Soft Computing (2019) 23:2129–2146
https://doi.org/10.1007/s00500-018-3423-3

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On linear varieties of MTL-algebras

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Published online: 7 August 2018
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Abstract

In this paper, we focus on those varieties of MTL-algebras whose lattice of subvarieties is totally ordered. Such varieties are called linear. We show that a variety $\mathbb{L}$ of MTL-algebras is linear if and only if each of its subvarieties is generated by one chain. We also study the order type of their lattices of subvarieties, and the structure of their generic chains. If $\mathbb{L}$ is a linear variety with the finite model property, we have that the class of chains in $\mathbb{L}$ is formed by either bipartite or simple chains. As a further result, we provide a complete classification of the linear varieties of BL-algebras. The more general case of MTL-algebras is out of reach, but nevertheless we classify all the linear varieties of WNM-algebras.

Keywords Linear varieties · MTL-algebras · Lattices of varieties · Single-chain completeness · Almost minimal varieties

1 Introduction

MTL-algebras and their corresponding logic MTL were firstly introduced in Esteva and Godo (2001), as a generalization of Hájek's basic logic BL, the logic that was proven in Cignoli et al. (2000) to be the logic of all continuous t-norms and their residua. On the other hand, MTL is the logic of all left-continuous t-norm and their residua (Jenei and Montagna 2002).

As pointed out in Noguera (2006), MTL and its axiomatic extensions are all algebraizable in the sense of Blok and Pigozzi (1989), and their corresponding semantics forms a variety of algebras. The variety of MTL-algebras and its subvarieties form an algebraic lattice: during the years, efforts have been devoted to analyze and classify parts of such a lattice, as well as the corresponding logics. Given an axiomatic extension L of MTL, we denote by $\mathbb{L}$ its corresponding variety.

In this paper, we focus on those subvarieties of $\mathbb{MTL}$ whose lattice of subvarieties forms a chain. We call such varieties *linear*. A notable example of linear variety of $\mathbb{MTL}$ is given by the variety $\mathbb{G}$ of Gödel algebras, as shown in Hecht and Katriňák (1972). A variety is called single chain generated whenever it is generated by one chain. A first result is that a variety is linear if and only if each of its subvarieties is single chain generated. In the first part of the paper, we provide an analysis of the general properties of linear varieties of MTL-algebras.

The second part of the paper provides a classification of all the linear varieties of $\mathbb{BL}$. Our result is based on the decomposition of any BL-chain into an ordinal sum of Wajsberg hoops (Aglianò and Montagna 2003). The more general case of $\mathbb{MTL}$ is out of reach, due to the lack of a sufficiently strong classification of the general structure of these algebras. However, we are able to classify all the linear varieties of WNM-algebras, firstly introduced in Esteva and Godo (2001). Finally, special cases of linear varieties are given by *almost minimal varieties*, previously studied in Galatos et al. (2007) and Aguzzoli and Bianchi (2017), and *maximally linear varieties*. We conclude the paper with two sections on them and with a final discussion of future research.

Communicated by C. Noguera.

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2 Preliminaries

We assume that the reader is acquainted with many-valued logics as developed by Hájek: see Hájek (1998), Esteva and Godo (2001) and Cintula et al. (2011) for details. In particular, we focus on some axiomatic extensions of the monoidal t-norm based logic MTL, firstly introduced by Esteva and Godo (2001) and proved to be the logic of all left-continuous t-norm and their residua in Jenei and Montagna (2002).

2.1 Syntax

The language of MTL is based over the set of connectives $\{\wedge, \&, \to, \bot\}$: the formulas are built in the usual inductive way from these connectives, and a denumerable set of variables.

Useful derived connectives are the following:

$$\neg\varphi \overset{\text{def}}{=} \varphi \to \bot \tag{negation}$$

$$\varphi \vee \psi \overset{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \tag{disjunction}$$

$$\varphi \leftrightarrow \psi \overset{\text{def}}{=} (\varphi \to \psi) \& (\psi \to \varphi) \tag{biconditional}$$

$$\top \overset{\text{def}}{=} \neg\bot. \tag{top}$$

With φ^n and $n\varphi$, we denote, respectively, $\varphi \& \ldots \& \varphi$ (n times) and $\varphi \veebar \ldots \veebar \varphi$ (n times), where $\varphi \veebar \psi \overset{\text{def}}{=} \neg(\neg\varphi \& \neg\psi)$.

MTL can be axiomatized with a Hilbert style calculus: for the reader's convenience, we list the axioms of MTL:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{A1}$$

$$(\varphi \& \psi) \to \varphi \tag{A2}$$

$$(\varphi \& \psi) \to (\psi \& \varphi) \tag{A3}$$

$$(\varphi \wedge \psi) \to \varphi \tag{A4}$$

$$(\varphi \wedge \psi) \to (\psi \wedge \varphi) \tag{A5}$$

$$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi) \tag{A6}$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi) \tag{A7a}$$

$$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \tag{A7b}$$

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \tag{A8}$$

$$\bot \to \varphi \tag{A9}$$

As inference rule, we have modus ponens:

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \tag{MP}$$

An axiomatic extension of MTL is a logic obtained by adding one or more axiom schemata to it. A theory is a set of formulas: the notion of proof and logical consequence are defined as in the classical case.

In this paper, we present results concerning several axiomatic extensions of MTL.

The logics BL, WNM, G, DP (Hájek 1998; Esteva and Godo 2001; Aguzzoli et al. 2014) are axiomatized as MTL plus, respectively:

$$(\varphi \wedge \psi) \leftrightarrow (\varphi \& (\varphi \to \psi)). \tag{div}$$

$$\neg(\varphi \& \psi) \vee ((\varphi \wedge \psi) \to (\varphi \& \psi)). \tag{wnm}$$

$$\varphi \to (\varphi \& \varphi). \tag{id}$$

$$\varphi \vee \neg(\varphi \& \varphi). \tag{dp}$$

NM is axiomatized as WNM plus:

$$\neg\neg\varphi \to \varphi. \tag{inv}$$

IMTL is axiomatized as MTL plus (inv). SMTL is axiomatized as MTL plus

$$\neg(\varphi \wedge \neg\varphi). \tag{s}$$

BP_0 is axiomatized as MTL plus:

$$\neg((\neg(\varphi^2))^2) \leftrightarrow (\neg((\neg\varphi)^2))^2. \tag{BP_0}$$

NM^- is axiomatized as NM plus (BP_0).

For $n \geq 2$, the logic S_nMTL is axiomatized as MTL plus

$$\varphi \vee \neg\left(\varphi^{n-1}\right) \tag{EM_n}$$

Note that S_3MTL coincides with DP, and S_2MTL is classical propositional logic B. For $n \geq 2$, $S_n\text{MTL}^-$ is axiomatized as S_nMTL plus

$$\neg\left((\varphi \leftrightarrow \neg\varphi)^{n-1}\right), \tag{$nofix_n$}$$

see Noguera (2006).

Product logic P is axiomatized as BL plus

$$\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi). \tag{canc}$$

Ł is axiomatized as BL plus (inv). The Chang's logic C is axiomatized as Ł plus (BP_0). For $n \geq 2$, $Ł_n$, G_n, DP_n, NM_n are axiomatized as, respectively, Ł, G, DP, NM plus:

$$\bigvee_{1 \leq i \leq n} (\varphi_i \to \varphi_{i+1}), \tag{n-elem}$$

with the addition of the following family of axioms, for $Ł_n$:

$$\left(p\left(\varphi^{p-1}\right)\right)^n \leftrightarrow n\left(\varphi^p\right), \tag{n-div}$$

where p does not divide $n-1$.

2.2 Semantics

An MTL-algebra is an algebra $\langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$ such that

1. $\langle A, \sqcap, \sqcup, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \Rightarrow\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \Rightarrow y$ for all $x, y, z \in A$.
4. The following identity holds, for all $x, y \in A$:

$$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1 \tag{Prelinearity}$$

A totally ordered MTL-algebra is called MTL-chain.

In the rest of the paper, the notation $\sim x$ denotes $x \Rightarrow 0$.

Let L be an axiomatic extension of MTL. It is known (see Noguera 2006; Cintula et al. 2009) that L is algebraizable in the sense of Blok and Pigozzi (1989) and that the equivalent algebraic semantics forms a subvariety of MTL-algebras, called $\mathbb{L}$. The members of $\mathbb{L}$ are called L-algebras. Following the traditional terminology, we make an exception for Ł, whose corresponding variety is called $\mathbb{MV}$: its members are called MV-algebras. Conversely, each subvariety $\mathbb{L}$ of $\mathbb{MTL}$ is algebraizable, and we denote by L the corresponding axiomatic extension of MTL. With $\mathbb{B}$, we denote the variety of Boolean algebras.

In particular, L is the extension of MTL via a set of axioms $\{\varphi\}_{i\in I}$ if and only if $\mathbb{L}$ is the subvariety of MTL-algebras satisfying the equations $\{\bar{\varphi} = 1\}_{i\in I}$, where $\bar{\varphi}$ is obtained from φ by replacing each occurrence of $\&, \to, \wedge, \vee, \neg, \bot, \top$ with $*, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1$, and every formula symbol occurring in φ with an individual variable. Given an MTL-chain $\mathcal{A}$, and an equation e, the notation $\mathcal{A} \models e$ ($\mathcal{A} \not\models e$) indicates that $\mathcal{A}$ satisfies (does not satisfy) e. If φ is a formula, the notations $\mathcal{A} \models \varphi$ and $\mathcal{A} \not\models \varphi$ are defined similarly.

A variety of MTL-algebras is said to be *n-contractive*, for some $n \geq 2$, whenever each of its algebras satisfies the equation $x^n = x^{n-1}$.

If $\mathcal{A}$ is an MTL-chain, with $\mathbf{V}(\mathcal{A})$ we denote the variety generated by $\mathcal{A}$, i.e., $\mathbf{HSP}(\mathcal{A})$ (Burris and Sankappanavar 1981): similarly, if K is a class of MTL-chains, then $\mathbf{V}(K)$ indicates the variety generated by them. For example, $\mathbf{V}(\mathbf{2}) = \mathbb{B}$, where $\mathbf{2}$ is the two-element Boolean algebra. With $\mathbf{Q}(\mathcal{A})$ and $\mathbf{Q}(K)$, we denote the quasivarieties generated by $\mathcal{A}$ and K, respectively (see Burris and Sankappanavar 1981). Given an MTL-chain $\mathcal{A}$, we define the sets $A^+ \overset{\text{def}}{=} \{a \in A : a > \sim a\}$ and $A^- \overset{\text{def}}{=} \{a \in A : a < \sim a\}$.

With $\mathbf{1}$, we denote the trivial, one-element MTL-algebra (up to isomorphisms).

We recall that a variety of MTL-algebras $\mathbb{L}$ is *locally finite* whenever for every algebra $\mathcal{A} \in \mathbb{L}$ and any finite subset X of A, the subalgebra of $\mathcal{A}$ generated by X is finite. A variety $\mathbb{L}$ of MTL-algebras has the *finite model property* (FMP), whenever it is generated by its finite chains. Similarly, an axiomatic extension L of MTL has the FMP whenever it is complete w.r.t. the class of finite L-chains.

As we shall implicitly use the following fact throughout the paper, we stress that the equation corresponding to the (n-elem) axiom is satisfied by an MTL-chain iff it has at most n elements. See Cintula et al. (2009, Proposition 4.18).

2.3 Simple, bipartite and semihoop simple MTL-chains

We first recall that, given an MTL-chain $\mathcal{A}$, a filter F on $\mathcal{A}$ is a non-empty set being upward closed (if $x \in F$, then $y \in F$, for every $y \geq x$) and such that if $x, y \in F$, then $x * y \in F$. Given a filter F, we define $\overline{F} \overset{\text{def}}{=} \{a \in A : \sim a \in F\}$.

Definition 1 – Given an MTL-chain $\mathcal{A}$, with $\text{Rad}(\mathcal{A})$ we denote the largest proper filter[1] of $\mathcal{A}$.

– An MTL-chain $\mathcal{A}$ is said to be *simple* whenever $\{1\}$ and A are its only filters.
– An MTL-chain $\mathcal{A}$ is said to be *bipartite* if $A = \text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}}(\mathcal{A})$.
– An MTL-chain $\mathcal{A}$ is said to be *semihoop simple* if $\text{Rad}(\mathcal{A})$ is the universe of a simple totally ordered semihoop, *i.e.* for all $1 \neq x, y \in \text{Rad}(\mathcal{A})$, there is a natural n such thath $x^n < y$.
– An MTL-chain $\mathcal{A}$ is said to be *strongly semihoop simple* if it is bipartite and semihoop simple.

We recall some useful properties of the radical.

Lemma 1 *For every MTL-chain $\mathcal{A}$, the following hold:*

[1] Usually $\text{Rad}(\mathcal{A})$ is defined as the intersection of all the maximal proper filters of $\mathcal{A}$. Since we work only on MTL-chains, the two definitions are equivalent.

- $\mathrm{Rad}(A) \subseteq A^+$.
- $\mathrm{Rad}(\mathcal{A}) = \{a \in A : a^n > {\sim}a$ *for every* $n \geq 1\}$.
- *For every filter* F *of* $\mathcal{A}$, $\overline{F}$ *is downward closed, and* $F \cup \overline{F}$ *is a subuniverse of* $\mathcal{A}$.

Proof For the first two items, see Noguera et al. (2005). For the third one, let F be a filter of an MTL-chain $\mathcal{A}$. Pick $x \in \overline{F}$: then, for every $y \leq x$ it holds that ${\sim}y \geq {\sim}x$. As ${\sim}x \in F$, then also ${\sim}y \in F$, whence $y \in \overline{F}$, which is then downward closed. Consider the set $S = F \cup \overline{F}$: since $F \subseteq \mathrm{Rad}(A) \subseteq A^+$, an easy check shows that $\overline{F} \in A^-$. Then, it is immediate to see that S is closed under $*$. To conclude the proof, we need to show that S is closed under $\Rightarrow$. Since F is upward closed, and for every $x, y \in F$, $x \Rightarrow y \geq y$, we conclude that F is closed also under $\Rightarrow$. Let $x \in \overline{F}$ and $y \in F \cup \overline{F}$: then, $x \Rightarrow y \geq {\sim}x$. As ${\sim}x \in F$, also $x \Rightarrow y \in F$. Finally, let $x \in F$ and $y \in \overline{F}$. We have $x \Rightarrow y \leq x \Rightarrow {\sim}{\sim}y = x \Rightarrow ((y \Rightarrow 0) \Rightarrow 0) = (x * (y \Rightarrow 0)) \Rightarrow 0 = {\sim}(x * {\sim}y)$. As $x * {\sim}y \in F$, then ${\sim}{\sim}(x * {\sim}y) \in F$, and hence ${\sim}(x * {\sim}y) \in \overline{F}$. As $\overline{F}$ is downward closed, $x \Rightarrow y \in \overline{F}$. The proof is settled. □

There is the following characterization, for bipartite MTL-algebras.

