Soft Computing (2019) 23:85–99
https://doi.org/10.1007/s00500-018-3411-7

METHODOLOGIES AND APPLICATION

Fuzzy multi-granulation decision-theoretic rough sets based on fuzzy preference relation

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Published online: 25 July 2018

Abstract

Preference analysis is a class of important issues in multi-criteria decision making. The rough set theory is a powerful approach to handle preference analysis. In order to solve the multi-criteria preference analysis, this work improves the fuzzy multi-granulation decision-theoretic rough set model with additive consistent fuzzy preference relation, and it is used to analyze data from different sources, i.e., multi-source (fuzzy) information system. More specifically, we introduce the models of optimistic and pessimistic fuzzy preference relation multi-granulation decision-theoretic rough sets. Then, their principal structure, basic properties and several kinds of uncertainty measure methods are investigated as well. An example is employed to illustrate the effectiveness of the proposed models, and comparisons are also offered according to different measures of our models and existing models.

Keywords Decision-theoretic rough set · Fuzzy preference relation · Multi-granulation · Granular computing

1 Introduction

After the introduction of rough set theory by Pawlak (1982), number of generalization have been proposed in terms of various requirements. For example, decision-theoretic rough sets (Deng and Yao 2014; Yao and Wong 1992; Yao 2003, 2008; Sun et al. 2016), variable precision rough sets (Ziarko 1993), Bayesian rough sets (Slezak and Ziarko 2005), game-theoretic rough sets (Herbert and Yao 2011), fuzzy rough sets/rough fuzzy sets (Dubois and Prade 2011), Pythagorean fuzzy decision-theoretic rough sets (Mandal and Ranadive 2018a), multi-granulation rough sets (Qian et al. 2010, 2014a), multi-granulation decision-theoretic rough sets (Qian et al. 2014b), multi-granulation rough sets based on covering (Lin et al. 2013), neighborhood-based multi-granulation rough sets (Lin et al. 2012), multi-granulation bipolar-valued fuzzy probabilistic rough sets (Mandal and Ranadive 2017), fuzzy multi-granulation decision-theoretic rough sets (Lin et al. 2016), Multi-granulation interval-valued fuzzy probabilistic rough sets based on interval-valued fuzzy preference relations (Mandal and Ranadive 2018b) and so on.

In the viewpoint of granular computing and multi-source information system, fuzzy multi-granulation decision-theoretic rough set model is an important generalization of rough set theory. It is based on fuzzy equivalence relation induced by a fuzzy attribute. However, they still cannot be used to analyze the information with preference relation, which limits its application in many problems under the framework of the preference analysis. This motives us to develop a new approximate strategy based on multi-granulation decision-theoretic rough sets to solve the multi-criteria preference analysis. And in this work, we combine multi-granulation decision-theoretic rough set with fuzzy preference relation and introduce a fuzzy preference relation multi-granulation rough set model.

Based on this idea, the contribution of this paper includes: (1) this paper is to present a new approach to approximate the decision class with a certain level of tolerance for errors through inclusion measure between two fuzzy preference granules; (2) process an additive consistent fuzzy preference relation multi-granulation decision-theoretic rough set

Communicated by V. Loia.

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model in order to solve the multi-criteria preference problem; (3) several kinds of uncertainty measure methods of proposed models are discussed; (4) furthermore, the comparisons of proposed models and existing models are also offered according to the given uncertainty measures.

The remainder of this paper is organized as follows: in Sect. 2 provides some basic concepts of fuzzy preference relations. In Sect. 3, fuzzy preference relation multi-granulation decision-theoretic rough set models are proposed. Then, we discussed their some properties. The uncertainties of the proposed models are measured in Sect. 4. In Sect. 5, an example is used to illustrate our method and comparison of existing methods. Finally, Sect. 6 concludes the paper.

2 Preliminaries

In this section, we will review some basic concepts such as fuzzy preference relations and inclusion measure, which have been addressed in Herrera-Viedma et al. (2004), Hu et al. (2010b), Pan et al. (2017) and Lin et al. (2016). Throughout, the paper, let U be a finite non-empty set called the universe of discourse. The class of all fuzzy sets in U will be denoted as $F(U)$. For a set A, $|A|$ denotes the cardinality of the set A.

Definition 1 (Lin et al. 2016) A multi-source fuzzy information system is

$$MS = \{IS_l \mid IS_l = (U, AT_l, \{(V_a)_{a \in AT_l}\}, f_l\},$$

where

(1) $U = \{x_1, x_2, \ldots, x_n\}$ is a finite non-empty set of objects, called the universe;
(2) $AT_l (1 \leq l \leq m)$ is a non-empty finite set of attributes of each subsystem;
(3) $\{V_a\}$ is the value of the attribute $a \in AT_l$; and
(4) $f_l : U \times AT_l \to \{(V_a)_{a \in AT_l}\}$ such that for all $x_i \in U$ and $a \in AT_l$, $f(x_i, a) \in V_a$, where $f(x_i, a)$ is the value of the attribute x_i with respect to the attribute a.

Particularly, if the attribute value is fuzzy, we call

$$MS = \{IS_l \mid IS_l = (U, AT_l, \{(V_a)_{a \in AT_l}\}, f_l\},$$

is a multi-source information system.

Definition 2 (Herrera-Viedma et al. 2004) Let R be a fuzzy preference relation (FPR) for the set $U = \{x_1, x_2, \ldots, x_n\}$, shown as follows:

$$R = (r_{ij})_{n \times n} = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix} \end{array}$$

where r_{ij} denotes the degree of preference of alternative x_i over alternative x_j, $r_{ij} \in [0, 1]$, $r_{ij} + r_{ji} = 1$, $\forall i, j \in \{1, 2, \ldots, n\}$. Especially,

$r_{ij} = 0.5$ indicates that there is no difference between alternative x_i and alternative x_j;

$r_{ij} > 0.5$ indicates that alternative x_i is better than alternative x_j;

$r_{ij} < 0.5$ indicates that alternative x_j is better than alternative x_i;

$r_{ij} = 1$ indicates that alternative x_i is absolutely better than alternative x_j;

$r_{ij} = 0$ indicates that alternative x_j is absolutely better than alternative x_i;

where $1 \leq i \leq n$ and $1 \leq j \leq n$.

