1d_2, b_1c_2, c_1b_2, d_1a_2); w_{\tilde{a}} \wedge w_{\tilde{b}}, u_{\tilde{a}} \vee u_{\tilde{b}}, y_{\tilde{a}} \vee y_{\tilde{b}}\rangle & (d_1 < 0, d_2 > 0) \\ \langle(d_1d_2, c_1c_2, b_1b_2, a_1a_2); w_{\tilde{a}} \wedge w_{\tilde{b}}, u_{\tilde{a}} \vee u_{\tilde{b}}, y_{\tilde{a}} \vee y_{\tilde{b}}\rangle & (d_1 < 0, d_2 < 0) \end{cases}$
4. $\gamma\tilde{a} = \begin{cases} \langle(\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle & (\gamma > 0) \\ \langle(\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle & (\gamma < 0) \end{cases}$
5. $\tilde{a}^{\gamma} = \begin{cases} \langle(a_1^{\gamma}, b_1^{\gamma}, c_1^{\gamma}, c_1^{\gamma}); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle & (\gamma > 0) \\ \langle(c_1^{\gamma}, c_1^{\gamma}, b_1^{\gamma}, a_1^{\gamma}); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle & (\gamma < 0) \end{cases}$

Definition 2.4 (Şubaş 2015) Let $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \triangle$ $(j \in I_n)$. Then, SVTN weighted arithmetic operator, denoted by G_{ao}, is defined as:

$$G_{ao}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\langle \left(\sum_{j=1}^{n} a_j \times w_j, \sum_{j=1}^{n} b_j \times w_j, \sum_{j=1}^{n} c_j \times w_j, \sum_{j=1}^{n} d_j \times w_j \right); \bigwedge_{j=1}^{n} w_{\tilde{a}_j}, \bigvee_{j=1}^{n} u_{\tilde{a}_j}, \bigvee_{j=1}^{n} y_{\tilde{a}_j} \right\rangle$$

where, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is a weight vector associated with the G_{go} operator, for every $j \in I_n$ such that, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

3 A brief note on existing ranking methods of SVTN-number

As mentioned in the introduction, ranking of SVTN-number is always a necessary step in solving the problems under SVTN-number environment and the literature review reckoned the existence of six to seven ranking methods. In this section, all the existing ranking methods of SVTN-numbers are examined with numerical examples.

Method 1 (Deli and Şubaş 2017a) Let $\tilde{a_1}$ and $\tilde{a_2}$ be two SVTN-numbers and $\theta \in [0, 1]$. Based on weighted values and ambiguities of the SVN-numbers, the ranking order of $\tilde{a_1}$ and $\tilde{a_2}$ is defined as:

1. If $V_\theta(\tilde{a_1}) > V_\theta(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \tilde{>} \tilde{a_2}$;
2. If $V_\theta(\tilde{a_1}) = V_\theta(\tilde{a_2})$, then
   (a) If $A_\theta(\tilde{a_1}) > A_\theta(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \tilde{>} \tilde{a_2}$
   (b) If $A_\theta(\tilde{a_1}) = A_\theta(\tilde{a_2})$, then $\tilde{a_1}$ is equal to $\tilde{a_2}$, denoted by $\tilde{a_1} \tilde{=} \tilde{a_2}$;

where for $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle$ (j=1,2)

1. the weighted value $V_\theta(\tilde{a_j})$is calculated as:

$$V_\theta(\tilde{a_j}) = \frac{a_j + 2b_j + 2c_j + d_j}{6} \left[\theta w_{\tilde{a}_j}^2 + (1-\theta)(1-u_{\tilde{a}_j})^2 + (1-\theta)(1-y_{\tilde{a}_j})^2\right]$$

2. the weighted ambiguity $A_\theta(\tilde{a_j})$ is calculated as:

$$A_\theta(\tilde{a_j}) = \frac{d_j - a_j + 2c_j - 2b_j}{6} \left[\theta w_{\tilde{a}_j}^2 + (1-\theta)(1-u_{\tilde{a}_j})^2 + (1-\theta)(1-y_{\tilde{a}_j})^2\right]$$

Table 1 Ranking for Method 1

SVTrN-numbers	θ	$V_\theta(\tilde{a}_1)$	$V_\theta(\tilde{a}_2)$	$A_\theta(\tilde{a}_1)$	$A_\theta(\tilde{a}_2)$	Ranking
	0.0	2.6916	5.0433	0.7083	1.4833	$\tilde{a}_1 \tilde{<} \tilde{a}_2$
	0.3	2.6536	3.6833	0.6983	1.0833	$\tilde{a}_1 \tilde{<} \tilde{a}_2$
$\tilde{a}_1 = \langle(3, 4, 5, 6); 0.9, 0.3, 0.4\rangle$	0.5	2.6283	2.7766	0.6916	0.8166	$\tilde{a}_1 \tilde{<} \tilde{a}_2$
$\tilde{a}_2 = \langle(5, 7, 9, 11); 0.3, 0.5, 0.2\rangle$	0.6	2.6156	2.3233	0.6883	0.6833	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
	0.8	2.5903	1.4166	0.6816	0.4166	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
	1.0	2.5650	0.5100	0.6750	0.1500	$\tilde{a}_1 \tilde{>} \tilde{a}_2$

Table 2 Ranking for Method 2

SVTN-numbers	(λ, μ, ν)	$V_{\lambda,\mu,\nu}(\tilde{a}_1)$	$V_{\lambda,\mu,\nu}(\tilde{a}_1)$	$A_{\lambda,\mu,\nu}(\tilde{a}_1)$	$A_{\lambda,\mu,\nu}(\tilde{a}_1)$	Ranking
	(0.1,0.2,0.3)	23.4	19.8	3	3	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
	(0.4,0.3,0.5)	46.8	39.6	6	6	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
$\tilde{a}_1 = \langle(3, 4, 5, 6); 1.0, 0.0, 0.0\rangle$	(0.0,0.3,1.0)	15.6	13.2	2	2	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
$\tilde{a}_2 = \langle(4, 5, 6, 7); 1.0, 0.0, 0.0\rangle$	(1.0,0.3,0.9)	85.8	72.6	11	11	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
	(1.0,1.0,1.0)	117	99	15	15	$\tilde{a}_1 \tilde{>} \tilde{a}_2$
	(0.0,0.0,0.0)	0	0	0	0	$\tilde{a}_1 \tilde{=} \tilde{a}_2$

Example 3.1 Let $\tilde{a}_1 = \langle(3, 4, 5, 6); 0.9, 0.3, 0.4\rangle$ and $\tilde{a}_2 = \langle(5, 7, 9, 11); 0.3, 0.5, 0.2\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a}_1$ and $\tilde{a}_2$ as Table 1.

Method 2 (Biswas et al. 2016a) Let $\tilde{a}_1$ and $\tilde{a}_2$ be two SVTN-numbers and $\theta \in [0, 1]$. For weighted values and ambiguities of the SVTN-numbers $\tilde{a}_1 = \langle[a_{11}, a_{21}, a_{31}, a_{41}], [b_{11}, b_{21}, b_{31}, b_{41}], [c_{11}, c_{21}, c_{31}, c_{41}]\rangle$ and $\tilde{a}_2 = \langle[a_{12}, a_{22}, a_{32}, a_{42}], [b_{12}, b_{22}, b_{32}, b_{42}], [c_{12}, c_{22}, c_{32}, c_{42}]\rangle$, the ranking order of $\tilde{a}_1$ and $\tilde{a}_2$, based on $\lambda, \mu, \nu \in [0, 1]$ is defined as:

1. If $V_{\lambda,\mu,\nu}(\tilde{a}_1) > V_{\lambda,\mu,\nu}(\tilde{a}_2)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{>} \tilde{a}_2$;
2. If $V_{\lambda,\mu,\nu}(\tilde{a}_1) = V_{\lambda,\mu,\nu}(\tilde{a}_2)$, then
   (a) If $A_{\lambda,\mu,\nu}(\tilde{a}_1) > A_{\lambda,\mu,\nu}(\tilde{a}_2)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{>} \tilde{a}_2$,
   (b) If $A_{\lambda,\mu,\nu}(\tilde{a}_1) = A_{\lambda,\mu,\nu}(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{=} \tilde{a}_2$;

where for $\tilde{a}_j = \langle[a_{1j}, a_{2j}, a_{3j}, a_{4j}], [b_{1j}, b_{2j}, b_{3j}, b_{4j}], [c_{1j}, c_{2j}, c_{3j}, c_{4j}]\rangle (j = 1, 2)$

1. The value index $V_{\lambda,\mu,\nu}(\tilde{a}_j)$is calculated as:

$$
\begin{aligned}
V_{\lambda,\mu,\nu}(\tilde{a}_j) = {} & \frac{\lambda}{6}(a_{1j} + 2a_{2j}, 2a_{3j} + a_{4j}) \\
& + \frac{\mu}{6}(b_{1j} + 2b_{2j} + 2b_{3j} + b_{4j}) \\
& + \frac{\nu}{6}(c_{1j} + 2c_{2j} + 2c_{3j} + c_{4j})
\end{aligned}
$$

2. The ambiguity index $A_{\lambda,\mu,\nu}(\tilde{a}_j)$ is calculated as:

$$
\begin{aligned}
A_{\lambda,\mu,\nu}(\tilde{a}_j) = {} & \frac{\lambda}{6}(-a_{1j} - 2a_{2j}, 2a_{3j} + a_{4j}) \\
& + \frac{\mu}{6}(-b_{1j} - 2b_{2j} + 2b_{3j} + b_{4j}) \\
& + \frac{\nu}{6}(-c_{1j} - 2c_{2j} + 2c_{3j} + c_{4j})
\end{aligned}
$$

Example 3.2 Let $\tilde{a}_1 = \langle(3, 4, 5, 6); 1.0, 0.0, 0.0\rangle$ and $\tilde{a}_2 = \langle(4, 5, 6, 7); 1.0, 0.0, 0.0\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a}_1$ and $\tilde{a}_2$ as Table 2.

Method 3 (Ye 2017) Let $\tilde{a}_1 = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a}_1}, u_{\tilde{a}_1}, y_{\tilde{a}_1}\rangle$ and $\tilde{a}_2 = \langle(a_2, b_2, c_2, d_2); w_{\tilde{a}_2}, u_{\tilde{a}_2}, y_{\tilde{a}_2}\rangle$ be two SVTN-numbers. Then,

1. If $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$
2. If $S(\tilde{a}_1) = S(\tilde{a}_2)$;
   (a) If $A(\tilde{a}_1) < A(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$
   (b) If $A(\tilde{a}_1) = A(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ are the same, denoted by $\tilde{a}_1 = \tilde{a}_2$

where for $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle (j = 1, 2)$

1. The score $S(\tilde{a}_j)$ is calculated as:

$$
\begin{aligned}
S(\tilde{a}_j) = {} & \frac{1}{12}[a_j + b_j + c_j + d_j] \\
& \times (2 + w_{\tilde{a}_j} - u_{\tilde{a}_j} - y_{\tilde{a}_j})
\end{aligned} \tag{1}
$$

Table 3 Ranking for Method 3

SVTN-numbers	$S(\tilde{a_1})$	$S(\tilde{a_2})$	$A(\tilde{a_1})$	$A(\tilde{a_2})$	Ranking
$\tilde{a_1} = \langle(1, 6, 10, 12); 0.5, 0.1, 0.3\rangle$ $\tilde{a_2} = \langle(0, 5, 7, 15); 0.8, 0.9, 0.7\rangle$	5.0750	2.7000	6.5250	5.8500	$\tilde{a_1} \gtrsim \tilde{a_2}$

Table 4 Ranking for Method 4

SVTN-numbers	$E(\tilde{a_1})$	$E(\tilde{a_2})$	$\bar{A}(\tilde{a_1})$	$\bar{A}(\tilde{a_1})$	$C(\tilde{a_1})$	$C(\tilde{a_2})$	Ranking
$\tilde{a_1} = \langle(5, 10, 15, 20); 0.5, 0.1, 0.3\rangle$ $\tilde{a_2} = \langle(0, 5, 10, 15); 0.8, 0.9, 0.7\rangle$	8.7500	3	2.5	0.7500	6.2500	6	$\tilde{a_1} \gtrsim \tilde{a_2}$

2. The accuracy $A(\tilde{a_j})$ is calculated as:

$$A(\tilde{a_j}) = \frac{1}{12}[a_j + b_j + c_j + d_j] \times (2 + w_{\tilde{a}_j} - u_{\tilde{a}_j} + y_{\tilde{a}_j})$$

Example 3.3 Let $\tilde{a_1} = \langle(1, 6, 10, 12); 0.5, 0.1, 0.3\rangle$ and $\tilde{a_2} = \langle(0, 5, 7, 15); 0.8, 0.9, 0.7\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a_1}$ and $\tilde{a_2}$ as Table 3.

