Soft Computing (2019) 23:251–267
https://doi.org/10.1007/s00500-018-3592-0

METHODOLOGIES AND APPLICATION

Pythagorean hesitant fuzzy Choquet integral aggregation operators and their application to multi-attribute decision-making

Muhammad Sajjad Ali Khan[1] · Saleem Abdullah[2] · Asad Ali[1] · Fazli Amin[1] · Fawad Hussain[3]

Published online: 10 November 2018

Abstract

Pythagorean hesitant fuzzy sets play a vital role in decision-making as it permits a set of possible elements in membership and non-membership degrees and satisfy the condition that the square sum of its memberships degree is less than or equal to 1. While aggregation operators are used to aggregate the overall preferences of the attributes, under Pythagorean hesitant fuzzy environment and fuzzy measure in the paper we develop Pythagorean hesitant fuzzy Choquet integral averaging operator, Pythagorean hesitant fuzzy Choquet integral geometric operator, generalized Pythagorean hesitant fuzzy Choquet integral averaging operator and generalized Pythagorean hesitant fuzzy Choquet integral geometric operator. We also discuss some properties such as idempotency, monotonicity and boundedness of the developed operators. Moreover, we apply the developed operators to multi-attribute decision-making problem to show the validity and effectiveness of the developed operators. Finally, a comparison analysis is given.

Keywords Pythagorean hesitant fuzzy sets · Generalized Pythagorean hesitant fuzzy Choquet integral averaging (GPHFCIA) operator · Generalized Pythagorean hesitant fuzzy Choquet integral geometric (GPHFCIG) operator · Multi-attribute decision-making

1 Introduction

Fuzzy set introduced by Zadeh (1965a, b) is one of the batter tools to deal with uncertainty and hesitancy. Many researchers applied the concept of fuzzy set to pure and applied mathematics. To obtain the exact and numerical solution of fuzzy Fredholm–Volterra integro-differential equation, Arqub (2017) proposed the reproducing kernel Hilbert space method. Arqub et al. (2017) proposed the analytic and approximate solutions of second-order, two-point fuzzy boundary value problems based on the reproducing kernel theory under the assumption of strongly generalized differentiability. Arqub et al. (2016) developed a new method for solving fuzzy differential equations based on the reproducing kernel theory under strongly generalized differentiability. The notion of intuitionistic fuzzy set was first introduced by Atanassov (1986, 1999), as a generalization of fuzzy set. Intuitionistic fuzzy set is more suitable to deal with uncertainty and fuzziness. The notion of intuitionistic fuzzy set is broadly applied by many authors in decision-making problems (Beg and Rashid 2014; Boran et al. 2009; De et al. 2001; Li 2005). Torra (2010) introduced another extension of fuzzy set, said to be hesitant fuzzy set. Hesitant fuzzy set permits the condition that the membership degree has a set of probable values. Based on hesitant fuzzy set, many researchers solved group decision-making problems (Liu and Sun 2013; Xia et al. 2013; Xu and Zhang 2013; Yu et al. 2011; Zhang 2013). Based on fuzzy measure (Sugeno 1974) and Choquet

Communicated by V. Loia.

✉ Muhammad Sajjad Ali Khan
sajjadalimath@yahoo.com

Saleem Abdullah
saleemabdullah81@yahoo.com

Asad Ali
asad_maths@hu.edu.pk

Fazli Amin
faminmath@yahoo.com

Fawad Hussain
fawadhussain998@hotmail.com

1 Department of Mathematics, Hazara University, Mansehra, KPK, Pakistan

2 Department of Mathematics, Abdul Wali Khan University, Mardan, KPK, Pakistan

3 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad, KPK, Pakistan

integral (Choquet 1954), Yu et al. (2011) developed hesitant fuzzy Choquet geometric (HFCG) operator. Wei et al. (2012) developed generalized hesitant fuzzy Choquet ordered averaging (GHFCOA) operator and generalized hesitant fuzzy Choquet ordered geometric (GHFCOG) operator.

Yager (2013, 2014) introduced the notion of Pythagorean fuzzy set (PFS) by generalizing the concept of intuitionistic fuzzy set, such that the sum of square of membership degree and non-membership degree is ≤ 1. Yager and Abbasov (2013) discuss the relation among Pythagorean membership degrees and complex numbers. Under the Pythagorean fuzzy set environments, Yager (2014) developed a series of aggregation operators. Zhang and Xu (2014) developed a method for order preference by similarity to a best solution to solve MCDM problem with Pythagorean fuzzy information. Peng and Yang (2015) introduced some new operations in Pythagorean fuzzy set and discussed superiority and inferiority ranking method to deal with multi-attribute group decision-making (MAGDM) problems with Pythagorean fuzzy environment. Based on Choquet integral, Peng and Yang introduced the concept of Pythagorean fuzzy Choquet integral (PFCI) operators and applied to multi-attribute decision-making problems (Peng and Yang 2016). Khan et al. (2018a, b) developed interval-valued Pythagorean fuzzy Choquet integral geometric (IVPFCIG) operator to deal with MADM problems.

Generalizing the concept of hesitant fuzzy set with intuitionistic fuzzy set, Qian et al. introduced the notion of generalized hesitant fuzzy set HFSs (Qian et al. 2013), which is an extension of the elements of HFSs from a real number to IFNs. Zhu et al. (2012) proposed the idea of dual hesitant fuzzy set (DHFS) and investigated some basic properties and operations. Peng et al. (2014) introduced a MCDM approach with hesitant interval-valued intuitionistic fuzzy sets (HIV-IFSs), which are an extension of dual IVHFSs. However, dual HFSs are defined in terms of sets of values, as opposed to precise numbers, for the membership degrees of IFSs. By using the concept of fuzzy cross-entropy in Peng et al. (2014), the authors deal with group decision-making (GDM) problems under intuitionistic hesitant fuzzy set (IHFS) environment. Generalizing the concept of IHFS, Khan et al. (2017) introduced the concept of Pythagorean hesitant fuzzy set (PHFS) and developed Pythagorean hesitant fuzzy weighted averaging (PHFWA) operator and Pythagorean hesitant fuzzy weighted geometric (PHFWG) operator for multi-attribute decision-making problem. Khan et al. (2018a, b) introduced Pythagorean hesitant fuzzy ordered weighted averaging (PHFOWA) operator and Pythagorean hesitant fuzzy ordered geometric (PHFOWG) operator for multi-attribute decision-making problems. In group decision-making problems, aggregation of decision-makers' opinions is very important to appropriately perform evaluation process. To overcome this limitation of above aggregation operator in this paper based on the Choquet integral (Choquet 1954; Denneberg 1994) with respect to fuzzy measure (Sugeno 1974), we propose two Pythagorean hesitant fuzzy Choquet integral operators, namely Pythagorean hesitant fuzzy Choquet integral averaging (PHFCIA) operator and Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator and their generalizations under Pythagorean fuzzy multi-attribute decision-making environment, where interactions phenomena among the decision-making problem are considered. In order to demonstrate this, the remainder of the paper is arranged as follows.

In Sect. 2, we briefly review some fundamental definitions and properties of PHFS. In Sect. 3, we develop Pythagorean hesitant fuzzy Choquet integral averaging (PHFCIA) operator, Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator, generalized Pythagorean hesitant fuzzy Choquet integral averaging (GPHFCIA) operator and generalized Pythagorean hesitant fuzzy Choquet integral geometric (GPHFCIG) operator. We also discuss some properties of the developed operators such as boundedness, monotonicity and idempotency. In Sect. 4, we develop a multi-attribute decision-making based on the developed operators. In Sect. 5, we give a numerical example to illustrate the effectiveness of the developed approach. Finally, we compare the proposed method with the existing methods. Section 6 concludes the paper.

2 Preliminaries

In this section, we briefly explain some basic definitions such as fuzzy measure, Choquet integral, Pythagorean hesitant fuzzy sets. Throughout the paper, we denote a fuzzy measure by ξ, Pythagorean hesitant fuzzy set by P_H and Pythagorean hesitant fuzzy number by $\hat{h}$.

2.1 Fuzzy measure and Choquet integral

The notion of fuzzy measure (nonadditive measure) introduced by Sugeno (1974) has only monotonicity instead of additivity property. The assumption that attributes are independent of one another does not need and is used as a useful tool for molding interaction phenomena in decision-making. In the Choquet integral model (Choquet 1954; Denneberg 1994), where attributes can be dependent, to define a weight on each combination of attributes a fuzzy measure is used, thus making it possible to model the interaction present among attributes. In this subsection, fuzzy measure, λ-fuzzy measure, discrete Choquet integral are defined as follows:

Definition 1 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse and $P(X)$ be the power set of X. A fuzzy measure ξ

on X is a set function $\xi : P(X) \rightarrow [0,1]$, satisfying the following conditions:

1. $\xi(\phi)=0, \xi(X)=1$.
2. If $S, T \in P(X)$ and $S \subseteq T$, then $\xi(S) \leq \xi(T)$.

Even though it is necessary to add the axiom of continuity when X is infinite, it is enough to consider a finite universal set in actual practice. $\xi(\{x_1, x_2, \ldots, x_n\})$ can be considered as the grade of subjective importance of decision attribute set $\{x_1, x_2, \ldots, x_n\}$. Thus, with the separate weights of attributes, weights of any combination of attributes can also be defined. Intuitively, we could say the following about any pair of attributes sets $S, T \in P(X), S \cap T = \phi$: S and T are considered to be without interaction (or to be independent) if

$$\xi(S \cup T) = \xi(S) + \xi(T) \tag{1}$$

which is called an additive measure. S and T exhibit a positive synergetic interaction between them (or are complementary) if

$$\xi(S \cup T) \succ \xi(S) + \xi(T) \tag{2}$$

which is called a superadditive measure. S and T exhibit a negative synergetic interaction between them (or are redundant or substitutive) if

$$\xi(S \cup T) \prec \xi(S) + \xi(T) \tag{3}$$

which is called a sub-additive measure.