In Definition 2, the FPR is considered, r_{ij} merely presents the degree of preference of alternative x_i is prior to the alternative x_j. However, in some practical applications, we need to show the degree of alternative x_i is poor than the alternative x_j. In order to satisfy all the two cases, we call the FPR in Definition 1 as upward fuzzy preference relation (UFPR), and call the other FPR as downward fuzzy preference relation (DFPR). We denote the UFPR as $R^{\uparrow} = (r_{ij}^{\uparrow})_{n \times n}$ and the DFPR as $R^{\downarrow} = (r_{ij}^{\downarrow})_{n \times n}$, and use $R = (r_{ij})_{n \times n}$ to denote all the two kinds of cases. In general, $r_{ij}^{\uparrow} + r_{ij}^{\downarrow} = 1$. Thus, for the DFPR,

$r_{ij}^{\downarrow} = 0.5$ indicates that there is no difference between alternative x_i and alternative x_j;

$r_{ij}^{\downarrow} > 0.5$ indicates that alternative x_i is poor than alternative x_j;

$r_{ij}^{\downarrow} < 0.5$ indicates that alternative x_j is poor than alternative x_i;

$r_{ij}^{\downarrow} = 1$ indicates that alternative x_i is absolutely poor than alternative x_j;

$r_{ij}^{\downarrow} = 0$ indicates that alternative x_j is absolutely poor than alternative x_i;

where $1 \leq i \leq n$ and $1 \leq j \leq n$.

As same as the UFPR, for the DFPR, $r_{ij}^{\downarrow} + r_{ij}^{\downarrow} = 1$ holds.

Obviously, FPRs not only reflect the fact that an object x_i is greater (less) than another x_j but also measure how much x_i is greater (less) than x_j. This shows fuzzy preference relations are more powerful in extracting information from fuzzy data than dominance relations.

Definition 3 A FPR $R = (r_{ij})_{n \times n}$ is called an additive consistent fuzzy preference relation, if it satisfies the following property

$r_{ij} = r_{ik} - r_{jk} + 0.5, \forall i, j, k \in \{1, 2, \ldots, n\}$.

Hu et al. (2010a) adopted the well-known Logis transfer function $\frac{1}{1+e^{k(f(x_i,a)-f(x_i,a))}}$ to compute the fuzzy preference degree of the alternative x_i to the alternative x_j

$$r_{ij}^{\uparrow} = \frac{1}{1 + e^{-k(f(x_i,a)-f(x_i,a))}}$$
$$r_{ij}^{\downarrow} = \frac{1}{1 + e^{k(f(x_i,a)-f(x_i,a))}}$$

where k is a positive constant. However, Pan et al. (2017) pointed out that this transfer fuzzy preference degree is not additive consistent and they suggest another transfer function. They compute the fuzzy preference degree of the alternative x_i to the alternative x_j

$$r_{ij}^{\uparrow} = 0.5 \times \left(\frac{f(x_i, a) - \wedge_{i=1}^{n} f(x_i, a)}{\wedge_{i=1}^{n} f(x_i, a) - \vee_{i=1}^{n} f(x_i, a)} - \frac{f(x_j, a) - \wedge_{i=1}^{n} f(x_i, a)}{\wedge_{i=1}^{n} f(x_i, a) - \vee_{i=1}^{n} f(x_i, a)} + 1 \right) \tag{1}$$

$$r_{ij}^{\downarrow} = 0.5 \times \left(\frac{f(x_j, a) - \wedge_{i=1}^{n} f(x_i, a)}{\wedge_{i=1}^{n} f(x_i, a) - \vee_{i=1}^{n} f(x_i, a)} - \frac{f(x_i, a) - \wedge_{i=1}^{n} f(x_i, a)}{\wedge_{i=1}^{n} f(x_i, a) - \vee_{i=1}^{n} f(x_i, a)} + 1 \right), \tag{2}$$

"$\wedge$" and "$\vee$" are the minimum and maximum value of $f(x_i, a)$, respectively.

According to the transfer functions (1) and (2), we give the following definition.

Definition 4 The upward and downward fuzzy preference classes $[x_i]_{R^\uparrow}$ and $[x_i]_{R^\downarrow}$ of x_i induced by the upward and downward additive fuzzy preference relations $R^\uparrow$ and $R^\downarrow$ are defined as follows:

$$[x_i]_{R^\uparrow} = \frac{r_{i1}^{\uparrow}}{x_1} + \frac{r_{i2}^{\uparrow}}{x_2} + \cdots + \frac{r_{in}^{\uparrow}}{x_n}$$

and

$$[x_i]_{R^\downarrow} = \frac{r_{i1}^{\downarrow}}{x_1} + \frac{r_{i2}^{\downarrow}}{x_2} + \cdots + \frac{r_{in}^{\downarrow}}{x_n},$$

where "+" means the union operation. Where $r_{ij}^{\uparrow}$ and $r_{ij}^{\downarrow}$ are defined in Eqs. (1) and (2). Obviously, $[x_i]_{R^\uparrow}$ and $[x_i]_{R^\downarrow}$ are the fuzzy information granules containing x_i.

The upward and downward additive preference relations generate a family of fuzzy information granules from the universe, which composes the upward and downward additive fuzzy preference granular structures, written by $P(R^\uparrow) = \{[x_1]_{R^\uparrow}, [x_2]_{R^\uparrow}, \ldots, [x_n]_{R^\uparrow}\}$ and $P(R^\downarrow) = \{[x_1]_{R^\downarrow}, [x_2]_{R^\downarrow}, \ldots, [x_n]_{R^\downarrow}\}$. Particularly,

(1) if $r_{ii}^{\uparrow} = r_{ii}^{\downarrow} = 1$ and $r_{ij}^{\uparrow} = r_{ij}^{\downarrow} = 0$, $j \neq i$, $i, j < n$, then $[x_i]_{R^\uparrow} = [x_i]_{R^\downarrow} = 1$, $i < n$ and $R^\uparrow = R^\downarrow = R$ is called a fuzzy preference identity relation.
(2) if $r_{ij}^{\uparrow} = r_{ij}^{\downarrow} = 1$, $i, j < n$, then $\left|[x_i]_{R^\uparrow}\right| = \left|[x_i]_{R^\downarrow}\right| = |U|$, $i < n$ and $R^\uparrow = R^\downarrow = R$ is called a fuzzy preference universal relation.