Method 4 (Liang et al. 2017b) Let $\tilde{a_1} = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a_1}}, u_{\tilde{a_1}}, y_{\tilde{a_1}}\rangle$ and $\tilde{a_2} = \langle(a_2, b_2, c_2, d_2); w_{\tilde{a_2}}, u_{\tilde{a_2}}, y_{\tilde{a_2}}\rangle$ be two SVTN-numbers. Then,

1. If $E(\tilde{a_1}) < E(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;
2. If $E(\tilde{a_1}) = E(\tilde{a_2})$,
   (a) $\bar{A}(\tilde{a_1}) < \bar{A}(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;,
   (b) If $\bar{A}(\tilde{a_1}) = \bar{A}(\tilde{a_2})$,
      i. $C(\tilde{a_1}) < C(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;

where for $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle$ (j=1,2)

1. The score function $E(\tilde{a_j})$ is given as

$$E(\tilde{a_j}) = \frac{(a_j + 2b_j + 2c_j + d_j)}{6} \cdot \frac{(2 + w_{\tilde{a}_j} - u_{\tilde{a}_j} - y_{\tilde{a}_j})}{3}$$

2. Accuracy function $\bar{A}(\tilde{a_j})$ is given as

$$\bar{A}(\tilde{a_j}) = \frac{(a_j + 2b_j + 2c_j + d_j)}{6} \cdot (w_{\tilde{a}_j} - y_{\tilde{a}_j})$$

3. Certainty function $C(\tilde{a_j})$ is given as

$$C(\tilde{a_j}) = \frac{(a_j + 2b_j + 2c_j + d_j)}{6} \cdot (w_{\tilde{a}_j})$$

Example 3.4 Let $\tilde{a_1} = \langle(5, 10, 55, 20); 0.5, 0.1, 0.3\rangle$ and $\tilde{a_2} = \langle(0, 5, 10, 15); 0.8, 0.9, 0.7\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a_1}$ and $\tilde{a_2}$ as Table 4.

Method 5 (Liang et al. 2017a) Let $\tilde{a_1} = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a_1}}, u_{\tilde{a_1}}, y_{\tilde{a_1}}\rangle$ and $\tilde{a_2} = \langle(a_2, b_2, c_2, d_2); w_{\tilde{a_2}}, u_{\tilde{a_2}}, y_{\tilde{a_2}}\rangle$ be two SVTN-numbers. Then,

1. If $E(\tilde{a_1}) < E(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;
2. If $E(\tilde{a_1}) = E(\tilde{a_2})$,
   (a) $\bar{A}(\tilde{a_1}) < \bar{A}(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;,
   (b) If $\bar{A}(\tilde{a_1}) = \bar{A}(\tilde{a_2})$,
      i. $C(\tilde{a_1}) < C(\tilde{a_2})$, then $\tilde{a_1}$ is bigger than $\tilde{a_2}$, denoted by $\tilde{a_1} \precsim \tilde{a_2}$;

where for $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle (j = 1, 2)$

1. The score function $E(\tilde{a_j})$ based on center of gravity is given as

$$E(\tilde{a_j}) = \begin{cases} \frac{1}{3}\left(a_j + b_j + c_j + d_j - \frac{d_j.c_j - b_j.a_j}{d_j + c_j - b_j - a_j}\right) \cdot \frac{(2 + w_{\tilde{a}_j} - u_{\tilde{a}_j} - y_{\tilde{a}_j})}{3} & \text{otherwise} \\ a \cdot \frac{(2 + w_{\tilde{a}_j} - u_{\tilde{a}_j} - y_{\tilde{a}_j})}{3} & \text{if } a = a_j = b_j = c_j = d_j \end{cases}$$

2. Accuracy function $\bar{A}(\tilde{a_j})$ based on center of gravity is given as

$$\bar{A}(\tilde{a_j}) = \begin{cases} \frac{1}{3}\left(a_j + b_j + c_j + d_j - \frac{d_j.c_j - b_j.a_j}{d_j + c_j - b_j - a_j}\right) \cdot (w_{\tilde{a}_j} - y_{\tilde{a}_j}) & \text{otherwise} \\ a \cdot (w_{\tilde{a}_j} - y_{\tilde{a}_j}) & \text{if } a = a_j = b_j = c_j = d_j \end{cases}$$

3. Certainty function $C(\tilde{a_j})$ based on center of gravity is given as

$$C(\tilde{a_j}) = \begin{cases} \frac{1}{3}\left(a_j + b_j + c_j + d_j - \frac{d_j.c_j - b_j.a_j}{d_j + c_j - b_j - a_j}\right) \cdot w_{\tilde{a}_j} & \text{otherwise} \\ a \cdot w_{\tilde{a}_j} & \text{if } a = a_j = b_j = c_j = d_j \end{cases}$$

Table 5 Ranking for Method 5

SVTN-numbers	$E(\tilde{a}_1)$	$E(\tilde{a}_2)$	$\bar{A}(\tilde{a}_1)$	$\bar{A}(\tilde{a}_1)$	$C(\tilde{a}_1)$	$C(\tilde{a}_2)$	Ranking
$\tilde{a}_1 = \langle(0.5, 0.10, 0.15, 0.20); 0.5, 0.1, 0.3\rangle$ $\tilde{a}_2 = \langle(0.0, 0.5, 0.10, 0.15); 0.8, 0.9, 0.7\rangle$	0.0875	0.0300	0.0250	0.0075	0.0625	0.0600	$\tilde{a}_1 \tilde{>} \tilde{a}_2$

Table 6 Ranking for Method 6

SVTN-numbers	$s(\tilde{a}_1)$	$s(\tilde{a}_2)$	$h(\tilde{a}_1)$	$h(\tilde{a}_1)$	Ranking
$\tilde{a}_1 = \langle(0.1, 0.2, 0.3, 0.4); 1.0, 0.0, 0.0\rangle$ $\tilde{a}_2 = \langle(1, 2, 3, 4); 1.0, 0.0, 0.0\rangle$	0.5833	-0.1666	0	0	$\tilde{a}_1 \tilde{>} \tilde{a}_2$

Example 3.5 Let $\tilde{a}_1 = \langle(0.5, 0.10, 0.15, 0.20); 0.5, 0.1, 0.3\rangle$ and $\tilde{a}_2 = \langle(0.0, 0.5, 0.10, 0.15); 0.8, 0.9, 0.7\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a}_1$ and $\tilde{a}_2$ as Table 5.

Method 6 (Ye 2015) Let $\tilde{a}_1$ and $\tilde{a}_2$ be two SVTN-numbers and $\theta \in [0, 1]$. For weighted values and ambiguities of the SVTN-numbers $\tilde{a}_1 = \langle[a_{11}, a_{21}, a_{31}, a_{41}], [b_{11}, b_{21}, b_{31}, b_{41}], [c_{11}, c_{21}, c_{31}, c_{41}]\rangle$ and $\tilde{a}_2 = \langle[a_{12}, a_{22}, a_{32}, a_{42}], [b_{12}, b_{22}, b_{32}, b_{42}], [c_{12}, c_{22}, c_{32}, c_{42}]\rangle$, the ranking order of $\tilde{a}_1$ and $\tilde{a}_2$ is defined as:

1. If $s(\tilde{a}_1) > s(\tilde{a}_2)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{>} \tilde{a}_2$;
2. If $s(\tilde{a}_1) = s(\tilde{a}_2)$, then
   (a) If $h(\tilde{a}_1) > h(\tilde{a}_2)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{>} \tilde{a}_2$,
   (b) If $h(\tilde{a}_1) = h(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, denoted by $\tilde{a}_1 \tilde{=} \tilde{a}_2$;

where for $\tilde{a}_j = \langle[a_{1j}, a_{2j}, a_{3j}, a_{4j}], [b_{1j}, b_{2j}, b_{3j}, b_{4j}], [c_{1j}, c_{2j}, c_{3j}, c_{4j}]\rangle$ $(j = 1, 2)$

1. The score degrees of $\tilde{a}_j$ is given by

$$s(\tilde{a}_j) = \frac{1}{3}\left(2 + \frac{a_{1j} + a_{2j} + a_{3j} + a_{4j}}{4} - \frac{b_{1j} + b_{2j} + b_{3j} + b_{4j}}{4} - \frac{c_{1j} + c_{2j} + c_{3j} + c_{4j}}{4}\right)$$

2. The accuracy degrees of $\tilde{a}_j$ is given by

$$a(\tilde{a}_j) = \frac{a_{1j} + a_{2j} + a_{3j} + a_{4j}}{4} - \frac{c_{1j} + c_{2j} + c_{3j} + c_{4j}}{4}$$

.

Example 3.6 Let $\tilde{a}_1 = \langle(0.1, 0.2, 0.3, 0.4); 1.0, 0.0, 0.0\rangle$ and $\tilde{a}_2 = \langle(1, 2, 3, 4); 1.0, 0.0, 0.0\rangle$ be two SVTN-numbers.

Then, we can compare the SVTN-numbers $\tilde{a}_1$ and $\tilde{a}_2$ as Table 6.

4 Centroid and distance-measure-based approach for ranking of SVTN-number

In this subsection, we derive the centroid point of the SVTN-number. The method of ranking SVTN-number with centroid index uses the geometric center of an SVTN-number which has $X(\tilde{a})$ value on the horizontal axis and $Y(\tilde{a})$ value on the vertical axis.

Consider an SVTN-number of the form $\tilde{a} = \langle(a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle$ on R characterized by a truth-membership function $T_{\tilde{a}} : R \to [0, 1]$, an indeterminacy-membership function $I_{\tilde{a}} : R \to [0, 1]$ and a falsity-membership function $F_{\tilde{a}} : R \to [0, 1]$ is defined as:

$$\tilde{a}(x) = \begin{cases} \tilde{a}^L(x), & a \le x < b \\ \tilde{a}^C(x), & b \le x \le c \\ \tilde{a}^L(x), & c < x \le d \\ \tilde{a}^B(x), & \text{otherwise.} \end{cases}$$

where

$$\tilde{a}^L(x) = \left(T^L_{\tilde{a}}(x), I^L_{\tilde{a}}(x), F^L_{\tilde{a}}(x)\right) = \left(\frac{(x-a)w_{\tilde{a}}}{b-a}, \frac{b-x+u_{\tilde{a}}(x-a)}{b-a}, \frac{b-x+y_{\tilde{a}}(x-a)}{b-a>}\right),$$

$$\tilde{a}^C(x) = \left(T^C_{\tilde{a}}(x), I^C_{\tilde{a}}(x), F^C_{\tilde{a}}(x)\right) = (w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}),$$

$$\tilde{a}^R(x) = \left(T^R_{\tilde{a}}(x), I^R_{\tilde{a}}(x), F^R_{\tilde{a}}(x)\right)$$

$$= \left(\frac{(d-x)w_{\tilde{a}}}{d-c}, \frac{x-c+u_{\tilde{a}}(d-x)}{d-c}, \frac{x-c+y_{\tilde{a}}(d-x)}{d-c} \right)$$

and

$$\tilde{a}^B(x) = \left(T_{\tilde{a}}^B(x), I_{\tilde{a}}^B(x), F_{\tilde{a}}^B(x) \right) = (0, 1, 1),$$

Here, the functions

1. $T_{\tilde{a}}^L : [a, b] \subseteq R \to [0, w_{\tilde{a}}]$, $I_{\tilde{a}}^L : [a, b] \subseteq R \to [u_{\tilde{a}}, 1]$ and $F_{\tilde{a}}^L : [a, b] \subseteq R \to [y_{\tilde{a}}, 1]$ are surjective and injective in $[a, b] \subseteq R$.
2. $T_{\tilde{a}}^R : [c, d] \subseteq R \to [0, w_{\tilde{a}}]$, $I_{\tilde{a}}^R : [c, d] \subseteq R \to [u_{\tilde{a}}, 1]$ and $F_{\tilde{a}}^R : [c, d] \subseteq R \to [y_{\tilde{a}}, 1]$ are surjective and injective in $[c, d] \subseteq R$.