Since it is difficult to determine the fuzzy measure according to Definition 1, therefore, to confirm a fuzzy measure in MAGDM problems, Sugeno (1974) presented the following λ-fuzzy measure:

$$\xi(S \cup T) = \xi(S) + \xi(T) + \lambda \xi(S) \xi(T) \tag{4}$$

$\lambda \in [-1, \infty)$, $S \cap T = \phi$. The parameter λ determines interaction between the attributes. In Eq. (4), if $\lambda = 0$, λ-fuzzy measure reduces to simply an additive measure. And for negative and positive λ, the λ-fuzzy measure reduces to sub-additive and superadditive measures, respectively. Meanwhile, if all the elements in X are independent, and we have

$$\xi(S) = \sum_{i=1}^{n} \xi(\{x_i\}). \tag{5}$$

If X is a finite set, then $\cup_{i=1}^{n} \{x_i\} = X$. The λ-fuzzy measure ξ satisfies Eq. (6).

$$\xi(X) = \xi(\cup_{i=1}^{n} \{x_i\})$$

$$= \begin{cases} \frac{1}{\lambda} \left(\prod_{i=1}^{n} [1 + \lambda \xi(x_i)] - 1 \right) & \text{if } \lambda \neq 0 \\ \sum_{i=1}^{n} \xi(x_i) & \lambda = 0 \end{cases}, \tag{6}$$

where $x_i \cap x_j = \phi$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$. It can be noted that $\xi(x_i)$ for a subset with a single element x_i is called a fuzzy density and can be denoted as $\xi_i = \xi(x_i)$.

Especially for every subset $S \in P(X)$, we have

$$\xi(S) = \begin{cases} \frac{1}{\lambda} \left(\prod_{i=1}^{n} [1 + \lambda \xi(x_i)] - 1 \right) & \text{if } \lambda \neq 0 \\ \sum_{i=1}^{n} \xi(x_i) & \text{if } \lambda = 0 \end{cases}. \tag{7}$$

Based on Eq. (7), the value λ can be uniquely determined from $\xi(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n} [1 + \lambda \xi_i]. \tag{8}$$

It should be noted that λ can be uniquely determined by $\xi(X) = 1$.

Definition 2 Let f be a positive real-valued function on X and ξ be a fuzzy measure on X. The discrete Choquet integral of f with respect to ξ is defined by

$$C_\mu(f) = \sum_{i=1}^{n} f_{\sigma(i)} [\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})], \tag{9}$$

where $\sigma(i)$ indicates a permutation on X such that $f_{\sigma(1)} \geq f_{\sigma(2)} \geq \cdots \geq f_{\sigma(n)}$, $A_{\sigma(i)} = \{1, 2, \ldots, i\}$, $A_{\sigma(0)} = \phi$.

It is seen that the discrete Choquet integral is a linear expression up to a reordering of the elements. Moreover, it identifies with the weighted mean (discrete Lebesgue integral) as soon as the fuzzy measure is additive. And in some conditions, the Choquet integral operator coincides with the OWA operator.

2.2 Pythagorean hesitant fuzzy sets

In this subsection, we define some basic definitions and properties of Pythagorean hesitant fuzzy sets (Khan et al. 2017).

Definition 3 (Khan et al. 2017) Let X be a fixed set. By a Pythagorean hesitant fuzzy set abbreviated as PHFS, we mean a structure P_H in X of the form.

$$P_H = \left\{ \left\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) \,|\, x \in X \right\rangle \right\}, \tag{10}$$

where $\Lambda_{P_H}(x)$ and $\Gamma_{P_H}(x)$ are mappings from X to $[0,1]$, denoting a possible degree of membership and non-membership degree of element $x \in X$ in P_H, respectively, and for each element $x \in X$, $\forall h_{P_H}(x) \in \Lambda_{P_H}(x)$, $\exists h'_{P_H}(x) \in \Gamma_{P_H}(x)$ such that $0 \le h^2_{P_H}(x) + h'^2_{P_H}(x) \le 1$, and $\forall\, h'_{P_H}(x) \in \Gamma_{P_H}(x)$, $\exists h_{P_H}(x) \in \Lambda_{P_H}(x)$ such that $0 \le h^2_{P_H}(x) + h'^2_{P_H}(x) \le 1$.

Moreover, PHFS(X) denotes the set of all elements of PHFSs. If X has only one element, $\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x)\rangle$ is said to be Pythagorean hesitant fuzzy number and is denoted by $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ for convenience. We denote the set of all PHFNs by PHFNS.

Definition 4 (Khan et al. 2017) Let $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$, $\hat{h}_1 = \left\langle \Lambda_{\hat{h}_1}, \Gamma_{\hat{h}_1}\right\rangle$, $\hat{h}_2 = \left\langle \Lambda_{\hat{h}_2}, \Gamma_{\hat{h}_2}\right\rangle$ are three PHFNs, and $\lambda > 0$, then their operations are defined as follows:

1. $\hat{h}_1 \cup \hat{h}_2 = \left\langle \max\left\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\right\}, \min\left\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\right\}\right\rangle$, (11)
2. $\hat{h}_1 \cap \hat{h}_2 = \left\langle \min\left\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\right\}, \max\left\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\right\}\right\rangle$, (12)
3. $\hat{h}^c = \langle \Gamma_{\hat{h}}, \Lambda_{\hat{h}}\rangle$, (13)
4. $\hat{h}_1 \oplus \hat{h}_2 = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{\sqrt{h^2_{\hat{h}_1} + h^2_{\hat{h}_2} - h^2_{\hat{h}_1} h^2_{\hat{h}_2}}\right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{h'_{\hat{h}_1} h'_{\hat{h}_2}\right\} \end{array}\right\rangle$, (14)
5. $\hat{h}_1 \otimes \hat{h}_2 = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{h_{\hat{h}_1} h_{\hat{h}_2}\right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{\sqrt{h'^2_{\hat{h}_1} + h'^2_{\hat{h}_2} - h'^2_{\hat{h}_1} h'^2_{\hat{h}_2}}\right\} \end{array}\right\rangle$, (15)
6. $\lambda \hat{h} = \left\langle \cup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{\sqrt{1 - \left(1 - (h_{\hat{h}})^2\right)^{\lambda}}\right\}, \cup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{\left(h'_{\hat{h}}\right)^{\lambda}\right\}\right\rangle$, $\lambda \succ 0$, (16)
7. $\hat{h}^{\lambda} = \left\langle \cup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{h^{\lambda}_{\hat{h}}\right\}, \cup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{\sqrt{1 - \left(1 - \left(h'_{\hat{h}}\right)^2\right)^{\lambda}}\right\}\right\rangle$, $\lambda \succ 0$. (17)

To compare the PHFNs, Khan et al. (2017) introduced score function and accuracy degree between PHFNs. The authors compare and rank among PHFNs as follows:

Definition 5 Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \beta_{\hat{h}_i}\right\rangle (i = 1, 2)$ be two PHFNs, $S\left(\hat{h}_1\right)$, $S\left(\hat{h}_2\right)$ be the score of $\hat{h}_1, \hat{h}_2$, respectively, defined by

$$S\left(\hat{h}_1\right) = \left(\frac{1}{l_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}}} \sum_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}} h_{\hat{h}_1}\right)^2 - \left(\frac{1}{l_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}}} \sum_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}} h'_{\hat{h}_1}\right)^2,$$

$$S\left(\hat{h}_2\right) = \left(\frac{1}{l_{h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}}} \sum_{h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} h_{\hat{h}_2}\right)^2 - \left(\frac{1}{l_{h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}}} \sum_{h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} h'_{\hat{h}_2}\right)^2$$

and $\overline{\sigma}\left(\hat{h}_1\right), \overline{\sigma}\left(\hat{h}_2\right)$ be the deviation degree of $\hat{h}_1, \hat{h}_2$, respectively, defined by

$$\overline{\sigma}\left(\hat{h}_1\right) = \left(\frac{1}{l_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}}} \sum_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}} h_{\hat{h}_1} - S\left(\hat{h}_1\right)\right)^2 + \left(\frac{1}{l_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}}} \sum_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}} h'_{\hat{h}_1} - S\left(\hat{h}_1\right)\right)^2$$

$$\overline{\sigma}\left(\hat{h}_2\right) = \left(\frac{1}{l_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}}} \sum_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}} h_{\hat{h}} - S\left(\hat{h}_1\right)\right)^2 + \left(\frac{1}{l_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}}} \sum_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}} h'_{\hat{h}_1} - S\left(\hat{h}_1\right)\right)^2,$$

where $l_{h_{\hat{h}_1}}, l_{h_{\hat{h}_2}}$ represent the number of elements in $\hat{h}_1, \hat{h}_2$, respectively. Then,

(a). If $S\left(\hat{h}_1\right) < S\left(\hat{h}_2\right)$, then $\hat{h}_1 < \hat{h}_2$.
(b). If $S\left(\hat{h}_1\right) > S\left(\hat{h}_2\right)$, then $\hat{h}_1 > \hat{h}_2$.
(c). If $S\left(\hat{h}_1\right) = S\left(\hat{h}_2\right)$, then $\hat{h}_1 \sim \hat{h}_2$.
(i). If $\overline{\sigma}\left(\hat{h}_1\right) < \overline{\sigma}\left(\hat{h}_2\right)$, then $\hat{h}_1 < \hat{h}_2$.
(ii). If $\overline{\sigma}\left(\hat{h}_1\right) > \overline{\sigma}\left(\hat{h}_2\right)$, then $\hat{h}_1 > \hat{h}_2$.
(ii). If $\overline{\sigma}\left(\hat{h}_1\right) = \overline{\sigma}\left(\hat{h}_2\right)$, then $\hat{h}_1 \sim \hat{h}_2$.

Based on the operation developed in Definition 4, Khan et al. (2017) introduced the following Pythagorean hesitant fuzzy weighted aggregation operators.