To aggregate the FPRs induced by multiple criteria, we use the following technique.

Definition 5 (Hu et al. 2010b) If r_{ij} and s_{ij} are the fuzzy preference degrees of the alternative x_i and the alternative x_j derived from the criteria a_1 and a_2, respectively, then the aggregate preference of a_1 and a_2 is defined as $\min(r_{ij}, s_{ij})$.

Definition 6 (Hu et al. 2010b) Let A and B be two fuzzy granules in the universe U, the inclusion measure $I(A, B)$ is defined as

$$I(A, B) = \frac{|A \wedge B|}{|A|},$$

where "$\wedge$" means the operation "min" and $|A| = \sum_{x \in U} A(x)$.

3 Fuzzy preference relation multi-granulation decision-theoretic rough sets

In this section, we will adopt the transfer functions (1) and (2) to compute the preference degree and introduce a fuzzy preference relation multi-granulation decision-theoretic rough sets (FPR-MG-DTRSs).

Let us consider $MS = \{IS_l \mid IS_l = (U, AT_l, \{(V_a)_{a \in AT_l}\}, f_l\}$ a multi-source fuzzy information system. In this paper, we assume MS is composed of m single-source information system. Similar to the granular method for the single-source information system, one gets

m upward fuzzy preference granular structures: $P(R_1^\uparrow)$, $P(R_2^\uparrow)$, ..., $P(R_m^\uparrow)$, where $P(R_l^\uparrow) = \{[x_1]_{R_l^\uparrow}, [x_1]_{R_l^\uparrow}, \ldots, [x_n]_{R_l^\uparrow}\}$ and $[x_i]_{R_l^\uparrow}$ $(i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, m)$ are fuzzy preference granules,

m downward fuzzy preference granular structures: $P(R_1^\downarrow)$, $P(R_2^\downarrow)$, ..., $P(R_m^\downarrow)$, where $P(R_l^\downarrow) = \{[x_1]_{R_l^\downarrow}, [x_2]_{R_l^\downarrow}, \ldots, [x_n]_{R_l^\downarrow}\}$ and $[x_i]_{R_l^\downarrow}$ $(i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, m)$ are fuzzy preference granules.

3.1 Optimistic fuzzy preference relation multi-granulation decision-theoretic rough sets

Definition 7 Given $P(R_1^\uparrow), P(R_2^\uparrow), \ldots, P(R_m^\uparrow)$ and $P(R_1^\downarrow), P(R_2^\downarrow), \ldots, P(R_m^\downarrow)$ are upward and downward m fuzzy granular structures. For a crisp decision class $X \subseteq U$, we can defined as.

Upward optimistic fuzzy preference relation multi-granulation lower approximation

$$\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X) = \Big\{x \in U \mid I([x]_{R_1^{\uparrow}}, X) \geq \alpha \vee I([x]_{R_2^{\uparrow}}, X) \geq \alpha \vee \cdots \vee I([x]_{R_m^{\uparrow}}, X) \geq \alpha\Big\},$$

Upward optimistic fuzzy preference relation multi-granulation upper approximation

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X) = U - \Big\{x \in U \mid I([x]_{R_1^{\uparrow}}, X) \leq \beta \vee I([x]_{R_2^{\uparrow}}, X) \leq \beta \vee \cdots \vee I([x]_{R_m^{\uparrow}}, X) \leq \beta\Big\},$$

Downward optimistic fuzzy preference relation multi-granulation lower approximation

$$\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X) = \Big\{x \in U \mid I([x]_{R_1^{\downarrow}}, X) \geq \alpha \vee I([x]_{R_2^{\downarrow}}, X) \geq \alpha \vee \cdots \vee I([x]_{R_m^{\downarrow}}, X) \geq \alpha\Big\},$$

Downward optimistic fuzzy preference relation multi-granulation upper approximation

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X) = U - \Big\{x \in U \mid I([x]_{R_1^{\downarrow}}, X) \leq \beta \vee I([x]_{R_2^{\downarrow}}, X) \leq \beta \vee \cdots \vee I([x]_{R_m^{\downarrow}}, X) \leq \beta\Big\},$$

where $[x]_{R_l^{\uparrow}}$ and $[x]_{R_l^{\downarrow}}$ are the upward and downward fuzzy preference classes of x induced by the upward and downward additive fuzzy preference relations $R_l^{\uparrow}$ and $R_l^{\downarrow}$; $I([x]_{R_l^{\uparrow}}, X)$ ($I([x]_{R_l^{\downarrow}}, X)$) is the fuzzy inclusion degree between $[x]_{R_l^{\uparrow}}$ ($[x]_{R_1^{\downarrow}}$) and X; α and β are two probability constraints with $0.5 \leq \alpha \leq 1$ and $0 \leq \beta < 0.5$.

Then, we call $(\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X), \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X))$ and $(\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X), \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X))$, the upward optimistic fuzzy preference relation multi-granulation decision-theoretic rough set (UOFPR-MG-DTRS) and downward optimistic fuzzy preference relation multi-granulation decision-theoretic rough set (DOFPR-MG-DTRS). The upward and downward optimistic fuzzy preference relation multi-granulation decision-theoretic boundary regions of X are defined as

$$BND^{\alpha(O),\beta(O)}_{\sum_{l=1}^{m} R_l^{\uparrow}}(X) = \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X) - \underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X)$$

and

$$BND^{\alpha(O),\beta(o)}_{\sum_{l=1}^{m} R_l^{\downarrow}}(X) = \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X) - \underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X).$$

According to Definition 7, we have the following propositions

Proposition 1 *Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow}), P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class $X \subseteq U$.*