Therefore, the inverse functions exist which are also of the same nature as:

1. ${T^{-1}}_{\tilde{a}}^L : [0, w_{\tilde{a}}] \to [a, b] \subseteq R$, ${I^{-1}}_{\tilde{a}}^L : [u_{\tilde{a}}, 1] \to [a, b] \subseteq R$ and ${F^{-1}}_{\tilde{a}}^L : [y_{\tilde{a}}, 1] \to [a, b] \subseteq R$,
2. ${T^{-1}}_{\tilde{a}}^R : [0, w_{\tilde{a}}] \to [c, d] \subseteq R$, ${I^{-1}}_{\tilde{a}}^R : [u_{\tilde{a}}, 1] \to [c, d] \subseteq R$ and ${F^{-1}}_{\tilde{a}}^R : [y_{\tilde{a}}, 1] \to [c, d] \subseteq R$,

Then, the inverse functions can be analytically expressed as follows:

1. ${T^{-1}}_{\tilde{a}}^L(y) = a + \frac{(b-a)}{w_{\tilde{a}}} y \ (0 \le y < w_{\tilde{a}})$, ${I^{-1}}_{\tilde{a}}^L(y) = \frac{(a-b)y+(b-au_{\tilde{a}})}{1-u_{\tilde{a}}} \ (u_{\tilde{a}} \le y < 1)$ and ${F^{-1}}_{\tilde{a}}^L(y) = \frac{(a-b)y+(b-ay_{\tilde{a}})}{1-y_{\tilde{a}}} \ (y_{\tilde{a}} \le y < 1)$,
2. ${T^{-1}}_{\tilde{a}}^R(y) = d - \frac{(d-c)}{w_{\tilde{a}}} y$, $(0 \le y < w_{\tilde{a}})$, ${I^{-1}}_{\tilde{a}}^R(y) = \frac{(d-c)y+(c-du_{\tilde{a}})}{1-u_{\tilde{a}}} \ (u_{\tilde{a}} \le y < 1)$ and ${F^{-1}}_{\tilde{a}}^R(y) = \frac{(d-c)y+(c-dy_{\tilde{a}})}{1-y_{\tilde{a}}} \ (y_{\tilde{a}} \le y < 1)$,

Definition 4.1 Let $\tilde{a} = \langle (a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be an SVTN-number. Then, the centroid point of the SVTN-number $\tilde{a}$ denoted by $C(\tilde{a})$ is defined as: $C(\tilde{a}) = ((X(\tilde{a})_T, Y(\tilde{a})_T), (X(\tilde{a})_I, Y(\tilde{a})_I), (X(\tilde{a})_F, Y(\tilde{a})_F))$,
where

1. $X(\tilde{a})_T = \dfrac{\int_a^b \frac{(x-a)w_{\tilde{a}}}{b-a} x\mathrm{d}x + \int_b^c w_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{(d-x)w_{\tilde{a}}}{d-c} x\mathrm{d}x}{\int_a^b \frac{(x-a)w_{\tilde{a}}}{b-a} \mathrm{d}x + \int_b^c w_{\tilde{a}} \mathrm{d}x + \int_c^d \frac{(d-x)w_{\tilde{a}}}{d-c} \mathrm{d}x}$
2. $X(\tilde{a})_I = \dfrac{\int_a^b \frac{b-x+u_{\tilde{a}}(x-a)}{b-a} x\mathrm{d}x + \int_b^c u_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{x-c+u_{\tilde{a}}(d-x)}{d-c} x\mathrm{d}x}{\int_a^b \frac{b-x+u_{\tilde{a}}(x-a)}{b-a} \mathrm{d}x + \int_b^c u_{\tilde{a}} \mathrm{d}x + \int_c^d \frac{x-c+u_{\tilde{a}}(d-x)}{d-c} \mathrm{d}x}$
3. $X(\tilde{a})_F = \dfrac{\int_a^b \frac{b-x+y_{\tilde{a}}(x-a)}{b-a} x\mathrm{d}x + \int_b^c y_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{x-c+y_{\tilde{a}}(d-x)}{d-c} x\mathrm{d}x}{\int_a^b \frac{b-x+y_{\tilde{a}}(x-a)}{b-a} \mathrm{d}x + \int_b^c y_{\tilde{a}} \mathrm{d}x + \int_c^d \frac{x-c+y_{\tilde{a}}(d-x)}{d-c} \mathrm{d}x}$
4. $Y(\tilde{a})_T = \dfrac{\int_0^{w_{\tilde{a}}} \left(a + \frac{(b-a)}{w_{\tilde{a}}} y \right) y\mathrm{d}y - \int_0^{w_{\tilde{a}}} \left(d - \frac{(d-c)}{w_{\tilde{a}}} y \right) y\mathrm{d}y}{\int_0^{w_{\tilde{a}}} \left(a + \frac{(b-a)}{w_{\tilde{a}}} y \right) \mathrm{d}y - \int_0^{w_{\tilde{a}}} \left(d - \frac{(d-c)}{w_{\tilde{a}}} y \right) \mathrm{d}y}$
5. $Y(\tilde{a})_I = \dfrac{\int_{u_{\tilde{a}}}^1 \left(\frac{(a-b)y+(b-au_{\tilde{a}})}{1-u_{\tilde{a}}} \right) y\mathrm{d}y - \int_{u_{\tilde{a}}}^1 \left(\frac{(d-c)y+(c-du_{\tilde{a}})}{1-u_{\tilde{a}}} \right) y\mathrm{d}y}{\int_{u_{\tilde{a}}}^1 \left(\frac{(a-b)y+(b-au_{\tilde{a}})}{1-u_{\tilde{a}}} \right) \mathrm{d}y - \int_{u_{\tilde{a}}}^1 \left(\frac{(d-c)y+(c-du_{\tilde{a}})}{1-u_{\tilde{a}}} \right) \mathrm{d}y}$ and
6. $Y(\tilde{a})_F = \dfrac{\int_{y_{\tilde{a}}}^1 \left(\frac{(a-b)y+(b-ay_{\tilde{a}})}{1-y_{\tilde{a}}} \right) y\mathrm{d}y - \int_{y_{\tilde{a}}}^1 \left(\frac{(d-c)y+(c-\mathrm{d}y_{\tilde{a}})}{1-y_{\tilde{a}}} \right) y\mathrm{d}y}{\int_{y_{\tilde{a}}}^1 \left(\frac{(a-b)y+(b-ay_{\tilde{a}})}{1-y_{\tilde{a}}} \right) \mathrm{d}y - \int_{y_{\tilde{a}}}^1 \left(\frac{(d-c)y+(c-\mathrm{d}y_{\tilde{a}})}{1-y_{\tilde{a}}} \right) \mathrm{d}y}$

Proposition 4.2 *Let $\tilde{a} = \langle (a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be an SVTN-number and $C(\tilde{a})$ be the centroid point of $\tilde{a}$. Then, $X(\tilde{a})_T, Y(\tilde{a})_T, (X(\tilde{a})_I, Y(\tilde{a})_I), (X(\tilde{a})_F$ and $Y(\tilde{a})_F$ are given as:*

1. $X(\tilde{a})_T = \dfrac{w_{\tilde{a}}}{3} \dfrac{a^3d - ad^3 + b^3c - a^3c - b^3c + a^3c}{a^2d - db^2 + b^2c - a^2c - bc^2 + ac^2 - ad^2}$
2. $X(\tilde{a})_I = \dfrac{\frac{1}{b-a}(b^3 + 2u_{\tilde{a}}b^3 - 3u_{\tilde{a}}ab^2 - 3ba^2 + 2a^3 + u_{\tilde{a}}a^3) + 3u_{\tilde{a}}c^2 - 3u_{\tilde{a}}b^2 + \frac{1}{d-c}(2d^3 - 3cd^2 + u_{\tilde{a}}d^3 + c^2 - 3u_{\tilde{a}}dc^2 + 2u_{\tilde{a}}c^3)}{\frac{1}{b-a}(-3b^2 + 3u_{\tilde{a}}b^2 - 6u_{\tilde{a}}ab) - 6ba + 3a^2 + 3u_{\tilde{a}}a^2)) + 6u_{\tilde{a}}c - 6u_{\tilde{a}}b + \frac{1}{d-c}(3d^2 - 6cd + 3u_{\tilde{a}}d^2 + 3c^2 - 6u_{\tilde{a}}dc + 3u_{\tilde{a}}c^2)}$
3. $X(\tilde{a})_F = \dfrac{\frac{1}{b-a}(b^3 + 2y_{\tilde{a}}b^3 - 3y_{\tilde{a}}ab^2 - 3ba^2 + 2a^3 + y_{\tilde{a}}a^3) + 3y_{\tilde{a}}c^2 - 3y_{\tilde{a}}b^2 + \frac{1}{d-c}(2d^3 - 3cd^2 + y_{\tilde{a}}d^3 + c^2 - 3y_{\tilde{a}}dc^2 + 2y_{\tilde{a}}c^3)}{\frac{1}{b-a}(-3b^2 + 3y_{\tilde{a}}b^2 - 6y_{\tilde{a}}ab) - 6ba + 3a^2 + 3y_{\tilde{a}}a^2)) + 6y_{\tilde{a}}c - 6y_{\tilde{a}}b + \frac{1}{d-c}(3d^2 - 6cd + 3y_{\tilde{a}}d^2 + 3c^2 - 6y_{\tilde{a}}dc + 3y_{\tilde{a}}c^2)}$
4. $Y(\tilde{a})_T = w_{\tilde{a}} \dfrac{a + 2b - d - 2c}{3b + 3a - 3d - 3c}$
5. $Y(\tilde{a})_I = \dfrac{2a + b - 3au_{\tilde{a}} - 2d - c + 3du_{\tilde{a}} + 2bu_{\tilde{a}}{}^3 - 3bu_{\tilde{a}}{}^2 + au_{\tilde{a}}{}^3 - 2cu_{\tilde{a}}{}^3 + 3cu_{\tilde{a}}{}^2 - du_{\tilde{a}}{}^3}{3a + 3b - 6au_{\tilde{a}} - 3d - 3c + 6du_{\tilde{a}} + 3bu_{\tilde{a}}{}^2 - 6bu_{\tilde{a}} + 3au_{\tilde{a}}{}^2 - 3du_{\tilde{a}}{}^2 - 3cu_{\tilde{a}}{}^2 + 6cu_{\tilde{a}}}$
6. $Y(\tilde{a})_F = \dfrac{2a + b - 3ay_{\tilde{a}} - 2d - c + 3dy_{\tilde{a}} + 2by_{\tilde{a}}{}^3 - 3by_{\tilde{a}}{}^2 + ay_{\tilde{a}}{}^3 - 2cy_{\tilde{a}}{}^3 + 3cy_{\tilde{a}}{}^2 - dy_{\tilde{a}}{}^3}{3a + 3b - 6ay_{\tilde{a}} - 3d - 3c + 6dy_{\tilde{a}} + 3by_{\tilde{a}}{}^2 - 6by_{\tilde{a}} + 3ay_{\tilde{a}}{}^2 - 3dy_{\tilde{a}}{}^2 - 3cy_{\tilde{a}}{}^2 + 6cy_{\tilde{a}}}$