Definition 6 (Khan et al. 2017) Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs and $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \ge 0$ $(i = 1, 2, 3, \ldots, n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using PHFWA operator is also a PHFN and

$$\mathrm{PHFWA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \\ \left\{\sqrt{1 - \prod_{i=1}^{n} \left(1 - h^2_{\hat{h}_i}\right)^{w_i}}\right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots, h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \\ \left\{\prod_{i=1}^{n} \left(h'_{\hat{h}_i}\right)^{w_i}\right\} \end{array}\right\rangle \quad (18)$$

Definition 7 (Khan et al. 2017) Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs and $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0$ $(i = 1, 2, 3, \ldots, n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using PHFWG operator is also a PHFN, and

$$\mathrm{PHFWG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \\ \left\{ \prod_{i=1}^{n} \left(h_{\hat{h}_i}\right)^{w_i} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}, \ldots, h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h'^2_{\hat{h}_i}\right)^{w_i}} \right\} \end{array} \right\rangle. \tag{19}$$

Definition 8 (Khan et al. 2018a, b) Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs $\hat{h}_{\sigma(i)}$ be the largest in them and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0$ $(i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using PHFOWA operator is also a PHFN, and

$$PHFOWA\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}} \in \Lambda_{\hat{h}_{\sigma(n)}}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h^2_{\hat{h}_{\sigma(i)}}\right)^{w_i}} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}} \in \Gamma_{\hat{h}_{\sigma(n)}}} \\ \left\{ \prod_{i=1}^{n} \left(h'_{\hat{h}_{\sigma(i)}}\right)^{w_i} \right\} \end{array} \right\rangle. \tag{20}$$

Definition 9 (Khan et al. 2018a, b) Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i=1, 2, 3, \ldots, n)$ be a collection of all $PHFNs$, $\hat{h}_{\sigma(i)}$ be the largest in them, $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0$ $(i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using PHFOWG operator is also a PHFN, and

$$\mathrm{PHFOWG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}} \in \Lambda_{\hat{h}_{\sigma(n)}}} \\ \left\{ \prod_{i=1}^{n} \left(h_{\hat{h}_{\sigma(i)}}\right)^{w_i} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}} \in \Gamma_{\hat{h}_{\sigma(n)}}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h'^2_{\hat{h}_{\sigma(i)}}\right)^{w_i}} \right\} \end{array} \right\rangle. \tag{21}$$

3 Pythagorean hesitant fuzzy Choquet integral aggregation operators

In this section, we develop some aggregation operators for Pythagorean hesitant fuzzy numbers and investigate some of its properties.

Definition 10 Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFN's and ξ be a fuzzy measure on X. Then, Pythagorean hesitant fuzzy Choquet integral average (PHFCIA) operator of dimension n is a mapping PHFCIA : $\Omega^n \rightarrow \Omega$ such that

$$\begin{aligned} \mathrm{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) &= (\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)}))(\hat{h}_{\sigma(1)}) \\ &\oplus (\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)}))(\hat{h}_{\sigma(2)}) \oplus \cdots \oplus (\xi(A_{\sigma(n)}) \\ &- \xi(A_{\sigma(n-1)}))(\hat{h}_{\sigma(n)}), \end{aligned} \tag{22}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $\hat{h}_{\sigma(0)} = \phi$.

From the above definition, we deduce four cases.

1. If Eq. (22) satisfies, then $\xi(\{x_{\sigma(i)}\}) = \xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)}) (i = 1, 2, \ldots, n)$ which shows that Eq. (22) reduces to PHFWA operator.
2. If $\xi(A) = \sum_{i=1}^{|A|} w_i$, for all $A \in X$ where $|A|$ is the number of elements in A, $w_i = \xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)}) (i = 1, 2, \ldots, n)$ where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, with $\sum_{i=1}^{n} w_i = 1$. This shows that Eq. (22) reduces to PHFOWA operator.
3. If $\xi(A) = 1$, for all $A \in X$, then $\mathrm{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \max(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \hat{h}_{\sigma(1)}$.
4. If $\xi(A) = 0$, for all $A \in X$, then $\mathrm{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \min(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \hat{h}_{\sigma(n)}$.

Theorem 1 *Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs. Then, the aggregation result using PHFCIA operator is also a PHFN and*

$$\mathrm{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h^2_{\hat{h}_i}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \\ \left\{ \prod_{i=1}^{n} \left(h'_{\hat{h}_i}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle \tag{23}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ *for* $k \geq 1$ *and* $\hat{h}_{\sigma(0)} = \phi$.

$$\oplus \left\langle \begin{array}{l} \cup_{h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{ \sqrt{1 - \left(1 - \left(h_{\hat{h}_{\sigma(2)}}\right)^2\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})}} \right\}, \\ \cup_{h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ \left(h'_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \\ \left\{ \sqrt{\begin{array}{l} 1 - \left(1 - h^2_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} + 1 - \left(1 - h^2_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \\ - \left(\left(1 - h^2_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \left(1 - h^2_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})}\right) \end{array}} \right\}, \\ \cup_{h'_{\hat{h}_1 s} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ \left(h'_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \left(h'_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{ \sqrt{1 - \prod_{i=1}^{2} \left(1 - h_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ \prod_{i=1}^{2} \left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle.$$

Proof First part of the theorem directly follows from Definition 10. Next, by mathematical induction we prove that Eq. (23) holds for all n. For this, first we show that Eq. (23) holds for $n = 2$. Since

$$\left(\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})\right) \hat{h}_1 = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}} \left\{ \sqrt{1 - \left(1 - \left(h_{\hat{h}_{\sigma(1)}}\right)^2\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}} \left\{ \left(h'_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \right\} \end{array} \right\rangle$$

and

$$\left(\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})\right) \hat{h}_2 = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{ \sqrt{1 - \left(1 - \left(h_{\hat{h}_{\sigma(2)}}\right)^2\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})}} \right\}, \\ \cup_{h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ \left(h'_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \right\} \end{array} \right\rangle$$

so

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2\right) = \left(\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})\right) \hat{h}_1 \oplus \left(\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})\right) \hat{h}_2$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}} \left\{ \sqrt{1 - \left(1 - \left(h_{\hat{h}_{\sigma(1)}}\right)^2\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}} \left\{ \left(h'_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \right\} \end{array} \right\rangle$$

Thus, Eq. (23) is true for $n = 2$. Assume that Eq. (23) holds for $n = k$. i.e.,

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_k\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_k} \in \Lambda_{\hat{h}_k}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_k} \in \Gamma_{\hat{h}_k}} \\ \left\{ \prod_{i=1}^{k} \left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle.$$

We show that Eq. (23) holds for $n = k + 1$. i.e.,

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{k+1}\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_k} \in \Lambda_{\hat{h}_k}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - h^2_{\hat{h}_i}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_k} \in \Gamma_{\hat{h}_k}} \\ \left\{ \prod_{i=1}^{k} \left(h'_{\hat{h}_i}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle$$

$$\oplus \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{k+1}} \in \Lambda_{\hat{h}_{k+1}}} \left\{ \sqrt{1 - \left(1 - h^2_{\hat{h}_{\sigma(k+1)}}\right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})}} \right\}, \\ h'_{\hat{h}_{k+1}} \in \Gamma_{\hat{h}_{k+1}} \left\{ \left(h'_{\hat{h}_{\sigma(k+1)}}\right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} \right\} \end{array} \right\rangle$$

$$
= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2},\dots,h_{\hat{h}_k}\in\Lambda_{\hat{h}_k}}, \cup_{h_{\hat{h}_{k+1}}\in\Lambda_{\hat{h}_{k+1}}} \\ \left\{ \left[\sqrt{\begin{array}{l} 1-\prod_{i=1}^{k}\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} + 1-\left(1-h^2_{\hat{h}_{\sigma(k+1)}}\right)^{\xi(A_{\sigma(k+1)})-\xi(A_{\sigma(k)})} \\ -\left(1-\prod_{i=1}^{k}\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right)\left(1-\left(1-h^2_{\hat{h}_{\sigma(k+1)}}\right)^{\xi(A_{\sigma(k+1)})-\xi(A_{\sigma(k)})}\right) \end{array}} \right] \right\}, \\ \cup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2},\dots,h'_{\hat{h}_k}\in\Gamma_{\hat{h}_k}}, \cup_{h'_{\hat{h}_{k+1}}\in\Gamma_{\hat{h}_{k+1}}} \\ \left\{ \prod_{i=1}^{k}\left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\} \left\{ \left(h'_{\hat{h}_{\sigma(k+1)}}\right)^{\xi(A_{\sigma(k+1)})-\xi(A_{\sigma(k)})} \right\} \end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2},\dots,h_{\hat{h}_{k+1}}\in\Lambda_{\hat{h}_{k+1}}} \left\{ \sqrt{1-\prod_{i=1}^{k+1}\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2},\dots h'_{\hat{h}_{k+1}}\in\Gamma_{\hat{h}_{k+1}}} \left\{ \prod_{i=1}^{k+1}\left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle .
$$

Thus, Eq. (23) is true for $n = k + 1$. Hence, the result holds for all n. □

In the following, we present some properties of the PHFCIA operator.

Theorem 2 *Let* $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \dots, n)$ *be a collection of all* $PHFN's$*, and* $\{\sigma(1), \sigma(2), \dots, \sigma(n)\}$ *is a permutation of* $(1, 2, \dots, n)$ *such that* $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \dots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ *for* $k \geq 1$ *and* $\hat{h}_{\sigma(0)} = \phi$.

