Soft Computing (2019) 23:7889–7900
https://doi.org/10.1007/s00500-018-3419-z

METHODOLOGIES AND APPLICATION

On Pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis

Abhishek Guleria[1] · Rakesh Kumar Bajaj[1]

Published online: 1 August 2018

Abstract
In the present communication, we introduce Pythagorean fuzzy soft matrix and its various possible types. Some binary operations and various properties over the matrices are also being defined with their proof of validity. Further, the Pythagorean fuzzy soft matrices have been taken into account for proposing a new algorithm for decision making by using choice matrix and weighted choice matrix. In addition to this, an algorithm for medical diagnosis problem by making use of score matrix and utility matrix has also been proposed. Numerical examples for each of the applications have been successfully illustrated. A comparative analysis with other existing methods has also been carried out.

Keywords Pythagorean fuzzy soft set · Pythagorean fuzzy soft matrix · Decision making · Medical diagnosis

1 Introduction

In the literature, there are many theories which are used to deal with vagueness and uncertainties of many problems arising in engineering, economics, social science, etc. But all the theories have their own limitations intuitively because of the parametrization tool involved in it. In order to overcome these difficulties, Molodstov (1999) introduced a novel mathematical tool for dealing with the uncertainties, called soft set, which is free from the inadequacy of parametrization and established various results based on this. In 2001, Maji et al. (2001) studied the theory of soft sets and defined soft binary operations such as AND, OR, union, intersection, equality, complement of the soft sets. Further, Maji et al. (2002, 2003) successfully extended the soft set to fuzzy soft set and intuitionistic fuzzy soft set and studied the application of these soft sets in decision-making problems. Peng et al. (2015) introduced the Pythagorean fuzzy soft set (PFSS) and studied various binary operations over PFSS and also proposed a decision-making algorithm based on the Pythagorean fuzzy soft set. Naim and Serdar (2010) introduced the soft matrices which are representations of the Molodtsov's soft sets and successfully applied the soft matrices in decision-making problems. Yong and Chenli (2011) and Chetia and Das (2012) extended the matrix representation of soft set to fuzzy soft set and intuitionistic fuzzy soft matrix, respectively, and applied it to decision-making problems. Various other researchers have also worked on the concept of soft set and soft matrices which are available in the literature.

In this paper, Pythagorean fuzzy soft matrices and its types are introduced and defined based on the Pythagorean fuzzy soft set. Also, the various binary operations are analogously proposed for the Pythagorean fuzzy soft matrices and a new decision-making algorithm has been proposed. We have studied some preliminaries and fundamental notions in Sect. 2. In Sect. 3, we have formally introduced the idea of the Pythagorean fuzzy soft matrix and extended various binary operations on them. In Sect. 4, we have framed choice and weighted choice matrix to propose an algorithm to solve decision-making problem by taking Pythagorean fuzzy soft matrix into account and illustrated through an example. In a similar way, an algorithm for medical diagnosis problem by using score and utility matrix has been proposed in view of the Pythagorean fuzzy soft matrices with a numerical example in Sect. 5. A detailed methodology and comparative analysis have also been provided to discuss the reliability of

Communicated by V. Loia.

✉ Rakesh Kumar Bajaj
rakesh.bajaj@juit.ac.in

Abhishek Guleria
abhishekguleriahappy@gmail.com

[1] Department of Mathematics, Jaypee University of Information Technology, Waknaghat, Solan, Himachal Pradesh 173 234, India

the proposed method. Finally, the paper has been concluded in Sect. 6.

2 Preliminaries

In this section, we recall and present some fundamental concepts in connection with the Pythagorean fuzzy set, which are well known in literature.

Definition 1 (Atanassov 1986) An intuitionistic fuzzy set (IFS) I in X (universe of discourse) is given by

$$I = \{< x, \mu_I(x), \nu_I(x) >| \ x \in X\};$$

where $\mu_I : X \to [0, 1]$ and $\nu_I : X \to [0, 1]$ denote the degree of membership and degree of non-membership, respectively, and for every $x \in X$ satisfy the condition

$$0 \leq \mu_I(x) + \nu_I(x) \leq 1$$

and the degree of indeterminacy for any IFS I and $x \in X$ is given by

$$\pi_I(x) = 1 - \mu_I(x) - \nu_I(x).$$

Definition 2 (Yager 2013) A Pythagorean fuzzy set (PFS) M in X (universe of discourse) is given by

$$M = \{< x, \mu_M(x), \nu_M(x) >| \ x \in X\};$$

where $\mu_M : X \to [0, 1]$ and $\nu_M : X \to [0, 1]$ denote the degree of membership and degree of non-membership, respectively, and for every $x \in X$ satisfy the condition

$$0 \leq \mu_M^2(x) + \nu_M^2(x) \leq 1$$

and the degree of indeterminacy for any Pythagorean fuzzy set M and $x \in X$ is given by

$$\pi_M(x) = \sqrt{1 - \mu_M^2(x) - \nu_M^2(x)}.$$

In case of PFS, the restriction corresponding to the degree of membership $\mu_M(x)$ and the degree of non-membership $\nu_M(x)$ is

$$0 \leq \mu_M^2(x) + \nu_M^2(x) \leq 1,$$

whereas the condition in case of IFS is

$$0 \leq \mu_I(x) + \nu_I(x) \leq 1$$

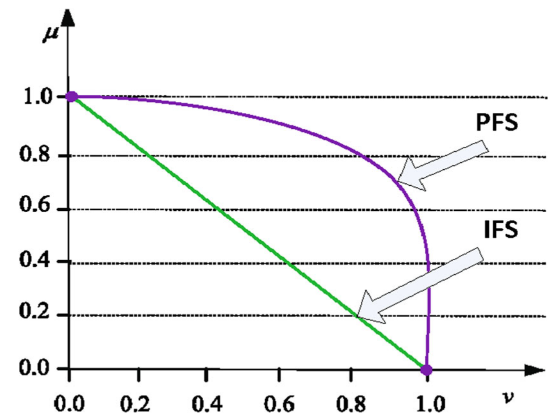


Fig. 1 IFS versus PFS

for $\mu_M(x), \nu_M(x) \in [0, 1]$. This difference in constraint conditions gives an additional advantage for a wider coverage of information span which can be geometrically shown in Fig. 1.

The generalization in terms of development of concepts from Soft Sets to Pythagorean fuzzy soft sets is available with explanatory examples in the literature (Molodstov 1999; Naim and Serdar 2010; Peng et al. 2015).
Let $X = \{x_1, x_2, \ldots, x_m\}$ be the universe of discourse and $E = \{e_1, e_2, \ldots, e_n\}$ be the set of parameters. Consider $A \subseteq E$. The basic notion of soft set, soft matrix and Pythagorean fuzzy soft set is presented below which is well known in the literature:

- The pair (F_A, E) is called *soft set* over X if and only if $F_A : A \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.
- Let $F(X)$ denotes the set of all fuzzy sets of X. A pair (F_A, E) is called a *fuzzy soft set* over $F(X)$, where F is a mapping given by $F_A : A \to \mathcal{P}(F(X))$.
- The pair (F_A, E) is called the *Pythagorean fuzzy soft set* (PFSS) over X if $F_A : A \to PFS(X)$ and can be represented as

  $$(F_A, E) = \{(e, F_A(e)) : e \in A, F_A(e) \in PFS(X)\},$$

  where $PFS(X)$ denotes the set of all Pythagorean fuzzy sets of X.
- Let (F_A, E) be a soft set over X. Then, the subset $X \times E$ is uniquely defined by $R_A = \{(x, e), e \in A, x \in F_A(e)\}$. The characteristic function of R_A is $\chi_{R_A} : X \times E \to [0, 1]$ given by

  $$\chi_{R_A}(x, e) = \begin{cases} 1 & \text{if } (x, e) \in A \\ 0 & \text{if } (x, e) \notin A \end{cases}.$$

  If $a_{ij} = \chi_{R_A}(x_i, e_j)$, then a matrix $[a_{ij}] = [\chi_{R_A}(x_i, e_j)]$ is called *soft matrix* of the soft set (F_A, E) over X of order $m \times n$.