Proof 1. For $\tilde{a} = \langle (a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $X(\tilde{a})_T$ is computed as:

$$
\begin{aligned}
X(\tilde{a})_T &= \frac{\int_a^b \frac{(x-a)w_{\tilde{a}}}{b-a} x\mathrm{d}x + \int_b^c w_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{(d-x)w_{\tilde{a}}}{d-c} x\mathrm{d}x}{\int_a^b \frac{(x-a)w_{\tilde{a}}}{b-a}\mathrm{d}x + \int_b^c w_{\tilde{a}}\mathrm{d}x + \int_c^d \frac{(d-x)w_{\tilde{a}}}{d-c}\mathrm{d}x} \\
&= \frac{\frac{w_{\tilde{a}}}{b-a}\int_a^b (x^2 - ax)\mathrm{d}x + w_{\tilde{a}}\int_b^c x\mathrm{d}x + \frac{w_{\tilde{a}}}{d-c}\int_c^d (dx - x^2)\mathrm{d}x}{\frac{w_{\tilde{a}}}{b-a}\int_a^b (x-a)\mathrm{d}x + w_{\tilde{a}}\int_b^c \mathrm{d}x + \frac{w_{\tilde{a}}}{d-c}\int_c^d (d-x)\mathrm{d}x} \\
&= \frac{\frac{w_{\tilde{a}}}{b-a}\left(\frac{x^3}{3} - a\frac{x^2}{2}\right)|_a^b + w_{\tilde{a}}\frac{x^2}{2}|_b^c + \frac{w_{\tilde{a}}}{d-c}\left(d\frac{x^2}{2} - \frac{x^3}{3}\right)|_c^d}{\frac{w_{\tilde{a}}}{b-a}\left(\frac{x^2}{2} - ax\right)|_a^b + w_{\tilde{a}}x|_b^c + \frac{w_{\tilde{a}}}{d-c}\left(dx - \frac{x^2}{2}\right)|_c^d} \\
&= \frac{\frac{w_{\tilde{a}}}{b-a}\left[\left(\frac{b^3}{3} - \frac{ab^2}{2}\right) - \left(\frac{a^3}{3} - \frac{a^3}{2}\right)\right] + w_{\tilde{a}}\left[\frac{c^2}{2} - \frac{b^2}{2}\right] + \frac{w_{\tilde{a}}}{d-c}\left[\left(\frac{d^3}{2} - \frac{d^3}{3}\right) - \left(\frac{dc^2}{2} - \frac{c^3}{3}\right)\right]}{\frac{w_{\tilde{a}}}{b-a}\left[\left(\frac{b^2}{2} - ab\right) - \left(\frac{a^2}{2} - a^2\right)\right] + w_{\tilde{a}}[c-b] + \frac{w_{\tilde{a}}}{d-c}\left[\left(d^2 - \frac{d^2}{2}\right) - \left(dc - \frac{c^2}{2}\right)\right]} \\
&= \frac{\frac{w_{\tilde{a}}}{b-a}\left[\frac{2b^3 - 3ab^2 + a^3}{6}\right] + w_{\tilde{a}}[\frac{c^2 - b^2}{2}] + \frac{w_{\tilde{a}}}{d-c}\left[\left(\frac{d^3 - 3dc^2 + 2c^3}{6}\right)\right]}{\frac{w_{\tilde{a}}}{b-a}\left[\frac{b^2 - 2ab + a^2}{2}\right] + w_{\tilde{a}}[c-b] + \frac{w_{\tilde{a}}}{d-c}\left[\frac{d^2 - 2dc + c^2}{2}\right]} \\
&= \frac{w_{\tilde{a}}}{3}\frac{a^3d - ad^3 + b^3c - a^3c - b^3c + a^3c}{a^2d - db^2 + b^2c - a^2c - bc^2 + ac^2 - ad^2}
\end{aligned}
$$

2. For $\tilde{a} = \langle (a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $X(\tilde{a})_I$ is computed as:

$$
\begin{aligned}
X(\tilde{a})_I &= \frac{\int_a^b \frac{b-x+u_{\tilde{a}}(x-a)}{b-a} x\mathrm{d}x + \int_b^c u_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{x-c+u_{\tilde{a}}(d-x)}{d-c} x\mathrm{d}x}{\int_a^b \frac{b-x+u_{\tilde{a}}(x-a)}{b-a}\mathrm{d}x + \int_b^c u_{\tilde{a}}\mathrm{d}x + \int_c^d \frac{x-c+u_{\tilde{a}}(d-x)}{d-c}\mathrm{d}x} \\
&= \frac{\frac{1}{b-a}\int_a^b (b - x + u_{\tilde{a}}(x-a))x\mathrm{d}x + u_{\tilde{a}}\int_b^c x\mathrm{d}x + \frac{1}{d-c}\int_c^d (x - c + u_{\tilde{a}}(d-x))x\mathrm{d}x}{\frac{1}{b-a}\int_a^b (b - x + u_{\tilde{a}}(x-a))\mathrm{d}x + u_{\tilde{a}}\int_b^c \mathrm{d}x + \frac{1}{d-c}\int_c^d (x - c + u_{\tilde{a}}(d-x))\mathrm{d}x} \\
&= \frac{\frac{1}{b-a}\int_a^b (bx - x^2 + u_{\tilde{a}}x^2 - u_{\tilde{a}}ax)\mathrm{d}x + u_{\tilde{a}}\int_b^c x\mathrm{d}x + \frac{1}{d-c}\int_c^d (x^2 - cx + u_{\tilde{a}}dx - u_{\tilde{a}}x^2)\mathrm{d}x}{\frac{1}{b-a}\int_a^b (b - x + u_{\tilde{a}}x - u_{\tilde{a}}a))\mathrm{d}x + u_{\tilde{a}}\int_b^c \mathrm{d}x + \frac{1}{d-c}\int_c^d (x - c + u_{\tilde{a}}d - u_{\tilde{a}}x)\mathrm{d}x} \\
&= \frac{\frac{1}{b-a}\left(b\frac{x^2}{2} - \frac{x^3}{3} + u_{\tilde{a}}\frac{x^3}{3} - u_{\tilde{a}}a\frac{x^2}{2}\right)|_a^b + u_{\tilde{a}}\frac{x^2}{2}|_b^c + \frac{1}{d-c}\left(\frac{x^3}{3} - c\frac{x^2}{2} + u_{\tilde{a}}d\frac{x^2}{2} - u_{\tilde{a}}\frac{x^3}{3}\right)|_c^d}{\frac{1}{b-a}\left(bx - \frac{x^2}{2} + u_{\tilde{a}}\frac{x^2}{2} - u_{\tilde{a}}ax)\right)|_a^b + u_{\tilde{a}}x|_b^c + \frac{1}{d-c}(\frac{x^2}{2} - cx + u_{\tilde{a}}dx - u_{\tilde{a}}\frac{x^2}{2})|_c^d} \\
&= \frac{\frac{1}{b-a}(3bx^2 - 2x^3 + 2u_{\tilde{a}}x^3 - 3u_{\tilde{a}}ax^2)|_a^b + 3u_{\tilde{a}}x^2|_b^c + \frac{1}{d-c}(2x^3 - 3cx^2 + 3u_{\tilde{a}}dx^2 - 2u_{\tilde{a}}x^3)|_c^d}{\frac{1}{b-a}(6bx - 3x^2 + 3u_{\tilde{a}}x^2 - 6u_{\tilde{a}}ax))|_a^b + 6u_{\tilde{a}}x|_b^c + \frac{1}{d-c}(3x^2 - 6cx + 6u_{\tilde{a}}dx - 3u_{\tilde{a}}x^2)|_c^d} \\
&= \frac{\frac{1}{b-a}(b^3 + 2u_{\tilde{a}}b^3 - 3u_{\tilde{a}}ab^2 - 3ba^2 + 2a^3 + u_{\tilde{a}}a^3) + 3u_{\tilde{a}}c^2 - 3u_{\tilde{a}}b^2 + \frac{1}{d-c}(2d^3 - 3cd^2 + u_{\tilde{a}}d^3 + c^2 - 3u_{\tilde{a}}dc^2 + 2u_{\tilde{a}}c^3)}{\frac{1}{b-a}(-3b^2 + 3u_{\tilde{a}}b^2 - 6u_{\tilde{a}}ab) - 6ba + 3a^2 + 3u_{\tilde{a}}a^2)) + 6u_{\tilde{a}}c - 6u_{\tilde{a}}b + \frac{1}{d-c}(3d^2 - 6cd + 3u_{\tilde{a}}d^2 + 3c^2 - 6u_{\tilde{a}}dc + 3u_{\tilde{a}}c^2)}
\end{aligned}
$$

3. For $\tilde{a} = \langle (a,b,c,d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $X(\tilde{a})_F$ is computed as:

$$
\begin{aligned}
X(\tilde{a})_F &= \frac{\int_a^b \frac{b-x+y_{\tilde{a}}(x-a)}{b-a} x\mathrm{d}x + \int_b^c y_{\tilde{a}} x\mathrm{d}x + \int_c^d \frac{x-c+y_{\tilde{a}}(d-x)}{d-c} x\mathrm{d}x}{\int_a^b \frac{b-x+y_{\tilde{a}}(x-a)}{b-a} \mathrm{d}x + \int_b^c y_{\tilde{a}} \mathrm{d}x + \int_c^d \frac{x-c+y_{\tilde{a}}(d-x)}{d-c} \mathrm{d}x} \\
&= \frac{\frac{1}{b-a} \int_a^b (b-x+y_{\tilde{a}}(x-a)) x\mathrm{d}x + y_{\tilde{a}} \int_b^c x\mathrm{d}x + \frac{1}{d-c} \int_c^d (x-c+y_{\tilde{a}}(d-x)) x\mathrm{d}x}{\frac{1}{b-a} \int_a^b (b-x+y_{\tilde{a}}(x-a)) \mathrm{d}x + y_{\tilde{a}} \int_b^c \mathrm{d}x + \frac{1}{d-c} \int_c^d (x-c+y_{\tilde{a}}(d-x)) \mathrm{d}x} \\
&= \frac{\frac{1}{b-a} \int_a^b (bx - x^2 + y_{\tilde{a}} x^2 - y_{\tilde{a}} ax) \mathrm{d}x + y_{\tilde{a}} \int_b^c x\mathrm{d}x + \frac{1}{d-c} \int_c^d (x^2 - cx + y_{\tilde{a}} \mathrm{d}x - y_{\tilde{a}} x^2) \mathrm{d}x}{\frac{1}{b-a} \int_a^b (b - x + y_{\tilde{a}} x - y_{\tilde{a}} a)) \mathrm{d}x + y_{\tilde{a}} \int_b^c \mathrm{d}x + \frac{1}{d-c} \int_c^d (x - c + y_{\tilde{a}} d - y_{\tilde{a}} x) \mathrm{d}x} \\
&= \frac{\frac{1}{b-a} \left(b\frac{x^2}{2} - \frac{x^3}{3} + y_{\tilde{a}} \frac{x^3}{3} - y_{\tilde{a}} a \frac{x^2}{2} \right) |_a^b + y_{\tilde{a}} \frac{x^2}{2} |_b^c + \frac{1}{d-c} \left(\frac{x^3}{3} - c\frac{x^2}{2} + y_{\tilde{a}} d \frac{x^2}{2} - y_{\tilde{a}} \frac{x^3}{3} \right) |_c^d}{\frac{1}{b-a} \left(bx - \frac{x^2}{2} + y_{\tilde{a}} \frac{x^2}{2} - y_{\tilde{a}} ax \right) |_a^b + y_{\tilde{a}} x |_b^c + \frac{1}{d-c} \left(\frac{x^2}{2} - cx + y_{\tilde{a}} dx - y_{\tilde{a}} \frac{x^2}{2} \right) |_c^d} \\
&= \frac{\frac{1}{b-a} (3bx^2 - 2x^3 + 2y_{\tilde{a}} x^3 - 3y_{\tilde{a}} ax^2) |_a^b + 3y_{\tilde{a}} x^2 |_b^c + \frac{1}{d-c} (2x^3 - 3cx^2 + 3y_{\tilde{a}} dx^2 - 2y_{\tilde{a}} x^3) |_c^d}{\frac{1}{b-a} (6bx - 3x^2 + 3y_{\tilde{a}} x^2 - 6y_{\tilde{a}} ax)) |_a^b + 6y_{\tilde{a}} x |_b^c + \frac{1}{d-c} (3x^2 - 6cx + 6y_{\tilde{a}} dx - 3y_{\tilde{a}} x^2) |_c^d} \\
&= \frac{\frac{1}{b-a} (b^3 + 2y_{\tilde{a}} b^3 - 3y_{\tilde{a}} ab^2 - 3ba^2 + 2a^3 + y_{\tilde{a}} a^3) + 3y_{\tilde{a}} c^2 - 3y_{\tilde{a}} b^2 + \frac{1}{d-c} (2d^3 - 3cd^2 + y_{\tilde{a}} d^3 + c^2 - 3y_{\tilde{a}} dc^2 + 2y_{\tilde{a}} c^3)}{\frac{1}{b-a} (-3b^2 + 3y_{\tilde{a}} b^2 - 6y_{\tilde{a}} ab) - 6ba + 3a^2 + 3y_{\tilde{a}} a^2)) + 6y_{\tilde{a}} c - 6y_{\tilde{a}} b + \frac{1}{d-c} (3d^2 - 6cd + 3y_{\tilde{a}} d^2 + 3c^2 - 6y_{\tilde{a}} dc + 3y_{\tilde{a}} c^2)}
\end{aligned}
$$