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X) \supseteq \underline{R_l}^{\uparrow,\alpha}(X)$, $l \leq m$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X) \subseteq \overline{R_l}^{\uparrow,\beta}(X)$, $l \leq m$;
(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X) \supseteq \underline{R_l}^{\downarrow,\alpha}(X)$, $l \leq m$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X) \subseteq \overline{R_l}^{\downarrow,\beta}(X)$, $l \leq m$;

where

$$\underline{R_l}^{\uparrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) \geq \alpha\},$$

$$\overline{R_l}^{\uparrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) < \beta\},$$

$$\underline{R_l}^{\downarrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) \geq \alpha\}$$

and

$$\overline{R_l}^{\downarrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) < \beta\}.$$

Proposition 2 *Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow}), P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class $X \subseteq U$.*

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X) = \cup_{l=1}^{m} \underline{R_l}^{\uparrow,\alpha}(X)$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X) = \cap_{l=1}^{m} \overline{R_l}^{\uparrow,\beta}(X)$;

(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X) = \cup_{l=1}^{m} \underline{R_l}^{\downarrow,\alpha}(X)$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X) = \cap_{l=1}^{m} \overline{R_l}^{\downarrow,\beta}(X)$;

where

$$\underline{R_l}^{\uparrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) \geq \alpha\},$$

$$\overline{R_l}^{\uparrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) < \beta\},$$

$$\underline{R_l}^{\downarrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) \geq \alpha\}$$

and

$$\overline{R_l}^{\downarrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) < \beta\}.$$

Proposition 3 *Given* $P(R_1^{\uparrow})$, $P(R_2^{\uparrow})$, ..., $P(R_m^{\uparrow})$ *and* $P(R_1^{\downarrow})$, $P(R_2^{\downarrow})$, ..., $P(R_m^{\downarrow})$ *are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class* $X \subseteq Y \subseteq U$.

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(X) \supseteq \underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(O)}(Y)$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(X) \subseteq \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(Y)$;
(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(X) \supseteq \underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(O)}(Y)$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(X) \subseteq \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(Y)$.

Similar to the classical decision-theoretic rough sets, when the thresholds $1 \geq \alpha > \beta \geq 0$, we can obtain the decision rules tie-broke:

For UOFPR-MG-DTRS

(UOP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \geq \alpha$, decide $POS(X)$;
(UON1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \leq \beta$, decide $NEG(X)$;
(UOB1) otherwise, we decide $BND(X)$.

For DOFPR-MG-DTRS

(DOP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \geq \alpha$, decide $POS(X)$;
(DON1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \leq \beta$, decide $NEG(X)$;
(DOB1) otherwise, we decide $BND(X)$.

When the thresholds $1 \geq \alpha = \gamma = \beta \geq 0$, we can get the following decision rules:

For UOFPR-MG-DTRS

(UOP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \geq \alpha$, decide $POS(X)$;
(UON1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \leq \alpha$, decide $NEG(X)$;
(UOB1) otherwise, we decide $BND(X)$.

For DOFPR-MG-DTRS

(DOP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \geq \alpha$, decide $POS(X)$;
(DON1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \leq \alpha$, decide $NEG(X)$;
(DOB1) otherwise, we decide $BND(X)$.

3.2 Pessimistic fuzzy preference relation multi-granulation decision-theoretic rough sets

Definition 8 Given $P(R_1^{\uparrow})$, $P(R_2^{\uparrow})$, ..., $P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow})$, ..., $P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. For a crisp decision class $X \subseteq U$, we can defined as.

Upward pessimistic fuzzy preference relation multi-granulation lower approximation

$$\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X) = \Big\{x \in U \mid I([x]_{R_1^{\uparrow}}, X) \geq \alpha \wedge I([x]_{R_2^{\uparrow}}, X) \geq \alpha \wedge \cdots \wedge I([x]_{R_m^{\uparrow}}, X) \geq \alpha\Big\},$$

Upward pessimistic fuzzy preference relation multi-granulation upper approximation

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X) = U - \Big\{x \in U \mid I([x]_{R_1^{\uparrow}}, X) \leq \beta \wedge I([x]_{R_2^{\uparrow}}, X) \leq \beta \wedge \cdots \wedge I([x]_{R_m^{\uparrow}}, X) \leq \beta\Big\},$$

Downward pessimistic fuzzy preference relation multi-granulation lower approximation

$$\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X) = \Big\{x \in U \mid I([x]_{R_1^{\downarrow}}, X) \geq \alpha \wedge I([x]_{R_2^{\downarrow}}, X) \geq \alpha \wedge \cdots \wedge I([x]_{R_m^{\downarrow}}, X) \geq \alpha\Big\},$$

Downward pessimistic fuzzy preference relation multi-granulation upper approximation

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X) = U - \Big\{x \in U \mid I([x]_{R_1^{\downarrow}}, X) \leq \beta \wedge I([x]_{R_2^{\downarrow}}, X) \leq \beta \wedge \cdots \wedge I([x]_{R_m^{\downarrow}}, X) \leq \beta\Big\},$$

where $[x]_{R_l^{\uparrow}}$ and $[x]_{R_l^{\downarrow}}$ are the upward and downward fuzzy preference classes of x induced by the upward and downward additive fuzzy preference relations $R_l^{\uparrow}$ and $R_l^{\downarrow}$; $I([x]_{R_l^{\uparrow}}, X)$ ($I([x]_{R_l^{\downarrow}}, X)$) is the fuzzy inclusion degree between $[x]_{R_l^{\uparrow}}$ ($[x]_{R_1^{\downarrow}}$) and X; α and β are two probability constraints with $0.5 \leq \alpha \leq 1$ and $0 \leq \beta < 0.5$.