Theorem 1 (Noguera et al. 2005, Theorem 3.20) *Let* $\mathcal{A}$ *be an MTL-chain. Then, the following conditions are equivalent:*

- $\mathcal{A}$ *is bipartite.*
- $\mathrm{Rad}(\mathcal{A}) = A^+$ *and* $\mathcal{A}$ *does not have a negation fixpoint.*
- $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \simeq \mathbf{2}$.
- $\mathcal{A}$ *satisfies* (BP_0).

It is very easy to check that the only non-trivial simple chain belonging to $\mathbb{BP}_0$ is (up to isomorphisms) $\mathbf{2}$.

2.4 Semihoops, hoops, ordinal sums

Definition 2 A *semihoop* is a structure $\mathcal{A} = \langle A, *, \sqcap, \Rightarrow, 1\rangle$ such that $\langle A, \sqcap, 1\rangle$ is an inf-semilattice with upper bound 1, $*$ is a binary operation on A with unit 1, and $\Rightarrow$ is a binary operation such that:

- $x \leq y$ iff $x \Rightarrow y = 1$,
- $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

A *bounded* semihoop is a semihoop with a minimum element; conversely, an *unbounded* hoop is a hoop without minimum.

- A *hoop* is a semihoop satisfying $x * (x \Rightarrow y) = y * (y \Rightarrow x)$.
- A *Wajsberg hoop* is a hoop satisfying $x \Rightarrow (x \Rightarrow y) = y \Rightarrow (y \Rightarrow x)$.
- A *cancellative hoop* is a hoop satisfying $x \Rightarrow (x * y) = y$.

Lemma 2 (Blok and Ferreirim 2000; Esteva et al. 2003) *Every totally ordered Wajsberg hoop is either bounded, or it is cancellative. Bounded Wajsberg hoops are the* 0*-free reducts of MV-algebras.*

Definition 3 Let $\langle I, \leq\rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered semihoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$. Assume also that $\mathcal{A}_0$ is bounded. Then, $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose top is 1, and whose operations are:

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y = \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\},\ x \in A_j) \end{cases}$$

$$x \sqcap y = \begin{cases} x \sqcap^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\},\ x \in A_j) \end{cases}$$

A totally ordered semihoop is indecomposable if it is not isomorphic to an ordinal sum of two non-trivial totally ordered semihoops.

An easy check shows that every ordinal sum of totally ordered semihoops, with the first component bounded, is an MTL-chain. Also, as shown in Aglianò and Montagna (2003) every totally ordered Wajsberg hoop is indecomposable.

The following result concerns the general structure of MTL-chains.

Theorem 2 (Noguera 2006, Theorem 4.5.4) *For every MTL-chain, there is a maximum decomposition as ordinal sum of indecomposable totally ordered semihoops, with the first one bounded.*

Observe that the idempotent elements in any ordinal sum of semihoops are exactly 1 and the bottoms of every bounded component.

2.5 BL-chains

Theorem 2 is significantly strengthened if restricted to BL-chains.

Theorem 3 (Aglianò and Montagna 2003) *Every BL-chain can be uniquely decomposed as an ordinal sum of Wajsberg hoops, whose first component is bounded.*

MV-algebras are exactly the bounded Wajsberg hoops (Esteva et al. 2003). The standard MV-algebra $[0, 1]_Ł$ equips the unit interval with the operations $x*y = \max\{0, x+y-1\}$, $x \sqcap y = \min\{x, y\}$, and $x \Rightarrow y = \min\{1, 1 - x + y\}$. For $k \geq 2$, $\mathbb{L}_k$ is the subvariety of $\mathbb{MV}$ generated by the k-element MV-chain $\mathbf{L}_k = (\{0, \frac{1}{k-1}, \ldots, \frac{k-1}{k-1}\}, *, \to, 0)$, whose operations are defined by restriction from those of $[0, 1]_Ł$. Notice that two finite MV-chains with the same cardinality are isomorphic. Clearly $\mathbf{L}_2 \simeq \mathbf{2}$.

Given an MV-chain $\mathcal{A}$, with $\mathrm{Rad}(\mathcal{A})$ we denote the maximal proper filter of $\mathcal{A}$. It is well known that $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is a simple MV-chain. An MV-chain has finite rank if $|\mathcal{A}/\mathrm{Rad}(\mathcal{A})|$ is finite, i.e., $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \simeq \mathbf{L}_k$, for some k. In particular, $\mathrm{Rank}(\mathcal{A}) = |\mathcal{A}/\mathrm{Rad}(\mathcal{A})|$ if $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is finite, while $\mathrm{Rank}(\mathcal{A}) = \infty$ otherwise. For $m \geq 2$, the MV-chain $\mathbf{K}_m$ is defined as $\Gamma(\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}, (m-1, 0))$, where Γ is Mundici's functor, which provides the categorical equivalence between MV-algebras and lattice-ordered Abelian groups with a distinguished strong order unit: see Cignoli et al. (1999) for details.

Notice that $\mathbf{K}_n/\mathrm{Rad}(\mathbf{K}_n)$ is isomorphic with $\mathbf{L}_n$. The MV-algebra $\mathbf{K}_2$ is the well-known Chang's MV-algebra. The variety related to the logic C is generated by $\mathbf{K}_2$: it will be denoted $\mathbb{C}$.

With $\mathbb{P}$, we denote the variety of product algebras.

Each totally ordered product algebra is isomorphic to the ordinal sum of $\mathbf{2} \oplus \mathcal{A}$, where $\mathcal{A}$ is a cancellative hoop, and each totally ordered cancellative hoop arises as the second and last component of a product chain (Esteva et al. 2003). Notice that each totally ordered cancellative hoop either is the singleton or has infinitely many elements.

2.6 WNM-chains

In this section, we provide some results concerning WNM-chains and their operations.

Lemma 3 (Gispert 2003) *In every WNM-chain $\mathcal{A}$, the operations $*$ and $\Rightarrow$ are defined as follows, for every $x, y \in A$:*

$$x * y = \begin{cases} 0 & \text{if } x \leq {\sim}y \\ \min\{x, y\} & \text{otherwise.} \end{cases} \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{{\sim}x, y\} & \text{otherwise.} \end{cases} \tag{1}$$

We now describe the structure and the operations of DP-chains.

Lemma 4 (Noguera 2006; Aguzzoli et al. 2014) *Every DP-chain $\mathcal{A}$ with more than two elements has a coatom c. Moreover, the operations $*$ and $\Rightarrow$ of such a chain are the following ones, for every $x, y \in A$:*

$$x * y = \begin{cases} 0 & \text{if } x, y < 1, \\ \min\{x, y\} & \text{otherwise.} \end{cases} \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ c & \text{if } 1 > x > y, \\ y & \text{if } x = 1. \end{cases} \tag{2}$$

Observe that the coatom c is a negation fixpoint, that is, ${\sim}c = c$.

3 Completeness properties

Definition 4 Let L be an axiomatic extension of MTL. Then:

- L enjoys the *single-chain completeness* (SCC) if there is an L-chain such that L is complete w.r.t. it.
- L enjoys the *finite strong single-chain completeness* (FSSCC) if there is an L-chain such that L is finitely strongly complete w.r.t. it.
- L enjoys the *strong single-chain completeness* (SSCC) if there is an L-chain such that L is strongly complete w.r.t. it.
- L enjoys the hereditary single-chain completeness (HSCC), if L and every of its (consistent) axiomatic extensions have the SCC.
- L enjoys the hereditary strong single-chain completeness (HSSCC), if L and every of its (consistent) axiomatic extensions have the SSCC.

Theorem 4 (Montagna 2011, Theorem 3) *Let L be an axiomatic extension of MTL having the FSSCC. Then, L has also the SSCC.*

Theorem 5 *For every finite MTL-chain $\mathcal{A}$, it holds that the class of chains in $\mathbf{V}(\mathcal{A})$ coincides with $\mathbf{HS}(\mathcal{A})$, and $\mathbf{HS}(\mathcal{A}) = \mathbf{SH}(\mathcal{A})$. As a consequence, if $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B})$, for some finite MTL-chain $\mathcal{B}$, then $\mathcal{A} \simeq \mathcal{B}$.*

Proof Let $\mathcal{A}$ be a finite MTL-chain. It is known that every variety of MTL-algebras is congruence distributive, and then by Burris and Sankappanavar (1981, Ch. IV, §6, Corollary 6.10), we have that all the subdirectly irreducible algebras in $\mathbf{V}(\mathcal{A})$ belong to $\mathbf{HS}(\mathcal{A})$. Note that every chain in $\mathbf{V}(\mathcal{A})$ is finite, as $\mathcal{A}$ satisfies the identity (n-elem) for some n, and hence subdirectly irreducible; moreover, every subdirectly irreducible MTL-algebra is totally ordered. Since every element of $\mathbf{HS}(\mathcal{A})$ is a chain, we conclude that the class of chains in $\mathbf{V}(\mathcal{A})$ coincides with $\mathbf{HS}(\mathcal{A})$. Since every variety of MTL-algebras has the congruence extension property (see Noguera 2006), we have that $\mathbf{HS}(\mathcal{A}) = \mathbf{SH}(\mathcal{A})$.

Suppose that $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B})$, for some finite MTL-chain $\mathcal{B}$. By Burris and Sankappanavar (1981, Ch. IV, §6, Exercise

5), we conclude that $\mathcal{A} \simeq \mathcal{B}$. As a matter of fact, if $\mathcal{A} \not\simeq \mathcal{B}$ and $|A| = |B|$, then $\mathcal{A} \notin \mathbf{HS}(\mathcal{B})$ and $\mathcal{B} \notin \mathbf{HS}(\mathcal{A})$, whence $\mathbf{V}(\mathcal{A}) \neq \mathbf{V}(\mathcal{B})$. □

4 Linear varieties of MTL-algebras: basic definitions and results

Lemma 5 *Let $\mathbb{L}$ be a variety of MTL-algebras. Then, the class of its non-trivial subvarieties, ordered by inclusion forms a lattice $\mathcal{L}_{\mathbb{L}}$, having $\mathbb{L}$ as maximum, and the variety of Boolean algebras $\mathbb{B}$ as minimum.*

Definition 5 Let $\mathbb{L}$ be a variety of MTL-algebras. Then

- $\mathbb{L}$ is said to be *single chain generated* whenever $\mathbf{V}(\mathcal{A}) = \mathbb{L}$, for some chain $\mathcal{A} \in \mathbb{L}$.
- $\mathbb{L}$ is said to be *strongly single chain generated* whenever L is strongly complete w.r.t $\mathcal{A}$, for some chain $\mathcal{A} \in \mathbb{L}$.
- $\mathbb{L}$ is said to be *linear* whenever the lattice of its subvarieties is totally ordered.
- $\mathbb{L}$ is said to be *strongly linear* whenever $\mathbb{L}$ is linear, and for every subvariety $\mathbb{L}' \subseteq \mathbb{L}$, there is a chain in $\mathcal{A} \in \mathbb{L}'$ such that every countable chain in $\mathbb{L}'$ is embeddable into $\mathcal{A}$.

$\mathbb{BL}, \mathbb{MV}, \mathbb{P}, \mathbb{G}$ are only few examples of varieties being single chain generated. $\mathbb{P}$ and $\mathbb{G}$ are linear.

We can state the following proposition which easily follows from the previous definitions.

Proposition 1 *Let $\mathcal{L}_{lin}$ be the class of all the linear varieties of MTL-algebras, ordered by inclusion. Then $\mathcal{L}_{lin}$ is a downward-closed sub-inf-semilattice of $\mathcal{L}_{\mathbb{MTL}}$.*

We also recall the notion of almost minimal varieties of MTL-algebras, a topic firstly introduced and studied in Galatos et al. (2007) and Katoh et al. (2006) for residuated lattices, and further analyzed in Aguzzoli and Bianchi (2017).

Definition 6 A variety $\mathbb{L}$ of MTL-algebras is said *almost minimal* whenever $\mathbb{B} \subsetneq \mathbb{L}$, and $\mathbb{B}$ is the only proper non-trivial subvariety of $\mathbb{L}$.

In other words, almost minimal (AM, for short) varieties are the atoms of $\mathcal{L}_{\mathbb{MTL}}$. It is immediate to check that every almost minimal variety is also linear. The converse, clearly, does not hold: $\mathbb{G}$ is a counterexample.

Moving to the logical side, we have the following facts, obtained by Definitions 4, 5 and Cintula et al. (2009, Theorem 3.5).

Remark 1 Let L be an axiomatic extension L of MTL. Then,

- L enjoys the SCC whenever $\mathbb{L}$ is single chain generated.
- L enjoys the SSCC whenever $\mathbb{L}$ is strongly single chain generated.

Theorem 6 *Let $\mathbb{L}$ be a variety of MTL-algebras. Then, $\mathbb{L}$ is linear if and only if every non-trivial subvariety of $\mathbb{L}$ is single chain generated. Equivalently, $\mathbb{L}$ is linear if and only if L enjoys the HSCC.*

Proof Let $\mathbb{L}$ be a variety of MTL-algebras.

Assume first that every subvariety of $\mathbb{L}$ is generated by a chain. By Aguzzoli and Bianchi (2016, Theorem 5.1), we have that every non-trivial subvariety of $\mathbb{L}$ is join irreducible, in $\mathcal{L}_{\mathbb{MTL}}$. Then, clearly $\mathbb{L}$ is linear, for otherwise it would contain a subvariety which is the join of other two subvarieties.