3 Pythagorean fuzzy soft matrices and various operations

Since matrices play an important role in many computational techniques, handling dimensionality feature of various problems of engineering, medical sciences, social sciences, etc., it motivates to extend the concept of Pythagorean fuzzy soft set to Pythagorean fuzzy soft matrices. In this section, we propose the concept of Pythagorean fuzzy soft matrix with various operations over it.

If (F_A, E) be a Pythagorean fuzzy soft set over X, then the subset $X \times E$ is uniquely defined by $R_A = \{(x, e), e \in A, x \in F_A(e)\}$. The R_A can be characterized by its membership function and non-membership function given by $\mu_{R_A} : X \times E \to [0, 1]$ and $\nu_{R_A} : X \times E \to [0, 1]$, respectively.

If $(\mu_{ij}, \nu_{ij}) = (\mu_{R_A}(x_i, e_j), \nu_{R_A}(x_i, e_j))$, where $\mu_{R_A}(x_i, e_j)$ is the membership of x_i in the Pythagorean fuzzy set $F(e_j)$ and $\nu_{R_A}(x_i, e_j)$ is the non-membership of x_i in the Pythagorean fuzzy set $F(e_j)$, respectively, then we define a matrix given by

$$[M] = [m_{ij}]_{m\times n} = \left[\left(\mu_{ij}^M, \nu_{ij}^M\right)\right]_{m\times n} = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{bmatrix}$$

which is called **Pythagorean fuzzy soft matrix** of order $m \times n$ over X.

For a better understanding, let us consider $X = \{x_1, x_2, x_3\}$ as a universal set and $E = \{e_1, e_2, e_3, e_4\}$ as a set of parameters. If $A = \{e_1, e_2, e_3\} \subseteq E$ and

$$F_A(e_1) = \{(x_1, 0.6, 0.5), (x_2, 0.5, 0.8), (x_3, 0.9, 0.2)\},$$
$$F_A(e_2) = \{(x_1, 0.8, 0.5), (x_2, 0.9, 0.3), (x_3, 0.6, 0.6)\},$$
$$F_A(e_3) = \{(x_1, 0.6, 0.7), (x_2, 0.5, 0.6), (x_3, 0.7, 0.5)\},$$

then (F_A, E) is the parameterized family of $F_A(e_1)$, $F_A(e_2)$, $F_A(e_3)$ over X.

Hence, the Pythagorean fuzzy soft matrix $[M(F_A, E)]$ can be written as

$$[M] = \left[\left(\mu_{ij}^M, \nu_{ij}^M\right)\right]_{m\times n} = \begin{bmatrix} (0.6, 0.5) & (0.8, 0.5) & (0.6, 0.7) \\ (0.5, 0.8) & (0.9, 0.3) & (0.5, 0.6) \\ (0.9, 0.2) & (0.6, 0.6) & (0.7, 0.5) \end{bmatrix}.$$

Suppose $PFSM_{m\times n}$ is a collection of all Pythagorean fuzzy soft matrices over X. Subsequently, various kinds of Pythagorean fuzzy soft matrices have been analogously proposed. A Pythagorean fuzzy soft matrix $M = [(\mu_{ij}^M, \nu_{ij}^M)] \in PFSM_{m\times n}$ is called:

- **Pythagorean fuzzy soft zero matrix** if

  $\mu_{ij}^M = 0$ and $\nu_{ij}^M = 0; \forall i, j$ and is denoted by $0 = [0, 0]$.

- **Pythagorean fuzzy soft square matrix** if $m = n$.
- **Pythagorean fuzzy soft row matrix** if $n = 1$.
- **Pythagorean fuzzy soft column matrix** if $m = 1$.
- **Pythagorean fuzzy soft diagonal matrix** if all its non-diagonal elements are zero $\forall\, i, j$.
- **Pythagorean fuzzy soft μ-universal matrix** if $\mu_{ij}^M = 1$ and $\nu_{ij}^M = 0 \;\forall\, i$ and j, denoted by P_μ.
- **Pythagorean fuzzy soft ν-universal matrix** if $\mu_{ij}^M = 0$ and $\nu_{ij}^M = 1 \;\forall\, i$ and j, denoted by P_ν.
- **Scalar multiplication of Pythagorean fuzzy soft matrix**: For any scalar k, we define $kA = [(k\mu_{ij}^M, k\nu_{ij}^M)], \forall\, i$ and j.

Further, we define the following relations over two Pythagorean fuzzy soft matrices $M = [(\mu_{ij}^M, \nu_{ij}^M)]$ and $N = [(\mu_{ij}^N, \nu_{ij}^N)] \in PFSM_{m\times n}$:

- **Sub matrix:** $M \subseteq N$ if $\mu_{ij}^M \le \mu_{ij}^N$ and $\nu_{ij}^M \ge \nu_{ij}^N \;\forall\, i$ and j.
- **Super matrix:** $M \supseteq N$ if $\mu_{ij}^M \ge \mu_{ij}^N$ and $\nu_{ij}^M \le \nu_{ij}^N \;\forall\, i$ and j.
- **Equal matrix:** $M = N$ if $\mu_{ij}^M = \mu_{ij}^N$ and $\nu_{ij}^M = \nu_{ij}^N \;\forall\, i$ and j.
- **Max–min product of Pythagorean fuzzy soft matrix:** Let $M = [a_{ij}] = [(\mu_{ij}^M, \nu_{ij}^M)] \in PFSM_{m\times n}$ & $N = [b_{jk}] = [(\mu_{jk}^N, \nu_{jk}^N)] \in PFSM_{n\times p}$ be two Pythagorean fuzzy soft matrices, then
  $$M * N = [c_{ik}]_{m\times p} = \left[\left\{\max\left(\min_j\left(\mu_{ij}^M, \mu_{jk}^N\right)\right), \min\left(\max_j\left(\nu_{ij}^M, \nu_{jk}^N\right)\right)\right\}\right] \;\forall\, i, j \text{ and } k.$$

Remark In the literature, various basic *triangular norm* (t-*norm*) and *triangular conorm* (t-*conorm*) along with their types and properties have been discussed by Klement et al. (2000, 2004) in detail. In order to define various operations over Pythagorean fuzzy soft matrices, some combinations of them have to be taken into account. However, in the present communication, we have considered the combination of maximum operator (t-*conorm*) and minimum operator (t-*norm*). Some other combinations using different types of t-*norm* and t-*conorm* given in the above stated literature may also be considered in the future.

Operations over Pythagorean fuzzy soft matrices

Various standard operations over two Pythagorean fuzzy soft matrices $A = [(\mu_{ij}^A, \nu_{ij}^A)]$ and $B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$ can be defined as follows:

- $A^c = \left[\left(\nu_{ij}^A, \mu_{ij}^A\right)\right]$ $\forall\, i$ and j.
- $A \cup B = \left[\max\left(\mu_{ij}^A, \mu_{ij}^B\right), \min\left(\nu_{ij}^A, \nu_{ij}^B\right)\right]$ $\forall\, i$ and j.
- $A \cap B = \left[\min\left(\mu_{ij}^A, \mu_{ij}^B\right), \max\left(\nu_{ij}^A, \nu_{ij}^B\right)\right]$ $\forall\, i$ and j.
- $A \cdot B = \left[\left(\mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A + \nu_{ij}^B - \nu_{ij}^A \cdot \nu_{ij}^B\right)\right]$ $\forall\, i$ and j.
- $A + B = \left[\left(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B\right)\right]$ $\forall\, i$ and j.
- $A \otimes B = \left[\left(\mu_{ij}^A \cdot \mu_{ij}^B, \sqrt{(\nu_{ij}^A)^2 + (\nu_{ij}^B)^2 - (\nu_{ij}^A)^2 \cdot (\nu_{ij}^B)^2}\right)\right]$ $\forall\, i$ and j.
- $A \oplus B = \left[\left(\sqrt{(\mu_{ij}^A)^2 + (\mu_{ij}^B)^2 - (\mu_{ij}^A)^2 \cdot (\mu_{ij}^B)^2}, \nu_{ij}^A \cdot \nu_{ij}^B\right)\right]$ $\forall\, i$ and j.
- $A\ @\ B = \left[\left(\frac{\mu_{ij}^A + \mu_{ij}^B}{2}, \frac{\nu_{ij}^A + \nu_{ij}^B}{2}\right)\right]$ $\forall\, i$ and j.
- $A @_w B = \left[\left(\frac{w_1 \mu_{ij}^A + w_2 \mu_{ij}^B}{w_1 + w_2}, \frac{w_1 \nu_{ij}^A + w_2 \nu_{ij}^B}{w_1 + w_2}\right)\right]$ $\forall\, i$ and j ; where $w_1, w_2 > 0$ are the weights.
- $A \$ B = \left[\left(\sqrt{\mu_{ij}^A \cdot \mu_{ij}^B}, \sqrt{\nu_{ij}^A \cdot \nu_{ij}^B}\right)\right]$ $\forall\, i$ and j.
- $A \$_w B = \left[\left((\mu_{ij}^A)^{w_1} \cdot (\mu_{ij}^B)^{w_2}\right)^{\frac{1}{w_1 + w_2}}, \left((\nu_{ij}^A)^{w_1} \cdot (\nu_{ij}^B)^{w_2}\right)^{\frac{1}{w_1 + w_2}})\right]$ $\forall\, i$ and j, where $w_1, w_2 > 0$ are the weights.
- $A \bowtie B = \left[\left(2 \cdot \frac{\mu_{ij}^A \cdot \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}, 2 \cdot \frac{\nu_{ij}^A \cdot \nu_{ij}^B}{\nu_{ij}^A + \nu_{ij}^B}\right)\right]$ $\forall\, i$ and j.
- $A \bowtie_w B = \left[\left(\frac{w_1 + w_2}{\frac{w_1}{\mu_{ij}^A} + \frac{w_2}{\mu_{ij}^B}}, \frac{w_1 + w_2}{\frac{w_1}{\nu_{ij}^A} + \frac{w_2}{\nu_{ij}^B}}\right)\right]$ $\forall\, i$ and j ; where $w_1, w_2 > 0$ are the weights.

Proposition 1 *Let A and B $\in PFSM_{m\times n}$ be two Pythagorean fuzzy soft matrices, then the following results hold:*

(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
(iii) $A + B = B + A$
(iv) $A \cdot B = B \cdot A$
(v) $(A \cup B)^c = A^c \cap B^c$
(vi) $(A \cap B)^c = A^c \cup B^c$
(vii) $(A^c \cap B^c)^c = A \cup B$
(viii) $(A^c \cup B^c)^c = A \cap B$
(ix) $(A^c + B^c)^c = A \cdot B$
(x) $(A^c \cdot B^c)^c = A + B$.

Proof Let $A = [(\mu_{ij}^A, \nu_{ij}^A)]$, $B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$. For all i and j we have,

(i)

$$\begin{aligned} A \cup B &= \left[\max\left(\mu_{ij}^A, \mu_{ij}^B\right), \min\left(\nu_{ij}^A, \nu_{ij}^B\right)\right] \\ &= \left[\max\left(\mu_{ij}^B, \mu_{ij}^A\right), \min\left(\nu_{ij}^B, \nu_{ij}^A\right)\right] = B \cup A. \end{aligned}$$

(ii)

$$\begin{aligned} A \cap B &= \left[\min\left(\mu_{ij}^A, \mu_{ij}^B\right), \max\left(\nu_{ij}^A, \nu_{ij}^B\right)\right] \\ &= \left[\min\left(\mu_{ij}^B, \mu_{ij}^A\right), \max\left(\nu_{ij}^B, \nu_{ij}^A\right)\right] = B \cap A. \end{aligned}$$

(iii)

$$\begin{aligned} A + B &= \left[\left(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B\right)\right] \\ &= \left[\left(\mu_{ij}^B + \mu_{ij}^A - \mu_{ij}^B \cdot \mu_{ij}^A, \nu_{ij}^B \cdot \nu_{ij}^A\right)\right] = B + A. \end{aligned}$$

(iv)

$$\begin{aligned} A \cdot B &= \left[\left(\mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A + \nu_{ij}^B - \nu_{ij}^A \cdot \nu_{ij}^B\right)\right] \\ &= \left[\left(\mu_{ij}^B \cdot \mu_{ij}^A, \nu_{ij}^B + \nu_{ij}^A - \nu_{ij}^B \cdot \nu_{ij}^A\right)\right] = B \cdot A. \end{aligned}$$

(v)

$$\begin{aligned} (A \cup B)^c &= \left(\left[\left(\mu_{ij}^A, \nu_{ij}^A\right)\right] \cup \left[\left(\mu_{ij}^B, \nu_{ij}^B\right)\right]\right)^c \\ &= \left[\max\left(\mu_{ij}^A, \mu_{ij}^B\right), \min\left(\nu_{ij}^A, \nu_{ij}^B\right)\right]^c \\ &= \left[\min\left(\nu_{ij}^A, \nu_{ij}^B\right), \max\left(\mu_{ij}^A, \mu_{ij}^B\right)\right] \\ &= \left[\left(\nu_{ij}^A, \mu_{ij}^A\right)\right] \cap \left[\left(\nu_{ij}^B, \mu_{ij}^B\right)\right] = A^c \cap B^c. \end{aligned}$$

Similarly, (vi), (vii), $(viii)$, (ix) and (x) can be proved easily. □

Proposition 2 *Let* $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$ *be a Pythagorean fuzzy soft matrix. Then, in view of the definitions, the following results may easily be verified:*

(i) $(A^c)^c = A$
(ii) $\left(P_\mu\right)^c = P_\nu$
(iii) $(P_\nu)^c = P_\mu$
(iv) $A \cup A = A$
(v) $A \cup P_\mu = P_\mu$
(vi) $A \cap P_\nu = A$
(vii) $A \cap A = A$
(viii) $A \cap P_\mu = A$
(ix) $A \cap P_\nu = P_\nu$.

Proposition 3 *Let A and B $\in PFSM_{m\times n}$ be two Pythagorean fuzzy soft matrices, then the following results with respect to the weighed operations hold:*

(i) $(A^c @_w B^c)^c = A @_w B$
(ii) $(A^c \$_w B^c)^c = A \$_w B$
(iii) $(A^c \bowtie_w B^c)^c = A \bowtie_w B$
(iv) $A @_w B = B @_w A$
(v) $A \$_w B = B \$_w A$

(vi) $A \bowtie_w B = B \bowtie_w A$.

Proof Let $A = [(\mu_{ij}^A, \nu_{ij}^A)]$, $B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$. For all i, j and $w_1, w_2 > 0$,, we have,

(i) $$(A^c @_w B^c)^c = \left(\left[\left(\nu_{ij}^A, \mu_{ij}^A\right) @_w \left(\nu_{ij}^B, \mu_{ij}^B\right)\right]\right)^c = \left(\left[\frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}, \frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}\right]\right)^c = \left[\frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}, \frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}\right] = A @_w B.$$

(ii) $$(A^c \$_w B^c)^c = \left(\left[\left(\nu_{ij}^A, \mu_{ij}^A\right) \$_w \left(\nu_{ij}^B, \mu_{ij}^B\right)\right]\right)^c = \left(\left[\left(\left(\nu_{ij}^A\right)^{w_1} \cdot \left(\nu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}, \left(\left(\mu_{ij}^A\right)^{w_1} \cdot \left(\mu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}\right]\right)^c = \left[\left(\left(\mu_{ij}^A\right)^{w_1} \cdot \left(\mu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}, \left(\left(\nu_{ij}^A\right)^{w_1} \cdot \left(\nu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}\right] = A\$_w B.$$

Similar proof for (iii).