Then, we call $(\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X), \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X))$ and $(\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X), \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X))$, the upward pessimistic fuzzy preference relation multi-granulation decision-theoretic rough set (UPFPR-MG-DTRS) and downward pessimistic fuzzy preference relation multi-granulation decision-theoretic rough set (DPFPR-MG-DTRS). The upward and downward pessimistic fuzzy preference relation multi-granulation decision-theoretic boundary regions of X are defined as

$$BND^{\alpha(P),\beta(P)}_{\sum_{l=1}^{m} R_l^{\uparrow}}(X) = \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X) - \underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X)$$

and

$$BND^{\alpha(P),\beta(P)}_{\sum_{l=1}^{m} R_l^{\downarrow}}(X) = \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X) - \underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X).$$

According to Definition 8, we have the following propositions

Proposition 4 *Given $P(R_1^{\uparrow})$, $P(R_2^{\uparrow})$, …, $P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow})$, …, $P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class $X \subseteq U$.*

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X) \subseteq \underline{R}_l^{\uparrow,\alpha}(X)$, $l \leq m$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X) \subseteq \overline{R}_l^{\uparrow,\beta}(X)$, $l \leq m$;
(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X) \subseteq \underline{R}_l^{\downarrow,\alpha}(X)$, $l \leq m$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X) \subseteq \overline{R}_l^{\downarrow,\beta}(X)$, $l \leq m$;

where

$$\underline{R}_l^{\uparrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) \geq \alpha\},$$
$$\overline{R}_l^{\uparrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) < \beta\},$$
$$\underline{R}_l^{\downarrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) \geq \alpha\}$$

and

$$\overline{R}_l^{\downarrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) < \beta\}.$$

Proposition 5 *Given $P(R_1^{\uparrow})$, $P(R_2^{\uparrow})$, …, $P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow})$, …, $P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class $X \subseteq U$.*

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X) = \cup_{l=1}^{m}\underline{R}_l^{\uparrow,\alpha}(X)$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X) = \cap_{l=1}^{m}\overline{R}_l^{\uparrow,\beta}(X)$;
(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X) = \cup_{l=1}^{m}\underline{R}_l^{\downarrow,\alpha}(X)$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X) = \cap_{l=1}^{m}\overline{R}_l^{\downarrow,\beta}(X)$;

where

$$\underline{R}_l^{\uparrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) \geq \alpha\},$$
$$\overline{R}_l^{\uparrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\uparrow}}, X) < \beta\},$$
$$\underline{R}_l^{\downarrow,\alpha}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) \geq \alpha\}$$

and

$$\overline{R}_l^{\downarrow,\beta}(X) = \{x \in U \mid I([x]_{R_l^{\downarrow}}, X) < \beta\}.$$

Proposition 6 *Given $P(R_1^{\uparrow})$, $P(R_2^{\uparrow})$, …, $P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow})$, …, $P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. Then, the following properties hold for a crisp decision class $X \subseteq Y \subseteq U$.*

(1) $\underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(X) \supseteq \underline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\alpha(P)}(Y)$;
(2) $\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(X) \subseteq \overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(Y)$;
(3) $\underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(X) \supseteq \underline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\alpha(P)}(Y)$;
(4) $\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(X) \subseteq \overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(Y)$.

Similar to the classical decision-theoretic rough sets, when the thresholds $1 \geq \alpha > \beta \geq 0$, we can obtain the decision rules tie-broke:

For UPFPR-MG-DTRS

(UPP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \geq \alpha$, decide $POS(X)$;
(UPN1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \leq \beta$, decide $NEG(X)$;
(UPB1) otherwise, we decide $BND(X)$.

For DPFPR-MG-DTRS

(DPP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \geq \alpha$, decide $POS(X)$;
(DPN1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \leq \beta$, decide $NEG(X)$;
(DPB1) otherwise, we decide $BND(X)$.

When the thresholds $1 \geq \alpha = \gamma = \beta \geq 0$, we can get the following decision rules:

For UPFPR-MG-DTRS

(UPP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \geq \alpha$, decide $POS(X)$;
(UPN1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\uparrow}}, X) \leq \alpha$, decide $NEG(X)$;
(UPB1) otherwise, we decide $BND(X)$.

For DPFPR-MG-DTRS

(DPP1) if $\exists l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \geq \alpha$, decide $POS(X)$;
(DPN1) if $\forall l \in \{1, 2, \ldots, m\}$ such that $I([x]_{R_l^{\downarrow}}, X) \leq \alpha$, decide $NEG(X)$;
(DPB1) otherwise, we decide $BND(X)$.

4 Uncertainty measures

In this section, several measures are utilized to calculate the uncertainty of these models which proposed in previous. The uncertainty of knowledge is caused by the boundary regions, in the view point of rough set approximation. The larger the boundary area is, the more uncertainly. According to the uncertainty measure method in classical rough set, we can define the accuracy, roughness and approximation quality for each model as follows.

Definition 9 Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. The accuracies degrees of X in terms of UOFPR-MG-DTRS, DOFPR-MG-DTRS, UPFPR-MG-DTRS and DPFPR-MG-DTRS are defined, respectively, as

$$\rho^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(O)}(X)\right|}{\left|\overline{\sum_{l=1}^m R_l^{\uparrow}}^{\beta(O)}(X)\right|}, \tag{3}$$

$$\rho^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(O)}(X)\right|}{\left|\overline{\sum_{l=1}^m R_l^{\downarrow}}^{\beta(O)}(X)\right|}, \tag{4}$$

$$\rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(P)}(X)\right|}{\left|\overline{\sum_{l=1}^m R_l^{\uparrow}}^{\beta(P)}(X)\right|}, \tag{5}$$

$$\rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(P)}(X)\right|}{\left|\overline{\sum_{l=1}^m R_l^{\downarrow}}^{\beta(P)}(X)\right|}. \tag{6}$$

The corresponding roughness degrees of X in terms of UOFPR-MG-DTRS, DOFPR-MG-DTRS, UPFPR-MG-DTRS and DPFPR-MG-DTRS are defined, respectively, as

$$\zeta^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = 1 - \rho^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X),$$
$$\zeta^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = 1 - \rho^{\alpha(O),\beta(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X),$$
$$\zeta^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = 1 - \rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X),$$
$$\zeta^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = 1 - \rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X).$$

Roughness measure is the well-known Marczewski–Steinhaus distance between the lower and upper approximations according to Yao (2001). Because some properties of expanded model have changed.