Assume now that $\mathbb{L}$ is linear: then, by Lemma 5 we have that every subvariety in $\mathcal{L}_{\mathbb{L}}$ is join irreducible. By Aguzzoli and Bianchi (2016, Theorem 5.1), we have that every one of these varieties is single chain generated. By Remark1, we conclude that $\mathbb{L}$ is linear if and only if L enjoys the HSCC. □

Theorem 7 *A variety $\mathbb{L}$ of MTL-algebras is strongly linear if and only if the logic corresponding to $\mathbb{L}$, say L, has the HSSCC.*

Proof Immediate by Theorem 6 and Cintula et al. (2009, Theorem 3.5). □

5 Logical characterization of linearity

In this section, we discuss some logical properties related to linear varieties.

Lemma 6 *Let L be an axiomatic extension of MTL having the HSCC. If $\mathbb{L}$ is generated by an infinite chain, then every variety generated by an infinite L-chain must contain all the finite L-chains.*

Proof Let L be an axiomatic extension of MTL having the HSCC, and assume that $\mathbb{L}$ is generated by an infinite chain. Suppose that there is an infinite L-chain $\mathcal{B}$ such that $\mathcal{C} \notin \mathbf{V}(\mathcal{B})$, for some finite L-chain $\mathcal{C}$. Since L has the HSCC, the lattice of the subvarieties of $\mathbb{L}$ must form a chain, and hence $\mathbf{V}(\mathcal{C}) \subseteq \mathbf{V}(\mathcal{B})$ or $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{C})$. However, $\mathbf{V}(\mathcal{C}) \nsubseteq \mathbf{V}(\mathcal{B})$, since $\mathcal{C} \notin \mathbf{V}(\mathcal{B})$. Moreover, $\mathbf{V}(\mathcal{B}) \nsubseteq \mathbf{V}(\mathcal{C})$, since $\mathcal{B}$ is infinite, while $\mathcal{C}$ is finite. Hence, we have a contradiction, and we must conclude that every variety generated by an infinite L-chain must contain all the finite L-chains. □

Theorem 8 *Let L be an axiomatic extension of MTL having the HSCC. Then, for every $n \geq 2$, all the L-chains of cardinality n are isomorphic.*

Moreover, if L has the finite model property,

– *If $\mathbb{L}$ is generated by an infinite L-chain, then every other infinite L-chain is generic. Moreover, the only proper subvarieties of $\mathbb{L}$, which are infinitely many, are those singly generated by a finite chain. As a consequence, the lattice of the subvarieties of $\mathbb{L}$ is isomorphic to $\omega + 1$.*
– *The only proper subvarieties of $\mathbb{L}$ are those singly generated by a finite chain.*

Proof Let L be an axiomatic extension of MTL having the HSCC.

Suppose by contradiction that there are two finite L-chains $\mathcal{A}, \mathcal{B}$ with the same cardinality, and such that $\mathcal{A} \not\cong \mathcal{B}$. Then, $\mathcal{B} \notin \mathbf{HS}(\mathcal{A})$ and $\mathcal{A} \notin \mathbf{HS}(\mathcal{B})$, and by Theorem 5, we conclude that $\mathcal{B} \notin \mathbf{V}(\mathcal{A})$ and $\mathcal{A} \notin \mathbf{V}(\mathcal{B})$. However, since L has HSCC, the lattice of the subvarieties of $\mathbb{L}$ forms a chain, which implies that $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$ or $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{A})$, a contradiction. We conclude that, for every $n \geq 2$, all the L-chains of cardinality n are isomorphic.

Suppose now that L has the finite model property: as a consequence, $\mathbb{L}$ is generated by the class of its finite chains.

If $\mathbb{L}$ is generated by a finite chain, then we immediately have that also every subvariety of $\mathbb{L}$ is generated by a finite chain, since L has the HSCC, and every L-chain is finite.

If $\mathbb{L}$ is generated by an infinite chain, by Lemma 6 and the FMP we immediately obtain that $\mathbb{L}$ is generated by every other infinite L-chain.

Since L has the HSCC, we have that the only proper subvarieties $\mathbb{L}$ are the ones generated by a finite chain. Suppose now by contradiction that $\mathbb{L}$ contains only a finite number of finite chains: let k be the cardinality of the largest one. As a consequence, every chain in $\mathbb{L}$ satisfies $\bigsqcup_{1\leq i\leq k}(x_i \Leftrightarrow x_{i+1}) = 1$, which would imply that there are no infinite L-chains, a contradiction. Then, there are infinitely many non-isomorphic finite L-chains, and since L has the HSCC, the lattice of the subvarieties of $\mathbb{L}$ is isomorphic to $\omega + 1$. This concludes the proof. □

Remark 2 Note that there are axiomatic extensions of MTL having the HSCC, but for which the FMP fails to hold. An example is given by product logic.

However, we have the following result.

Theorem 9 *Let L be an axiomatic extension of MTL having the HSCC. Then, L has the FMP if and only if one of the following two cases holds.*

1. *$\mathbb{L}$ is generated by a finite chain.*
2. *$\mathbb{L}$ is generated by an infinite chain, and*
   2.1. $\mathbf{V}(\mathcal{A}) = \mathbb{L}$ for every infinite L-chain $\mathcal{A}$.
   2.2. There is no finite upper bound on the cardinality of non-trivial finite chains in $\mathbb{L}$.

Proof Let L be an axiomatic extension of MTL having the HSCC. If 1 holds, then it is immediate to check that L has the FMP. If 2 holds, then since $\mathbb{L}$ is linear, there is an L-chain $\mathcal{C}$ such that $\mathbf{V}(K) = \mathbf{V}(\mathcal{C})$. Note that $\mathcal{C}$ cannot be finite, as otherwise every chain in $\mathbf{V}(K)$ would have at most $|\mathcal{C}|$ elements, in contrast with 2. So $\mathcal{C}$ must be infinite, and by 2 we conclude that $\mathbf{V}(K) = \mathbf{V}(\mathcal{C}) = \mathbb{L}$. Then, L has the FMP.

Assume now that L has the FMP, and suppose by contradiction that neither 1 nor 2 hold. Let us call K the class of non-trivial finite L-chains. Then, $\mathbb{L}$ is generated by an infinite chain $\mathcal{C}$ and either there is an infinite L-chain $\mathcal{D}$ such that $\mathbf{V}(\mathcal{D}) \subsetneq \mathbf{V}(\mathcal{C})$ or there is $k \in \mathbb{N}$ such that $|\mathcal{E}| \leq k$, for every finite L-chain $\mathcal{E}$. Observe that such $\mathcal{D}$ cannot exist, as it would be in contrast to Theorem 8: then, $\mathbf{V}(\mathcal{F}) = \mathbb{L}$, for every infinite L-chain $\mathcal{F}$. So, the only possibility is that $|\mathcal{E}| \leq k$, for every $\mathcal{E} \in K$, but—since $\mathbb{L}$ is linear and $\mathcal{C}$ is infinite—this would imply that $\mathbf{V}(K) \subsetneq \mathbf{V}(\mathcal{C}) = \mathbb{L}$, and hence, the FMP would not hold, a contradiction. Then, we conclude that if L has the FMP, either 1 or 2 must hold. The proof is settled. □

Clearly, the HSSCC implies the HSCC. Does the converse hold? We are only able to provide a partial answer.

Theorem 10 *Let L be an axiomatic extension having the HSCC, SSCC, and FMP. If for every finite L-chain $\mathcal{A}$ it holds that $\mathbf{HS}(\mathcal{A}) = \mathbf{IS}(\mathcal{A} \cup \{\mathbf{1}\})$, then L enjoys also the HSSCC.*

Proof Let L be an axiomatic extension having the HSCC, SSCC, and FMP, and assume that for every finite L-chain $\mathcal{A}$, it holds that $\mathbf{HS}(\mathcal{A}) = \mathbf{IS}(\mathcal{A} \cup \{\mathbf{1}\})$. Since L has the SSCC by hypothesis, it is enough to show that the same holds for all the axiomatic extensions of L. By Theorem 8, we know that the only proper subvarieties of $\mathbb{L}$ are the ones generated by a single finite chain. Let $\mathcal{A}$ be a finite chain, and let us call M the logic related to $\mathbf{V}(\mathcal{A})$. By hypothesis $\mathbf{HS}(\mathcal{A}) = \mathbf{IS}(\mathcal{A} \cup \{\mathbf{1}\})$, and hence by Theorem 5, we have that every non-trivial chain in $\mathbf{V}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{A}$. By Cintula et al. (2009, Theorem 3.5), M has the SSCC. Then, we conclude that L enjoys the HSSCC. □

6 Linear varieties generated by a finite chain: general results

We start with some general results concerning finite MTL-chains.

Proposition 2 *Let $\mathcal{A}$ be a finite MTL-chain. Then, for every filter F in A there is an idempotent $c \in A$ such that $F = \{x \in A : x \geq c\}$.*

Proof Let $\mathcal{A}$ be a finite MTL-chain. Then, a filter F is a finite set, and hence, it has a minimum, say m. Since F is closed under the monoidal operation, m must necessarily be idempotent. Finally, F is a totally ordered upward closed set, and hence, we conclude that $F = \{x \in A : x \geq m\}$. □

Theorem 11 *Let $\mathcal{A}$ be a finite MTL-chain such that every non-trivial chain in $\mathbf{H}(\mathcal{A})$ belongs to $\mathbf{IS}(\mathcal{A})$. Then, $\mathbf{V}(\mathcal{A})$ is linear if and only if for every $\mathcal{B}, \mathcal{C} \in \mathbf{S}(\mathcal{A})$ it holds that $\mathcal{B} \hookrightarrow \mathcal{C}$ or $\mathcal{C} \hookrightarrow \mathcal{B}$.*

Proof Let $\mathcal{A}$ be a finite MTL-chain such that every non-trivial chain in $\mathbf{H}(\mathcal{A})$ belongs to $\mathbf{IS}(\mathcal{A})$. By Theorem 5, every non-trivial chain in $\mathbf{V}(\mathcal{A})$ belongs to $\mathbf{SH}(\mathcal{A}) = \mathbf{IS}(\mathcal{A}(\mathcal{A} \cup \{\mathbf{1}\}))$. Then, for every $\mathcal{B}, \mathcal{C} \in \mathbf{S}(\mathcal{A})$, we have that $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{C})$ if and only if $\mathcal{B} \hookrightarrow \mathcal{C}$. From this fact, we obtain the theorem's claim. □

Corollary 1 *Let $\mathcal{A}$ be a finite MTL-chain being simple or strongly semihoop simple. Then, $\mathbf{V}(\mathcal{A})$ is linear if and only if for every $\mathcal{B}, \mathcal{C} \in \mathbf{S}(\mathcal{A})$ it holds that $\mathcal{B} \hookrightarrow \mathcal{C}$ or $\mathcal{C} \hookrightarrow \mathcal{B}$.*

Proof Let $\mathcal{A}$ be a finite MTL-chain being simple or strongly semihoop simple. Then, $\{1\}$ and $\mathrm{Rad}(A)$ are the only proper filters of $\mathcal{A}$ whence every non-trivial chain in $\mathbf{H}(\mathcal{A})$ belongs to $\mathbf{IS}(\mathcal{A})$. By Theorem 11, $\mathbf{V}(\mathcal{A})$ is linear if and only if for every $\mathcal{B}, \mathcal{C} \in \mathbf{S}(\mathcal{A})$ it holds that $\mathcal{B} \hookrightarrow \mathcal{C}$ or $\mathcal{C} \hookrightarrow \mathcal{B}$. □

As a consequence, we have the following result.

Theorem 12 *Let $\mathbb{L}$ be a linear variety generated by a finite MTL-chain being simple or strongly semihoop simple. Then, $\mathbb{L}$ is also strongly linear.*

Proof Immediate by Corollary 1 and Cintula et al. (2009, Theorem 3.5). □

We can also provide the following characterization, for the linear varieties generated by a finite MTL-chain.

Lemma 7 *Let $\mathcal{A}$ be a non-simple MTL-chain with negation fixpoint. Then, there is a chain $\mathcal{B} \in \mathbf{V}(\mathcal{A})$ without negation fixpoint, and such that $\mathcal{B} \not\cong \mathbf{2}$. As a consequence, $\mathbf{V}(\mathcal{A})$ is not linear.*

Proof Let $\mathcal{A}$ be a non-simple MTL-chain with negation fixpoint f. Then, $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A$. By Lemma 1, $\mathrm{Rad}(\mathcal{A}) \cup \overline{\mathrm{Rad}}(\mathcal{A})$ is the carrier of a subalgebra of $\mathcal{A}$, which is bipartite by construction. Let us call $\mathcal{B}$ such chain. As $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A$, we have that $|\mathcal{B}| > 2$. Note that the subalgebra of $\mathcal{A}$ generated by f is isomorphic to $\mathbf{L}_3$, and then, we have that $\mathcal{B} \notin \mathbf{V}(\mathbf{L}_3)$, which implies $\mathbf{V}(\mathcal{B}) \nsubseteq \mathbf{V}(\mathbf{L}_3)$. As $\mathcal{B}$ is bipartite by Theorem 1, we have that $\mathcal{B} \models$ (BP$_0$) and $\mathbf{L}_3 \not\models$ (BP$_0$). Then, $\mathbf{V}(\mathbf{L}_3) \nsubseteq \mathbf{V}(\mathcal{B})$, which implies that $\mathbf{V}(\mathcal{A})$ is not linear. □

Lemma 8 *Let $\mathcal{A}$ be a finite MTL-chain such that $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A^+$. Then, $\mathbf{V}(\mathcal{A})$ is not linear.*

Proof Let $\mathcal{A}$ be a finite MTL-chain such that $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A^+$. By Lemma 1, $\mathrm{Rad}(\mathcal{A}) \cup \overline{\mathrm{Rad}}(\mathcal{A})$ is the carrier of a subalgebra of $\mathcal{A}$, and by the construction, it is bipartite. Let us call $\mathcal{B}$ such chain. As $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A^+$, we have that $\mathcal{B}$ is bipartite and not simple. Since $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A^+$, by Theorem 1 $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is not bipartite.