4. For $\tilde{a} = \langle (a,b,c,d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $Y(\tilde{a})_T$ is computed as:

$$
\begin{aligned}
Y(\tilde{a})_T &= \frac{\int_0^{w_{\tilde{a}}} \left(a + \frac{(b-a)}{w_{\tilde{a}}} y \right) y\mathrm{d}y - \int_0^{w_{\tilde{a}}} \left(d - \frac{(d-c)}{w_{\tilde{a}}} y \right) y\mathrm{d}y}{\int_0^{w_{\tilde{a}}} \left(a + \frac{(b-a)}{w_{\tilde{a}}} y \right) \mathrm{d}y - \int_0^{w_{\tilde{a}}} \left(d - \frac{(d-c)}{w_{\tilde{a}}} y \right) \mathrm{d}y} \\
&= \frac{\int_0^{w_{\tilde{a}}} (aw_{\tilde{a}} + (b-a)y) y\mathrm{d}y - \int_0^{w_{\tilde{a}}} (dw_{\tilde{a}} - (d-c)y) y\mathrm{d}y}{\int_0^{w_{\tilde{a}}} (aw_{\tilde{a}} + (b-a)y) \mathrm{d}y - \int_0^{w_{\tilde{a}}} (dw_{\tilde{a}} - (d-c)y) \mathrm{d}y} \\
&= \frac{(3ay^2 w_{\tilde{a}} + 2by^3 - 2ay^3)|_0^{w_{\tilde{a}}} - (3\mathrm{d}y^2 w_{\tilde{a}} - 2\mathrm{d}y^3 + 2cy^3)|_0^{w_{\tilde{a}}}}{(6ayw_{\tilde{a}} + 3by^2 - 3ay^2)|_0^{w_{\tilde{a}}} - (6dyw_{\tilde{a}} - 3\mathrm{d}y^2 + 3cy^2)|_0^{w_{\tilde{a}}}} \\
&= \frac{aw_{\tilde{a}}^3 + 2bw_{\tilde{a}}^3 - dw_{\tilde{a}}^3 - 2cw_{\tilde{a}}^3}{3bw_{\tilde{a}}^2 + 3aw_{\tilde{a}}^2 - 3dw_{\tilde{a}}^2 - 3cw_{\tilde{a}}^2} \\
&= w_{\tilde{a}} \frac{a + 2b - d - 2c}{3b + 3a - 3d - 3c}
\end{aligned}
$$

5. For $\tilde{a} = \langle (a,b,c,d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $Y(\tilde{a})_I$ is computed as:

$$
\begin{aligned}
Y(\tilde{a})_I &= \frac{\int_{u_{\tilde{a}}}^1 \left(\frac{(a-b)y + (b - au_{\tilde{a}})}{1-u_{\tilde{a}}} \right) y\mathrm{d}y - \int_{u_{\tilde{a}}}^1 \left(\frac{(d-c)y + (c - du_{\tilde{a}})}{1-u_{\tilde{a}}} \right) y\mathrm{d}y}{\int_{u_{\tilde{a}}}^1 \left(\frac{(a-b)y + (b - au_{\tilde{a}})}{1-u_{\tilde{a}}} \right) \mathrm{d}y - \int_{u_{\tilde{a}}}^1 \left(\frac{(d-c)y + (c - du_{\tilde{a}})}{1-u_{\tilde{a}}} \right) \mathrm{d}y} \\
&= \frac{\int_{u_{\tilde{a}}}^1 (ay^2 - by^2 + by - au_{\tilde{a}} y) \mathrm{d}y - \int_{u_{\tilde{a}}}^1 (\mathrm{d}y^2 - cy^2 + cy - dyu_{\tilde{a}}) \mathrm{d}y}{\int_{u_{\tilde{a}}}^1 (ay - by + b - au_{\tilde{a}}) \mathrm{d}y - \int_{u_{\tilde{a}}}^1 (\mathrm{d}y - cy + c - du_{\tilde{a}}) \mathrm{d}y} \\
&= \frac{(2ay^3 - 2by^3 + 3by^2 - 3au_{\tilde{a}} y^2)|_{u_{\tilde{a}}}^1 - (2\mathrm{d}y^3 - 2cy^3 + 3cy^2 - 3\mathrm{d}y^2 u_{\tilde{a}})|_{u_{\tilde{a}}}^1}{(3ay^2 - 3by^2 + 6by - 6ayu_{\tilde{a}})|_{u_{\tilde{a}}}^1 - (3\mathrm{d}y^2 - 3cy^2 + 6cy - 6dyu_{\tilde{a}})|_{u_{\tilde{a}}}^1} \\
&= \frac{2a + b - 3au_{\tilde{a}} - 2d - c + 3du_{\tilde{a}} + 2bu_{\tilde{a}}{}^3 - 3bu_{\tilde{a}}{}^2 + au_{\tilde{a}}{}^3 - 2cu_{\tilde{a}}{}^3 + 3cu_{\tilde{a}}{}^2 - du_{\tilde{a}}{}^3}{3a + 3b - 6au_{\tilde{a}} - 3d - 3c + 6du_{\tilde{a}} + 3bu_{\tilde{a}}{}^2 - 6bu_{\tilde{a}} + 3au_{\tilde{a}}{}^2 - 3du_{\tilde{a}}{}^2 - 3cu_{\tilde{a}}{}^2 + 6cu_{\tilde{a}}}
\end{aligned}
$$

6. For $\tilde{a} = \langle (a,b,c,d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, the $Y(\tilde{a})_F$ is computed as:

$$Y(\tilde{a})_F = \frac{\int_{y_{\tilde{a}}}^{1} \left(\frac{(a-b)y+(b-ay_{\tilde{a}})}{1-y_{\tilde{a}}}\right) y\mathrm{d}y - \int_{y_{\tilde{a}}}^{1} \left(\frac{(d-c)y+(c-\mathrm{d}y_{\tilde{a}})}{1-y_{\tilde{a}}}\right) y\mathrm{d}y}{\int_{y_{\tilde{a}}}^{1} \left(\frac{(a-b)y+(b-ay_{\tilde{a}})}{1-y_{\tilde{a}}}\right) \mathrm{d}y - \int_{y_{\tilde{a}}}^{1} \left(\frac{(d-c)y+(c-\mathrm{d}y_{\tilde{a}})}{1-y_{\tilde{a}}}\right) \mathrm{d}y}$$
$$= \frac{\int_{y_{\tilde{a}}}^{1} (ay^2 - by^2 + by - ay_{\tilde{a}}y)\mathrm{d}y - \int_{y_{\tilde{a}}}^{1} (\mathrm{d}y^2 - cy^2 + cy - dyy_{\tilde{a}})\mathrm{d}y}{\int_{y_{\tilde{a}}}^{1} (ay - by + b - ay_{\tilde{a}})\mathrm{d}y - \int_{y_{\tilde{a}}}^{1} (\mathrm{d}y - cy + c - \mathrm{d}y_{\tilde{a}})\mathrm{d}y}$$
$$= \frac{(2ay^3 - 2by^3 + 3by^2 - 3ay_{\tilde{a}}y^2)|_{y_{\tilde{a}}}^{1} - (2dy^3 - 2cy^3 + 3cy^2 - 3\mathrm{d}y^2 y_{\tilde{a}})|_{y_{\tilde{a}}}^{1}}{(3ay^2 - 3by^2 + 6by - 6ayy_{\tilde{a}})|_{y_{\tilde{a}}}^{1} - (3dy^2 - 3cy^2 + 6cy - 6dyy_{\tilde{a}})|_{y_{\tilde{a}}}^{1}}$$
$$= \frac{2a + b - 3ay_{\tilde{a}} - 2d - c + 3dy_{\tilde{a}} + 2by_{\tilde{a}}^3 - 3by_{\tilde{a}}^2 + ay_{\tilde{a}}^3 - 2cy_{\tilde{a}}^3 + 3cy_{\tilde{a}}^2 - dy_{\tilde{a}}^3}{3a + 3b - 6ay_{\tilde{a}} - 3d - 3c + 6dy_{\tilde{a}} + 3by_{\tilde{a}}^2 - 6by_{\tilde{a}} + 3ay_{\tilde{a}}^2 - 3dy_{\tilde{a}}^2 - 3cy_{\tilde{a}}^2 + 6cy_{\tilde{a}}}$$

Example 4.3 Let $\tilde{a} = \langle (1,3,5,7); 0.5, 0.7, 0.3 \rangle$ be SVTN-numbers and $C(\tilde{a})$ be centroid point of $\tilde{a}$. Then, $X(\tilde{a})_T$, $Y(\tilde{a})_T$, $(X(\tilde{a})_I, Y(\tilde{a})_I)$, $(X(\tilde{a})_F$ and $Y(\tilde{a})_F$ are given as:

1. $X(\tilde{a})_T = \frac{w_{\tilde{a}}}{3} \frac{a^3d - ad^3 + b^3c - a^3c - b^3c + a^3c}{a^2d - db^2 + b^2c - a^2c - bc^2 + ac^2 - ad^2} = 0.0658$
2. $X(\tilde{a})_I = \frac{\frac{1}{b-a}(b^3 + 2u_{\tilde{a}}b^3 - 3u_{\tilde{a}}ab^2 - 3ba^2 + 2a^3 + u_{\tilde{a}}a^3) + 3u_{\tilde{a}}c^2 - 3u_{\tilde{a}}b^2 + \frac{1}{d-c}(2d^3 - 3cd^2 + u_{\tilde{a}}d^3 + c^2 - 3u_{\tilde{a}}dc^2 + 2u_{\tilde{a}}c^3)}{\frac{1}{b-a}(-3b^2 + 3u_{\tilde{a}}b^2 - 6u_{\tilde{a}}ab) - 6ba + 3a^2 + 3u_{\tilde{a}}a^2)) + 6u_{\tilde{a}}c - 6u_{\tilde{a}}b + \frac{1}{d-c}(3d^2 - 6cd + 3u_{\tilde{a}}d^2 + 3c^2 - 6u_{\tilde{a}}dc + 3u_{\tilde{a}}c^2)} = 9.3722$
3. $X(\tilde{a})_F = \frac{\frac{1}{b-a}(b^3 + 2y_{\tilde{a}}b^3 - 3y_{\tilde{a}}ab^2 - 3ba^2 + 2a^3 + y_{\tilde{a}}a^3) + 3y_{\tilde{a}}c^2 - 3y_{\tilde{a}}b^2 + \frac{1}{d-c}(2d^3 - 3cd^2 + y_{\tilde{a}}d^3 + c^2 - 3y_{\tilde{a}}dc^2 + 2y_{\tilde{a}}c^3)}{\frac{1}{b-a}(-3b^2 + 3y_{\tilde{a}}b^2 - 6y_{\tilde{a}}ab) - 6ba + 3a^2 + 3y_{\tilde{a}}a^2)) + 6y_{\tilde{a}}c - 6y_{\tilde{a}}b + \frac{1}{d-c}(3d^2 - 6cd + 3y_{\tilde{a}}d^2 + 3c^2 - 6y_{\tilde{a}}dc + 3y_{\tilde{a}}c^2)} = -1.6705$
4. $Y(\tilde{a})_T = w_{\tilde{a}} \frac{a + 2b - d - 2c}{3b + 3a - 3d - 3c} = 0.2083$
5. $Y(\tilde{a})_I = \frac{2a + b - 3au_{\tilde{a}} - 2d - c + 3du_{\tilde{a}} + 2bu_{\tilde{a}}^3 - 3bu_{\tilde{a}}^2 + au_{\tilde{a}}^3 - 2cu_{\tilde{a}}^3 + 3cu_{\tilde{a}}^2 - du_{\tilde{a}}^3}{3a + 3b - 6au_{\tilde{a}} - 3d - 3c + 6du_{\tilde{a}} + 3bu_{\tilde{a}}^2 - 6bu_{\tilde{a}} + 3au_{\tilde{a}}^2 - 3du_{\tilde{a}}^2 - 3cu_{\tilde{a}}^2 + 6cu_{\tilde{a}}} = 0.0355$
6. $Y(\tilde{a})_F = \frac{2a + b - 3ay_{\tilde{a}} - 2d - c + 3dy_{\tilde{a}} + 2by_{\tilde{a}}^3 - 3by_{\tilde{a}}^2 + ay_{\tilde{a}}^3 - 2cy_{\tilde{a}}^3 + 3cy_{\tilde{a}}^2 - dy_{\tilde{a}}^3}{3a + 3b - 6ay_{\tilde{a}} - 3d - 3c + 6dy_{\tilde{a}} + 3by_{\tilde{a}}^2 - 6by_{\tilde{a}} + 3ay_{\tilde{a}}^2 - 3dy_{\tilde{a}}^2 - 3cy_{\tilde{a}}^2 + 6cy_{\tilde{a}}} = 0.2149$

Remark 4.4 Let $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (a_2, b_2, c_2, d_2); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}} \rangle$ be two SVTN-numbers. Then, we have $C(\tilde{a}+\tilde{b}) \neq C(\tilde{a}) + C(\tilde{b})$ and $C(\tilde{a}.\tilde{b}) \neq C(\tilde{a}).C(\tilde{b})$.