Definition 10 Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow})$, $P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. The approximated degrees of X in terms of UOFPR-MG-DTRS, DOFPR-MG-DTRS, UPFPR-MG-DTRS and DPFPR-MG-DTRS are defined, respectively, as

$$\pi^{\alpha(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(O)}(X)\right|}{|X|}, \tag{7}$$

$$\pi^{\alpha(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(O)}(X)\right|}{|X|}, \tag{8}$$

$$\pi^{\alpha(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(P)}(X)\right|}{|X|}, \tag{9}$$

$$\pi^{\alpha(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(P)}(X)\right|}{|X|}. \tag{10}$$

Definition 11 Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow}), P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. The degree of dependency of X in terms of UOFPR-MG-DTRS, DOFPR-MG-DTRS, UPFPR-MG-DTRS and DPFPR-MG-DTRS is defined, respectively, as

$$\omega^{\alpha(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(O)}(X)\right|}{|U|}, \tag{11}$$

$$\omega^{\alpha(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(O)}(X)\right|}{|U|}, \tag{12}$$

$$\omega^{\alpha(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\uparrow}}^{\alpha(P)}(X)\right|}{|U|}, \tag{13}$$

$$\omega^{\alpha(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) = \frac{\left|\underline{\sum_{l=1}^m R_l^{\downarrow}}^{\alpha(P)}(X)\right|}{|U|}. \tag{14}$$

By Definitions 9, 10 and 11, we get the following properties.

Proposition 7 *Given $P(R_1^{\uparrow}), P(R_2^{\uparrow}), \ldots, P(R_m^{\uparrow})$ and $P(R_1^{\downarrow}), P(R_2^{\downarrow}), \ldots, P(R_m^{\downarrow})$ are upward and downward m fuzzy granular structures. For any $X \subseteq U$, the following properties hold.*

(1) $\rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) \leq \rho^{\alpha(P),\beta(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X)$;
(2) $\rho^{\alpha(P),\beta(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) \leq \rho^{\alpha(P),\beta(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X)$;
$\pi^{\alpha(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X) \leq \pi^{\alpha(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X)$;
(3) $\pi^{\alpha(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) \leq \pi^{\alpha(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X)$;
(4) $0 \leq \omega^{\alpha(O)}_{\sum_{l=1}^m R_l^{\uparrow}}(X), \omega^{\alpha(P)}_{\sum_{l=1}^m R_l^{\uparrow}}(X), \omega^{\alpha(O)}_{\sum_{l=1}^m R_l^{\downarrow}}(X), \omega^{\alpha(P)}_{\sum_{l=1}^m R_l^{\downarrow}}(X) \leq 1$.

5 An illustrative example

Table 1 (Lin et al. 2016) depicts a fuzzy multi-source decision information system about the evaluation problem of credit card applicants. Suppose that $U = \{x_1, x_2, \ldots, x_9\}$ is a set of nine applicant. Every applicant in each sub-information system, denoted by EC_1, EC_2 and EC_3, is described by three fuzzy conditional attributes. They are $a_1 = best\ education$, $a_2 = better\ education$, $a_3 = good\ education$, $a_4 = high\ salary$, $a_5 = middle\ salary$, $a_6 = low\ salary$, $a_7 = older\ age$, $a_8 = middle\ age$, $a_9 = young\ age$, respectively. The member ship degrees of every applicant are given in Table 1. A decision partition is $D_1 = \{x_1, x_2, x_4, x_7\}$ and $D_2 = \{x_3, x_5, x_6, x_8, x_9\}$.

In the following, we will describe the process of computing in detail.

Table 1 A multi-source fuzzy information system

	EC_1			EC_2			EC_3			Decision
	a_1	a_4	a_7	a_2	a_5	a_8	a_3	a_6	a_9	D
x_1	0.8	0.1	0.2	0.1	0.1	0.5	0.1	0.2	0.3	Accept
x_2	0.3	0.5	0.2	0.3	0.3	0.7	0.4	0.2	0.1	Accept
x_3	0.2	0.1	0.6	0.6	0.3	0.2	0.2	0.6	0.2	Decline
x_4	0.6	0.3	0.5	0.2	0.2	0.2	0.2	0.5	0.3	Accept
x_5	0.4	0.4	0.3	0.3	0.4	0.3	0.3	0.2	0.4	Decline
x_6	0.2	0.3	0.5	0.2	0.3	0.6	0.2	0.4	0.2	Decline
x_7	0.3	0.3	0.6	0.3	0.4	0.2	0.3	0.3	0.2	Accept
x_8	0.3	0.4	0.3	0.3	0.2	0.2	0.3	0.4	0.5	Decline
x_9	0.3	0.2	0.4	0.3	0.4	0.4	0.4	0.4	0.2	Decline

In the following, we will describe the process of computing in detail.

(1) We make use of Eq. (1) to compute the fuzzy preference degree of the alternative x_i ($i = 1, 2, \ldots, 9$) to the alternative x_j ($j = 1, 2, \ldots, 9$) by each attribute of every source. Then for a source EC_1 of the multi-source information system, one obtains three upward additive consistent fuzzy preference relations from conditional attributes a_1, a_4, and a_7, respectively, which represent in Eqs. (15)–(17).

(2) We use Definition 5 to aggregate the three upward additive consistent fuzzy preference relations $R^{\uparrow}_{a_1}$, $R^{\uparrow}_{a_4}$, and $R^{\uparrow}_{a_7}$ and gets a upward additive consistent fuzzy preference relation on three attributes of EC_1, which generates a fuzzy partition called a upward additive fuzzy preference granular structure on U, represent in Eq. (18).