By Theorem 1 $\mathbf{V}(\mathcal{A}/\mathrm{Rad}(\mathcal{A})) \nsubseteq \mathbf{V}(\mathcal{B})$, as $\mathcal{B} \models$(BP$_0$) and $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \not\models$ (BP$_0$). As $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is finite and simple, by Theorem 5, every chain in $\mathbf{V}(\mathcal{A}/\mathrm{Rad}(\mathcal{A}))$ belongs to $\mathbf{HS}(\mathcal{A}/\mathrm{Rad}(\mathcal{A})) = \mathbf{IS}(\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \cup \{\mathbf{1}\})$. Since $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is simple (i.e., there are no idempotent elements between 0 and 1), the same holds for every of its subalgebras, and this implies that $\mathbf{V}(\mathcal{B}) \nsubseteq \mathbf{V}(\mathcal{A}/\mathrm{Rad}(\mathcal{A}))$, as $\mathcal{B}$ is not simple (in particular, it has at least one idempotent element between 0 and 1). Then, $\mathbf{V}(\mathcal{A})$ is not linear, and the proof is settled. □

Remark 3 One can ask if Lemma 8 holds true, when $\mathcal{A}$ is infinite. Actually, we do not know the answer, as we cannot use the same proof strategy. Indeed, if $\mathcal{A}$ is infinite, it is not necessarily true that $\mathbf{V}(\mathcal{A}/\mathrm{Rad}(\mathcal{A})) \subsetneq \mathbf{V}(\mathcal{A})$. Just take $\mathcal{A} = \Gamma(\mathbb{Q} \times_{\mathrm{lex}} \mathbb{Z}, (1, 0))$.

A direct inspection shows that $\{1\} \subsetneq \mathrm{Rad}(\mathcal{A}) \subsetneq A^+$, and $\mathbf{V}(\mathcal{A}/\mathrm{Rad}(\mathcal{A})) = \mathbf{V}(\mathcal{A}) = \mathbb{MV}$, as $\mathcal{A}$ and $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ are both infinite MV-chains with infinite rank.

Theorem 13 *Let $\mathcal{A}$ be a finite MTL-chain such that $\mathbf{V}(\mathcal{A})$ is linear. Then, $\mathcal{A}$ is simple or bipartite.*

Proof Immediate by Lemmas 7 and 8. □

As a consequence,

Corollary 2 *Let $\mathbb{L}$ be a linear variety of MTL-algebras generated by a finite chain. Then, the chains in $\mathbb{L}$ are either all bipartite or all simple.*

Proof Let $\mathbb{L}$ be a linear variety of MTL-algebras generated by a finite chain. By Theorem 13, every chain in $\mathbb{L}$ is simple or bipartite. By contradiction, assume $\mathbb{L}$ contains two chains $\mathcal{A}, \mathcal{B}$, being, respectively, simple and bipartite, and having both at least three elements (this to avoid trivialities, as $\mathbf{2}$ is both simple and bipartite). Clearly, $\mathcal{A} \notin \mathbf{V}(\mathcal{B})$, as $\mathcal{B} \models$ (BP$_0$) and $\mathcal{A} \not\models$ (BP$_0$). Also, by Theorem 5 $\mathcal{B} \notin \mathbf{V}(\mathcal{A})$, as $\mathcal{B} \notin \mathbf{HS}(\mathcal{A})$. This because $\mathcal{B}$ contains at least a non-Boolean idempotent, being finite (with at least three elements) and bipartite, and $\mathcal{A}$ does not, being finite and simple. But then $\mathbb{L}$ is not linear, a contradiction. The proof is settled. □

Then,

Theorem 14 *Let $\mathbb{L}$ be a linear variety of MTL-algebras generated by a finite chain. Then, either $\mathbb{L} \subsetneq \mathbb{BP}_0$ or $\mathbb{L} \subseteq \mathbb{S}_n\mathbb{MTL}$, for some $k \geq 3$.*

Proof Immediate by Corollary 2, and the fact that every finite and simple MTL-chain satisfies $\sim(x^n) \sqcup x = 1$, for some $n \geq 0$. □

7 Linear varieties and finite model property

Theorem 15 *Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP. Then, every chain in $\mathbb{L}$ is either simple or bipartite.*

Proof Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP. If $\mathbb{L}$ is generated by a finite chain, then the result follows by Theorem 13. Suppose now that $\mathbb{L}$ is generated by an infinite chain, and assume by contradiction that there is a chain $\mathcal{A} \in \mathbb{L}$ being neither simple nor bipartite. Then, $\mathcal{A}$ is necessarily infinite. As $\mathbb{L}$ is linear, by Lemma 7 $\mathcal{A}$ cannot have a negation fixpoint. Since $\mathcal{A}$ is neither simple nor bipartite, we must have that $\{1\} \subsetneq \text{Rad}(\mathcal{A}) \subsetneq A^+$. By Lemma 1, $\text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}}(\mathcal{A})$ is the carrier of a subalgebra of $\mathcal{A}$, say $\mathcal{C}$. By construction, $\mathcal{C}$ is bipartite, and $|\mathcal{C}| \geq 3$. By Theorem 9, we have that $\mathbb{L} = \mathbf{V}(\mathcal{A})$.

Suppose that $\mathcal{C}$ is infinite. As $\mathcal{C}$ is bipartite, we have $\mathcal{C} \models$(BP$_0$), and since $\mathcal{A} \not\models$ (BP$_0$), this would imply that $\mathbf{V}(\mathcal{C}) \neq \mathbf{V}(\mathcal{A}) = \mathbb{L}$, in contrast to Theorem 9.

So, the only possibility is that $\mathcal{C}$ is finite. As $\mathbb{L}$ is linear, by Corollary 2 a finite chain in $\mathbb{L}$ is either simple or bipartite. Let $\mathcal{B} \in \mathbb{L}$ be a finite chain. If $|\mathcal{B}| \leq |\mathcal{C}|$, then $\mathcal{B} \in \mathbf{V}(\mathcal{C})$, and then $\mathcal{B}$ is necessarily bipartite. If $|\mathcal{B}| > |\mathcal{C}|$, then $\mathcal{C} \in \mathbf{V}(\mathcal{B})$: this implies that $\mathcal{B}$ cannot be simple, as otherwise $\mathcal{C} \notin \mathbf{HS}(\mathcal{B})$. Then, by Theorem 13, we must conclude that $\mathcal{B}$ is bipartite. As a consequence, all the finite chains in $\mathbb{L}$ must be bipartite. However, $\mathcal{L}$ has the finite model property, and since every finite chain satisfies (BP$_0$), this implies that every chain in $\mathbb{L}$ is bipartite, in contrast to the fact that $\mathcal{A} \not\models$ (BP$_0$).

Since in all the cases we obtain a contradiction, we must conclude that either every chain in $\mathbb{L}$ is simple or it is bipartite, and the proof is settled. □

This result can be strengthened:

Lemma 9 *Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP. If there is a chain in $\mathbb{L}$ being bipartite and not simple, then every chain in $\mathbb{L}$ is bipartite.*

Proof Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP, and assume that there is a chain $\mathcal{A} \in \mathbb{L}$ being bipartite and not simple. This means that $|\mathcal{A}| \geq 3$. If $\mathcal{A}$ is infinite, then by Theorem 9 $\mathbf{V}(\mathcal{A}) = \mathbb{L}$, and hence, every chain in $\mathbb{L}$ is bipartite.

Finally, assume that $\mathcal{A}$ is finite. Because of the linearity of $\mathbb{L}$ every chain $\mathcal{B} \in \mathbb{L}$ with $|\mathcal{B}| \leq |\mathcal{A}|$ is such that $\mathcal{B} \in \mathbf{V}(\mathcal{A})$, and hence $\mathcal{B}$ is bipartite. Moreover, every finite chain $\mathcal{C} \in \mathbb{L}$ with $|\mathcal{C}| > |\mathcal{A}|$ is such that $\mathcal{A} \in \mathbf{V}(\mathcal{C})$: by Theorem 13, $\mathcal{C}$ is simple or bipartite. However, $\mathcal{C}$ cannot be simple, as otherwise $\mathcal{A} \notin \mathbf{HS}(\mathcal{C})$ (remember that $|\mathcal{A}| \geq 3$), and hence, it must be bipartite. As a consequence, all the finite chains in $\mathbb{L}$ are bipartite, and by the FMP, we have that every chain in $\mathbb{L}$ is bipartite. □

Theorem 16 *Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP. Then,*

– *every chain in $\mathbb{L}$ is simple or*
– *every chain in $\mathbb{L}$ is bipartite.*

Proof Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP. If there is at least one $\mathcal{A} \in C$ which is bipartite, then by Lemma 9 every chain in $\mathbb{L}$ is bipartite.

If no chain in $\mathbb{L}$ is bipartite, then by Theorem 15 every chain in $\mathbb{L}$ must be simple. □

8 Linear varieties of WNM-algebras

In this section, we classify all the linear subvarieties of $\mathbb{WNM}$. We begin by introducing a new variety, called $\mathbb{F}$.

Definition 7 Let us call $\mathbb{F}$ the variety of WNM-algebras axiomatized as $\mathbb{WNM}$ plus:

$$\sim\left(\left(\sim\left(x^2\right)\right)^2\right) \Leftrightarrow \left(\sim\left((\sim x)^2\right)\right)^2 = 1, \tag{bp$_0$}$$

$$x \sqcup \sim\left(x^2\right) \sqcup y \sqcup ((y \Rightarrow x) \sqcap (\sim\sim x \Rightarrow x)) = 1. \tag{F}$$

The aim of this section is to show that the only linear varieties of WNM-algebras are the following ones:

– $\mathbb{G}$ and its subvarieties.
– $\mathbb{DP}$ and its subvarieties.
– $\mathbb{NM}^-$ and its subvarieties.
– $\mathbb{F}$ and its subvarieties, where every chain $\mathcal{A} \in \mathbb{F}$ is such that if $|A| > 2$, then $\mathcal{A}$ has a coatom c, $\sim\sim c = c$, and $\sim c$ is the predecessor of c.

We also show that all these varieties are strongly linear.

Lemma 10 *Let $\mathcal{A}$ be a WNM-chain, and let B be a subset of A, containing $\{0, 1\}$ and which is closed under negation. Then, B is the carrier of a subalgebra of $\mathcal{A}$.*

Proof Immediate by (1). □

Theorem 17 *Let $\mathbb{V}$ be a subvariety of $\mathbb{WNM}$ containing at least two among $\mathbf{G}_3$ or $\mathbf{NM}_4$ or $\mathbf{L}_3$. Then, $\mathbb{V}$ is not linear.*

Proof Immediate by the fact that if $\mathcal{A}, \mathcal{B} \in \{\mathbf{G}_3, \mathbf{NM}_4, \mathbf{L}_3\}$, and $\mathcal{A} \neq \mathcal{B}$, then $\mathbf{V}(\mathcal{A}) \not\subseteq \mathbf{V}(\mathcal{B})$, and $\mathbf{V}(\mathcal{B}) \not\subseteq \mathbf{V}(\mathcal{A})$. □

Theorem 18 *Let $\mathcal{A}$ be a WNM-chain having an element $0 < x < 1$ with $\sim x = 0$. If there is $0 < y < 1$ with $\sim y > 0$ (i.e., $\mathcal{A} \notin \mathbb{G}$), then $\mathbf{V}(\mathcal{A})$ is not linear.*

Proof Let $\mathcal{A}$ be a WNM-chain having an element $0 < x < 1$ with $\sim x = 0$, and $0 < y < 1$ with $\sim y > 0$. Then, by Lemma 10 the subalgebra of $\mathcal{A}$ generated by x is isomorphic to $\mathbf{G}_3$. Note that $\sim y$ is involutive, as $\sim\sim\sim y = \sim y$, and since $0 < y < 1$, we have $0 < \sim y < 1$, and $0 < y \leq \sim\sim y < 1$. As a consequence, by Lemma 10 the subalgebra of $\mathcal{A}$ generated by $\sim y$ is isomorphic to $\mathbf{L}_3$ or $\mathbf{NM}_4$, depending on whether $\sim y = \sim\sim y$ or $\sim y \neq \sim\sim y$. By Theorem 17, $\mathbf{V}(\mathcal{A})$ is not linear. □

Lemma 11 *Let $\mathcal{A}$ be a WNM-chain with negation fixpoint f. Then, $\mathcal{A}$ is a DP-chain if and only if for every $0 < x < 1$ it holds that $\sim x = f$.*

Proof Let $\mathcal{A}$ be a DP-chain with a negation fixpoint f. By Lemma 4, f is the coatom of $\mathcal{A}$, and for every $0 < x < 1$ it holds $\sim x = f$. Let now $\mathcal{A}$ be a WNM-chain with negation fixpoint f, and assume that for every $0 < x < 1$ it holds that $\sim x = f$. This implies that $x \leq f$, for every $0 < x < 1$: indeed, if x is greater than f, and $\sim x = f$, then $x * f = 0$, in contrast to the fact that $f = \sim f = \max\{z : z * f = 0\}$. Then, $x \leq f$, for every $0 < x < 1$, and by Lemma 4, we immediately see that $\mathcal{A}$ is a DP-chain. □

Theorem 19 *Let $\mathcal{A}$ be a WNM-chain with a negation fixpoint. If $|A| > 3$ and $\mathcal{A}$ is not a DP-chain, then $\mathbf{V}(\mathcal{A})$ is not linear.*

Proof Let $\mathcal{A}$ be a WNM-chain with a negation fixpoint f. Clearly if $|A| = 3$, then $\mathcal{A} \simeq \mathbf{L}_3$, and $\mathbf{V}(\mathcal{A})$ is linear. Assume now that $|A| > 4$ and that $\mathcal{A}$ is not a DP-chain. Then, by Lemma 11 there is $0 < x < 1$ such that $\sim x \neq f$: if $\sim x = 0$, then by Theorem 18 $\mathbf{V}(\mathcal{A})$ is not linear. Suppose then $0 < x, \sim x < 1$: since $\sim\sim\sim x = \sim x$, we have that $\sim x$ is involutive, and clearly $0 < x \leq \sim\sim x < 1$. Since $\sim x \neq f$ by hypothesis, we have $\sim\sim x \neq f$, and hence, $\sim\sim x$ is not a negation fixpoint. By Lemma 10, the subalgebra of $\mathcal{A}$ generated by $\sim x$ has $\{0, \sim x, \sim\sim x, 1\}$ as carrier, and it is isomorphic to $\mathbf{NM}_4$. On the other hand the subalgebra of $\mathcal{A}$ generated by f is isomorphic to $\mathbf{L}_3$, and by Theorem 17 $\mathbf{V}(\mathcal{A})$ is not linear. □

Lemma 12 *Let $\mathcal{A}$ be an MTL-chain without negation fixpoint such that A^+ has a minimum m, and $\sim\sim m = m$. Then, $\sim m = \max(A^-)$.*

Proof Let $\mathcal{A}$ be an MTL-chain such that A^+ has a minimum m, and $\sim\sim m = m$: then, clearly $\sim m < \sim\sim m = m$, and $\sim m \in A^-$. Suppose by contradiction that there is $a \in A^-$ such that $a > \sim m$. Then, $\sim a \in A^+$, and $\sim a \leq \sim\sim m = m$, which implies $\sim a = m$, and $\sim a * m = 0$. This, however, is a contradiction, as $\sim a > \sim m = \max\{z : z * m = 0\}$. □

Proposition 3 *The only chains in $\mathbb{F}$ are those WNM-chains having a coatom c with $\sim\sim c = c$, and $\sim c$ is the predecessor of c.*

Proof Let $\mathcal{A}$ be a WNM-chain belonging to $\mathbb{F}$. Since $\mathcal{A}$ satisfies bp_0, by Gispert (2003, Theorem 2) there is no negation fixpoint. Note that if $x = 1$ or $y = 1$ or $x \leq \sim x$ or $y \leq \sim y$, then F holds.