(iv) $$A @_w B = \left[\frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}, \frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}\right] = \left[\frac{w_2\mu_{ij}^B + w_1\mu_{ij}^A}{w_2+w_1}, \frac{w_2\nu_{ij}^B + w_1\nu_{ij}^A}{w_2+w_1}\right] = B @_w A.$$

(v) $$A\$_w B = \left[\left(\left(\mu_{ij}^A\right)^{w_1} \cdot \left(\mu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}, \left(\left(\nu_{ij}^A\right)^{w_1} \cdot \left(\nu_{ij}^B\right)^{w_2}\right)^{\frac{1}{w_1+w_2}}\right] = \left[\left(\left(\mu_{ij}^B\right)^{w_2} \cdot \left(\mu_{ij}^A\right)^{w_1}\right)^{\frac{1}{w_2+w_1}}, \left(\left(\nu_{ij}^B\right)^{w_2} \cdot \left(\nu_{ij}^A\right)^{w_1}\right)^{\frac{1}{w_2+w_1}}\right] = B\$_w A$$

Similar proof for (vi). □

Proposition 4 *Let A, B and $C \in PFSM_{m\times n}$ be three Pythagorean fuzzy soft matrices, then the following results related to associativity of operations hold:*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$
(iii) $(A + B) + C = A + (B + C)$
(iv) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
(v) $(A@B)@C = A@(B@C)$
(vi) $(A\$B)\$C = A\$(B\$C)$
(vii) $(A \bowtie B) \bowtie C = A \bowtie (B \bowtie C)$.

Proof For all i and j, we have

(i) $$(A \cup B) \cup C = \left[\left(\max\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \min\left\{\nu_{ij}^A, \nu_{ij}^B\right\}\right)\right] \cup \left[\left(\mu_{ij}^C, \nu_{ij}^C\right)\right] = \left[\left(\max\left\{\left(\mu_{ij}^A, \mu_{ij}^B\right), \mu_{ij}^C\right\}, \min\left\{\left(\nu_{ij}^A, \nu_{ij}^B\right), \nu_{ij}^C\right\}\right)\right] = \left[\left(\max\left\{\left(\mu_{ij}^A, \left(\mu_{ij}^B, \mu_{ij}^C\right)\right)\right\}, \min\left\{\nu_{ij}^A, \left(\nu_{ij}^B, \nu_{ij}^C\right)\right\}\right)\right] = A \cup (B \cup C).$$

(ii) $$(A \cap B) \cap C = \left[\left(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \max\left\{\nu_{ij}^A, \nu_{ij}^B\right\}\right] \cup \left(\mu_{ij}^C, \nu_{ij}^C\right)\right] = \left[\left(\min\left\{\left(\mu_{ij}^A, \mu_{ij}^B\right), \mu_{ij}^C\right\}, \max\left\{\left(\nu_{ij}^A, \nu_{ij}^B\right), \nu_{ij}^C\right\}\right)\right] = \left[\left(\min\left\{\left(\mu_{ij}^A, \left(\mu_{ij}^B, \mu_{ij}^C\right)\right)\right\}, \max\left\{\nu_{ij}^A, \left(\nu_{ij}^B, \nu_{ij}^C\right)\right\}\right)\right] = A \cap (B \cap C).$$

(iii) $$(A + B) + C = \left[\left(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B\right)\right] + \left[\left(\mu_{ij}^C, \nu_{ij}^C\right)\right] = \left[\left(\mu_{ij}^A + \mu_{ij}^B\right) + \mu_{ij}^C - \left(\mu_{ij}^A \cdot \mu_{ij}^B\right) \cdot \mu_{ij}^C, \left(\nu_{ij}^A \cdot \nu_{ij}^B\right) \cdot \nu_{ij}^C\right] = \left[\mu_{ij}^A + \left(\mu_{ij}^B + \mu_{ij}^C\right) - \mu_{ij}^A \cdot \left(\mu_{ij}^B \cdot \mu_{ij}^C\right), \nu_{ij}^A \cdot \left(\nu_{ij}^B \cdot \nu_{ij}^C\right)\right] = A + (B + C).$$

Similar proof for (iv), (v), (vi) and (vii). □

Proposition 5 *Let A, B and $C \in PFSM_{m\times n}$ be three Pythagorean fuzzy soft matrices, then the following results related to distributivity of operations hold:*

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(ii) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(iv) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
(v) $(A \cap B)@C = (A@C) \cap (B@C)$
(vi) $(A \cap B) \bowtie C = (A \bowtie C) \cap (B \bowtie C)$
(vii) $(A \cup B) + C = (A + C) \cup (B + C)$
(viii) $(A \cup B) \cdot C = (A \cdot C) \cup (B \cdot C)$
(ix) $A \cup (B@C) = (A \cup B)@(A \cup C)$
(x) $(A \cup B) \bowtie C = (A \bowtie C) \cup (B \bowtie C)$
(xi) $A@(B \cup C) = (A@B) \cup (A@C)$
(xii) $A@(B \cap C) = (A@B) \cap (B@C)$
(xiii) $A\$(B \cup C) = (A\$B) \cup (A\$C)$
(xiv) $(A \cup B)\$C = (A\$C) \cup (B\$C)$
(xv) $A \cdot (B@C) = (A \cdot B)@(A \cdot C)$
(xvi) $A \cup (B \bowtie C) = (A \cup B) \bowtie (A \cup C)$
(xvii) $A \bowtie (B \cup C) = (A \bowtie B) \cup (A \bowtie C)$
(xviii) $A\$(B \cap C) = (A\$B) \cap (B\$C)$
(xix) $(A \cap B)\$C = (A\$C) \cap (B\$C)$.

Proof For all i and j, we have

(i)
$$A \cap (B \cup C) = \left[\left(\mu_{ij}^A, \nu_{ij}^A\right)\right] \cap \left[\left(\max\left\{\mu_{ij}^B, \mu_{ij}^C\right\}, \min\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\right)\right] = \left[\left(\min\left\{\mu_{ij}^A, \max\left\{\mu_{ij}^B, \mu_{ij}^C\right\}\right\}, \max\left\{\nu_{ij}^A, \min\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\right\}\right)\right].$$

Now,

$$\begin{aligned}(A \cap B) \cup (A \cap C) &= \left[\left(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \max\left\{\nu_{ij}^A, \nu_{ij}^B\right\}\right]\right. \cup \left[\left(\min\left\{\mu_{ij}^A,\mu_{ij}^C\right\},\max\left\{\nu_{ij}^A,\nu_{ij}^C\right\}\right]\right.\\ &= \Big[\max\Big(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \min\left\{\mu_{ij}^A, \mu_{ij}^C\right\}\Big), \min\Big(\max\left\{\nu_{ij}^A,\nu_{ij}^B\right\},\max\left\{\nu_{ij}^A,\nu_{ij}^C\right\}\Big)\Big]\\ &= \Big[\max\Big(\mu_{ij}^A, \min\left\{\mu_{ij}^B, \mu_{ij}^C\right\}\Big), \min\Big(\nu_{ij}^A, \max\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\Big)\Big]\\ &= \Big[\min\Big(\mu_{ij}^A, \max\left\{\mu_{ij}^B, \mu_{ij}^C\right\}\Big), \max\Big(\nu_{ij}^A, \min\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\Big)\Big]\\ &= A \cap (B \cup C).\end{aligned}$$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ holds.

(ii)
$$\begin{aligned}(A \cap B) \cup C &= \left[\left(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \max\left\{\nu_{ij}^A, \nu_{ij}^B\right\}\right]\right. \cup \left[\left(\mu_{ij}^C, \nu_{ij}^C\right)\right]\\ &= \left[\max\left(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \mu_{ij}^C\right), \min\left(\max\left\{\nu_{ij}^A, \nu_{ij}^B\right\}, \nu_{ij}^C\right]\right..\end{aligned}$$

Now,

$$\begin{aligned}(A \cup C) \cap (B \cup C) &= \left[\max\left\{\mu_{ij}^A, \mu_{ij}^C\right\}, \min\left\{\nu_{ij}^A, \nu_{ij}^C\right\}\right] \cap \left[\max\left\{\mu_{ij}^B, \mu_{ij}^C\right\}, \min\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\right]\\ &= \left[\min\left(\max\left\{\mu_{ij}^A, \mu_{ij}^C\right\}, \max\left\{\mu_{ij}^B, \mu_{ij}^C\right\}\right), \max\left(\min\left\{\nu_{ij}^A, \nu_{ij}^C\right\}, \min\left\{\nu_{ij}^B, \nu_{ij}^C\right\}\right)\right]\\ &= \left[\min\left(\max\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \mu_{ij}^C\right\}\right), \max\left(\min\left\{\nu_{ij}^A, \nu_{ij}^B\right\}, \nu_{ij}^C\right\}\right)\right]\\ &= \left[\max\left(\min\left\{\mu_{ij}^A, \mu_{ij}^B\right\}, \mu_{ij}^C\right\}\right), \min\left(\max\left\{\nu_{ij}^A, \nu_{ij}^B\right\}, \nu_{ij}^C\right\}\right)\right]\\ &= (A \cap B) \cup C\end{aligned}$$