Example 4.5 Let $\tilde{a} = \langle (4,5,6,8); 0.5, 0.2, 0.9 \rangle$ and $\tilde{b} = \langle (2,5,6,10); 1, 0.2, 0.1 \rangle$ be two SVTN-numbers. Then, we have $\tilde{a} + \tilde{b} = \langle (6,10,12,18); 1, 0.2, 0.1 \rangle$.

Now, we compute the $C(\tilde{a}), C(\tilde{b}), C(\tilde{a}+\tilde{b}), C(\tilde{a})+C(\tilde{b})$ and $C(\tilde{a}.\tilde{b})$ and $C(\tilde{a}).C(\tilde{b})$ as:

$$C(\tilde{a}) = ((0.9142, 0.2000), (0.5478, 0.1673), (0.1152, 0.0018))$$
$$C(\tilde{b}) = ((1.7679, 0.3703), (-3.7033, 0.2604), (-2.7262, 0.3308))$$
$$C(\tilde{a}+\tilde{b}) = ((3.8918, 0.3809), (-0.2652,)0.2181, (0.1658, 0.2850))$$
$$C(\tilde{a}) + C(\tilde{b}) = ((2.6822, 0.5703), (-3.1555, 0.4278), (-2.6109, 0.3326))$$
$$C(\tilde{a}\cdot\tilde{b}) = ((15.3572, 0.3775), (788.3783, 0.3467), (-682.4267, 0.4162))\text{and}$$
$$C(\tilde{a})\cdot C(\tilde{b}) = ((1.6164, 0.0740), (-2.0288, 0.0436), (-0.3143, 0.0005))\text{as;}$$

Clearly, $C(\tilde{a}+\tilde{b}) \neq C(\tilde{a}) + C(\tilde{b})$ and $C(\tilde{a}\cdot\tilde{b}) \neq C(\tilde{a})\cdot C(\tilde{b})$.

Definition 4.6 Let $\tilde{a} = \langle (a,b,c,d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be an SVTN-number and $((X(\tilde{a})_T, Y(\tilde{a})_T), (X(\tilde{a})_I, Y(\tilde{a})_I), (X(\tilde{a})_F, Y(\tilde{a})_F))$ be a centroid point of the SVTN-number $\tilde{a}$. Then,

1. Hamming ranking value of $\tilde{a}$ denoted $H(\tilde{a})$ is defined by

$$H(\tilde{a}) = |X(\tilde{a})_T - Y(\tilde{a})_T| + |X(\tilde{a})_I - Y(\tilde{a})_I| + |X(\tilde{a})_F, Y(\tilde{a})_F|$$

Table 7 Ranking of $\tilde{a}$ and $\tilde{b}$ based on centroid point

SVTN-numbers	$H(\tilde{a})$	$H(\tilde{b})$	$E(\tilde{a})$	$E(\tilde{b})$	Ranking
$\tilde{a} = \langle(1, 3, 5, 7); 0.5, 0.7, 0.3\rangle$					
$\tilde{b} = \langle(2, 3, 4, 5); 0.3, 0.7, 0.6\rangle$	11.3645	1.0482	9.5262	0.6244	$\tilde{a} \gtrsim \tilde{b}$

2. Euclidean ranking value of $\tilde{a}$ denoted $E(\tilde{a})$ is defined by

$$E(\tilde{a}) = \sqrt{|X(\tilde{a})_T - Y(\tilde{a})_T|^2 + |X(\tilde{a})_I - Y(\tilde{a})_I|^2 + |X(\tilde{a})_F, Y(\tilde{a})_F|^2}$$

Definition 4.7 Let $\tilde{a} = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle$ and $\tilde{b} = \langle(a_2, b_2, c_2, d_2); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}\rangle$ be two SVTN-numbers and $K \in \{H, E\}$. Then, an approach of ranking $\tilde{a}$ and $\tilde{b}$ is given as:

1. $K(\tilde{a}) > K(\tilde{b})$ if and only if $\tilde{a} > \tilde{b}$
2. $K(\tilde{a}) < K(\tilde{b})$ if and only if $\tilde{a} < \tilde{b}$
3. $K(\tilde{a}) = K(\tilde{b})$ if and only if $\tilde{a} = \tilde{b}$

Example 4.8 Let $\tilde{a} = \langle(1, 3, 5, 7); 0.5, 0.7, 0.3\rangle$ and $\tilde{b} = \langle(2, 3, 4, 5); 0.3, 0.7, 0.6\rangle$ be two SVTN-numbers and $C(\tilde{a})$ and $C(\tilde{b})$ be centroid point of $\tilde{a}$ and $\tilde{b}$, respectively. Then, $C(\tilde{a})$ and $C(\tilde{b})$ are computed as:

$C(\tilde{a}) = ((0.0658, 0.2083), (9.3722, 0.0355), (-1.6705, 0.2149))$

and

$C(\tilde{b}) = ((0.3559, 0.125), (0.3939, 0.0216), (0.4852, 0.0402))$.

Therefore, we can compare as Table 7.

Remark 4.9 Let $\tilde{a} = \langle(a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}\rangle$ and $\tilde{b} = \langle(a_2, b_2, c_2, d_2); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}\rangle$ be two SVTN-numbers. Then, we have

1. $H(\tilde{a} + \tilde{b}) \neq H(\tilde{a}) + H(\tilde{b})$
2. $E(\tilde{a} + \tilde{b}) \neq E(\tilde{a}) + E(\tilde{b})$
3. $H(\tilde{a} \cdot \tilde{b}) \neq H(\tilde{a}) \cdot H(\tilde{b})$
4. $E(\tilde{a} \cdot \tilde{b}) \neq E(\tilde{a}) \cdot E(\tilde{b})$

Example 4.10 Let we consider Example 4.5. Then, we compute the $H(\tilde{a})$, $H(\tilde{b})$, $H(\tilde{a} + \tilde{b})$, $H(\tilde{a} + \tilde{b})$, $H(\tilde{a} \cdot \tilde{b})$ and $H(\tilde{a}) \cdot H(\tilde{b})$ as: $H(\tilde{a}) = 1.2082$, $H(\tilde{b}) = 8.4185$, $H(\tilde{a} + \tilde{b}) = 4.1135$, $H(\tilde{a}) + H(\tilde{b}) = 9.6267$, $H(\tilde{a} \cdot \tilde{b}) = 1485.8542$ and $H(\tilde{a}) \cdot H(\tilde{b}) = 10.1714$

Clearly, $H(\tilde{a} + \tilde{b}) \neq H(\tilde{a}) + H(\tilde{b})$ and $H(\tilde{a} \cdot \tilde{b}) \neq H(\tilde{a}) \cdot H(\tilde{b})$.

Similarly, we compute the $E(\tilde{a})$, $E(\tilde{b})$, $E(\tilde{a} + \tilde{b})$, $E(\tilde{a} + \tilde{b})$, $E(\tilde{a} \cdot \tilde{b})$ and $E(\tilde{a}) \cdot E(\tilde{b}) =$ as:

$E(\tilde{a}) = 0.8172$, $E(\tilde{b}) = 5.1972$, $E(\tilde{a} + \tilde{b}) = 3.5460$, $E(\tilde{a}) + E(\tilde{b}) = 6.0144$, $E(\tilde{a} \cdot \tilde{b}) = 1042.8291$ and $E(\tilde{a}) \cdot E(\tilde{b}) = 4.2472$

Clearly, $E(\tilde{a} + \tilde{b}) \neq E(\tilde{a}) + E(\tilde{b})$ and $E(\tilde{a} \cdot \tilde{b}) \neq E(\tilde{a}) \cdot E(\tilde{b})$

5 Multi-attribute decision-making problem based on centroid points of SVNT-numbers

5.1 Problem description

Regarding a multi-attribute decision-making problem, assume $X = (A_1, A_2, \ldots, A_m)$ be a set of alternatives, $U = (u_1, u_2, \ldots, u_n)$ is the attribute set, and $\tilde{a}_{ij} = \langle(a_{ij}, b_{ij}, c_{ij}, d_{ij}); T_{\tilde{a}_{ij}}, I_{\tilde{a}_j}, F_{\tilde{a}_{ij}}\rangle$ is the decision-making matrix based on Table 8, where $\tilde{a}_{ij}$ represents the SVTN-number attribute value of the jth attribute u_j of the ith alternative A_i ,$1 \leq i \leq m, 1 \leq j \leq n$. Moreover, the attribute weight is given in the form of an SVTN-number, that is, the attribute weight vector $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots \tilde{w}_n)$, where $\tilde{w}_j = \langle(a_j, b_j, c_j, d_j); T_{\tilde{w}_j}, I_{\tilde{w}_j}, F_{\tilde{w}_j}\rangle$ represents the weight of the jth attribute u_j.

5.2 Decision-making steps

Step 1 Give the linguistic decision-making matrix $[\tilde{a}_{ij}]_{m\times n}$ according to the attribute value of the jth attribute u_j of the ith alternative A_i ,$1 \leq i \leq m, 1 \leq j \leq n$.
Step 2 Convert the linguistic decision matrix $[\tilde{a}_{ij}]_{m\times n}$ into the SVTN-number decision matrix.
Step 3 Give the linguistic weights of the attributes.
Step 4 Find the weights of the attributes.

(a) Convert the linguistic weights of the attributes. The weight information of the attribute weight is given in the form of an SVTN-number, that is, the attribute weight vector is $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots \tilde{w}_n)$, where $\tilde{w}_j = \langle(a_j, b_j, c_j, d_j); T_{\tilde{w}_j}, I_{\tilde{w}_j}, F_{\tilde{w}_j}\rangle$.
(b) Obtain the Hamming ranking value $H(\tilde{w}_j)$ based on $C(\tilde{w}_j)$, using Definition 4.1, of the SVTN-number $\tilde{w}_j$ (or Euclidean ranking value $E(\tilde{w}_j)$),
(c) The normalized weight w_j of the jth attribute u_j can be computed as:

$$w_j = \frac{H(\tilde{w}_j)}{\sum_{j=1}^{n} H(\tilde{w}_j)} \left(\text{or } w_j = \frac{E(\tilde{w}_j)}{\sum_{j=1}^{n} E(\tilde{w}_j)}\right)$$

for $1 \leq j \leq n$.