Suppose that $|A^+| \geq 3$: then, we can find $x, y \in A^+ \setminus \{1\}$ with $x > y$. Since $x \Rightarrow y < 1$, $\sim(x^2) < 1$, $\sim(y^2) < 1$, we immediately have that F is not satisfied. So we must have that $|A^+| \leq 2$. If $|A^+| = 1$, then $\mathcal{A} \simeq \mathbf{2}$ or $\mathcal{A}$ must be a DP-chain; however, we exclude this last case, as $\mathcal{A}$ does not have a negation fixpoint. If $|A^+| = 2$, then $\mathcal{A}$ has a coatom $0 < c < 1$, and $A^+ = \{c, 1\}$. Then, F is satisfied if $\{x, y\} = A^+$. The only case left is $x = y = c$, and a direct inspection shows that F holds true if and only if $\sim\sim c = c$. As $c = \min(A^+)$ by Lemma 12 we have $\sim c = \max(A^-)$ (remember that $\mathcal{A}$ does not have negation fixpoint), and hence $\sim c$ is the predecessor of c. Then, we conclude that every chain in $\mathbb{F}$ is a WNM-chain having a coatom c with $\sim\sim c = c$, and $\sim c$ is the predecessor of c. □

By Proposition 3, (1) and the fact that $\sim$ is order reversing, we also have that:

Lemma 13 *Let $\mathcal{A}$ be a chain in $\mathbb{F}$, and let c be its coatom. Then, for every $0 < x < \sim c$ (if any) it holds that $\sim x = c$. In particular, the operations $*$ and $\Rightarrow$ are the following ones, for every $x, y \in A$:*

$$x * y = \begin{cases} \min\{x, y\} & \text{if } x, y \geq c \text{ or} \\ & x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ c & \text{if } 0 \leq y < x \leq \sim c, \\ \sim c & \text{if } x = c \text{ and } y < c, \\ y & \text{otherwise.} \end{cases}$$

Theorem 20 *Every infinite chain in $\mathbb{F}$ is generates the whole variety.*

Proof As $\mathbb{F} \subsetneq \mathbb{WNM}$, we have that $\mathbb{F}$ is locally finite, and then it has the finite model property. By Proposition 3, we have that every chain $\mathcal{A} \in \mathbb{F}$ has a coatom c, and $\sim c$ is its predecessor. It follows that if $\mathcal{A}$ has more than 4 elements, then there is $0 < x < \sim c$: in particular, if $\mathcal{A}$ is infinite, then it has infinitely many elements between 0 and $\sim c$. As $\sim$ is order reversing, for every $0 < x < \sim c$ it holds that $\sim x \geq \sim\sim c = c$, and hence, $\sim x = c$. So, if $\mathcal{B}$ is an infinite chain in $\mathbb{F}$, it is immediate to check that every finite chain in $\mathbb{F}$ can be embedded into $\mathcal{B}$, and hence $\mathbf{V}(\mathcal{B}) = \mathbb{F}$. □

Theorem 21 *Let $\mathcal{A}, \mathcal{B}$ be two chains in $\mathbb{F}$. Then*

- *If $|\mathcal{A}| < |\mathcal{B}|$, then $\mathcal{A} \hookrightarrow \mathcal{B}$.*
- *If $\langle A, \leq_A \rangle \simeq \langle B, \leq_B \rangle$, then $\mathcal{A} \simeq \mathcal{B}$.*

– *The only proper subvarieties of* $\mathbb{F}$ *are those singly generated by a finite chain.*

Proof Let $\mathcal{A}, \mathcal{B}$ be two chains in $\mathbb{F}$.

Assume first that $|\mathcal{A}| < |\mathcal{B}|$: observe that by Proposition 3 a chain in $\mathbb{F}$ has more than four elements if and only if there is at least one $0 < b < \sim c$, where c is its coatom. Hence by Proposition 3 and Lemma 13, we immediately have that $\mathcal{A} \hookrightarrow \mathcal{B}$. As an immediate consequence of Lemma 13 if $\langle A, \leq_A\rangle \simeq \langle B, \leq_B\rangle$, then $\mathcal{A} \simeq \mathcal{B}$. From this last fact and Theorem 20, we have that the only proper subvarieties of $\mathbb{F}$ are the ones generated by a finite chain. □

In the following Theorem 22, we shall use the WNM-chain $\mathcal{Q}$, first introduced in Aguzzoli and Bianchi (2016).

Definition 8 Let us call $\mathcal{Q}$ the WNM-chain $\langle\{0, a, b, c, 1\}, *, \Rightarrow, \min, \max, 0, 1\rangle$, with $0 < a < b < c < 1$, and whose negation $\sim$ is defined as follows.

x	$\sim x$
0	1
a	c
b	a
c	a
1	0

Let us call $\mathbb{Q}$ the variety generated by $\mathcal{Q}$.

Theorem 22 *Let* $\mathcal{A}$ *be a WNM-chain such that there is* $0 < x < 1$ *with* $\sim\sim x = x$ *and* $\sim x \neq x$. *If* $\mathcal{A} \notin \mathbb{NM} \cup \mathbb{F}$, *then* $\mathbf{V}(\mathcal{A})$ *is not linear.*

Proof Let $\mathcal{A}$ be a WNM-chain such that there is $0 < x < 1$ with $\sim\sim x = x$, $\sim x \neq x$, and $\mathcal{A} \notin \mathbb{NM} \cup \mathbb{F}$. Since $\mathcal{A} \notin \mathbb{NM}$, there is $0 < y < 1$ with $\sim\sim y > y$. We have two cases.

Assume first that there is $z \notin \{x, \sim x\}$ with $0 < z < 1$ and $\sim\sim z = z$. If $\sim z = z$, then by Theorem 19 $\mathbf{V}(\mathcal{A})$ is not linear. Assume now $\sim z \neq z$: note that the subalgebra of $\mathcal{A}$ generated by $\{x, z\}$ has $\{0, x, z, \sim x, \sim z, 1\}$ as carrier, and it is isomorphic to $\mathbf{NM}_6$. However, the subalgebra of $\mathcal{A}$ generated by y, say $\mathcal{B}$, has at most 5 elements, and it is not an NM-chain (since y is not involutive). It follows that $\mathbf{V}(\mathcal{B}) \nsubseteq \mathbb{NM}_6$, $\mathbb{NM}_6 \nsubseteq \mathbf{V}(\mathcal{B})$, and $\mathbf{V}(\mathcal{A})$ is not linear.

The second case is when x and $\sim x$ are the only involutive elements between 0 and 1: without loss of generality assume that $\sim x < x$. Since $\mathcal{A} \notin \mathbb{F}$, by Proposition 3 there is $x < a < 1$ or there is $\sim x < b < x$. Assume first that there is $x < a < 1$: if $\sim a = 0$ by Theorem 18 $\mathbf{V}(\mathcal{A})$ is not linear. We exclude the case $\sim\sim a = a$, as x and $\sim x$ are the only involutive elements between 0 and 1. Then, suppose $\sim\sim a > a$, and $\sim a > 0$: as $\sim\sim\sim a = \sim a$, $\sim a$ is involutive, and we must have $\sim a \in \{x, \sim x\}$, as $1 > a, \sim a > 0$. However, $\sim\sim a > a > x > \sim x$, and hence, we have a contradiction.

Suppose now that there is $\sim x < b < x$. since $\sim$ is order reversing, we have $\sim b > \sim x > 0$, and we exclude the case $\sim\sim b = b$, as x and $\sim x$ are the only involutive elements between 0 and 1. So $\sim\sim b > b$, and $\sim b > 0$: since $\sim\sim\sim b = \sim b$, we have that $\sim b$ is involutive, and since $1 > b, \sim b > 0$, we have $\sim b \in \{x, \sim x\}$. Because $\sim\sim b > b > \sim x$, the only possibility is $\sim b = \sim x$, and hence $\sim x = \sim b < b < x$. This implies that b is idempotent. An easy check shows that the subalgebra of $\mathcal{A}$ generated by b is isomorphic to the chain $\mathcal{Q}$ described in Definition 8. It is easy to show that $\mathcal{Q}/\{x, 1\} \simeq \mathbf{G}_3$, while the subalgebra of $\mathcal{A}$ generated by x is isomorphic to $\mathbf{NM}_4$. Since $\mathbb{NM}_4 \nsubseteq \mathbb{G}_3$ and $\mathbb{G}_3 \nsubseteq \mathbb{NM}_4$, we conclude that $\mathbf{V}(\mathcal{A})$ is not linear. Since there are no other cases, this concludes the proof. □

Theorem 23 *The only linear subvarieties of* $\mathbb{WNM}$ *are the following ones.*

– $\mathbb{G}$ *and its subvarieties.*
– $\mathbb{DP}$ *and its subvarieties.*
– $\mathbb{NM}^-$ *and its subvarieties.*
– $\mathbb{F}$ *and its subvarieties.*

Proof First of all, by the results of Hecht and Katriňák (1972), Aguzzoli et al. (2014), Bianchi (2015) and Gispert (2003) we have that $\mathbb{G}$, $\mathbb{DP}$, $\mathbb{NM}^-$ and their subvarieties are linear. By Theorems 20 and 21, we have that also $\mathbb{F}$ and its subvarieties are linear.

Take now a WNM-chain $\mathcal{A}$ in $\mathbb{WNM}\setminus(\mathbb{G}\cup\mathbb{DP}\cup\mathbb{NM}^-\cup\mathbb{F})$: we show that $\mathbf{V}(\mathcal{A})$ is not linear. Notice first that $|\mathcal{A}| \geq 4$, as all the WNM-chains with two or three elements are contained in $\mathbb{G} \cup \mathbb{DP}$. If $\mathcal{A}$ has a negation fixpoint, then since $\mathcal{A} \notin \mathbb{DP}$ by Theorem 19 $\mathbf{V}(\mathcal{A})$ is not linear. So, assume that $\mathcal{A}$ does not have negation fixpoint, and pick $0 < x < 1$, with $x \in \mathcal{A}$. If $\sim x = 0$, since $\mathcal{A} \notin \mathbb{G}$ by Theorem 18 $\mathbf{V}(\mathcal{A})$ is not linear. Suppose now that $\sim\sim x = x$ and $\sim x \neq x$: since $\mathcal{A} \notin \mathbb{NM}^- \cup \mathbb{F}$, and it has no negation fixpoint, then $\mathcal{A} \notin \mathbb{NM} \cup \mathbb{F}$. By Theorem 22, $\mathbf{V}(\mathcal{A})$ is not linear. The last case is when $\sim\sim x > x$ and $\sim x > 0$. Note that $\sim\sim\sim x = \sim x$, and clearly $0 < \sim\sim x < 1$. Again, by Theorem 22 $\mathbf{V}(\mathcal{A})$ is not linear. □

We now show that all these varieties are also strongly linear.

Lemma 14 *Let* $\mathbb{L} \in \{\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}\}$. *Then, for every pair of chains* $\mathcal{A}, \mathcal{B} \in \mathbb{L}$ *the following holds:*

– *If* $|\mathcal{A}| < |\mathcal{B}|$, *then* $\mathcal{A} \hookrightarrow \mathcal{B}$.
– *If* $\langle A \leq_A\rangle \simeq \langle B, \leq_B\rangle$, *then* $\mathcal{A} \simeq \mathcal{B}$.

Proof Immediate by the results of Hecht and Katriňák (1972), Bianchi (2015), Aguzzoli et al. (2014) and Gispert (2003) and Theorem 21. □

Theorem 24 *The varieties $\mathbb{G}$, $\mathbb{DP}$, $\mathbb{NM}^-$, and $\mathbb{F}$ are strongly linear.*

Proof Let $\mathbb{L} \in \{\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}\}$. Since by Theorem 23 $\mathbb{L}$ is linear, and it is locally finite, by Theorem 8 we have that for every infinite chain $\mathcal{A}$, $\mathbf{V}(\mathcal{A}) = \mathbb{L}$, and every subvariety of $\mathbb{L}$ is generated by a finite chain. By Lemma 14, given a finite L-chain $\mathcal{A}$ it holds that every chain in $\mathbf{V}(\mathcal{A})$ is a subalgebra of $\mathcal{A}$. Hence by Cintula et al. (2009, Theorem 3.5) the logic related to $\mathbf{V}(\mathcal{A})$ has the SSCC, and as a consequence, every proper subvariety of $\mathbb{L}$ is strongly linear. To conclude the proof, it remains to show that L has the SSCC: let $\mathcal{A}$ be an infinite L-chain. By Lemma 14, we have that every finite L-chain is embeddable into $\mathcal{A}$, and since $\mathbb{L}$ is locally finite it follows that every L-chain is partially embeddable into $\mathcal{A}$. By Cintula et al. (2009, Theorem 3.8) L has the FSSCC, and by Theorem 4, we conclude that L has the SSCC, and $\mathbb{L}$ is strongly linear. □

9 Linear varieties of BL-algebras

In this section, we classify all the linear varieties of BL-algebras.