Hence, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

It may be observed that the results pointed in $(iii) - (xix)$ can be easily proved on similar lines as stated above. □

4 Application of Pythagorean fuzzy soft matrix in decision making

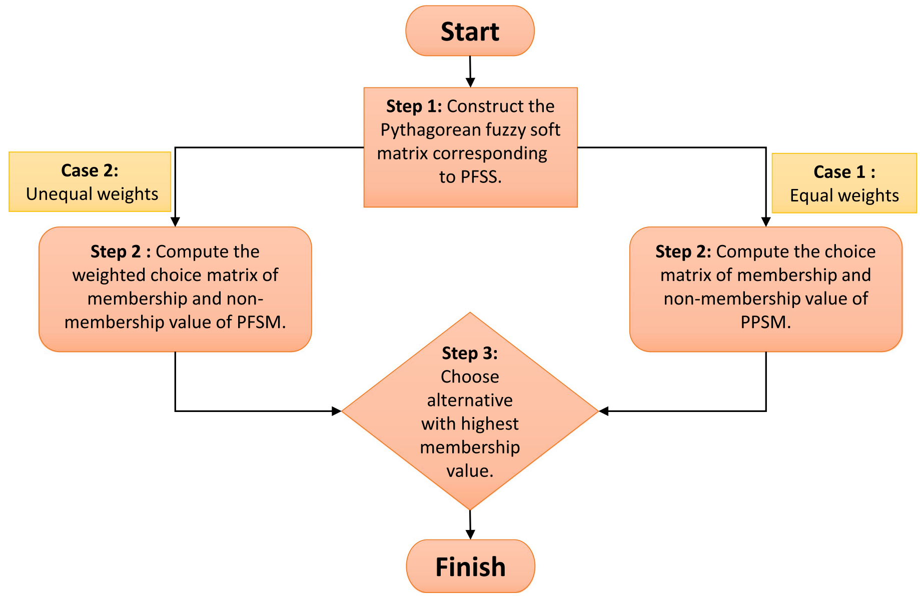
In view of a general decision-making problem and taking the idea of Pythagorean fuzzy soft matrix into account, we propose the following revised definitions for choice matrix and weighted choice matrix:

Definition 3 If $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$, then the choice matrix of Pythagorean fuzzy soft matrix A is given by

$$C(A) = \left[\left(\frac{\sum_{j=1}^{n}(\mu_{ij}^A)^2}{n}, \frac{\sum_{j=1}^{n}(\nu_{ij}^A)^2}{n}\right)\right]_{m\times 1}$$
$\forall\ i$ when weights are equal.

Definition 4 If $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$, then the weighted choice matrix of Pythagorean fuzzy soft matrix A is given by

Fig. 2 Flowchart of the algorithm for decision making



$$C_w(A) = \left[\left(\frac{\sum_{j=1}^{n} w_j (\mu_{ij}^A)^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j (\nu_{ij}^A)^2}{\sum w_j} \right) \right]_{m\times 1}$$
$\forall\ i$ where $w_j > 0$ are weights.

Based on the Pythagorean fuzzy soft matrix, choice matrix and weighted choice matrix, we propose an algorithm for solving a standard decision-making problem with the help of the following flowchart (Fig. 2):

Example 1 Suppose an automobile company produces three different types of car c_1, c_2, c_3, i.e., $U = \{c_1, c_2, c_3\}$, and let a set of parameters $E = \{e_1, e_2, e_3\}$ represent good mileage, comfort, good power steering, respectively. Suppose that a customer has to decide which car to be purchased? We model the problem by considering the Pythagorean fuzzy soft set (G, E) over U, where G is mapping $G : E \to \mathcal{P}(U)$ which represents the description of car on the basis of different parameters.

- *Step* 1 Construct the Pythagorean fuzzy soft matrix on the basis of the parameter as follows:

$$A = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ c_1 & (0.8, 0.5) & (0.6, 0.6) & (0.8, 0.2) \\ c_2 & (0.6, 0.5) & (0.7, 0.4) & (0.8, 0.4) \\ c_3 & (0.5, 0.7) & (0.7, 0.6) & (0.9, 0.3) \end{array}$$

It may be noted that the first element (0.8, 0.5) in the above Pythagorean fuzzy soft matrix A gives the degree to which the alternative c_1 satisfies the criterion e_1 is 0.8 while the degree to which the alternative c_1 does not match with the criterion e_1 is 0.5. Similar meaning is for the other entries of the above matrix A.

- *Step* 2
  - *Case 1: equal weights*
    Compute the choice matrix for the Pythagorean fuzzy soft matrix A as:

$$C(A) = \begin{bmatrix} (0.5467, 0.2167) \\ (0.4967, 0.19) \\ (0.5167, 0.3133) \end{bmatrix}$$

  - *Case 2: unequal weights*
    If the weights 0.2, 0.6, 0.2 are given for the parameters good mileage, comfort, good power steering, respectively, then the weighted choice matrix for A is as

$$C_w(A) = \begin{bmatrix} (0.472, 0.274) \\ (0.494, 0.178) \\ (0.506, 0.332) \end{bmatrix}$$

- *Step* 3
  - *Case 1 (equal weights)* It is clear from the choice matrix obtained in Step 2 that if we give equal preference for all the parameters, we have 0.5467 as the highest membership value, i.e., of car c_1. Therefore, in this case the most suitable car for the customer would be c_1.
  - *Case 2 (unequal weights)* However, it may also be observed that if the customer gives preference for the parameter "comfort" over the other parameters, then 0.506 is the highest membership value for car c_3.

Therefore, in this case the most suitable car for the customer would be c_3.

5 Application of Pythagorean soft matrix in medical diagnosis

In view of a general medical diagnosis problem and taking the idea of Pythagorean fuzzy soft matrix into account, we propose the following revised definitions for score matrix and utility:

Definition 5 If $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$, then the score matrix of Pythagorean fuzzy soft matrix A is given by $S(A) = [s_{ij}] = [((\mu_{ij}^A)^2 - (\nu_{ij}^A)^2)]$ for all i and j. In the literature, the (i, j)th entry of the score matrix is considered to be an important index for measuring the optimized magnitude of the belongingness/non-belongingness of ith patient having a chance of jth disease.

Definition 6 If $A = [(\mu_{ij}^A, \nu_{ij}^A)]$, $B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$ then the utility matrix of Pythagorean fuzzy soft matrices A and B is given by $U(A, B) = [u_{ij}]_{m\times n} = [S(A) - S(B)]\ \forall\, i$ and j. It may also be noted that the $(i, j)^{\text{th}}$ entry of the utility matrix represents another important index for measuring the mixed magnitude of the belongingness in connection with its non-belongingness of ith patient having a chance of jth disease.

Based on the Pythagorean fuzzy soft matrices, score matrix and utility matrix, we propose an algorithm for solving a standard diagnosis problem with the help of the following flowchart (Fig. 3):

In view of Step 6 of the above diagnosis algorithm, in case there are more than one instances where the value of max U_i is repeating, then to break the tie, we have to reassess the symptoms. In order to have a proper implementation and understanding of the algorithm, we present a formal methodology followed by a numerical example in the following subsection:

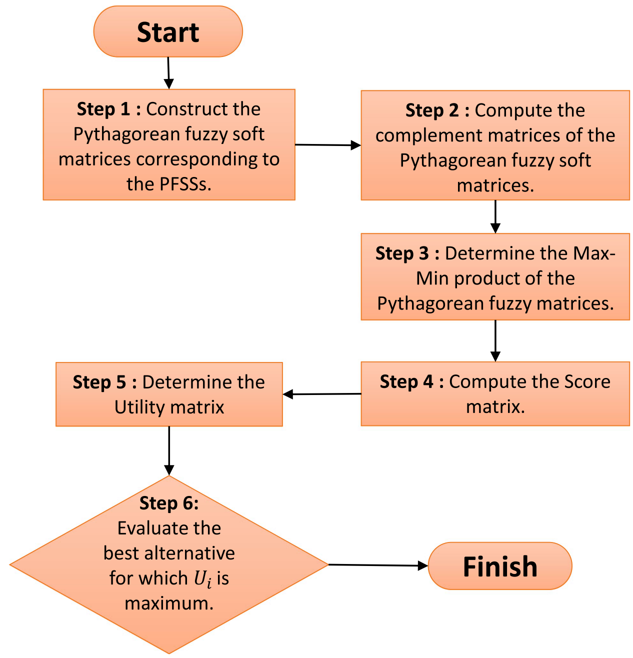


Fig. 3 Flowchart of the algorithm for medical diagnosis

5.1 Methodology

Let us assume that there is a set of m patients $A = \{a_1, a_2, \ldots, a_m\}$ with a set of symptoms $S = \{s_1, s_2, \ldots, s_n\}$ related to as set of k disease $Q = \{q_1, q_2, \ldots, q_k\}$. We apply the concept of Pythagorean fuzzy soft set to diagnose which patient is suffering from which disease. We construct a Pythagorean fuzzy soft set (F, S) over A, where F is a mapping $F : S \to \mathcal{P}(A)$ (collection of all the Pythagorean fuzzy subsets of A), which gives a collection of an approximate description of patient's symptoms. This Pythagorean fuzzy soft set gives a relation matrix M called patient symptoms matrix. Then, we construct another Pythagorean fuzzy soft set (G, Q) over S, where G is mapping $G : Q \to \mathcal{P}(S)$, (collection of all the Pythagorean fuzzy subsets of S), which gives an approximate description of Pythagorean fuzzy soft medical knowledge of diseases and their symptoms. This Pythagorean fuzzy soft set gives another relation matrix N called symptoms disease matrix. We compute the complement of matrices, i.e., M^c and N^c given by $(F, S)^c$ and $(G, Q)^c$, respectively. Further, the max-min product matrix $M * N$, denoted by R_1, is computed which gives the maximum membership of occurrence of the symptoms of the disease and consequently, the matrix $M^c * N^c$, denoted by R_2, which gives the maximum membership of non-occurrence of the symptoms of the disease. Using Definition (5), we compute the score matrices $S(R_1)$ and $S(R_2)$ which represent the respective optimized magnitude of the sense of belongingness and non-belongingness of a patient in a certain disease. Next, using Definition (6), the utility matrix is computed based on taking the above two score matrices into account. The elements of the utility matrix correspondingly represent that how well one alternative satisfies the decision maker's opinion. Since the utilities are real numbers, the preferred alternatives are those which are having higher values.

Example 2 Let $A = \{a_1, a_2, a_3, a_4\}$ be the universal set where a_1, a_2, a_3, a_4 represent the patients in a hospital. Consider a set of symptoms $S = \{s_1, s_2, s_3, s_4\}$ which represent temperature, headache, body pain and cough, respectively, and a set of diagnoses $Q = \{q_1, q_2, q_3\}$, which represent malaria, viral fever and typhoid, respectively. We model the problem by considering the Pythagorean fuzzy soft set (F, S)

over A, where F is mapping $F : S \to \mathcal{P}(A)$, which represents the description of patient symptoms in the hospital.

- *Step* 1

$$(F, S) = \left\{ \begin{array}{l} F(s_1) = \{(a_1, 0.8, 0.3), (a_2, 0.6, 0.5), (a_3, 0.2, 0.9), (a_4, 0.3, 0.8)\} \\ F(s_2) = \{(a_1, 0.3, 0.8), (a_2, 0.4, 0.7), (a_3, 0.6, 0.5), (a_4, 0.7, 0.4)\} \\ F(s_3) = \{(a_1, 0.9, 0.3), (a_2, 0.8, 0.5), (a_3, 0.7, 0.4), (a_4, 0.5, 0.6)\} \\ F(s_4) = \{(a_1, 0.8, 0.3), (a_2, 0.5, 0.6), (a_3, 0.4, 0.8), (a_4, 0.4, 0.7)\} \end{array} \right\}$$

By considering the Pythagorean fuzzy soft set, we can convert the problem into the following Pythagorean fuzzy soft matrix as follows:

$$M = \begin{array}{c|cccc} & s_1 & s_2 & s_3 & s_4 \\ a_1 & (0.8, 0.3) & (0.3, 0.8) & (0.9, 0.3) & (0.8, 0.3) \\ a_2 & (0.6, 0.5) & (0.4, 0.7) & (0.8, 0.5) & (0.5, 0.6) \\ a_3 & (0.2, 0.9) & (0.6, 0.5) & (0.7, 0.4) & (0.4, 0.8) \\ a_4 & (0.3, 0.8) & (0.7, 0.4) & (0.5, 0.6) & (0.4, 0.7) \end{array}$$

It may be noted that the first element (0.8, 0.3) in the above Pythagorean fuzzy soft matrix M gives the degree to which the patient a_1 satisfies the symptom s_1 is 0.8 while the degree to which the patient a_1 does not match with the symptom s_1 is 0.3. Similar meaning is for the other entries of the above matrix M.

Next, consider Pythagorean fuzzy soft set (G, Q) over S, where G is mapping $G : Q \to \mathcal{P}(S)$ which represents the medical knowledge of the diagnosis and their symptoms.

$$(G, Q) = \left\{ \begin{array}{l} G(q_1) = \{(s_1, 0.5, 0.7), (s_2, 0.7, 0.4), (s_3, 0.4, 0.8), (s_4, 0.4, 0.9)\} \\ G(q_2) = \{(s_1, 0.8, 0.5), (s_2, 0.4, 0.7), (s_3, 0.3, 0.9), (s_4, 0.82, 0.46)\} \\ G(q_3) = \{(s_1, 0.4, 0.7), (s_2, 0.5, 0.6), (s_3, 0.7, 0.4), (s_4, 0.9, 0.4)\} \end{array} \right\}$$

We model this problem into the Pythagorean fuzzy soft matrix as follows:

$$N = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ s_1 & (0.5, 0.7) & (0.8, 0.5) & (0.4, 0.7) \\ s_2 & (0.7, 0.4) & (0.4, 0.7) & (0.5, 0.6) \\ s_3 & (0.4, 0.8) & (0.3, 0.9) & (0.7, 0.4) \\ s_4 & (0.4, 0.9) & (0.82, 0.46) & (0.9, 0.4) \end{array}$$

Here, the first element (0.5, 0.7) in the above Pythagorean fuzzy soft matrix N gives the degree to which the symptom s_1 satisfies the disease q_1 is 0.5 while the degree to which the symptom s_1 does not have the disease q_1 is 0.7. Similar meaning is for the other entries of the above matrix N.

- *Step* 2 The Pythagorean fuzzy soft complement matrices of the Pythagorean fuzzy soft matrices obtained in Step 1 are computed as follows:

$$M^c = \begin{array}{c|cccc} & s_1 & s_2 & s_3 & s_4 \\ a_1 & (0.3, 0.8) & (0.8, 0.3) & (0.3, 0.9) & (0.3, 0.8) \\ a_2 & (0.5, 0.6) & (0.7, 0.4) & (0.5, 0.8) & (0.6, 0.5) \\ a_3 & (0.9, 0.2) & (0.5, 0.6) & (0.4, 0.7) & (0.8, 0.4) \\ a_4 & (0.8, 0.3) & (0.4, 0.7) & (0.6, 0.5) & (0.7, 0.4) \end{array}$$

$$N^c = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ s_1 & (0.7, 0.5) & (0.5, 0.8) & (0.7, 0.4) \\ s_2 & (0.4, 0.7) & (0.7, 0.4) & (0.6, 0.5) \\ s_3 & (0.8, 0.4) & (0.9, 0.3) & (0.4, 0.7) \\ s_4 & (0.9, 0.4) & (0.46, 0.82) & (0.4, 0.9) \end{array}$$