Step 5 Find the expected value $\mathrm{EV}(A_i) = G_{ao}(a_{i1}, a_{i2}, \ldots, a_{in})$ (for $1 \leq i \leq m$) of ith alternative A_i, according to

Table 8 SVNT-numbers for linguistic terms of the decision matrix

Linguistic terms	Linguistic values of SVNT-numbers	$H(\tilde{a})$	$E(\tilde{a})$
Very very absolutely low (VVAL)	$\langle(4, 5, 6, 8); 0.5, 0.2, 0.9\rangle$	1.2082	0.8172
Very absolutely low (VAL)	$\langle(2, 3, 4, 5); 0.8, 0.7, 0.6\rangle$	1.4331	0.8460
Absolutely low (AL)	$\langle(3, 4, 5, 6); 0.5, 1, 0.3\rangle$	2.1297	1.2777
Very very low (VVL)	$\langle(4, 5, 6, 7); 0.60.2, 0.3\rangle$	3.0916	1.8525
Very low (VL)	$\langle(1, 2, 3, 4); 0.1, 0.5, 0.9\rangle$	3.1401	2.5715
Low (L)	$\langle(6, 10, 12, 18); 1, 0.2, 0.1\rangle$	4.1135	3.5460
Medium (M)	$\langle(8, 9, 10, 11); 0.9, 0.7, 0.1\rangle$	5.2685	3.1347
Very medium (VM)	$\langle(5, 7, 8, 15); 0.6, 0.7, 0.1\rangle$	7.2209	4.4017
Very very medium (VVM)	$\langle(5, 7, 8, 9); 1, 0.2, 1\rangle$	7.7674	4.7667
Absolutely medium (AM)	$\langle(2, 5, 6, 10); 1, 0.2, 0.1\rangle$	8.4185	5.1972
Very absolutely medium (VAM)	$\langle(-4, -3, 0, 7); 0.6, 0.2, 0.3\rangle$	8.4117	5.6475
Very very absolutely medium (VVAM)	$\langle(0, 4, 5, 6); 1, 0.7, 0.6\rangle$	9.1213	6.6929
High (H)	$\langle(-1, 0, 10, 15); 0.1, 0.9, 0.3\rangle$	9.3703	6.1970
Very high (VH)	$\langle(3, 5, 10, 16); 0.5, 0, 0.3\rangle$	10.7485	7.1842
Very very high (VVH)	$\langle(-10, -4, 5, 6); 0.4, 0.7, 0.6\rangle$	19.6502	11.5976
Absolutely high (AH)	$\langle(2, 7, 8, 9); 0.3, 0.5, 0.6\rangle$	23.7577	16.6756
Very absolutely high (VAH)	$\langle(1, 5, 10, 15); 0.5, 1, 0.3\rangle$	36.1111	26.9757
Very very absolutely high (VVAH)	$\langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$	1485.8542	1042.8291

Table 9 SVNT-numbers for linguistic terms of attribute weights

Linguistic terms	Linguistic values of SVNT-numbers	$H(\tilde{a})$	$E(\tilde{a})$
Absolutely high (AH)	$\langle(0.1, 0.2, 0.3, 0.4); 0.1, 0.2, 0.1\rangle$	1.5386	1.0655
High (H)	$\langle(0.3, 0.5, 0.8, 1); 0.60.20.3\rangle$	1.0359	0.6696
Absolutely medium (AM)	$\langle(0, 0.1, 0.3, 0.9); 0.5, 0, 0.3\rangle$	0.8592	0.5098
Medium (M)	$\langle(0.4, 0.5, 0.8, 0.9); 0.6, 0.7, 0.1\rangle$	0.6440	0.4096
Low (L)	$\langle(0.5, 0.6, 0.7, 0.8); 1, 0.2, 1\rangle$	0.5249	0.3273
Absolutely low (AL)	$\langle(0.5, 0.6, 0.7, 0.9); 0.8, 0.7, 0.6\rangle$	0.3262	0.2007

$$\mathrm{EV}(A_i) = \Big\langle\Big(\sum_{j=1}^{n} a_{ij} * w_j, \sum_{j=1}^{n} b_{ij} * w_j, \sum_{j=1}^{n} c_{ij} * w_j, \sum_{j=1}^{n} d_{ij} * w_j\Big); \bigwedge_{j=1}^{n} w_{\tilde{a}_j}, \bigvee_{j=1}^{n} u_{\tilde{a}_j}, \bigvee_{j=1}^{n} y_{\tilde{a}_j}\Big\rangle$$

Step 6 Compute the Hamming ranking value $H(\mathrm{EV}(A_i))$ based on $C(\mathrm{EV}(A_i))$ of the SVTN-number $\mathrm{EV}(A_i)$ for $1 \le i \le m$; (or Euclidean ranking value $E(\mathrm{EV}(A_i))$,
Step 7 Rank all alternatives A_i for all i, $1 \le i \le m$ by using the $H(\mathrm{EV}(A_i))$ (or $E(\mathrm{EV}(A_i))$ and determine the best alternative.

Now, we give an example from Chan and Kumar (2007), Wang et al. (2012) and Zhang (2018) to illustrate the proposed methods.

5.2.1 Application

Suppose that the problem discussed here is concerned with a manufacturing company, search the best global suppliers for one of its most critical parts used in assembling process (adapted from Chan and Kumar 2007, Wang et al. 2012, Zhang 2018). There are potential global suppliers to be evaluated with four attributes: (1)u_1 : quality of the product, (2) u_2 : risk factor, (3) u_3: service performance of supplier and (4) u_4 : suppliers profile. An expert uses the linguistic terms shown in Table 8 to represent the characteristics of the potential global suppliers A_i (i = 1,2,3,4,5) with respect to different attributes u_i (i = 1,2,3,4). Also, same expert uses the linguistic terms shown in Table 9 to represent attribute weights.

Table 10 The linguistic decision matrix provided by expert

	u_1	u_2	u_3	u_4
A_1	VVM	VAM	VVAH	VVH
A_2	VVL	VVAL	VVM	AM
A_3	VM	VAH	VVAH	VL
A_4	VVL	VVAM	VVAH	VVM
A_5	VVL	VVAH	VH	AL

Table 11 The decision matrix provided by expert

	u_1	u_2	u_3	u_4
A_1	$\langle(5, 7, 8, 9); 1, 0.2, 1\rangle$	$\langle(-4, -3, 0, 7); 0.6, 0.2, 0.3\rangle$	$\langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$	$\langle(-10, -4, 5, 6); 0.4, 0.7, 0.6\rangle$
A_2	$\langle(4, 5, 6, 7); 0.60.2, 0.3\rangle$	$\langle(4, 5, 6, 8); 0.5, 0.2, 0.9\rangle$	$\langle(5, 7, 8, 9); 1, 0.2, 1\rangle$	$\langle(2, 5, 6, 10); 1, 0.2, 0.1\rangle$
A_3	$\langle(5, 7, 8, 15); 0.6, 0.7, 0.1\rangle$	$\langle(1, 5, 10, 15); 0.5, 1, 0.3\rangle$	$\langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$	$\langle(1, 2, 3, 4); 0.1, 0.5, 0.9\rangle$
A_4	$\langle(4, 5, 6, 7); 0.60.2, 0.3\rangle$	$\langle(0, 4, 5, 6); 1, 0.7, 0.6\rangle$	$\langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$	$\langle(5, 7, 8, 9); 1, 0.2, 1\rangle$
A_5	$\langle(4, 5, 6, 7); 0.60.2, 0.3\rangle$	$\langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$	$\langle(3, 5, 10, 16); 0.5, 0, 0.3\rangle$	$\langle(3, 4, 5, 6); 0.5, 1, 0.3\rangle$

Table 12 The linguistic weights of the attributes evaluated by the expert

u_1	u_2	u_3	u_4
M	AL	AH	AM

Table 13 The weights of the attributes

u_1	u_2	u_3	u_4
$\langle(0.4, 0.5, 0.8, 0.9); 0.6, 0.7, 0.1\rangle$	$\langle(0.5, 0.6, 0.7, 0.9); 0.8, 0.7, 0.6\rangle$	$\langle(0.1, 0.2, 0.3, 0.4); 0.1, 0.2, 0.1\rangle$	$\langle(0, 0.1, 0.3, 0.9); 0.5, 0, 0.3\rangle$

Step 1 We give the linguistic decision-making matrix $[\tilde{a}_{ij}]_{5\times4}$ in Table 10.
Step 2 We converted the linguistic decision matrix $[\tilde{a}_{ij}]_{5\times4}$ into the SVTN-number decision matrix in Table 11.
Step 3 We give the linguistic weights of the attributes in Table 12.
Step 4 We found the weights of the attributes as:

(a) We converted the linguistic weights of the attributes in Table 13.

(b) We obtained the Hamming ranking value $H(\tilde{w}_j)$ (for $1 \le j \le 4$) of the SVTN-number $\tilde{w}_j$ based on $C(\tilde{w}_j)$, using Definition 4.1 as:

$$H(\tilde{w}_1) = 0.6440$$
$$H(\tilde{w}_2) = 0.3262$$
$$H(\tilde{w}_3) = 1.5386$$
$$H(\tilde{w}_4) = 0.8592$$

(c) The normalized weight w_j of the jth attribute u_j based on centroid point $C(\tilde{w}_j)$ of the SVTN-number $\tilde{w}_j$ using Definition 4.1 can be computed as:

$$W = (w_1, w_2, w_3, w_4) = (0.1912, 0.0968, 0.4568, 0.2551)$$

Step 5 We calculated the expected value $\text{EV}(A_i) = G_{ao}(a_{i1}, a_{i2}, \ldots, a_{in})$ (for $1 \le i \le 5$) of ith alternative A_i as:

$$\text{EV}(A_1) = \langle(1.6718, 11.2564, 19.2504, 40.4746); 0.4, 0.7, 1\rangle$$
$$\text{EV}(A_2) = \langle(3.9465, 5.9136, 6.9136, 8.7758); 0.5, 0.2, 0.1\rangle$$
$$\text{EV}(A_3) = \langle(4.9625, 13.7533, 19.7089, 41.8866); 0.5, 1, 0.9\rangle$$
$$\text{EV}(A_4) = \langle(5.6949, 14.5496, 20.1177, 40.7607); 0.6, 0.7, 1\rangle$$
$$\text{EV}(A_5) = \langle(3.6755, 6.6822, 10.4781, 17.9275); 0.5, 1, 0.3\rangle$$

Step 6 We computed the Hamming ranking value $H(\text{EV}(A_i))$ based on $C(\text{EV}(A_i))$ of the SVTN-number $\text{EV}(A_i)$ for $1 \le i \le 5$ as:

$$H(\text{EV}(A_1)) = 64.9339$$
$$H(\text{EV}(A_2)) = 2.4027$$
$$H(\text{EV}(A_3)) = 109.5658$$
$$H(\text{EV}(A_4)) = 183.4800$$
$$H(\text{EV}(A_5)) = 208.0349$$

Step 7 We ranked all alternatives A_i for all i, $1 \le i \le 5$ by using the $H(\text{EV}(A_i))$ as:

$$A_5 > A_4 > A_3 > A_1 > A_2$$

Similarly, we use the Euclidean ranking value $E(\text{EV}(A_i)$, then we have

$$A_5 > A_4 > A_3 > A_1 > A_2$$

The best alternative is A_5.