(1) *(Idempotency) If all* $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \dots, n)$ *are equal, i.e.,* $\hat{h}_i\ (i = 1, 2, 3, \dots, n) = \hat{h}$*, then*

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n\right) = \hat{h}. \tag{24}$$

(2) *(Boundedness)*

$$\hat{h}^- \leq \text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n\right) \leq \hat{h}^+ \tag{25}$$

where $\hat{h}^- = \left\langle h^-, h^{+'} \right\rangle$, $\hat{h}^+ = \left\langle h^+, h^{-'} \right\rangle$, $h^- = \cup_{h_i \in \Lambda_{\hat{h}_i}} \min_i \{h_i\}$,

$$h^+ = \cup_{h_i\in\Lambda_{\hat{h}_i}} \max_i \{h_i\}, h^{-'} = \cup_{h'_i\in\Gamma_{\hat{h}_i}} \min_i \left\{h'_i\right\}, h^{+'} = \cup_{h'_i\in\Gamma_{\hat{h}_i}} \max_i \left\{h'_i\right\}.$$

(3) *(Monotonicity) If* $\hat{h}_i > \hat{h}^*_i$*, then*

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n\right) \leq \text{PHFCIA}\left(\hat{h}^*_1, \hat{h}^*_2, \dots, \hat{h}^*_n\right) \tag{26}$$

Proof (1) From Theorem 1, we have

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n\right)$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(i)}}\in\Lambda_{\hat{h}_{\sigma(i)}}} \left\{ \sqrt{1-\prod_{i=1}^{n}\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(i)}}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{ \prod_{i=1}^{n}\left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(i)}}\in\Lambda_{\hat{h}_{\sigma(i)}}} \left\{ \sqrt{1-\prod_{i=1}^{n}\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(i)}}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{ \prod_{i=1}^{n}\left(h'_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(i)}}\in\Lambda_{\hat{h}_{\sigma(i)}}} \left\{ \sqrt{1-\left(1-h^2_{\hat{h}_{\sigma(i)}}\right)^{\sum_{i=1}^{n}\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(i)}}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{ \left(h'_{\hat{h}_{\sigma(i)}}\right)^{\sum_{i=1}^{n}\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\} \end{array} \right\rangle .$$

Since $\sum_{i=1}^{n} \xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)}) = 1$, we have

$$\text{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_n\right)$$

$$= \left\langle \cup_{h_{\hat{h}}\in\Lambda_{\hat{h}}} \left\{ \sqrt{1-\left(1-h^2_{\hat{h}}\right)} \right\}, \cup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} \left\{h'_{\hat{h}}\right\} \right\rangle$$

$$= \left\langle \cup_{h_{\hat{h}}\in\Lambda_{\hat{h}}} \left\{h_{\hat{h}}\right\}, \cup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} \left\{h'_{\hat{h}}\right\} \right\rangle = \hat{h}.$$

(2) Since

$$\cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \min_i \left\{h_{\sigma(i)}\right\} \leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \left\{h_{\sigma(i)}\right\} \leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \max_i \left\{h_{\sigma(i)}\right\} \tag{27}$$

and

$$\cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \min_i \left\{h'_{\sigma(i)}\right\} \leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{h'_{\sigma(i)}\right\} \leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \max_i \left\{h'_{\sigma(i)}\right\}, \tag{28}$$

from Eq. (27) we have

$$\begin{aligned}
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\min_i \left\{h^2_{\sigma(i)}\right\}} \leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\sum_{i=1}^{n}\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\sum_{i=1}^{n}\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{-1+\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{-\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{-1+\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-1+\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-1+\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\min_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\max_i \left\{\left(h_{\sigma(i)}\right)^2\right\}}\\
&\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \min_i \left\{h_{\sigma(i)}\right\}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\\
&\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \max_i \left\{h_{\sigma(i)}\right\}.
\end{aligned}$$

Now, from Eq. (28) we have

$$\begin{aligned}
&\cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \min_i \left\{h'_{\sigma(i)}\right\} \leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{h'_{\sigma(i)}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \max_i \left\{h'_{\sigma(i)}\right\}\\
&\Leftrightarrow \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \min_i \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \max_i \left\{\left(h'_{i}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\Leftrightarrow \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \prod_{i=1}^{n}\min_i \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \prod_{i=1}^{n} \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \prod_{i=1}^{n}\max_i \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\Leftrightarrow \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \min_i \left\{\left(h'_{\sigma(i)}\right)^{\sum_{i=1}^{n}\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}\\
&\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \prod_{i=1}^{n} \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}
\end{aligned}$$

$$\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \max_i \left\{ \left(h'_{\sigma(i)}\right)^{\sum_{i=1}^{n} \xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\}$$

$$\Leftrightarrow \cup_{h'_i\in\Gamma_{\hat{h}_i}} \min_i \left\{\left(h'_i\right)\right\}$$

$$\leq \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \prod_{i=1}^{n} \left\{ \left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})} \right\}$$

$$\leq \cup_{h'_i\in\Gamma_{\hat{h}_i}} \max_i \left\{\left(h'_i\right)\right\}.$$

According to the score function, we have PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \geq \hat{h}^-$ with equality if and only if the $\hat{h}^- = PHFCIA(\hat{h})$.

Similarly, PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \leq \hat{h}^+$, with equality if and only if the PHFCIA$(\hat{h})$ is the same as $\hat{h}^+$. Hence, $\hat{h}^- \leq$ PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \leq \hat{h}^+$.

(3) If $\hat{h}_i > \hat{h}_i^*$, then PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \leq$ PHFCIA$\left(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*\right)$. Since $A_{\sigma(i)} \subseteq A_{\sigma(i-1)}$, $\lambda(A_{\sigma(i)}) - \lambda(A_{\sigma(i-1)}) \geq 0$. For all i, $\Lambda_{\hat{h}^*_{\sigma(i)}} \geq \Lambda_{\hat{h}_{\sigma(i)}}$, $\Gamma_{\hat{h}_{\sigma(i)}} \geq \Gamma_{\hat{h}^*_{\sigma(i)}}$. If $\Lambda_{\hat{h}_{\sigma(i)}} \leq \Lambda_{\hat{h}^*_{\sigma(i)}}$, then $\cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}}\left\{h_{\sigma(i)}\right\} \leq \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}}\left\{h^*_{\sigma(i)}\right\} \Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}}\left\{\left(h_{\sigma(i)}\right)^2\right\} \leq \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}}\left\{\left(h^*_{\sigma(i)}\right)^2\right\}$

$$\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left\{\left(h_{\sigma(i)}\right)^2\right\}} \leq \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}} \sqrt{\left\{\left(h^*_{\sigma(i)}\right)^2\right\}}$$

$$\Leftrightarrow \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}} \sqrt{1-\left\{\left(h^*_{\sigma(i)}\right)^2\right\}}$$

$$\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\left\{\left(h_{\sigma(i)}\right)^2\right\}}$$

$$\Leftrightarrow \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}} \sqrt{\left(1-\left\{\left(h^*_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}$$

$$\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}$$

$$\Leftrightarrow \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h^*_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}$$

$$\leq \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}$$

$$\Leftrightarrow \cup_{h_{\sigma(i)}\in\Lambda_{\hat{h}_{\sigma(i)}}} \sqrt{1-\prod_{i=1}^{n}\left(1-\left\{\left(h_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}$$

$$\leq \cup_{h^*_{\sigma(i)}\in\Lambda_{\hat{h}^*_{\sigma(i)}}} \sqrt{\prod_{i=1}^{n}\left(1-\left\{\left(h^*_{\sigma(i)}\right)^2\right\}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}. \tag{29}$$

Now, if $\Gamma_{\hat{h}_{\sigma(i)}} \geq \Gamma_{\hat{h}^*_{\sigma(i)}}$, then $\cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}}\left\{h'_{\sigma(i)}\right\} \geq \cup_{h^{*'}_{\sigma(i)}\in\Gamma_{\hat{h}^*_{\sigma(i)}}}\left\{h^{*'}_{\sigma(i)}\right\}$

$$\Leftrightarrow \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}$$

$$\geq \cup_{h^{*'}_{\sigma(i)}\in\Gamma_{\hat{h}^*_{\sigma(i)}}} \left\{\left(h^{*'}_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}$$

$$\Leftrightarrow \cup_{h'_{\sigma(i)}\in\Gamma_{\hat{h}_{\sigma(i)}}} \left\{\prod_{i=1}^{n}\left(h'_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}$$

$$\geq \cup_{h^{*'}_{\sigma(i)}\in\Gamma_{\hat{h}^*_{\sigma(i)}}} \left\{\prod_{i=1}^{n}\left(h^{*'}_{\sigma(i)}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}. \tag{30}$$

Let $\hat{h}$ = PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right)$ and $\hat{h}^*$ = PHFCIA$\left(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*\right)$. Then, from Eqs. (29) and (30) we have $S(\hat{h}) \leq S(\hat{h}^*)$.

If $S(\hat{h}) < S(\hat{h}^*)$, then PHFCIA$\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) <$ PHFCIA$\left(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*\right)$. If $S(\hat{h}) = S(\hat{h}^*)$, then

$$\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}}\right)^2 - \left(\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}}\right)^2$$

$$= \left(\frac{1}{l_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}}}\sum_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}} h^*_{\hat{h}^*}\right)^2 - \left(\frac{1}{l_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}}}\sum_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}} h^{*'}_{\hat{h}^*}\right)^2$$

$$\Leftrightarrow \left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}}\right)^2 = \left(\frac{1}{l_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}}}\sum_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}} h^*_{\hat{h}^*}\right)^2$$

and $\left(\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}}\right)^2 = \left(\frac{1}{l_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}}}\sum_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}} h^{*'}_{\hat{h}^*}\right)^2$

$$\Leftrightarrow \frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}} = \frac{1}{l_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}}}\sum_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}} h^*_{\hat{h}^*}$$

and $\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}} = \frac{1}{l_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}}}\sum_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}} h^{*'}_{\hat{h}^*}$.