We start with the following result.

Theorem 25 (Bianchi and Montagna 2009, Lemma 7) *Let L be an axiomatic extension of BL which is not n-contractive, for any n. Then, $\mathbb{L}$ contains the variety of product algebras $\mathbb{P}$ or the variety generated by Chang's MV-algebra, $\mathbf{K}_2$.*

Hence, in the rest of the section we analyze three cases. Specifically, the n-contractive linear varieties of BL-algebras, and the cases of linear varieties of BL-algebras containing $\mathbb{P}$ or $\mathbb{C}$.

We begin by presenting a family of varieties that will be shown to be the only linear varieties of BL-algebras containing $\mathbb{P}$.

Definition 9 – Let $\mathbb{P}_\infty$ be the variety axiomatized as $\mathbb{BL}$ plus (BP_0) and

$$\sim x \sqcup (x \Rightarrow (x * x)) \Rightarrow x = 1. \tag{SC}$$

– For $k \geq 1$, let $\mathbb{P}_k$ be the variety axiomatized as $\mathbb{P}_\infty$ plus:

$$\left(\prod_{i=0}^{k}((x_{i+1} \Rightarrow x_i) \Rightarrow x_i)\right) \Rightarrow \left(\bigsqcup_{i=0}^{k+1} x_i\right) = 1. \tag{λ_{k+1}}$$

The rest of the section is devoted to prove the following results.

– The only n-contractive varieties of BL-algebras which are linear are $\mathbb{G}$ and its subvarieties, the family of varieties $\{\mathbb{L}_k : k-1 = h^n, \text{ where } n \geq 1, \text{ and } 1 \leq h \text{ is prime}\}$, and the family of varieties $\{\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k) : k-1 = h^n, \text{ where } n \geq 1, \text{ and } 1 \leq h \text{ is prime}\}$.
– The only variety of BL-algebras which is linear and contains $\mathbb{C}$ is $\mathbb{C}$ itself.
– The only varieties of BL-algebras which are linear and contain $\mathbb{P}$ are the variety $\mathbb{P}_\infty$, and the family $\{\mathbb{P}_k\}_{k\geq 2}$. In particular, $\mathbb{P} = \mathbb{P}_1 \subsetneq \mathbb{P}_2 \subsetneq \mathbb{P}_3 \subsetneq \ldots \subsetneq \mathbb{P}_\infty$, where every chain in $\mathbb{P}_\infty$ ($\mathbb{P}_k, k \geq 1$, respectively) has the form $\mathbf{2} \oplus \mathcal{C}$, where $\mathcal{C}$ is an ordinal sum of cancellative hoops ($\mathcal{C}$ is an ordinal sum of at most k cancellative hoops, respectively).

We show that all these varieties are also strongly linear.

9.1 Linear *n*-contractive varieties of BL-algebras

This first subsection is devoted to the classification of linear n-contractive varieties of BL-algebras.

Lemma 15 *Let $\mathcal{A}$ be a finite BL-chain. Then, the class of non-trivial chains in $\mathbf{V}(\mathcal{A})$ coincides with $\mathbf{IS}(\mathcal{A})$.*

Proof Let $\mathcal{A}$ be a finite BL-chain. By Theorem 5, the class of chains in $\mathbf{V}(\mathcal{A})$ coincides with $\mathbf{HS}(\mathcal{A}) = \mathbf{SH}(\mathcal{A})$. Note that every finite MTL-chain is n-contractive, for some n, and by Bianchi and Montagna (2011, Proposition 1) every n-contractive BL-chain is isomorphic to an ordinal sum of finite MV-chains with at most n-elements, which are all simple. It follows that $\mathcal{A}$ is simple or it is isomorphic to an ordinal sum of finite simple MV-chains, and in this last case, the filters are given by $\{1\}$ and the upper sets of the bottom elements of each component. Hence, by Definition 3 we have that $\mathbf{H}(\mathcal{A}) \subseteq \mathbf{IS}(A \cup \{\mathbf{1}\})$, and then $\mathbf{HS}(\mathcal{A}) = \mathbf{SH}(\mathcal{A}) = \mathbf{IS}(A \cup \{\mathbf{1}\})$. We conclude that every non-trivial chain in $\mathbf{V}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{A}$. □

Theorem 26 *Let $\mathbb{L}$ be a variety generated by an n-contractive BL-chain $\mathcal{A} \simeq \bigoplus_{i\in I} \mathcal{A}_i$ such that $\mathcal{A} \notin \mathbb{G}$, and $|I| > 2$. Then $\mathbb{L}$ is not linear.*

Proof Let $\mathbb{L}$ be a variety generated by an n-contractive BL-chain $\mathcal{A} \simeq \bigoplus_{i\in I} \mathcal{A}_i$ such that $\mathcal{A} \notin \mathbb{G}$, and $|I| > 2$. By Bianchi and Montagna (2011, Proposition 1) every $\mathcal{A}_i$ is isomorphic to an ordinal sum of finite MV-chains with at most n-elements. Since $\mathcal{A} \notin \mathbb{G}$, there is $k \in I$ such that $\mathcal{A}_k \simeq \mathbf{L}_h$, with $h > 2$. Since $\mathcal{A}$ has more than two components, it has $\mathbf{G}_4$ as a subalgebra (just take the subalgebra generated by the bottom elements of two components different from the first one). Let now $\mathcal{B}$ be the subalgebra generated by $\mathcal{A}_k$: we have that $\mathcal{B} \simeq \mathbf{L}_h$ or $\mathcal{B} \simeq \mathbf{2} \oplus \mathbf{L}_h$. In both the cases, we have that $\mathbf{G}_4 \notin \mathbf{V}(\mathcal{B})$: indeed by Lemma 15 the non-trivial chains in $\mathbf{V}(\mathcal{B})$ are isomorphic to the subalgebras of

$\mathcal{B}$. Since $\mathcal{B} \not\models x = x^2$, we have $\mathcal{B} \notin \mathbf{V}(\mathbf{G}_4)$, and hence $\mathbf{V}(\mathcal{B}) \nsubseteq \mathbf{V}(\mathbf{G}_4)$ and $\mathbf{V}(\mathbf{G}_4) \nsubseteq \mathbf{V}(\mathcal{B})$. As a consequence, $\mathbb{L}$ is not linear, and this concludes the proof. □

Theorem 27 *A variety of the form $\mathbb{L}_k$ is linear if and only if $k-1 = h^n$, where $n \geq 1$, and $1 \leq h$ is prime.*

Proof By the results of Grigolia (1977), we have that for $2 \leq l < m$, $\mathbf{L}_l \in \mathbb{L}_m$ if and only if $l-1$ divides $m-1$, and by Lemma 15, this happens if and only if $\mathbf{L}_l \hookrightarrow \mathbf{L}_m$, which implies $\mathbb{L}_l \subseteq \mathbb{L}_m$.

It is then immediate to check that for each integer $k > 0$, the variety $\mathbb{L}_k$ is linear if and only if $k = h^n + 1$ for some prime number h and some integer $n > 0$, as the only divisors of $k-1$ are the numbers of the form h^m for $0 < m \leq n$. Indeed if k is not of that form, there exist two divisors of $k-1$, say r, s, which are not one the multiple of the other. But then $\mathbf{L}_r \notin \mathbb{L}_s$, $\mathbf{L}_s \notin \mathbb{L}_r$, and $\mathbb{L}_k$ would not be linear. □

We conclude the section with the classification of linear n-contractive varieties of BL-algebras.

Theorem 28 *The only linear n-contractive varieties of BL-algebras are the following ones.*

- *The variety of Gödel algebras.*
- *Every variety generated by a chain of the form $\mathbf{G}_k$, for $k \geq 2$.*
- *Every variety generated by a chain of the form $\mathbf{L}_k$, with $1 \leq k-1 = h^n$, where h is prime and $n \geq 1$.*
- *Every variety generated by a chain of the form $\mathbf{2} \oplus \mathbf{L}_k$, with $1 \leq k-1 = h^n$, where h is prime and $n \geq 1$.*

Proof As shown in Hecht and Katriňák (1972), $\mathbb{G}$ is linear, as well as $\mathbb{G}_k$, with $k \geq 2$. By Theorem 27 we have that if $1 \leq k-1 = h^n$, where h is prime, and $n \geq 1$, then $\mathbb{L}_k$ is linear.

Pick now $\mathcal{B} \simeq \mathbf{2} \oplus \mathbf{L}_k$, with $1 \leq k-1 = h^n$, where h is prime and $n \geq 1$. By Lemma 15 the only non-trivial chains in $\mathbf{V}(\mathcal{B})$ are the subalgebras of $\mathbf{2} \oplus \mathbf{L}_k$, that is, the chains isomorphic to $\mathbf{2} \oplus \mathbf{L}_h$, where $h-1$ divides $k-1$. By Theorem 27, for every pair of subalgebras of $\mathcal{B}$, say $\mathbf{2} \oplus \mathbf{L}_i, \mathbf{2} \oplus \mathbf{L}_j$, with $i < j$, we have $\mathbf{2} \oplus \mathbf{L}_i \hookrightarrow \mathbf{2} \oplus \mathbf{L}_j$, and hence, $\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_i) \subsetneq \mathbf{V}(\mathbf{2} \oplus \mathbf{L}_j)$. We conclude that $\mathbf{V}(\mathcal{B})$ is linear.

Finally, pick an n-contractive BL-chain $\mathcal{C}$ which is not of the type specified in the theorem.

If $\mathcal{C}$ is an MV-chain, then by Theorem 27 $\mathbf{V}(\mathcal{C})$ is not linear. If $\mathcal{C}$ has more than two components, since by hypothesis $\mathcal{C} \notin \mathbb{G}$, by Theorem 26 we have that $\mathbf{V}(\mathcal{C})$ is not linear. If $\mathcal{C}$ is of the form $\mathbf{2} \oplus \mathbf{L}_i$ with i not being of the form $h^n + 1$ for any prime number h and any integer $n > 0$, then $i-1$ has two divisors j_1 and j_2 which are not multiple the one of the other, as in the proof of Theorem 27. Then, both $\mathbf{2} \oplus \mathbf{L}_{j_1+1}$ and $\mathbf{2} \oplus \mathbf{L}_{j_2+1}$ embeds into $\mathbf{2} \oplus \mathbf{L}_i$, which is not linear, as $\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_{j_1+1}) \nsubseteq \mathbf{V}(\mathbf{2} \oplus \mathbf{L}_{j_2+1})$ and $\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_{j_2+1}) \nsubseteq \mathbf{V}(\mathbf{2} \oplus \mathbf{L}_{j_1+1})$. The last case is when $\mathcal{C}$ is isomorphic to an ordinal sum of exactly two (non-trivial) components, and the first one, say $\mathcal{C}_0$ has more than two elements, say h. So, $\mathcal{C}_0 \simeq \mathbf{L}_h$, but notice that the subalgebra generated by the bottom element of the second component of $\mathcal{C}$ is isomorphic to $\mathbf{G}_3$. Since $\mathbf{V}(\mathbf{G}_3) \nsubseteq \mathbf{V}(\mathbf{L}_h)$, and $\mathbf{V}(\mathbf{L}_h) \nsubseteq \mathbf{V}(\mathbf{G}_3)$, we conclude that $\mathbf{V}(\mathcal{C})$ is not linear. Since there are no other cases, this concludes the proof. □

9.2 Linear varieties of BL-algebras containing $\mathbb{P}$

We start with the following characterization.

Theorem 29 *Let $\mathbb{L}$ be a variety of BL-algebras such that $\mathbb{P} \subsetneq \mathbb{L}$. If $\mathbb{L}$ contains a BL-chain $\mathcal{A} \simeq \bigoplus_{i \in I} \mathcal{A}_i$ such that:*

- *$\mathcal{A}_0$ is an MV-chain with more than two elements or*
- *there is $i > 0$ such that $\mathcal{A}_i$ is bounded, that is, $\mathcal{A}_i$ is an MV-chain,*

then $\mathbb{L}$ is not linear.

Proof Let $\mathbb{L}$ be a variety of BL-algebras such that $\mathbb{P} \subsetneq \mathbb{L}$,and that contains a BL-chain $\mathcal{A} \simeq \bigoplus_{i \in I} \mathcal{A}_i$ with the properties mentioned in the statement.

If $\mathcal{A}_0$ is an MV-chain with more than two elements, then observe $\mathcal{A}_0 \in \mathbb{L}$, since it is a subalgebra of $\mathcal{A}$. Since $\mathbf{V}(\mathcal{A}_0) \nsubseteq \mathbb{P}$ and $\mathbb{P} \nsubseteq \mathbf{V}(\mathcal{A}_0)$, we have that $\mathbb{L}$ is not linear.