Table 14 Comparative examples of the proposed ranking process with the existing ranking process

Methods	The final ranking	The best alternative(s)	The worst alternative(s)
Method 1 (Deli and Şubaş 2017a) ($\theta = 0.1$)	$A_5 > A_2 > A_4 > A_1 > A_3$	A_5	A_1
Method 2 (Biswas et al. 2016a) ($(\lambda, \mu, \nu) = (0.5, 0.5, 0.5)$)	$A_4 > A_3 > A_1 > A_5 > A_2$	A_4	A_2
Method 3 (Ye 2017)	$A_4 > A_1 > A_3 > A_5 > A_2$	A_4	A_2
Method 4 (Liang et al. 2017b)	$A_4 > A_1 > A_3 > A_5 > A_2$	A_4	A_2
Method 5 (Liang et al. 2017a)	$A_4 > A_1 > A_3 > A_5 > A_2$	A_4	A_2
Method 6 (Ye 2015)	$A_4 > A_1 > A_3 > A_5 > A_2$	A_4	A_2
Proposed method	$A_5 > A_4 > A_3 > A_1 > A_2$	A_5	A_2

5.3 Significance of the proposed method

Here, we are using for, an SVTN-number $\tilde{a} = \langle (a, b, c, d); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, a truth-membership function $T_{\tilde{a}} : R \to [0, 1]$, an indeterminacy-membership function $I_{\tilde{a}} : R \to [0, 1]$ and a falsity-membership function $F_{\tilde{a}} : R \to [0, 1]$ to obtain a single score. If we obtain a truth-membership score, an indeterminacy-membership score and falsity-membership for an SVTN-number ranking of an SVTN-number fail if truth-membership score of $\tilde{a_1}$, truth-membership score of $\tilde{a_2}$, indeterminacy-membership score of $\tilde{a_1}$, indeterminacy-membership score of $\tilde{a_2}$ and falsity-membership score of $\tilde{a_1}$, falsity-membership score of $\tilde{a_2}$ where $\tilde{a_1}$ and $\tilde{a_2}$ are two SVTN-numbers. Getting a single score involving truth-membership, an indeterminacy-membership and falsity-membership degree overcome that situation. This new scoring method has wide applications in various fields such as: economics, decision making, game theory,…for extending equilibrium in the fuzzy sense to single-valued neutrosophic sense, these notions can be used.

5.4 Comparative analysis of the proposed method with the existing methods

In application Sect. 5.2.1, we compare the proposed method with other existing methods in the literature and comparison results are presented in Table 14.

As can be seen from Table 14, the ranking of the alternatives found by different methods is often similar. Therefore, the results are verified through comparative analysis.

The comparative results are given for the justification of the proposed method with the existing ranking in Table 15. The advantages of the proposed ranking method are:

1. The proposed ranking method can be applied for SVTN-numbers, which reflects the uncertainty suitably.
2. The proposed ranking technique based on centroid point, hamming ranking value and Euclidean ranking value is a flexible method when compared with other existing methods.

6 Conclusion

In this paper, we gave all the existing ranking methods of SVTN-numbers. Then, we firstly defined the concept of centroid point and examined several useful properties of the concept. Secondly, we developed hamming ranking value and Euclidean ranking value of SVTN-numbers to compare the SVTN-numbers. Thirdly, we proposed a novel defuzzification method to MADM with linguistic information and gave a real example deal with manufacturing company to illustrate the feasibility and effectiveness of the developed approach. Finally, we presented some examples to compare the proposed method with the existing ranking results and the results verified through comparative analysis. With the proposed method, many MADM and optimization problems of uncertain nature can be solved. In spite of existing ranking methods, no one can rank SVTN-numbers with human intuition consistently in all cases. Therefore, we pointed out the shortcoming of some recent ranking methods and introduced a novel defuzzification method for ranking SVTN-numbers. The developed hamming ranking value and Euclidean ranking value are simple and have consistent expression on the horizontal axis and vertical axis. In future, we will apply the proposed ranking technique to linear programming, game theory, medical diagnosis, and so on.

Table 15 Comparative Examples of the proposed ranking process with the existing ranking process

No.	Examples	Method 1 (Deli and Şubaş 2017a) ($\theta = 0.5$)	Method 2 (λ, μ, ν) (Biswas et al. 2016a) (0.5, 0.5, 0.5)	Method 3 (Ye 2017)	Method 4 (Liang et al. 2017b)	Method 5 (Liang et al. 2017a)	Method 6 (Ye 2015)	Proposed method
	$A = \langle(8, 9, 10, 11); 0.9, 0.7, 0.1\rangle$ $B = \langle(2, 3, 4, 5); 0.8, 0.7, 0.6\rangle$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$
2	$A = \langle(5, 7, 8, 9); 1, 0.2, 1\rangle$ $B = \langle(3, 5, 10, 16); 0.5, 0, 0.3\rangle$	$A < B$	$A < B$	$A < B$	$A < B$	$A < B$	$A < B$	$A < B$
3	$A = \langle(5, 7, 8, 15); 0.6, 0.7, 0.1\rangle$ $B = \langle(4, 5, 6, 7); 0.60.2, 0.3\rangle$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$
4	$A = \langle(1, 5, 10, 15); 0.5, 1, 0.3\rangle$ $B = \langle(0, 4, 5, 6); 1, 0.7, 0.6\rangle$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$
5	$A = \langle(1, 2, 3, 4); 0.1, 0.5, 0.9\rangle$ $B = \langle(3, 4, 5, 6); 0.5, 1, 0.3\rangle$	$A < B$	$A < B$	$A < B$	$A < B$	$A > B$	$A < B$	$A > B$
6	$A = \langle(2, 7, 8, 9); 0.3, 0.5, 0.6\rangle$ $B = \langle(4, 5, 6, 8); 0.5, 0.2, 0.9\rangle$	$A < B$	$A > B$	$A < B$	$A > B$	$A > B$	$A < B$	$A > B$
7	$A = \langle(2, 5, 6, 10); 1, 0.2, 0.1\rangle$ $B = \langle(6, 10, 12, 18); 1, 0.2, 0.1\rangle$	$A < B$	$A < B$	$A < B$	$A < B$	$A < B$	$A < B$	$A > B$
8	$A = \langle(8, 25, 36, 80); 1, 0.2, 0.1\rangle$ $B = \langle(-4, -3, 0, 7); 0.6, 0.2, 0.3\rangle$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A < B$
9	$A = \langle(-1, 0, 10, 15); 0.1, 0.9, 0.3\rangle$ $B = \langle(-10, -4, 5, 6); 0.4, 0.7, 0.6\rangle$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A > B$	$A < B$

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by the authors.

References

Arun K, Prakash KA, Suresh M, Vengataasalam S (2016) A new approach for ranking of intuitionistic fuzzy numbers using a centroid concept. Math Sci 10:177184

Atanassov KT (1999) Intuitionistic fuzzy sets. Springer, New York

Basset MA, Mohamed M, Sangaiah AK (2018a) Neutrosophic AHP-Delphi group decision making model based on trapezoidal neutrosophic numbers. J Intell Hum Comput Ambient. https://doi.org/10.1007/s12652-017-0548-7

Basset MA, Mohamed M, Hussien AN, Sangaiah AK (2018b) A novel group decision-making model based on triangular neutrosophic numbers. Soft Comput 22:6629–6643

Biswas P, Pramanik S, Giri BC (2016a) Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets Syst 12:127–138

Biswas P, Pramanik S, Giri BC (2016b) Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets Syst 12:20–38

Broumi S, Deli I, Smarandache F (2014) Distance and similarity measures of interval neutrosophic soft sets. Crit Rev Cent Math Uncertain Creighton Univ U S A 8(2014):14–31

Broumi S, Deli I, Smarandache F (2014) Interval valued neutrosophic parameterized soft set theory and its decision making. J New Results Sci 7(2014):58–71

Broumi S, Talea M, Bakali A, Smarandache F, Vladareanu L (2016a) Shortest path problem under triangular fuzzy neutrosophic information. In: 10th international conference on software, knowledge, information management and applications (SKIMA)

Broumi S, Talea M, Bakali A, Smarandache F, Vladareanu L (2016b) Computation of shortest path problem in a network with SV-trapezoidal neutrosophic numbers. In: Proceedings of the 2016 international conference on advanced mechatronic systems, Melbourne, Australia, 30 November–3 December

Chan FTS, Kumar N (2007) Global supplier development considering risk factors using fuzzy extended AHP-based approach. Omega Int J Manag Sci 35(4):417431

Das S, Guha D (2013) Ranking of intuitionistic fuzzy number by centroid point. J Ind Intell Inf 1(2):107–110

Das S, Guha D (2016) A centroid-based ranking method of trapezoidal intuitionistic fuzzy numbers and its application to MCDM problems. Fuzzy Inf Eng 8(1):41–74

Deli I, Şubaş Y (2017a) A ranking method of single valued neutrosophic numbers and its applications to multiattribute decision making problems. Int J Mach Learn Cybern. https://doi.org/10.1007/s13042-016-0505-3

Deli I, Şubaş Y (2017b) Some weighted geometric operators with SVTrN-numbers and their application to multi-criteria decision making problems. J Intell Fuzzy Syst 32(1):291–301. https://doi.org/10.3233/JIFS-151677

Hajek P, Olej V (2014) Defuzzification methods in intuitionistic fuzzy inference systems of Takagi–Sugeno type. In: 11th International conference on fuzzy systems and knowledge discovery

Kahraman C, Otay I (2019a) Fuzzy multi criteria decision making using neutrosophic sets. Studies in fuzziness and soft computing. A special issue on extensions of fuzzy sets in decision-making. Soft Comput 22:4851–4853

Kahraman C, Otay I (eds) (2019b) Fuzzy multi criteria decision making using neutrosophic sets, studies in fuzziness and soft computing, vol 369. Springer, Basel

Liang RX, Wang JQ, Zhang HY (2017a) A multi-criteria decision-making method based on single-valued trapezoidal neutrosophic preference relations with complete weight information. Neural Comput Appl. https://doi.org/10.1007/s00521-017-2925-8

Liang RX, Wang JQ, Li L (2017b) Multi-criteria group decision-making method based on interdependent inputs of single-valued trapezoidal neutrosophic information. Neural Comput Appl. https://doi.org/10.1007/s00521-016-2672-2

Liu PD, Li DF (2017) Some muirhead mean operators for intuitionistic fuzzy numbers and their applications to group decision making. PLoS ONE 12(1):423–431

Liu P, Zhang XH (2018) Some Maclaurin symmetric mean operators for single-valued trapezoidal neutrosophic numbers and their applications to group decision making. Int J Fuzzy Syst. https://doi.org/10.1007/s40815-017-0335-9

Peng JJ, Wang JQ, Wang J, Zhang HY, Chen XH (2014) An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. Appl Soft Comput 25(2014):336–346

Peng JJ, Wang JQ, Wang J, Zhang HY, Chen XH (2016) Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. Int J Syst Sci 47(2016):2342–2358

Rao PPB, Shankar NR (2011) Ranking fuzzy numbers with a distance method using circumcenter of centroids and an index of modality. Adv Fuzzy Syst. https://doi.org/10.1155/2011/178308

Smarandache F (1998) A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic. American Research Press, Rehoboth

Şubaş Y (2015) Neutrosophic numbers and their application to multi-attribute decision making problems. Masters thesis, Kilis 7 Aralık University, Graduate School of Natural and Applied Science, 2015. **(in Turkish)**

Varghese A, Kuriakose S (2012) Centroid of an intuitionistic fuzzy number. Notes Intuitionistic Fuzzy Sets 18(1):1924

Wan SP, Lin LL, Dong JY (2017) MAGDM based on triangular Atanassovs intuitionistic fuzzy information aggregation. Neural Comput Appl 28(9):2687–2702

Wang H, Smarandache F, Zhang Q, Sunderraman R (2010) Single valued neutrosophic sets. Multispace Multistruct 4:410–413

Wang WZ, Liu XW, Qin Y (2012) Multi-attribute group decision making models under interval type-2 fuzzy environment. Knowl Based Syst 30:121128

Wei GW (2010) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 10(2):423–431

Yager RR, Zadeh LA (1992) An introduction to fuzzy logic applications in intelligent systems. Kluwer Academic Publishers, Dordrecht

Ye J (2015) Trapezoidal neutrosophic set and its application to multiple attribute decision making. Neural Comput Appl 26(5):1157–1166

Ye J (2017) Some weighted aggregation operators of trapezoidal neutrosophic numbers and their multiple attribute decision making method. Informatica 28(2):387402

Ye J (2018) Neutrosophic number linear programming method and its application under neutrosophic number environments. Soft Comput 22:4639–4646

Ying H (2000) Fuzzy control and modeling analytical foundations and applications. IEEE Press, New York

Zadeh LA (1965) Fuzzy Sets. Inf Control 8:338–353

Zadeh LA (2005) From imprecise to granular probabilities. Fuzzy Sets Syst 154(2005):370–374

Zhang ZM (2017) Intuitionistic fuzzy aggregation operators. Neural Comput Appl 28(12):3781–3800

Zhang Z (2018) Trapezoidal interval type-2 fuzzy aggregation operators and their application to multiple attribute group decision making. Neural Comput Appl 29:1031054