Since

$$\overline{\sigma}\left(\hat{h}\right) = \left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}} - S\left(\hat{h}\right)\right)^2$$

$$+\left(\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}} - s\left(\hat{h}\right)\right)^2$$
$$=\left(\frac{1}{l_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}}}\sum_{h^*_{\hat{h}^*}\in\Lambda_{\hat{h}^*}} h^*_{\hat{h}^*} - s\left(\hat{h}^*\right)\right)^2$$
$$+\left(\frac{1}{l_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}}}\sum_{h^{*'}_{\hat{h}^*}\in\Gamma_{\hat{h}^*}} h^{*'}_{\hat{h}^*} - s\left(\hat{h}^*\right)\right)^2$$
$$=\overline{\sigma}\left(\hat{h}^*\right),$$

Therefore, $\mathrm{PHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \mathrm{PHFCIA}\left(\hat{h}^*_1, \hat{h}^*_2, \ldots, \hat{h}^*_n\right)$. □

Definition 11 Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFN's and λ be a fuzzy measure on X. Then, Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator of dimension n is a mapping $\mathrm{PHFCIG} : \Omega^n \to \Omega$ such that

$$\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = (\hat{h}_{\sigma(1)})^{(\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)}))} \otimes (\hat{h}_{\sigma(2)})^{(\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)}))} \otimes \cdots \otimes (\hat{h}_{\sigma(n)})^{(\xi(A_{\sigma(n)})-\xi(A_{\sigma(n-1)}))} \quad (31)$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $\hat{h}_{\sigma(0)} = \phi$.

From the above definition, we deduce four cases.

1. If Eq. (31) satisfies, then $\xi(\{x_{\sigma(i)}\}) = \xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$ which shows that Eq. (31) reduces to PHFWA operator.
2. If $\xi(A) = \sum_{i=1}^{|A|} w_i$, for all $A \in X$ where $|A|$ is the number of elements in A, $w_i = \xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$ where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, with $\sum_{i=1}^{n} w_i = 1$. This shows that Eq. (31) reduces to PHFOWG operator.
3. If $\xi(A) = 1$, for all $A \in X$, then $\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \max(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \hat{h}_{\sigma(1)}$.
4. If $\xi(A) = 0$, for all $A \in X$, then $\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \min(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \hat{h}_{\sigma(n)}$.

Theorem 3 *Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs. Then, the aggregation result using PHFCIG operator is also a PHFN and*

$$\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}}\in\Lambda_{\hat{h}_{\sigma(n)}}} \left\{\prod_{i=1}^{n}\left(h_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}\right\}, \\ \cup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_1}\in\Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_n}\in\Gamma_{\hat{h}_n}} \left\{\sqrt{1-\prod_{i=1}^{n}\left(1-h'^2_{\hat{h}_{\sigma(i)}}\right)^{\xi(A_{\sigma(i)})-\xi(A_{\sigma(i-1)})}}\right\} \end{array}\right\rangle \quad (32)$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $\hat{h}_{\sigma(0)} = \phi$.

Proof First part of the theorem directly follows from Definition 11. Next, by mathematical induction we prove that Eq. (32) holds for all n. For this, first we show that Eq. (32) holds for $n = 2$. Since

$$\hat{h}_{\sigma(1)}^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})} = \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}} \left\{\left(h_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})}\right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}}} \left\{\sqrt{1-\left(1-h'^2_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})}}\right\} \end{array}\right\rangle$$

and

$$\hat{h}_{\sigma(2)}^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})} = \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}} \left\{\left(h_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})}\right\}, \\ \cup_{h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}}} \left\{\sqrt{1-\left(1-h'^2_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})}}\right\} \end{array}\right\rangle,$$

$$\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2\right) = \hat{h}_{\sigma(1)}^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})} \otimes \hat{h}_{\sigma(2)}^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})}$$
$$= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}} \left\{\left(h_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})}\right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}}} \left\{\sqrt{1-\left(1-h'^2_{\hat{h}_{\sigma(1)}}\right)^{\xi(A_{\sigma(1)})-\xi(A_{\sigma(0)})}}\right\} \end{array}\right\rangle \otimes \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}} \left\{\left(h_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})}\right\}, \\ \cup_{h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}}} \left\{\sqrt{1-\left(1-h'^2_{\hat{h}_{\sigma(2)}}\right)^{\xi(A_{\sigma(2)})-\xi(A_{\sigma(1)})}}\right\} \end{array}\right\rangle$$

$$
= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}} \\ \left\{ \left(h_{\hat{h}_{\sigma(1)}} \right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \left(h_{\hat{h}_{\sigma(2)}} \right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}} \\ \left\{ \sqrt{ \begin{array}{c} 1 - \left(1 - h'^2_{\hat{h}_{\sigma(1)}} \right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} + 1 \\ - \left(1 - h'^2_{\hat{h}_{\sigma(2)}} \right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \\ - \left(\left(1 - h'^2_{\hat{h}_{\sigma(1)}} \right)^{\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)})} \left(1 - h'^2_{\hat{h}_{\sigma(2)}} \right)^{\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)})} \right) \end{array} } \right\} \end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}} \\ \left\{ \prod_{i=1}^{2} \left(h_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}} \\ \left\{ \sqrt{ 1 - \prod_{i=1}^{2} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} } \right\} \end{array} \right\rangle.
$$

Thus, Eq. (32) is true for $n = 2$. Assume that Eq. (32) holds for $n = k$, i.e.,

$$
\text{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_k \right) = \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(k)}} \in \Lambda_{\hat{h}_{\sigma(k)}}} \\ \left\{ \prod_{i=1}^{k} \left(h_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(k)}} \in \Gamma_{\hat{h}_{\sigma(k)}}} \\ \left\{ \sqrt{ 1 - \prod_{i=1}^{k} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} } \right\} \end{array} \right\rangle.
$$

We show that Eq. (8) holds for $n = k + 1$.

$$
\text{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{k+1} \right) = \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(k)}} \in \Lambda_{\hat{h}_{\sigma(k)}}} \\ \left\{ \prod_{i=1}^{k} \left(h_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(k)}} \in \Gamma_{\hat{h}_{\sigma(k)}}} \\ \left\{ \sqrt{ 1 - \prod_{i=1}^{k} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} } \right\} \end{array} \right\rangle \otimes \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(k+1)}} \in \Lambda_{\hat{h}_{\sigma(k+1)}}} \left\{ \left(h_{\hat{h}_{\sigma(k+1)}} \right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(k+1)}} \in \Gamma_{\hat{h}_{\sigma(k+1)}}} \left\{ \sqrt{ 1 - \left(1 - h'^2_{\hat{h}_{\sigma(k+1)}} \right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} } \right\} \end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(k)}} \in \Lambda_{\hat{h}_{\sigma(k)}}}, \cup_{h_{\hat{h}_{\sigma(k+1)}} \in \Lambda_{\hat{h}_{\sigma(k+1)}}} \\ \left\{ \prod_{i=1}^{k} \left(h_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \left(h_{\hat{h}_{\sigma(k+1)}} \right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(k)}} \in \Gamma_{\hat{h}_{\sigma(k)}}}, \cup_{h'_{\hat{h}_{k+1}} \in \Gamma_{\hat{h}_{k+1}}} \\ \left\{ \sqrt{ \begin{array}{c} 1 - \prod_{i=1}^{k} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} + 1 \\ - \left(1 - h'^2_{\hat{h}_{\sigma(k+1)}} \right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} - \\ \left(\begin{array}{c} \sqrt{ 1 - \prod_{i=1}^{k} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} } \\ \left(1 - \left(1 - h'^2_{\hat{h}_{\sigma(k+1)}} \right)^{\xi(A_{\sigma(k+1)}) - \xi(A_{\sigma(k)})} \right) \end{array} \right) \end{array} } \right\} \end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(k+1)}} \in \Lambda_{\hat{h}_{\sigma(k+1)}}} \\ \left\{ \prod_{i=1}^{k+1} \left(h_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(k+1)}} \in \Gamma_{\hat{h}_{\sigma(k+1)}}} \\ \left\{ \sqrt{ 1 - \prod_{i=1}^{k+1} \left(1 - h'^2_{\hat{h}_{\sigma(i)}} \right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})} } \right\} \end{array} \right\rangle.
$$

Thus, Eq. (32) is true for $n = k + 1$. Hence, the result holds for all n. □

Theorem 4 *Let* $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right\rangle (i = 1, 2, 3, \ldots, n)$ *be a collection of all PHFN's and* $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$ *such that* $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ *for* $k \geq 1$ *and* $\hat{h}_{\sigma(0)} = \phi$. *Then,*

1) *(Idempotency) If all* $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right)$ $(i = 1, 2, 3, \ldots, n)$ *are equal i.e.,* $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n) = \hat{h}$, *then*

$$PHFCIG\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) = \hat{h}. \tag{33}$$

2) *(Boundedness)*

$$\hat{h}^- \leq \mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \leq \hat{h}^+. \tag{34}$$

where $\hat{h}^- = \left\langle h^-, h^{+'}\right\rangle$, $\hat{h}^+ = \left\langle h^+, h^{-'}\right\rangle$, $h^- = \cup_{h_i \in \Lambda_{\hat{h}_i}} \min_i \{h_i\}$,

$$h^+ = \cup_{h_i \in \Lambda_{\hat{h}_i}} \max_i \{h_i\}, h^{-'} = \cup_{h_i' \in \Gamma_{\hat{h}_i}} \min_i \left\{h_i'\right\}, h^{+'} = \cup_{h_i' \in \Gamma_{\hat{h}_i}} \max_i \left\{h_i'\right\}.$$

3) *(Monotonicity) If* $\hat{h}_i > \hat{h}_i^*$, *then*

$$\mathrm{PHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) \leq \mathrm{PHFCIG}\left(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*\right). \tag{35}$$

Proof Proof of the theorem follows from Theorem 2. □

Definition 12 Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$ and λ be a fuzzy measure on X. Then, Pythagorean hesitant fuzzy Choquet integral average (PHFCIA) operator of dimension n is a mapping PHFCIA : $\Omega^n \rightarrow \Omega$ such that

$$\begin{aligned}\mathrm{GPHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) &= ((\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)}))(\hat{h}_{\sigma(1)})^{\delta} \\ &\oplus(\xi(A_{\sigma(2)}) - \xi(A_{\sigma(1)}))(\hat{h}_{\sigma(2)})^{\delta} \oplus \cdots \oplus (\xi(A_{\sigma(n)}) \\ &-\xi(A_{\sigma(n-1)}))(\hat{h}_{\sigma(n)})^{\delta})^{\frac{1}{\delta}}\end{aligned} \tag{36}$$

where $\xi > 0$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $\hat{h}_{\sigma(0)} = \phi$.