From the granular structure $R^{\uparrow}_{EC_1}(x_i, x_j)$, one can get nine fuzzy preference granules on U as follows:

$$\begin{aligned}
[x_1]_{R^{\uparrow}_{EC_1}} &= \frac{0.5000}{x_1} + \frac{0.0000}{x_2} + \frac{0.0000}{x_3} + \frac{0.1250}{x_4} + \frac{0.1250}{x_5} + \frac{0.1250}{x_6} + \frac{0.0000}{x_7} + \frac{0.1250}{x_8} + \frac{0.2500}{x_9};\\
[x_2]_{R^{\uparrow}_{EC_1}} &= \frac{0.0833}{x_1} + \frac{0.5000}{x_2} + \frac{0.0000}{x_3} + \frac{0.1250}{x_4} + \frac{0.3750}{x_5} + \frac{0.1250}{x_6} + \frac{0.0000}{x_7} + \frac{0.3750}{x_8} + \frac{0.2500}{x_9};\\
[x_3]_{R^{\uparrow}_{EC_1}} &= \frac{0.0000}{x_1} + \frac{0.0000}{x_2} + \frac{0.5000}{x_3} + \frac{0.1667}{x_4} + \frac{0.1250}{x_5} + \frac{0.2500}{x_6} + \frac{0.2500}{x_7} + \frac{0.1250}{x_8} + \frac{0.3750}{x_9};
\end{aligned}$$

$$[x_4]_{R^{\uparrow}_{EC_1}} = \frac{0.3333}{x_1} + \frac{0.2500}{x_2} + \frac{0.3750}{x_3} + \frac{0.5000}{x_4} + \frac{0.3750}{x_5} + \frac{0.5000}{x_6} + \frac{0.3750}{x_7} + \frac{0.3750}{x_8} + \frac{0.6250}{x_9};$$

$$[x_5]_{R^{\uparrow}_{EC_1}} = \frac{0.1667}{x_1} + \frac{0.3750}{x_2} + \frac{0.1250}{x_3} + \frac{0.2500}{x_4} + \frac{0.5000}{x_5} + \frac{0.2500}{x_6} + \frac{0.1250}{x_7} + \frac{0.5000}{x_8} + \frac{0.3750}{x_9};$$

$$[x_6]_{R^{\uparrow}_{EC_1}} = \frac{0.0000}{x_1} + \frac{0.2500}{x_2} + \frac{0.3750}{x_3} + \frac{0.1667}{x_4} + \frac{0.3333}{x_5} + \frac{0.5000}{x_6} + \frac{0.3750}{x_7} + \frac{0.3750}{x_8} + \frac{0.4167}{x_9};$$

$$[x_7]_{R^{\uparrow}_{EC_1}} = \frac{0.0833}{x_1} + \frac{0.2500}{x_2} + \frac{0.5000}{x_3} + \frac{0.2500}{x_4} + \frac{0.3750}{x_5} + \frac{0.5000}{x_6} + \frac{0.5000}{x_7} + \frac{0.3750}{x_8} + \frac{0.5000}{x_9};$$

$$R^{\uparrow}_{d_1}(x_i, x_j) = \begin{array}{c|ccccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ \hline x_1 & 0.5000 & 0.9167 & 1.0000 & 0.6667 & 0.8333 & 1.0000 & 0.9167 & 0.9167 & 0.9167 \\ x_2 & 0.0833 & 0.5000 & 0.5833 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ x_3 & 0.0000 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ x_4 & 0.3333 & 0.7500 & 0.8333 & 0.5000 & 0.6667 & 0.8333 & 0.7500 & 0.7500 & 0.7500 \\ x_5 & 0.1667 & 0.5833 & 0.6667 & 0.3333 & 0.5000 & 0.6667 & 0.5833 & 0.5833 & 0.5833 \\ x_6 & 0.0000 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ x_7 & 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ x_8 & 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ x_9 & 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \end{array} \tag{15}$$

$$R^{\uparrow}_{d_4}(x_i, x_j) = \begin{array}{c|ccccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ \hline x_1 & 0.5000 & 0.0000 & 0.5000 & 0.2500 & 0.1250 & 0.2500 & 0.2500 & 0.1250 & 0.3750 \\ x_2 & 1.0000 & 0.5000 & 1.0000 & 0.7500 & 0.6250 & 0.7500 & 0.7500 & 0.6250 & 0.87500 \\ x_3 & 0.5000 & 0.0000 & 0.5000 & 0.2500 & 0.1250 & 0.2500 & 0.2500 & 0.1250 & 0.3750 \\ x_4 & 0.7500 & 0.2500 & 0.7500 & 0.5000 & 0.3750 & 0.5000 & 0.5000 & 0.3750 & 0.6250 \\ x_5 & 0.8750 & 0.3750 & 0.8750 & 0.6250 & 0.5000 & 0.6250 & 0.6250 & 0.5000 & 0.7500 \\ x_6 & 0.7500 & 0.2500 & 0.7500 & 0.5000 & 0.3750 & 0.5000 & 0.5000 & 0.3750 & 0.6250 \\ x_7 & 0.7500 & 0.2500 & 0.7500 & 0.5000 & 0.3750 & 0.5000 & 0.5000 & 0.3750 & 0.6250 \\ x_8 & 0.8750 & 0.3750 & 0.8750 & 0.6250 & 0.5000 & 0.6250 & 0.6250 & 0.5000 & 0.7500 \\ x_9 & 0.6250 & 0.1250 & 0.6250 & 0.3750 & 0.2500 & 0.3750 & 0.3750 & 0.2500 & 0.5000 \end{array} \tag{16}$$

$$R_{a_7}^{\uparrow}(x_i,x_j)= \begin{array}{c} \\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8\\ x_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array}\\ \begin{pmatrix} 0.5000 & 0.5000 & 0.0000 & 0.1250 & 0.3750 & 0.1250 & 0.0000 & 0.3750 & 0.2500\\ 0.5000 & 0.5000 & 0.0000 & 0.1250 & 0.3750 & 0.1250 & 0.0000 & 0.3750 & 0.2500\\ 1.0000 & 1.0000 & 0.5000 & 0.6250 & 0.8750 & 0.6250 & 0.5000 & 0.8750 & 0.7500\\ 0.8750 & 0.8750 & 0.3750 & 0.5000 & 0.7500 & 0.5000 & 0.3750 & 0.7500 & 0.6250\\ 0.6250 & 0.6250 & 0.1250 & 0.2500 & 0.5000 & 0.2500 & 0.1250 & 0.5000 & 0.3750\\ 0.8750 & 0.8750 & 0.3750 & 0.5000 & 0.7500 & 0.5000 & 0.3750 & 0.7500 & 0.6250\\ 1.0000 & 1.0000 & 0.5000 & 0.6250 & 0.8750 & 0.6250 & 0.5000 & 0.8750 & 0.7500\\ 0.6250 & 0.6250 & 0.1250 & 0.2500 & 0.5000 & 0.2500 & 0.1250 & 0.5000 & 0.3750\\ 0.7500 & 0.7500 & 0.2500 & 0.3750 & 0.6250 & 0.3750 & 0.2500 & 0.6250 & 0.5000 \end{pmatrix} \end{array} \tag{17}$$