The last case is when there is $i > 0$ such that $\mathcal{A}_i$ is bounded: let us call 0_i the minimum of this component. Note that the algebra generated by 0_i is isomorphic to the three elements Gödel chain $\mathbf{G}_3$. Since $\mathbf{V}(\mathbf{G}_3) \nsubseteq \mathbb{P}$, and $\mathbb{P} \nsubseteq \mathbf{V}(\mathbf{G}_3)$, we conclude that $\mathbb{L}$ is not linear. □

Corollary 3 *Let $\mathbb{L}$ be a variety of BL-algebras such that $\mathbb{P} \subsetneq \mathbb{L}$. If $\mathbb{L}$ is linear, then every chain has the form $\mathbf{2} \oplus \bigoplus_{I \in i} \mathcal{C}_i$, where every $\mathcal{C}_i$ is a cancellative hoop.*

Proposition 4 *Let $\mathcal{A}$ be an infinite totally ordered cancellative hoop. Then, every other totally ordered cancellative hoop can be partially embedded into it.*

Proof Let $\mathcal{C}_\infty = \langle \{a_i : i \in \mathbb{N}\}, *, \sqcap, \Rightarrow \rangle, 1$ be a hoop such that $a_0 < a_1 < a_2 < \ldots$, and for every $a_i, a_j \in \mathcal{C}_\infty$, $a_i * a_j = a_{i+j}, a_i \Rightarrow a_j = a_{\max\{0, j-i\}}$. As shown in Aglianò et al. (2007, Theorem 6.2), the chain $\mathbf{2} \oplus \mathcal{C}_\infty$ generates the variety of product algebras as quasivariety, and hence by Cintula et al. (2009, Theorem 3.8) every product chain is partially embeddable into $\mathbf{2} \oplus \mathcal{C}_\infty$. Since a BL-chain is a product chain if and only if it has the form $\mathbf{2} \oplus \mathcal{B}$, where $\mathcal{B}$ is a totally ordered cancellative hoop, we conclude that every totally ordered cancellative hoop is partially embeddable into $\mathcal{C}_\infty$. Moreover, in Aglianò et al. (2007, Lemma 6.1) it is shown that $\mathcal{C}_\infty$

is embeddable into every infinite totally ordered cancellative hoop. Hence, we conclude that if $\mathcal{A}$ is an infinite totally ordered cancellative hoop, then every other totally ordered cancellative hoop can be partially embedded into it. □

Proposition 5 – *Given $k \geq 2$, every BL-chain of the form $\mathbf{2} \oplus \mathcal{C}$, where $\mathcal{C}$ is an ordinal sum of k infinite cancellative hoops, generates the same variety.*

– *Every BL-chain of the form $\mathbf{2} \oplus \mathcal{C}$, where $\mathcal{C}$ is an ordinal sum of infinitely many infinite cancellative hoops, generates the same variety.*

Proof Pick two BL-chains $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$ and $\mathcal{B} \simeq \mathbf{2} \oplus \bigoplus_{j \in J} \mathcal{D}_i$ such that for every i, j, $\mathcal{C}_i$ and $\mathcal{D}_j$ are infinite totally ordered cancellative hoops.

Assume first that $I = J = \{1, 2, \ldots, k\}$. By Proposition 4, we have that, for every $i \in \{1, \ldots, k\}$, $\mathcal{C}_i$ is partially embeddable in $\mathcal{D}_i$, and viceversa.

Hence, $\mathbf{V}\left(\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i\right) = \mathbf{V}\left(\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{D}_i\right)$. This shows the first claim of the theorem.

Suppose now that I, J are both infinite, and suppose by contradiction that there is an equation $e = 1$ that is satisfied in $\mathcal{A}$ but not in $\mathcal{B}$. Clearly, in e there occur only finitely many variables, say $x_1, \ldots, x_n$. Then, there is an n-tuple $(j_1, \ldots, j_n)$ of elements in B such that $e^{\mathcal{B}}(j_1, \ldots, j_n) < 1^{\mathcal{B}}$. By Definition 3, an easy and well-known inductive argument shows that for all subterms t of e we have that the value $t^{\mathcal{B}}(j_1, \ldots, j_n)$ is either 0 or it lies in a summand $\mathcal{C}_k$ for some index k belonging to a finite subset K of J such that $|K| \leq n$. Recall now that by Proposition 4, for every $h, k > 0$, $\mathcal{D}_h$ is partially embeddable into $\mathcal{C}_k$ whence $\mathbf{2} \oplus \bigoplus_{j \in J} \mathcal{D}_j$ partially embeds into $\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$ and the equation $e = 1$ fails in $\mathcal{A}$, too: we have reached a contradiction. We conclude that $\mathbf{V}\left(\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i\right) = \mathbf{V}\left(\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{D}_i\right)$, and the proof is settled. □

We now provide a characterization of the chains in $\mathbb{P}_\infty$ and in $\mathbb{P}_k$, for $k \geq 1$:

Theorem 30 *1. A BL-chain belongs to $\mathbb{P}_\infty$ if and only if it has the form $\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where each $\mathcal{C}_i$ is a totally ordered cancellative hoop.*

2. *$\mathbb{P}_\infty$ is generated by each BL-chain of the form $\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where I is infinite, and each $\mathcal{C}_i$ is a totally ordered infinite cancellative hoop.*
3. *For $k \geq 1$, a BL-chain belongs to $\mathbb{P}_k$ if and only if it has the form $\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where $0 \leq |I| \leq k$.*
4. *For $k \geq 1$, $\mathbb{P}_k$ is the variety generated by each BL-chain of the form $\mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where $|I| = k$, and each $\mathcal{C}_i$ is a totally ordered infinite cancellative hoop.*
5. *The only subvarieties of $\mathbb{P}_\infty$ are the ones of the form $\mathbb{P}_k$, for $k \geq 1$, and the variety of Boolean algebras.*
6. *If $i < j$, $\mathbb{P}_i \subsetneq \mathbb{P}_j$. In particular, $\mathbb{P}_1$ coincides with $\mathbb{P}$.*

Proof 1. Let $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where each $\mathcal{C}_i$ is a totally ordered cancellative hoop. Clearly $\mathcal{A}$ does not have a negation fixpoint, and $\text{Rad}(\mathcal{A})$, the largest filter of $\mathcal{A}$ coincides with $A^+ = \bigoplus_{i \in I} \mathcal{C}_i$. Then, by Noguera et al. (2005, Theorem 3.20) $\mathcal{A} \models (\text{BP}_0)$. An easy check shows also that $\mathcal{A} \models \text{SC}$.

Pick now $\mathcal{A} \in \mathbb{P}_\infty$. Since $\mathcal{A} \models \text{SC}$, there is no idempotent element $0 < x < 1$. Indeed, if not then $\sim x = 0$, as the non-Boolean idempotents of a BL-chain are the bottom elements of the Wajsberg components, and $(x \Rightarrow (x * x)) \Rightarrow x = 1 \Rightarrow x = x < 1$. However, this would imply $\mathcal{A} \not\models \text{SC}$, a contradiction. So, if $\mathcal{A} \simeq \bigoplus_{i \in I} \mathcal{A}_i$, with $|I| > 1$, then for every $i > 0$ $\mathcal{A}_i$ must be an unbounded hoop, as otherwise its minimum would be idempotent between 0 and 1. If $\mathcal{A}$ is an MV-chain, then it cannot have more than three elements: indeed, if $|\mathcal{A}| > 3$, then there would be at least one element $0 < a < 1$ with $0 < a < \sim a < 1$, and $(a \Rightarrow (a * a)) \Rightarrow a = \sim a \Rightarrow a < 1$, a contradiction. Moreover, $\mathcal{A} \models (\text{BP}_0)$, and by Noguera et al. (2005, Theorem 3.20) $\mathcal{A}$ does not have a negation fixpoint. Hence, if $\mathcal{A}$ is an MV-chain, then $\mathcal{A} \simeq \mathbf{2}$. Since every BL-chain has an MV-chain as first component of its decomposition as ordinal sum, we conclude that if $\mathcal{A} \in \mathbb{P}_\infty$, then $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where each $\mathcal{C}_i$ is a totally ordered cancellative hoop.

2. Let $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$ and $\mathcal{B} \simeq \mathbf{2} \oplus \bigoplus_{j \in J} \mathcal{D}_i$ where for every i, j, $\mathcal{C}_i$ and $\mathcal{D}_j$ are infinite totally ordered cancellative hoops. Assume also that I is infinite. By the proof of Proposition 5, we have that $\mathcal{B}$ is partially embeddable into $\mathcal{A}$. Hence, by point 1 of the present theorem we have that $\mathbf{V}(\mathcal{A}) = \mathbb{P}_\infty$.
3. By definition, $\mathbb{P}_k$ $(k \geq 1)$ is a subvariety of $\mathbb{P}_\infty$. Pick $\mathcal{A} \in \mathbb{P}_k$, with $k \geq 1$. As $\mathcal{A} \models \lambda_{k+1}$, by 1 and Aglianò and Montagna (2003, Lemma 4.2) we have that $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$, where $0 \leq |I| \leq k$.
4. Let $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i \in I} \mathcal{C}_i$ and $\mathcal{B} \simeq \mathbf{2} \oplus \bigoplus_{j \in J} \mathcal{D}_i$ where for every i, j, $\mathcal{C}_i$ and $\mathcal{D}_j$ are infinite totally ordered cancellative hoops. Assume also that $|J| \leq |I| = k - 1$. By the proof of Proposition 5, we have that $\mathcal{B}$ is partially embeddable into $\mathcal{A}$, and hence by 3 we have that $\mathbf{V}(\mathcal{A}) = \mathbb{P}_k$.
5. Immediate by 1–4.
6. Immediate by 3, 4, 5.

□

Finally,

Theorem 31 *The only linear varieties of BL-algebras containing $\mathbb{P}$ are $\mathbb{P}_\infty$, and $\{\mathbb{P}_k\}_{k \geq 2}$.*

Proof Immediate by Corollary 3 and Theorem 30. □

9.3 Linear varieties of BL-algebras containing $\mathbb{C}$

Theorem 32 *Let $\mathbb{L}$ be a variety of BL-algebras containing the variety $\mathbb{C}$ generated by Chang's MV-algebra. If $\mathbb{L}$ contains a BL-chain $\mathcal{A}$ such that:*

- *$\mathcal{A}$ is not an MV-chain or*
- *$\mathcal{A}$ is a simple MV-chain with more than two elements,*

then $\mathbb{L}$ is not linear.

Proof $\mathbb{L}$ be a variety of BL-algebras containing the variety $\mathbb{C}$ generated by Chang's MV-algebra.

Assume first that $\mathbb{L}$ contains a BL-chain $\mathcal{A}$ which is not an MV-chain. Then, we have $\mathcal{A} \simeq \bigoplus_{i\in I} \mathcal{A}_i$, where $|I| \geq 2$. If there is $i > 0$ such that $\mathcal{A}_i$ is bounded, let 0_i be the minimum of such component. Note that the algebra generated by 0_i is isomorphic to the three elements Gödel chain $\mathbf{G}_3$. Since $\mathbf{V}(\mathbf{G}_3) \nsubseteq \mathbb{C}$, and $\mathbb{C} \nsubseteq \mathbf{V}(\mathbf{G}_3)$, we have that $\mathbb{L}$ is not linear. If there is $i > 0$ such that $\mathcal{A}_i$ is unbounded, then the algebra generated by $\mathcal{A}_i$ is $\mathbf{2} \oplus \mathcal{A}_i$, and it generates $\mathbb{P}$. Since $\mathbb{P} \nsubseteq \mathbb{C}$ and $\mathbb{C} \nsubseteq \mathbb{P}$, we conclude that $\mathbb{L}$ is not linear.

Suppose now that $\mathbb{L}$ contains a simple MV-chain $\mathcal{A}$ with more than two elements. Then, there is $k \geq 3$ such that $\mathbf{L}_k$ is a subalgebra of $\mathcal{A}$, and hence, $\mathbf{L}_k \in \mathbb{L}$. Since $\mathbf{V}(\mathbf{L}_k) \nsubseteq \mathbb{C}$, and $\mathbb{C} \nsubseteq \mathbf{V}(\mathbf{L}_k)$, we conclude that $\mathbb{L}$ is not linear. □

We now show that the only linear variety of BL-algebras containing $\mathbb{C}$ is $\mathbb{C}$ itself. We start with Komori's classification.

Theorem 33 (Cignoli et al. 1999, Theorem 8.4.4) *A class C of MV-algebras is a proper variety of MV-algebras (i.e., $\mathbb{B} \subseteq C \subsetneq \mathbb{MV}$) iff there are two finite sets I and J of integers greater or equal than 2 such that $I \cup J \neq \emptyset$ and $C = \mathbf{V}(\{\mathbf{L}_i\}_{i\in I}, \{\mathbf{K}_j\}_{j\in J})$.*

Lemma 16 (Cignoli et al. 1999) *For every $n \geq 2$, $\mathbf{K}_n$ has rank n.*

Then, we can finally state the following result.

Theorem 34 *The only linear variety of BL-algebras containing $\mathbb{C}$ is $\mathbb{C}$ itself.*

Proof Let $\mathbb{L}$ be a linear variety of BL-algebras containing $\mathbb{C}$. By Theorem 32 we have that $\mathbb{L} \subsetneq \mathbb{MV}$, and $\mathbb{L}$ does not contain simple MV-chains with more than two elements. Suppose by contradiction that $\mathbb{L} \neq \mathbb{C}$. Since as pointed out in Cignoli et al. (1999), $\mathbf{K}_2$ is Chang's MV-algebra, by Theorem 33 there is $\mathcal{A} \in \mathbb{L}$ such that $\mathcal{A} \in \{\mathbf{L}_h, \mathbf{K}_h\}$, with $h > 2$. However, this is a contradiction, since by hypothesis $\mathbb{L}$ does not contain simple MV-chains with more than two elements, and by Lemma 16 $\mathbf{K}_h/\mathrm{Rad}(\mathbf{K}_h) \simeq \mathbf{L}_h \in \mathbf{V}(\mathcal{A}) \subseteq \mathbb{L}$. Then, we conclude that $\mathbb{L} = \mathbb{C}$. □

From this theorem and the results of the previous two sections, we are able to classify all the linear subvarieties of $\mathbb{BL}$.

Theorem 35 *The only linear varieties of BL-algebras are the following ones.*

- *$\mathbb{G}$ and $\{\mathbb{G}_k\}_{k\geq 2}$.*
- *The family of varieties $\{\mathbb{L}_k : k-1 = h^n, 1 \leq h$ is prime and $n \geq 1\}$*
- *The family of varieties $\{\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k) : k-1 = h^n, 1 \leq h$ is prime and $n \geq 1\}$.*
- *The variety $\mathbb{C}$.*
- *$\mathbb{P}$, $\mathbb{P}_\infty$, and $\{\mathbb{P}_k\}_{k\geq 2}$.*

All these varieties are also strongly linear.

Proof In Theorem 28, it is shown that the only linear n-contractive varieties of BL-algebras are $\mathbb{G}$, $\{\mathbb{G}_k\}_{k\geq 2}$, $\{\mathbb{L}_k : k-1 = h^n, 1 \leq h$ is prime and $n \geq 1\}$, and the family of varieties $\{\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k) : k-1 = h^n, 1 \leq h$ is prime and $n \geq 1\}$. The fact that they are also strongly linear easily follows from the results of Hájek (1998) and Grigolia (1977) and Montagna (2011, Theorem 9).