Theorem 5 *Let* $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ *be a collection of all PHFNs. Then, the aggregation result using GPHFCIA operator is also a PHFN and*

$$\mathrm{GPHFCIA}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right)$$

$$= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \\ \left\{\left(\sqrt{1 - \prod_{i=1}^{n}\left(1 - h_{\hat{h}_i}^{\delta}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}}\right)^{\frac{1}{\delta}}\right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \\ \left\{\sqrt{1 - \left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - h'^{2}_{\hat{h}_i}\right)^{\delta}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}\right)^{\frac{1}{\delta}}}\right\} \end{array} \right\rangle \tag{37}$$

where $\delta > 0$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ *for* $k \geq 1$ *and* $\hat{h}_{\sigma(0)} = \phi$.

Proof Easy to prove. □

Definition 13 Let $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$ and λ be a fuzzy measure on X. Then, Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator of dimension n is a mapping PHFCIG : $\Omega^n \rightarrow \Omega$ such that

$$\begin{aligned}\mathrm{GPHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right) &= \frac{1}{\delta}(\delta(\hat{h}_{\sigma(1)})^{(\xi(A_{\sigma(1)}) - \xi(A_{\sigma(0)}))} \\ &\otimes\delta(\hat{h}_{\sigma(2)})^{(\xi(A_{\sigma(2)}) - \lambda(A_{\sigma(1)}))} \\ &\otimes \cdots \otimes \delta(\hat{h}_{\sigma(n)})^{(\xi(A_{\sigma(n)}) - \xi(A_{\sigma(n-1)}))})\end{aligned} \tag{38}$$

where $\delta > 0$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $\hat{h}_{\sigma(0)} = \phi$.

Theorem 6 *Let* $\hat{h}_i = \left\langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right\rangle (i = 1, 2, 3, \ldots, n)$ *be a collection of all PHFNs. Then, the aggregation result using* $GPHFCIG$ *operator is also a PHFN and*

$$\mathrm{GPHFCIG}\left(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\right)$$

$$= \left\langle \begin{array}{c} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \\ \left\{\sqrt{1 - \left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - h^{2}_{\hat{h}_i}\right)^{\delta}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}\right)^{\frac{1}{\delta}}}\right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \\ \left\{\left(\sqrt{1 - \prod_{i=1}^{n}\left(1 - h'^{\delta}_{\hat{h}_i}\right)^{\xi(A_{\sigma(i)}) - \xi(A_{\sigma(i-1)})}}\right)^{\frac{1}{\delta}}\right\} \end{array} \right\rangle \tag{39}$$

where $\delta > 0$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\hat{h}_{\sigma(1)} \geq \hat{h}_{\sigma(2)} \geq \cdots \geq \hat{h}_{\sigma(n)}$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ *for* $k \geq 1$ *and* $\hat{h}_{\sigma(0)} = \phi$.

Proof Easy to prove. □

4 Decision-making based on Pythagorean hesitant fuzzy information

In this section, we apply the Pythagorean hesitant fuzzy aggregation operators to multi-attribute decision-making with anonymity. Suppose that there are n alternatives $X = \{x_{1,}x_2, \ldots, x_n\}$ and m attributes $A = \{A_{1,}A_2, \ldots, A_m\}$ to be evaluated having weight vector $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$ such that $w_j \in [0, 1],\ j = 1, 2, \ldots, m$ and $\sum_{j=1}^{m} w_j = 1$. To evaluate the performance of the alternative X_i under the attributes A_j, the decision-maker is required to provide not only the information that the alternative X_i satisfies the attributes A_j, but also the information that the alternative X_i does not satisfy the attributes A_j. These two part information can be expressed by Λ_{ij} and Γ_{ij} which denote the degrees that the alternative X_i satisfies the criterion A_j and does not satisfy the criterion A_j; then, the performance of the alternative X_i under the criteria A_j can be expressed by an PHFN $\hat{h}_{ij} = \langle \Lambda_{ij}, \Gamma_{ij} \rangle$ with the condition that for all $h_{ij} \in \Lambda_{ij}$, $\exists\ h'_{ij} \in \Gamma_{ij}$ such that $0 \le (h_{ij})^2 + (h'_{ij})^2 \le 1$, and for all $h_{ij} \in \Gamma_{ij}$, $\exists\ h'_{ij} \in \Lambda_{ij}$ such that $0 \le (h_{ij})^2 + (h'_{ij})^2 \le 1$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots .t$.

To obtain the ranking of the alternatives, the following steps are given:

Step 1. In this step, we construct the Pythagorean hesitant fuzzy decision matrices $C = (\hat{h}_{ij})_{m \times n}$ for decision where $\hat{h}_{ij} = \langle \Lambda_{ij}, \Gamma_{ij} \rangle\ (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$.

If the attribute has two types, such as cost and benefit attributes, then the Pythagorean hesitant decision matrix can be converted into the normalized Pythagorean hesitant fuzzy decision matrix.

$$D_N = (\gamma_{ij})_{m \times n},\ \text{where } \gamma_{ij} = \begin{cases} \hat{h}_{ij} & \text{if the attribute is of benefit type} \\ \hat{h}_{ij}^c & \text{if the attribute is of cost type} \end{cases}$$

where $\hat{h}_{ij}^c = \langle \Lambda_{ij}, \Gamma_{ij} \rangle\ (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$. If all the attributes have the same type, then there is no need to normalize the decision matrix.

Step 2. Confirm fuzzy density ξ of each attribute. According to Eq. (8), parameter λ of attributes can be determine.

Step 3. Utilize the developed aggregation operators to obtain the PHFN $\hat{h}_i\ (i = 1, 2, \ldots, n)$ for the alternatives X_i, that is, the developed operators to derive the collective overall preference values $\hat{h}_i\ (i = 1, 2, \ldots, n)$ of the alternative X_i.

Step 4. By using Definition 5, we calculate the scores $S(\hat{h}_i)\ (i = 1, 2, \ldots, n)$ and the deviation degree $\overline{\sigma}(\hat{h}_i)\ (i = 1, 2, \ldots, n)$ of all the overall values $\hat{h}_i\ (i = 1, 2, \ldots, n)$.

Step 5. Rank the alternatives $X_i\ (i = 1, 2, \ldots, n)$ and then select the best one.

5 Numerical example

In order to illustrate the developed method in this section, we present a numerical example to show the potential evaluation of emerging technology commercialization with Pythagorean hesitant fuzzy information. Suppose the decision-makers select four possible attributes to evaluate the emerging technology enterprises:

A_1 is the technical advancement;
A_2 is the potential market and market risk;
A_3 is the industrialization infrastructure, human resources and financial conditions;
A_4 is the employment creation and the development of science and technology.

The five possible emerging technology enterprises $X_i (i = 1, 2, 3, 4, 5)$ are to be evaluated using the Pythagorean hesitant fuzzy numbers by the decision-maker under the above four attributes and the decision matrices $C = [\hat{h}_{ij}]_{5 \times 4}$ as follows (Table 1):

Step 1. We construct the Pythagorean hesitant fuzzy decision matrix as shown in Table 1.

Step 2. We determine the fuzzy density ξ of each attribute and its λ parameter. Suppose that $\xi(A_1) = 0.25$, $\xi(A_2) = 0.35$, $\xi(A_3) = 0.3$, $\xi(A_4) = 0.4$, then $\lambda = -0.54$.

Now, $\xi(A_1, A_2) = 0.55$, $\xi(A_1, A_3) = 0.67$, $\xi(A_1, A_4) = 0.59$, $\xi(A_2, A_3) = 0.60$, $\xi(A_2, A_4) = 0.51$, $\xi(A_3, A_4) = 0.64$, $\xi(A_1, A_2, A_3) = 0.83$, $\xi(A_1, A_2, A_4) = 0.76$, $\xi(A_1, A_3, A_4) = 0.87$, $\xi(A_2, A_3, A_4) = 0.80$, $\xi(A_1, A_2, A_3, A_4) = 1$.

Step 3. Utilized Pythagorean hesitant fuzzy Choquet integral averaging (PHFCIA) operator to derive the collective overall preference values $\hat{h}_i\ (i = 1, 2, \ldots, n)$ of the alternative X_i. We get

$\hat{h}_1 = \langle$\{ 0.5620, 0.5692, 0.5895, 0.5816, 0.5875, 0.6065, 0.6058, 0.6112, 0.6287, 0.6064, 0.6117, 0.6293, 0.6224, 0.6275, 0.6441, 0.6435, 0.6482, 0.6635, 0.6115, 0.6167, 0.6340, 0.6273, 0.6322, 0.6485, 0.6479, 0.6525, 0.6676, 0.6484, 0.6530, 0.6681, 0.6622, 0.6666, 0.6809, 0.6804, 0.6844, 0.6978 \},\{ 0.5428, 0.5523, 0.5635, 0.5734, 0.5775, 0.5877, 0.5995, 0.6100, 0.6086, 0.6193, 0.6318, 0.6429, 0.5740, 0.5840, 0.5958, 0.6063, 0.61067, 0.6214, 0.6340, 0.6450, 0.6435, 0.6548, 0.6680, 0.6797, 0.6007, 0.6113, 0.6236, 0.6345, 0.6392, 0.6503, 0.6635, 0.6751, 0.6735, 0.6853, 0.6992, 0.7114 \}$\rangle$

$\hat{h}_2 = \langle$\{ 0.6313, 0.6608, 0.6830, 0.6392, 0.6678, 0.6895, 0.6522, 0.6794, 0.7001, 0.6595, 0.7071, 0.7062, 0.6812, 0.7056, 0.7242, 0.6877, 0.7115, 0.7297, 0.7205, 0.7413,