- *Step* 3 The max-min products of the Pythagorean fuzzy soft matrices obtained in Step 1 and Step 2 have been obtained as below:

$$R_1 = M * N = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ a_1 & (0.5, 0.7) & (0.8, 0.46) & (0.8, 0.4) \\ a_2 & (0.5, 0.7) & (0.6, 0.46) & (0.7, 0.5) \\ a_3 & (0.6, 0.5) & (0.4, 0.7) & (0.7, 0.4) \\ a_4 & (0.7, 0.4) & (0.4, 0.7) & (0.5, 0.6) \end{array}$$

$$R_2 = M^c * N^c = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ a_1 & (0.4, 0.7) & (0.7, 0.4) & (0.6, 0.5) \\ a_2 & (0.6, 0.5) & (0.7, 0.4) & (0.6, 0.5) \\ a_3 & (0.8, 0.4) & (0.5, 0.6) & (0.7, 0.4) \\ a_4 & (0.5, 0.6) & (0.7, 0.4) & (0.7, 0.5) \end{array}$$

- *Step* 4 The corresponding score matrices of the Pythagorean fuzzy soft matrices R_1 and R_2 obtained in Step 3 are:

$$S(R_1) = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ a_1 & -0.24 & 0.4284 & 0.48 \\ a_2 & -0.24 & 0.1484 & 0.24 \\ a_3 & 0.11 & -0.33 & 0.33 \\ a_4 & 0.33 & -0.33 & -0.11 \end{array}$$

$$S(R_2) = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ \hline a_1 & -0.33 & 0.33 & 0.11 \\ a_2 & 0.11 & 0.33 & 0.11 \\ a_3 & 0.48 & -0.11 & 0.33 \\ a_4 & -0.11 & 0.33 & 0.24 \end{array}$$

- *Step* 5 The utility matrix of the score matrices $S(R_1)$ and $S(R_2)$ obtained in Step 4 is:

$$U = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ \hline a_1 & 0.09 & 0.0984 & 0.37 \\ a_2 & -0.35 & -0.1816 & 0.13 \\ a_3 & -0.37 & -0.22 & 0 \\ a_4 & 0.44 & -0.66 & -0.35 \end{array}$$

- *Step* 6 Therefore, in view of the values of the elements of utility matrix obtained in Step 5, it reflects that the patients $\{a_1, a_2, a_3\}$ are suffering from typhoid (q_3) and patient a_4 is suffering from malaria (q_1).

5.2 Comparative study

In order to have the comparative analysis of the proposed methodology in the decision-making process, we investigate the example taken from the related literature and compare the results with the recent existing methods of medical diagnosis.

Example 3 (Szmidt and Kacprzyk 2004) Suppose a doctor wants to make a proper diagnosis $D = \{d_1, d_2, d_3, d_4, d_5\}$; where d_1 is viral fever, d_2 is Malaria, d_3 is typhoid, d_4 is stomach problem and d_5 is chest problem, for a set of patients $P = \{\text{Ted, Al, Bob, Joe}\}$ with the values of symptoms $V = \{v_1, v_2, v_3, v_4, v_5\}$; where v_1 is temperature, v_2 is headache, v_3 is stomach pain, v_4 is cough and v_5 is chest pain. We model the problem by considering the Pythagorean fuzzy soft set (F, V) over P, where F is mapping $F : V \to \mathcal{P}(P)$, which represents the description of patient's symptoms in the hospital.

- *Step* 1

$$(F, V) = \left\{ \begin{array}{l} F(v_1) = \{(\text{Al}, 0.8, 0.1), (\text{Bob}, 0.0, 0.8), (\text{Joe}, 0.8, 0.1), (\text{Ted}, 0.6, 0.1)\} \\ F(v_2) = \{(\text{Al}, 0.6, 0.1), (\text{Bob}, 0.4, 0.4), (\text{Joe}, 0.8, 0.1), (\text{Ted}, 0.5, 0.4)\} \\ F(v_3) = \{(\text{Al}, 0.2, 0.8), (\text{Bob}, 0.6, 0.1), (\text{Joe}, 0.0, 0.6), (\text{Ted}, 0.3, 0.4)\} \\ F(v_4) = \{(\text{Al}, 0.6, 0.1), (\text{Bob}, 0.1, 0.7), (\text{Joe}, 0.2, 0.7), (\text{Ted}, 0.7, 0.2)\} \\ F(v_5) = \{(\text{Al}, 0.1, 0.6), (\text{Bob}, 0.1, 0.8), (\text{Joe}, 0.0, 0.5), (\text{Ted}, 0.3, 0.4)\} \end{array} \right\}$$

By considering the Pythagorean fuzzy soft set, we can convert the problem into the following Pythagorean fuzzy soft matrix as follows:

$$M = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline \text{Al} & (0.8, 0.1) & (0.6, 0.1) & (0.2, 0.8) & (0.6, 0.1) & (0.1, 0.6) \\ \text{Bob} & (0.0, 0.8) & (0.4, 0.4) & (0.6, 0.1) & (0.1, 0.7) & (0.1, 0.8) \\ \text{Joe} & (0.8, 0.1) & (0.8, 0.1) & (0.0, 0.6) & (0.2, 0.7) & (0.0, 0.5) \\ \text{Ted} & (0.6, 0.1) & (0.5, 0.4) & (0.3, 0.4) & (0.7, 0.2) & (0.3, 0.4) \end{array}$$

Next, consider Pythagorean fuzzy soft set (G, D) over V, where G is mapping $G : D \to \mathcal{P}(V)$, which represents the medical knowledge of the diagnosis and their symptoms.

$$(G, D) = \left\{ \begin{array}{l} G(d_1) = \{(v_1, 0.4, 0.0), (v_2, 0.3, 0.5), (v_3, 0.1, 0.7), (v_4, 0.4, 0.3), (v_5, 0.1, 0.7)\} \\ G(d_2) = \{(v_1, 0.7, 0.0), (v_2, 0.2, 0.6), (v_3, 0.0, 0.9), (v_4, 0.7, 0.0), (v_5, 0.1, 0.8)\} \\ G(d_3) = \{(v_1, 0.3, 0.3), (v_2, 0.6, 0.1), (v_3, 0.2, 0.7), (v_4, 0.2, 0.6), (v_5, 0.1, 0.9)\} \\ G(d_4) = \{(v_1, 0.1, 0.7), (v_2, 0.2, 0.4), (v_3, 0.8, 0.0), (v_4, 0.2, 0.7), (v_5, 0.2, 0.7)\} \\ G(d_5) = \{(v_1, 0.1, 0.8), (v_2, 0.0, 0.8), (v_3, 0.2, 0.8), (v_4, 0.2, 0.8), (v_5, 0.8, 0.1)\} \end{array} \right\}$$

We model this problem into the Pythagorean fuzzy soft matrix as follows:

$$N = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline v_1 & (0.4, 0.0) & (0.7, 0.0) & (0.3, 0.3) & (0.1, 0.7) & (0.1, 0.8) \\ v_2 & (0.3, 0.5) & (0.2, 0.6) & (0.6, 0.1) & (0.2, 0.4) & (0.0, 0.8) \\ v_3 & (0.1, 0.7) & (0.0, 0.9) & (0.2, 0.7) & (0.8, 0.0) & (0.2, 0.8) \\ v_4 & (0.4, 0.3) & (0.7, 0.0) & (0.2, 0.6) & (0.2, 0.7) & (0.2, 0.8) \\ v_5 & (0.1, 0.7) & (0.1, 0.8) & (0.1, 0.9) & (0.2, 0.7) & (0.8, 0.1) \end{array}$$

- *Step* 2 The Pythagorean fuzzy soft complement matrices of the Pythagorean fuzzy soft matrices obtained in Step 1 are computed as follows:

$$M^c = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline \text{Al} & (0.1, 0.8) & (0.1, 0.6) & (0.8, 0.2) & (0.1, 0.6) & (0.6, 0.1) \\ \text{Bob} & (0.8, 0.0) & (0.4, 0.4) & (0.1, 0.6) & (0.7, 0.1) & (0.8, 0.1) \\ \text{Joe} & (0.1, 0.8) & (0.1, 0.8) & (0.6, 0.0) & (0.7, 0.2) & (0.5, 0.0) \\ \text{Ted} & (0.1, 0.6) & (0.4, 0.5) & (0.4, 0.3) & (0.2, 0.7) & (0.4, 0.3) \end{array}$$

$$N^c = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline v_1 & (0.0, 0.4) & (0.0, 0.7) & (0.3, 0.3) & (0.7, 0.1) & (0.8, 0.1) \\ v_2 & (0.5, 0.3) & (0.6, 0.2) & (0.1, 0.6) & (0.4, 0.2) & (0.8, 0.0) \\ v_3 & (0.7, 0.1) & (0.9, 0.0) & (0.7, 0.2) & (0.0, 0.8) & (0.8, 0.2) \\ v_4 & (0.3, 0.4) & (0.0, 0.7) & (0.6, 0.2) & (0.7, 0.2) & (0.8, 0.2) \\ v_5 & (0.7, 0.1) & (0.8, 0.1) & (0.9, 0.1) & (0.7, 0.2) & (0.1, 0.8) \end{array}$$