By Theorem 31, the only linear varieties of BL-algebras containing $\mathbb{P}$ are $\mathbb{P}$, $\mathbb{P}_\infty$, and $\{\mathbb{P}_k\}_{k\geq 2}$. The fact that $\mathbb{P}$ is strongly linear follows by Montagna (2011, Theorem 9). Pick now $k > 1$, and two chains $\mathcal{A} \simeq \mathbf{2} \oplus \bigoplus_{i\in I} \mathcal{A}_i$ and $\mathcal{B} \simeq \mathbf{2} \oplus \bigoplus_{j\in J} \mathcal{B}_j$, where I is infinite, $|J| = k$, and all the $\mathcal{A}_i$'s and $\mathcal{B}_j$'s are infinite cancellative hoops. By Theorem 30 and Propositions 5 and 4, we have that $\mathcal{A} \in \mathbb{P}_\infty$, $\mathcal{B} \in \mathbb{P}_k$, and every chain in $\mathbb{P}_\infty$ ($\mathbb{P}_k$) is partially embeddable into $\mathcal{A}$ ($\mathcal{B}$, respectively). By Cintula et al. (2009, Theorem 3.8), Theorems 4, and 7 we conclude that $\mathbb{P}_\infty$ and $\{\mathbb{P}_k\}_{k\geq 2}$ are strongly linear.

By Theorem 34, the only linear variety of BL-algebras containing $\mathbb{C}$ is $\mathbb{C}$ itself. By Bianchi (2012, Theorem 3.12), $\mathbb{C}$ is strongly linear, as its only subvariety is the one of Boolean algebras.

By Theorem 25, there are no other cases, and hence, the proof is settled. □

10 Almost minimal varieties of MTL-algebras

In this section, we focus on the almost minimal varieties of MTL-algebras.

By definition, every almost minimal variety is an atom, in $\mathcal{L}_{\mathbb{MTL}}$.

In Aguzzoli and Bianchi (2017), it has been provided a classification of the AM varieties of $\mathbb{BL}$ and $\mathbb{WNM}$. Using Theorems 35 and 23, the following result becomes an easy corollary.

Theorem 36 *((Aguzzoli and Bianchi 2017, Theorems 6,8)) The AM varieties in $\mathbb{BL}$ are $\mathbb{G}_3$, $\mathbb{P}$, $\mathbb{C}$, and $\{\mathbb{L}_k : k-1$ is prime$\}$. The AM varieties in $\mathbb{WNM}$ are $\mathbb{G}_3$, $\mathbb{NM}_4$ and $\mathbb{L}_3$.*

We now recall some results from Aguzzoli and Bianchi (2017).

Theorem 37 (Aguzzoli and Bianchi 2017, Theorem 4) *Let $\mathcal{A}$ be a finite MTL-chain. Then, the variety $\mathbb{L} = \mathbf{V}(\mathcal{A})$ is almost minimal if and only if $|\mathcal{A}| > 2$, and every element $0 < a < 1$ singly generates $\mathcal{A}$.*

Theorem 38 (Aguzzoli and Bianchi 2017, Theorem 2) *Let $\mathbb{L}$ be an almost minimal variety of MTL-algebras. Then either every L-chain is simple or every L-chain is bipartite.*

Compare the following result with Theorem 14. Note that, for $k \geq 4$ $\mathbb{DP}_k$ is linear but not AM.

Theorem 39 (Aguzzoli and Bianchi 2017, Theorem 3) *Let $\mathbb{L}$ be an almost minimal variety of MTL-algebras. Then, either $\mathbb{L} \subsetneq \mathbb{BP}_0$ or $\mathbb{L} = \mathbb{L}_3$ or $\mathbb{L} \subsetneq \mathbb{S}_n^-\mathbb{MTL}$, for some $n \geq 4$.*

Theorem 40 *Let $\mathbb{L}$ be an almost minimal variety generated by a finite chain. Then, $\mathbb{L}$ is strongly linear.*

Proof Let $\mathbb{L}$ be an almost minimal variety generated by a finite chain, say $\mathcal{A}$. By Theorem 5, up to isomorphisms, the only chains in $\mathbb{L}$ are $\mathbf{2}$ and $\mathcal{A}$. Since $\mathbf{2} \hookrightarrow \mathcal{A}$, and classical propositional logic is strongly complete w.r.t. $\mathbf{2}$ (i.e., $\mathbb{B}$ is strongly linear), by Theorem 7 and Cintula et al. (2009, Theorem 3.5) we have that $\mathbb{L}$ is strongly linear. □

Clearly every AM variety is linear, but is it also strongly linear? We have a partial answer, in the light of Theorem 38.

Theorem 41 *Let $\mathcal{A}$ be an infinite and simple MTL-chain such that $\mathbb{L} = \mathbf{V}(\mathcal{A})$ is almost minimal. Then $\mathbf{Q}(\mathcal{A}) = \mathbf{V}(\mathcal{A})$, and L has the SSCC. Whence $\mathbb{L}$ is strongly linear.*

Proof Let $\mathcal{A}$ be an infinite and simple MTL-chain such that $\mathbb{L} = \mathbf{V}(\mathcal{A})$ is almost minimal. Then, $\mathcal{A} \in \mathbb{S}_k\mathbb{MTL}$, for some $k \geq 4$, and by Aguzzoli et al. (2014, Theorem 5), the Δ operator is definable in the language. As a consequence, by Cintula et al. (2009, Theorem 3.2, Proposition 3.18), we have that $\mathbf{Q}(\mathcal{A}) = \mathbf{V}(\mathcal{A})$. Then, by Cintula et al. (2009, Theorem 3.2) and Montagna (2011, Theorem 3), L has the SSCC. □

Notice that up to now we are not able to exhibit an MTL-chain satisfying the conditions of Theorem 41.

We now provide two theorems, related to AM varieties generated by a finite IMTL or SMTL-chain.

Lemma 17 (Aguzzoli and Bianchi 2017, Lemma 1) *Let $\mathcal{A}$ be an MTL-chain containing $0 < a < 1$ such that $a * a = a$ and $\sim\sim a = a$. Then, the subalgebra of $\mathcal{A}$ generated by a is isomorphic to $\mathbf{NM}_4$.*

Theorem 42 *The only almost minimal variety of IMTL-algebras being generated by a finite and not simple IMTL-chain is $\mathbb{NM}_4$.*

Proof Let $\mathbb{L}$ be a almost minimal variety of IMTL-algebras, and assume that $\mathbb{L} = \mathbf{V}(\mathcal{A})$, where $\mathcal{A}$ is a finite and not simple IMTL-chain. By Theorem 13 $\mathcal{A}$ is also bipartite. Then, by Proposition 2 $\mathcal{A}$ has an idempotent element $0 < a < 1$, namely $\min(\mathrm{Rad}(A))$. By Lemma 17, we have that $\mathbf{NM}_4 \hookrightarrow \mathcal{A}$. Then, since $\mathbb{L}$ is almost minimal we must have that $\mathcal{A} \simeq \mathbf{NM}_4$. □

Theorem 43 *The only almost minimal variety of SMTL-algebras being generated by a finite SMTL-chain is $\mathbb{G}_3$.*

Proof Let $\mathbb{L} = \mathbf{V}(\mathcal{A})$ be an almost minimal variety of SMTL-algebras, where $\mathcal{A}$ is a finite SMTL-chain. As $|\mathcal{A}| > 2$, and $\mathbb{SMTL} \subsetneq \mathbb{BP}_0$ (see Cignoli and Torrens 2006) it is easy to check that $\mathcal{A}$ is bipartite, not simple, and $0 < m = \min(\mathrm{Rad}(\mathcal{A})) < 1$ is an idempotent element. Then, as $\mathcal{A}$ has a strict negation, it follows that the algebra generated by m is isomorphic to $\mathbf{G}_3$. If $\mathcal{A} \not\simeq \mathbf{G}_3$, then $\mathbf{V}(\mathbf{2}) \subsetneq \mathbb{G}_3 \subsetneq \mathbf{V}(\mathcal{A})$, which would imply that $\mathbf{V}(\mathcal{A})$ is not almost minimal, a contradiction. Then, we conclude that the only almost minimal variety of SMTL-algebras being generated by a finite SMTL-chain is $\mathbb{G}_3$. □

We end this section with a result connecting almost-minimality with local finiteness.

Theorem 44 *Let $\mathbb{L}$ be a locally finite variety of MTL-algebras such that $\mathbb{B} \subsetneq \mathbb{L}$. Then, the class of almost minimal subvarieties of $\mathbb{L}$ is finite, and each of them is generated by one finite chain.*

Proof Let $\mathbb{L}$ be a locally finite variety of MTL-algebras such that $\mathbb{B} \subsetneq \mathbb{L}$. Due to the local finiteness, we immediately have that every almost minimal subvariety of $\mathbb{L}$ must be generated by one finite chain. Assume by contradiction that the class $C = \{\mathbb{L}_i\}_{i\in I}$ of all the almost minimal subvarieties of $\mathbb{L}$ is infinite. Being AM, by Theorem 37 each $\mathbb{L}_i$ is generated by exactly one chain, up to isomorphisms, say $\mathcal{A}_i$. Note that, up to isomorphisms, there are only finitely many MTL-chains with cardinality $k \in \mathbb{N}$. Since $C = \{\mathbb{L}_i\}_{i\in I}$ is infinite, we must conclude that $\{\mathcal{A}_i\}_{i\in I}$ is a(n infinite) set of finite chains of unbounded cardinality. Then, the direct product $\mathcal{B} = \Pi_{i\in I}\mathcal{A}_i$ is a member of $\mathbb{L}$. Since every $\mathcal{A}_i$ is finite and has more than two elements, it must have a coatom $0 < c_i < 1$. Take the element $x = (c_i)_{i\in I} \in \mathcal{B}$.

By Theorem 37, given $i \in I$, c_i generates $\mathcal{A}_i$. Since $\{\mathcal{A}_i\}_{i\in I}$ is a set of finite chains of unbounded cardinality, we easily see that x generates infinitely many elements. Then, x generates an infinite subalgebra of $\mathcal{B}$, in contrast to the fact that $\mathbb{L}$ is locally finite, a contradiction.

Then, we conclude that the class of almost minimal subvarieties of $\mathbb{L}$ is finite, and each of them is generated by one finite chain. □

It will be interesting to settle whether the converse of Theorem 44 holds:

Problem 1 Let $\mathbb{L}$ be an AM variety containing only finitely many AM varieties, each of them generated by a finite chain. Is it always true that $\mathbb{L}$ is locally finite?

11 Maximally linear varieties

Recall from Proposition 1 that the class $Lin_{\mathbb{L}}$ of linear subvarieties of a variety $\mathbb{L}$ of MTL-algebras forms a downward-closed poset which inherits from $\mathcal{L}_{\mathbb{L}}$ the structure of sub-inf-semilattice. Moreover, it is obvious that the poset $Lin_{\mathbb{L}}$ is a tree rooted in the variety of Boolean algebras $\mathbb{B}$. Then $Lin_{\mathbb{L}}$ constitutes a faithful and complete description of the structure of a downward-closed fragment of $\mathbb{L}$. In this light, it is useful to observe that we cannot expand the sub-inf-semilattice $Lin_{\mathbb{L}}$ without losing the property that its shape is a tree. Particular interest lies in the leaves of this tree, when they exist. We call them *maximally linear varieties*.

Definition 10 Let $\mathbb{L}$ be a variety of MTL-algebras. Given a variety $\mathbb{L} \subseteq \mathbb{M} \subseteq \mathbb{MTL}$, we say that $\mathbb{L}$ is *maximally linear* within $\mathbb{M}$ whenever every variety $\mathbb{N}$ such that $\mathbb{L} \subsetneq \mathbb{N} \subseteq \mathbb{M}$ is not linear.

The following are corollaries of our classification of all the linear varieties of WNM- and BL-algebras.

Theorem 45 *The maximally linear varieties of WNM-algebras are exactly* $\mathbb{G}$, $\mathbb{NM}^-$, $\mathbb{DP}$, $\mathbb{F}$.

So, $Lin_{\mathbb{WNM}}$ has a maximum antichain, formed by its maximally linear varieties.

The situation is not so amenable for the BL case.

Theorem 46 *The maximally linear varieties of BL-algebras are exactly* $\mathbb{G}$, $\mathbb{C}$, $\mathbb{P}_\infty$.

whence the linear varieties of the form $\mathbb{L}_k$ or $\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k)$ have not a maximally linear varieties of BL-algebras above them.

An interesting problem is then the following:

Problem 2 Are there any examples of maximally linear varieties in $Lin_{\mathbb{MTL}}$?

We conclude this section stressing a property that identifies Chang's logic $\mathbb{C}$ inside the logics based on BL-algebras. Compare with Aguzzoli and Bianchi (2017, Theorem 9), where we characterized product logic as the only logic which is both continuous-t-norm based and minimally many-valued.

Theorem 47 *If a variety $\mathbb{V}$ of BL-algebras is both almost minimal and maximally linear, then $\mathbb{V} = \mathbb{C}$. Furthermore, if a variety $\mathbb{V}$ of MV-algebras is maximally linear then $\mathbb{V} = \mathbb{C}$.*

Proof Immediate, from Theorems 34 and 35. □

12 Conclusions and future research topics

Our future research will be aimed at further investigating the structure and classification of linear varieties of MTL-algebras, being well aware that a complete classification is probably out of reach. We shall address Problem 1 and Problem 2, but many other questions have been left unanswered in this work. We list some of them.

Problem 3 Let $\mathbb{L}$ be a linear variety of MTL-algebras. Is it always true that $\mathbb{L}$ is strongly linear?

Theorem 41 provides a partial answer.

Problem 4 Let $\mathbb{L}$ be a linear variety of MTL-algebras. Is it always true that every chain in $\mathbb{L}$ is either simple or bipartite?

Every almost minimal variety is linear. Some open problems besides Problem 1 are:

Problem 5 Can we find a characterization for the infinite MTL-chains that generates AM varieties?

Problem 6 Is it true that every AM variety is strongly linear?

Finally, we plan to investigate other topics, like the amalgamation property for linear varieties, as well as the connection between linear varieties and strictly join irreducible varieties of MTL-algebras (a notion firstly introduced in Bianchi 2018). Also, it would be interesting to find a classification of which linear varieties of MTL-algebras are finitely axiomatizable, from $\mathbb{MTL}$.

Compliance with ethical standards

Conflict of interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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