Table 1 Pythagorean hesitant fuzzy decision matrix $C = [\hat{h}_{ij}]_{5\times 4}$

	A_1	A_2	A_3	A_4
X_1	$\langle\{0.7, 0.8\}, \{0.4, 0.5, 0.6\}\rangle$	$\langle\{0.4, 0.5, 0.6\}, \{0.7, 0.8\}\rangle$	$\langle\{0.3, 0.4, 0.6\}, \{0.7, 0.8\}\rangle$	$\langle\{0.6, 0.7\}, \{0.5, 0.6, 0.7\}\rangle$
X_2	$\langle\{0.6, 0.7, 0.8\}, \{0.5, 0.6\}\rangle$	$\langle\{0.8, 0.9\}, \{0.2, 0.3, 0.4\}\rangle$	$\langle\{0.4, 0.6, 0.7\}, \{0.5, 0.7\}\rangle$	$\langle\{0.3, 0.4\}, \{0.7, 0.8, 0.9\}\rangle$
X_3	$\langle\{0.3, 0.4\}, \{0.7, 0.8, 0.9\}\rangle$	$\langle\{0.5, 0.6, 0.7\}, \{0.6, 0.7\}\rangle$	$\langle\{0.5, 0.6\}, \{0.6, 0.7, 0.8\}\rangle$	$\langle\{0.6, 0.7, 0.8\}, \{0.5, 0.6\}\rangle$
X_4	$\langle\{0.5, 0.6, 0.7\}, \{0.7, 0.8\}\rangle$	$\langle\{0.7, 0.8\}, \{0.4, 0.5, 0.6\}\rangle$	$\langle\{0.6, 0.7, 0.8\}, \{0.5, 0.6\}\rangle$	$\langle\{0.8, 0.9\}, \{0.2, 0.3, 0.4\}\rangle$
X_5	$\langle\{0.7, 0.8, 0.9\}, \{0.3, 0.4\}\rangle$	$\langle\{0.6, 0.7, 0.8\}, \{0.5, 0.6\}\rangle$	$\langle\{0.4, 0.5, 0.6\}, \{0.7, 0.8\}\rangle$	$\langle\{0.7, 0.8\}, \{0.4, 0.5, 0.6\}\rangle$

0.7572, 0.7261, 0.7463, 0.7619, 0.7352, 0.7547, 0.7696, 0.7404, 0.7747, 0.7741, 0.7559, 0.7736, 0.7873, 0.7606, 0.7780, 0.7913 }, { 0.3894, 0.4221, 0.4005, 0.4341, 0.4105, 0.4450, 0.4039, 0.4378, 0.4153, 0.4503, 0.4448, 0.4615, 0.4488, 0.4865, 0.4615, 0.5003, 0.4731, 0.5129, 0.4654, 0.5046, 0.4787, 0.5189, 0.5126, 0.5319, 0.4963, 0.5380, 0.5104, 0.5533, 0.5232, 0.5672, 0.5147, 0.5580, 0.5294, 0.5739, 0.5669, 0.5883 }⟩

$\hat{h}_3$ = ⟨{ 0.5180, 0.5291, 0.5489, 0.5589, 0.5301, 0.5408, 0.5598, 0.5694, 0.5466, 0.5567, 0.5747, 0.5839, 0.5761, 0.5852, 0.6016, 0.6100, 0.5860, 0.5949, 0.6107, 0.6188, 0.5997, 0.6081, 0.6232, 0.6310, 0.6471, 0.6542, 0.6670, 0.6736, 0.6548, 0.6617, 0.6742, 0.6806, 0.6655, 0.6721, 0.6841, 0.6902 }, { 0.5753, 0.5908, 0.6049, 0.6016, 0.6179, 0.6326, 0.6253, 0.6422, 0.6576, 0.5851, 0.6009, 0.6153, 0.6119, 0.6284, 0.6434, 0.6360, 0.6532, 0.6688, 0.6188, 0.6355, 0.6507, 0.6471, 0.6646, 0.6804, 0.6726, 0.6908, 0.7073, 0.6294, 0.6464, 0.6618, 0.6581, 0.6760, 0.6921, 0.6841, 0.7026, 0.7194 }⟩

$\hat{h}_4$ = ⟨{ 0.6987, 0.7100, 0.7254, 0.7216, 0.7319, 0.7459, 0.7529, 0.7618, 0.7739, 0.7123, 0.7230, 0.7375, 0.7340, 0.7438, 0.7570, 0.7636, 0.7721, 0.7836, 0.7770, 0.7848, 0.7956, 0.7930, 0.8003, 0.8102, 0.8152, 0.8216, 0.8303, 0.7865, 0.7940, 0.8042, 0.8018, 0.8087, 0.8181, 0.8229, 0.8290, 0.8373 }, { 0.3617, 0.3715, 0.3814, 0.3917, 0.3707, 0.3807, 0.3908, 0.4014, 0.3782, 0.3884, 0.3987, 0.4095, 0.4254, 0.4369, 0.4485, 0.4606, 0.4360, 0.4478, 0.4596, 0.4721, 0.4448, 0.4568, 0.4690, 0.4817, 0.4773, 0.4902, 0.5032, 0.5168, 0.4891, 0.5024, 0.5157, 0.5297, 0.4991, 0.5126, 0.5261, 0.5404 }⟩

$\hat{h}_5$ = ⟨{ 0.6257, 0.6386, 0.6557, 0.6520, 0.6637, 0.6793, 0.6878, 0.6980, 0.7116, 0.6665, 0.6776, 0.6923, 0.6891, 0.6993, 0.7128, 0.7202, 0.7291, 0.7410, 0.6650, 0.6762, 0.6910, 0.6878, 0.6980, 0.7116, 0.7190, 0.7280, 0.7400, 0.7004, 0.7101, 0.7230, 0.7202, 0.7291, 0.7410, 0.7475, 0.7554, 0.7660, 0.7243, 0.7331, 0.7448, 0.7422, 0.7503, 0.7611, 0.7670, 0.7742, 0.7838, 0.7522, 0.7599, 0.7702, 0.7680, 0.7751, 0.7847, 0.7899, 0.7963, 0.8048 }, { 0.4502, 0.4648, 0.4712, 0.4865, 0.4771, 0.4926, 0.4993, 0.5156, 0.5002, 0.5165, 0.5236, 0.5406, 0.4838, 0.4995, 0.5063, 0.5228, 0.5126, 0.5293, 0.5366, 0.5540, 0.5375, 0.5550, 0.5626, 0.5809 }⟩

Step 4. By Using Definition 5 the score values of the alternatives are given in Table 2.

Step 5. Ranking of the alternatives are given in Table 2.

Next we utilize the Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator to obtain the overall Pythagorean hesitant fuzzy preference values. For this we proceed as follows:

Step 1′. Step 1′ is same as step 1 above.

Step 2′. Step 2′ is same as step 2 above.

Step 3′. Utilize Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator to derive the collective overall preference values $\hat{h}_i$ $(i = 1, 2, \ldots, n)$ of the alternative X_i. We get

$\hat{h}_1$ = ⟨{ 0.5087, 0.5281, 0.5567, 0.5415, 0.5621, 0.5925, 0.5698, 0.5916, 0.6236, 0.5361, 0.5565, 0.5866, 0.5706, 0.5924, 0.6244, 0.6005, 0.6234, 0.6571, 0.5260, 0.5460, 0.5756, 0.5599, 0.5812, 0.6127, 0.5892, 0.6116, 0.6447, 0.5543, 0.5754, 0.6065, 0.5900, 0.6125, 0.6456, 0.6209, 0.6446, 0.6794 }, { 0.5842, 0.6087, 0.6344, 0.6549, 0.6131, 0.6352, 0.6587, 0.6774, 0.6498, 0.6692, 0.6899, 0.7065, 0.5997, 0.6229, 0.6475, 0.6670, 0.6271, 0.6482, 0.6706, 0.6885, 0.6621, 0.6806, 0.7004, 0.7163, 0.6202, 0.6417, 0.6647, 0.6830, 0.6457, 0.6654, 0.6864, 0.7032, 0.6784, 0.6958, 0.7144, 0.7294 }⟩]

$\hat{h}_2$ = ⟨{ 0.5205, 0.5737, 0.5953, 0.5529, 0.6094, 0.6324, 0.5368, 0.5916, 0.6139, 0.5702, 0.6285, 0.6522, 0.5513, 0.6077, 0.6306, 0.5856, 0.6455, 0.6698, 0.5424, 0.5978, 0.6204, 0.5762, 0.6350, 0.6590, 0.5594, 0.6165, 0.6398, 0.5942, 0.6549, 0.6796, 0.5745, 0.6332, 0.6571, 0.6103, 0.6727, 0.6980 }, { 0.4959, 0.5591, 0.5469, 0.6009, 0.6222, 0.6643, 0.5191, 0.5780, 0.5666, 0.6173, 0.6374, 0.6773, 0.5098, 0.5704, 0.5587, 0.6107, 0.6313, 0.6721, 0.5320, 0.5886, 0.5776, 0.6265, 0.6460, 0.6847, 0.5295, 0.5865,

Table 2 Score values and ranking of alternatives by using GPHFCIA operator

	A_1	A_2	A_3	A_4	A_5	Ranking
GPHFCIA$_1$	0.0114	0.2942	− 0.0467	0.3981	0.2590	$X_4 > X_2 > X_5 > X_1 > X_3$
GPHFCIA$_2$	0.2539	0.5063	0.2022	0.5812	0.4683	$X_4 > X_2 > X_5 > X_1 > X_3$
GPHFCIA$_5$	0.3066	0.5665	0.2535	0.6143	0.5093	$X_4 > X_2 > X_5 > X_1 > X_3$
GPHFCIA$_{10}$	0.3670	0.6241	0.3041	0.6523	0.5569	$X_4 > X_2 > X_5 > X_1 > X_3$