- *Step* 3 The max-min products of the Pythagorean fuzzy soft matrices obtained in Step 1 and Step 2 have been obtained as below:

Table 1 A comparison of the results of existing methods in medical diagnosis

	Al	Bob	Joe	Ted
S^Y (Ye 2011)	Viral fever	Stomach problem	Typhoid	Viral fever
Szmidt and Kacprzyk (2004)	Viral fever	Stomach problem	Typhoid	Malaria
S_{CC} (Chen and Chang 2015)	Malaria	Stomach problem	Typhoid	Malaria
Wei et al. (2011)	Malaria	Stomach problem	Typhoid	Viral fever
$p = 1$ S_M (Mitchell 2013)	Malaria	Stomach problem	Typhoid	Viral fever
S_1 (Peng et al. 2017)	Malaria	Stomach problem	Typhoid	Viral fever
Proposed algorithm	Malaria	Stomach problem	Typhoid	Viral fever

$$R_1 = M * N = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline \text{Al} & (0.4, 0.1) & (0.7, 0.1) & (0.6, 0.1) & (0.2, 0.4) & (0.2, 0.6) \\ \text{Bob} & (0.3, 0.5) & (0.4, 0.6) & (0.4, 0.4) & (0.6, 0.1) & (0.2, 0.8) \\ \text{Joe} & (0.4, 0.1) & (0.6, 0.1) & (0.7, 0.1) & (0.2, 0.4) & (0.2, 0.5) \\ \text{Ted} & (0.7, 0.1) & (0.7, 0.1) & (0.5, 0.3) & (0.3, 0.4) & (0.3, 0.4) \end{array}$$

$$R_2 = M^c * N^c = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline \text{Al} & (0.7, 0.1) & (0.7, 0.2) & (0.7, 0.1) & (0.6, 0.2) & (0.8, 0.2) \\ \text{Bob} & (0.7, 0.1) & (0.8, 0.1) & (0.8, 0.1) & (0.7, 0.1) & (0.8, 0.1) \\ \text{Joe} & (0.6, 0.1) & (0.6, 0.1) & (0.6, 0.1) & (0.7, 0.2) & (0.7, 0.2) \\ \text{Ted} & (0.4, 0.3) & (0.6, 0.3) & (0.4, 0.3) & (0.4, 0.3) & (0.4, 0.3) \end{array}$$

- *Step* 4 The corresponding score matrices of the Pythagorean fuzzy soft matrices R_1 and R_2 obtained in Step 3 are:

$$S(R_1) = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline \text{Al} & 0.15 & 0.48 & 0.35 & -0.12 & -0.32 \\ \text{Bob} & -0.16 & -0.20 & 0.0 & 0.35 & -0.60 \\ \text{Joe} & 0.15 & 0.35 & 0.48 & -0.12 & -0.21 \\ \text{Ted} & 0.48 & 0.48 & 0.16 & -0.07 & -0.07 \end{array}$$

$$S(R_2) = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline \text{Al} & 0.48 & 0.45 & 0.48 & 0.32 & 0.60 \\ \text{Bob} & 0.48 & 0.63 & 0.63 & 0.48 & 0.63 \\ \text{Joe} & 0.35 & 0.35 & 0.35 & 0.45 & 0.45 \\ \text{Ted} & 0.07 & 0.27 & 0.07 & 0.07 & 0.07 \end{array}$$

- *Step* 5 The utility matrix of the score matrices $S(R_1)$ & $S(R_2)$ obtained in Step 4 is:

$$U = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline \text{Al} & -0.33 & 0.03 & -0.13 & -0.44 & -0.92 \\ \text{Bob} & -0.64 & -0.83 & -0.63 & -0.13 & -1.23 \\ \text{Joe} & -0.20 & 0.0 & 0.13 & -0.57 & -0.66 \\ \text{Ted} & 0.41 & 0.21 & 0.09 & -0.14 & -0.14 \end{array}$$

- *Step* 6 Therefore, in view of the values of the elements of utility matrix obtained in Step 5, it is quite probable that the Al is suffering from malaria (d_2), Bob is suffering stomach problem (d_4), Joe is suffering from Typhoid (d_3) and Ted is suffering form viral fever (d_1).

Observations Comparing the obtained results with the results of various other researchers who investigated the same diagnosis problem, we find that the proposed methodology is completely consistent with the various existing methods. This has been summarized in the following table with their references (Table 1):

Similar comparative analysis can also be carried out by taking example in case of decision-making problem of Sect. 4. In the future, the proposed idea of Pythagorean fuzzy soft matrices may be used in various other aspects, such as group decision making, information retrieval, pattern recognition, dimensionality reduction, data mining.

6 Conclusions

The concept of the Pythagorean fuzzy soft matrix has been well established along with its various types and properties. Valid proofs for the proposed properties over the matrices have also been provided. Further, the proposed algorithms for decision making by using choice matrix and weighted choice matrix and for medical diagnosis problem by using score matrix and utility matrix have been successfully implemented with the help of numerical example for each. Further, the comparative analysis shows that the results of the proposed methodology are equally consistent with the results of various other existing methods available in the literature.

Acknowledgements We are very much thankful to the anonymous reviewers for suggesting the points/mistakes which have been well implemented/corrected for the necessary improvement of the manuscript. We sincerely acknowledge our deep sense of gratitude to the reviewers for giving their valuable time to the manuscript.

Compliance with ethical standards

Conflict of interest: Authors declare that he/she has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Chen SM, Chang CH (2015) A novel similarity measure between Atanssovs intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition. Inf Sci 291:96–114

Chetia B, Das PK (2012) Some results of Intuitionistic fuzzy soft matrix theory. Adv Appl Sci Res 3:412–423

Klement EP, Mesiar R, Pap E (2000) Triangular norms. Kluwer Academic Publishers, Dordrecht

Klement EP, Mesiar R, Pap E (2004) Triangular norms. Position paper I: basic analytical and algebraic properties. Fuzzy Sets Syst 143:5–26

Maji PK, Biswas R, Roy AR (2001) Intuitionistic fuzzy soft sets. J Fuzzy Math 9:677–692

Maji PK, Biswas R, Roy AR (2002) An application of soft sets in a decision making problem. Comput Math Appl 44:1077–1083

Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45:555–562

Mitchell HB (2013) On the Dengfeng–Chuntian similarity measure and its application to pattern recognition. Pattern Recogn Lett 24:3101–3104

Molodstov DA (1999) Soft set theory-first result. Comput Math Appl 27:19–31

Naim C, Serdar E (2010) Soft matrix theory and its decision making. Comput Math Appl 59:3308–3314

Peng X, Yang Y, Song J, Jiang Y (2015) Pythagorean fuzzy soft set and its application. Comput Eng 41:224–229

Peng X, Yuan H, Yang Y (2017) Pythagorean fuzzy information measures and their applications. Int J Intell Syst 32(10):991–1029

Szmidt E, Kacprzyk J (2004) A similarity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning. In: International conference artificial intelligence and soft computing, Zakopane, Poland, pp 388–393

Wei CP, Wang P, Zhang YZ (2011) Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications. Inf Sci 181:4273–4286

Yager RR (2013) Pythagorean fuzzy subsets. In: Proceedings of joint IFSA world congress and NAFIPS annual meeting, Edmonton, Canada, pp 57–61

Ye J (2011) Cosine similarity measures for intuitionistic fuzzy sets and their applications. Math Comput Model 53:91–97

Yong Y, Chenli J (2011) Fuzzy soft matrices and their applications. Lect Notes Comput Sci 7002:618–627