Table 3 Score values and ranking of alternatives by using GPHFCIG operator

	A_1	A_2	A_3	A_4	A_5	Ranking
GPHFCIG$_1$	− 0.0919	0.0439	− 0.1675	0.2439	0.1287	$X_4 > X_5 > X_2 > X_1 > X_3$
GPHFCIG$_2$	− 0.3261	− 0.2399	− 0.3851	− 0.0191	− 0.1370	$X_4 > X_5 > X_2 > X_1 > X_3$
GPHFCIG$_5$	− 0.3764	− 0.3547	− 0.4354	− 0.1224	− 0.2251	$X_4 > X_5 > X_2 > X_1 > X_3$
GPHFCIG$_{10}$	− 0.4395	− 0.4583	− 0.5005	− 0.1976	− 0.3183	$X_4 > X_5 > X_2 > X_1 > X_3$

0.5754, 0.6247, 0.6443, 0.6833, 0.5503, 0.6037, 0.5933, 0.6398, 0.6583, 0.6953 }⟩

$\hat{h}_3$ = ⟨{ 0.4856, 0.5144, 0.5120, 0.5423, 0.4954, 0.5248, 0.5223, 0.5533, 0.5039, 0.5338, 0.5313, 0.5627, 0.5165, 0.54707, 0.5445, 0.5423, 0.5269, 0.5582, 0.5556, 0.5533, 0.5360, 0.5677, 0.5651, 0.5627, 0.5448, 0.5771, 0.5744, 0.5423, 0.5559, 0.5888, 0.5860, 0.5533, 0.5654, 0.5988, 0.5961, 0.5627 }, { 0.5902, 0.6263, 0.6820, 0.6244, 0.6565, 0.7066, 0.6697, 0.6969, 0.7398, 0.6037, 0.6381, 0.6916, 0.6363, 0.6671, 0.7152, 0.6798, 0.7059, 0.7473, 0.6232, 0.6554, 0.70570, 0.6538, 0.6826, 0.7280, 0.6946, 0.7192, 0.7584, 0.6352, 0.6661, 0.7144, 0.6645, 0.6922, 0.7359, 0.7037, 0.7274, 0.7653 }⟩

$\hat{h}_4$ = ⟨{ 0.6602, 0.6847, 0.7061, 0.6904, 0.7160, 0.7384, 0.7176, 0.7160, 0.7676, 0.6699, 0.6948, 0.7166, 0.7006, 0.7266, 0.7493, 0.7282, 0.7266, 0.7789, 0.6920, 0.7177, 0.7402, 0.7237, 0.7505, 0.7740, 0.7522, 0.7505, 0.8046, 0.7023, 0.7283, 0.7511, 0.7344, 0.7616, 0.7855, 0.7634, 0.7616, 0.8165 }, { 0.5623, 0.5688, 0.5781, 0.5890, 0.5949, 0.6035, 0.5698, 0.5761, 0.5852, 0.5959, 0.6016, 0.6100, 0.5800, 0.5860, 0.5949, 0.6051, 0.6107, 0.6188, 0.6365, 0.6417, 0.6487, 0.6572, 0.6618, 0.6685, 0.6423, 0.6471, 0.6542, 0.6625, 0.6670, 0.6736, 0.6501, 0.6548, 0.6617, 0.6698, 0.6742, 0.6806 }⟩

$\hat{h}_5$ = ⟨{ 0.5889, 0.6213, 0.6491, 0.6120, 0.6457, 0.7000, 0.6328, 0.6676, 0.6975, 0.6097, 0.6432, 0.6720, 0.6337, 0.6685, 0.7247, 0.6552, 0.6912, 0.7221, 0.6089, 0.6424, 0.6711, 0.6328, 0.6676, 0.7238, 0.6543, 0.6903, 0.7212, 0.6304, 0.6651, 0.6948, 0.6552, 0.6912, 0.7493, 0.6774, 0.7147, 0.7466, 0.6271, 0.6616, 0.6912, 0.6517, 0.6876, 0.7454, 0.6738, 0.7109, 0.7427, 0.6492, 0.6849, 0.7156, 0.6747, 0.7119, 0.7717, 0.6976, 0.7360, 0.7689 }, {0.5109, 0.5659, 0.5383, 0.5888, 0.5315, 0.5831, 0.5571, 0.6047, 0.5581, 0.6055, 0.5816, 0.6255, 0.5250, 0.5777, 0.5512, 0.5997, 0.5447, 0.5942, 0.5692, 0.6150, 0.5702,0.6158, 0.5927,0.6351 }⟩

Table 4 Pythagorean fuzzy decision matrix

	A_1	A_2	A_3	A_4
X_1	⟨0.75, 0.50⟩	⟨0.50, 0.75⟩	⟨0.43, 0.75⟩	⟨0.65, 0.60⟩
X_2	⟨0.70, 0.55⟩	⟨0.85, 0.30⟩	⟨0.57, 0.60⟩	⟨0.35, 0.85⟩
X_3	⟨0.35, 0.80⟩	⟨0.60, 0.65⟩	⟨0.55, 0.70⟩	⟨0.70, 0.55⟩
X_4	⟨0.60, 0.75⟩	⟨0.75, 0.50⟩	⟨0.70, 0.55⟩	⟨0.85, 0.30⟩
X_5	⟨0.80, 0.35⟩	⟨0.75, 0.55⟩	⟨0.50, 0.75⟩	⟨0.75, 0.50⟩

Step 4′. By using Definition 5 the score values of the alternatives are given in Table 3.

Step 5′. Ranking of the alternatives are given in Table 3.

5.1 Comparison analysis

In this subsection, we compare our approach to the existing methods of PFNs, introduced by Yager (2013), and HFNs, introduced by Torra (2010), which are the special cases of PHFNs to verify the validity and effectiveness of the proposed approach.

5.1.1 A comparison analysis with the existing MCDM method with PFNs

PFNs can be considered as a special case of PHFNs when there is only one element in membership and non-membership degree. For comparison, the PHNs can be transformed to PFNs by calculating the average value of the membership and non-membership degrees. After transformation, the Pythagorean information is given in Table 4.

Now, we calculate the comprehensive evaluation values using the Pythagorean fuzzy Choquet integral average (PFCIA) operator and the Pythagorean fuzzy Choquet integral geometric (PFCIG) operator (Peng and Yang 2016). The score values and the ranking of the alternatives using

Table 5 Hesitant fuzzy decision matrix

	A_1	A_2	A_3	A_4
X_1	{0.7, 0.8}	{0.6, 0.7}	{0.4, 0.5, 0.6}	{0.3, 0.4, 0.6}
X_2	{0.8, 0.9}	{0.6, 0.7, 0.8}	{0.4, 0.6, 0.7}	{0.3, 0.4}
X_3	{0.6, 0.7, 0.8}	{0.5, 0.6, 0.7}	{0.5, 0.6}	{0.3, 0.4}
X_4	{0.8, 0.9}	{0.7, 0.8}	{0.6, 0.7, 0.8}	{0.5, 0.6, 0.7}
X_5	{0.7, 0.8, 0.9}	{0.7, 0.8}	{0.6, 0.7, 0.8}	{0.4, 0.5, 0.6}

Table 6 Comparison analysis with existing methods

	X_1	X_2	X_3	X_4	X_5	Ranking
PHFCIA	0.0114	0.2831	– 0.0467	0.3981	0.2590	$X_4 > X_2 > X_5 > X_1 > X_3$
PHFCIG	– 0.0919	0.0105	– 0.1651	0.2439	0.1287	$X_4 > X_5 > X_2 > X_1 > X_3$
PFCIA (Peng and Yang 2016)	– 0.0009	0.2849	– 0.0575	0.3821	0.2407	$X_4 > X_2 > X_5 > X_1 > X_3$
PFCIG (Peng and Yang 2016)	– 0.0784	0.0453	– 0.1413	0.5245	0.4454	$X_4 > X_5 > X_2 > X_1 > X_3$
HFCIA (Wei et al. 2012)	0.6259	0.7107	0.5988	0.7703	0.7168	$X_4 > X_2 > X_5 > X_1 > X_3$
HFCIG (Wei et al. 2012)	0.5888	0.6225	0.5473	0.7333	0.6792	$X_4 > X_5 > X_2 > X_1 > X_3$

PFCIA operator and PFCIG operator are given in Table 6, respectively, which are the same as the proposed approach. But PHFSs are more flexible than PFSs because they consider the situations where decision-makers would like to use several possible values to express the membership and non-membership degrees.

5.1.2 A comparison analysis with the existing MCDM method with HFNs

HFNs can be considered as a special case of PHFNs when decision-makers only consider membership degrees in evaluation. For comparison, the PHFNs can be transformed to HFNs by taking only the membership degrees, and the hesitant fuzzy information is represented in Table 5.

Now, we calculate the comprehensive evaluation values using the hesitant fuzzy Choquet integral average (HFCIA) operator and the hesitant fuzzy Choquet integral geometric (HFCIG) operator (Wei et al. 2012). The score values and the ranking of the alternatives using HFCIA operator and HFCIG operator are given in Table 6, which are the same as the proposed approach. But PHFSs are more flexible than HFSs because they consider the situations where decision-makers would like to use several possible values to express the membership and non-membership degrees.

6 Conclusion

Pythagorean hesitant fuzzy sets fulfill the condition that the square sum of its memberships degrees is less than or equal to 1, which is more flexible than Pythagorean fuzzy sets and hesitant fuzzy sets. Under the Pythagorean hesitant fuzzy set environments, in this paper, we developed Pythagorean hesitant fuzzy Choquet integral averaging (PHFCIA) operator, Pythagorean hesitant fuzzy Choquet integral geometric (PHFCIG) operator, generalized Pythagorean hesitant fuzzy Choquet integral averaging (GPHFCIA) operator and generalized Pythagorean hesitant fuzzy Choquet integral geometric (GPHFCIG) operator. We also discussed some of its properties of the developed operators such as idempotency, monotonicity and boundedness. Moreover, we apply these operators to multi-attribute decision-making problem. Finally, we compare our developed approach with the existing methods.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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