$$R_{EC_1}^{\uparrow}(x_i,x_j)= \begin{array}{c} \\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8\\ x_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array}\\ \begin{pmatrix} 0.5000 & 0.0000 & 0.0000 & 0.1250 & 0.1250 & 0.1250 & 0.0000 & 0.1250 & 0.2500\\ 0.0833 & 0.5000 & 0.0000 & 0.1250 & 0.3750 & 0.1250 & 0.0000 & 0.3750 & 0.2500\\ 0.0000 & 0.0000 & 0.5000 & 0.1667 & 0.1250 & 0.2500 & 0.2500 & 0.1250 & 0.3750\\ 0.3333 & 0.2500 & 0.3750 & 0.5000 & 0.3750 & 0.5000 & 0.3750 & 0.3750 & 0.6250\\ 0.1667 & 0.3750 & 0.1250 & 0.2500 & 0.5000 & 0.2500 & 0.1250 & 0.5000 & 0.3750\\ 0.8750 & 0.8750 & 0.3750 & 0.5000 & 0.7500 & 0.5000 & 0.3750 & 0.7500 & 0.6250\\ 0.0833 & 0.2500 & 0.5000 & 0.2500 & 0.3750 & 0.5000 & 0.5000 & 0.3750 & 0.5000\\ 0.0833 & 0.3750 & 0.1250 & 0.2500 & 0.4167 & 0.2500 & 0.1250 & 0.5000 & 0.3750\\ 0.0833 & 0.1250 & 0.2500 & 0.2500 & 0.2500 & 0.3750 & 0.2500 & 0.2500 & 0.5000 \end{pmatrix} \end{array} \tag{18}$$

$$[x_8]_{R_{EC_1}^{\uparrow}} = \frac{0.0833}{x_1}+\frac{0.3750}{x_2}+\frac{0.1250}{x_3} +\frac{0.2500}{x_4}+\frac{0.4167}{x_5}+\frac{0.2500}{x_6} +\frac{0.1250}{x_7}+\frac{0.5000}{x_8}+\frac{0.3750}{x_9}$$

and

$$[x_9]_{R_{EC_1}^{\uparrow}} = \frac{0.0833}{x_1}+\frac{0.1250}{x_2}+\frac{0.2500}{x_3} +\frac{0.2500}{x_4}+\frac{0.2500}{x_5}+\frac{0.3750}{x_6} +\frac{0.2500}{x_7}+\frac{0.2500}{x_8}+\frac{0.5000}{x_9}.$$

Based on the inclusion measure of fuzzy sets, we get that

$$I([x_1]_{R_{EC_1}^{\uparrow}},D_1)=0.50,\ I([x_2]_{R_{EC_1}^{\uparrow}},D_1)=0.56,$$
$$I([x_3]_{R_{EC_1}^{\uparrow}},D_1)=0.23,\ I([x_4]_{R_{EC_1}^{\uparrow}},D_1)=0.39,$$
$$I([x_5]_{R_{EC_1}^{\uparrow}},D_1)=0.34,\ I([x_6]_{R_{EC_1}^{\uparrow}},D_1)=0.28,$$
$$I([x_7]_{R_{EC_1}^{\uparrow}},D_1)=0.32,\ I([x_8]_{R_{EC_1}^{\uparrow}},D_1)=0.33,$$
$$I([x_9]_{R_{EC_1}^{\uparrow}},D_1)=0.30.$$

Similarly, one gets

$$I([x_1]_{R_{EC_1}^{\uparrow}},D_2)=0.50,\ I([x_2]_{R_{EC_1}^{\uparrow}},D_2)=0.44,$$
$$I([x_3]_{R_{EC_1}^{\uparrow}},D_2)=0.77,\ I([x_4]_{R_{EC_1}^{\uparrow}},D_2)=0.61,$$
$$I([x_5]_{R_{EC_1}^{\uparrow}},D_2)=0.66,\ I([x_6]_{R_{EC_1}^{\uparrow}},D_2)=0.72,$$
$$I([x_7]_{R_{EC_1}^{\uparrow}},D_2)=0.68,\ I([x_8]_{R_{EC_1}^{\uparrow}},D_2)=0.67,$$
$$I([x_9]_{R_{EC_1}^{\uparrow}},D_2)=0.70.$$

By using the same method, one obtains the other two upward additive consistent fuzzy preference relations induced by the attributes of EC_2 and EC_3, which represent in Eqs. (19) and (20).

Similarly, based on the inclusion degree from (19), we have that

$$I([x_1]_{R_{EC_2}^{\uparrow}},D_1)=0.68,\ I([x_2]_{R_{EC_2}^{\uparrow}},D_1)=0.53,$$
$$I([x_3]_{R_{EC_2}^{\uparrow}},D_1)=0.37,\ I([x_4]_{R_{EC_2}^{\uparrow}},D_1)=0.48,$$
$$I([x_5]_{R_{EC_2}^{\uparrow}},D_1)=0.45,\ I([x_6]_{R_{EC_2}^{\uparrow}},D_1)=0.52,$$
$$I([x_7]_{R_{EC_2}^{\uparrow}},D_1)=0.44,\ I([x_8]_{R_{EC_2}^{\uparrow}},D_1)=0.43,$$
$$I([x_9]_{R_{EC_2}^{\uparrow}},D_1)=0.46. \text{ And}$$
$$I([x_1]_{R_{EC_2}^{\uparrow}},D_2)=0.32,\ I([x_2]_{R_{EC_2}^{\uparrow}},D_2)=0.47,$$
$$I([x_3]_{R_{EC_2}^{\uparrow}},D_2)=0.63,\ I([x_4]_{R_{EC_2}^{\uparrow}},D_2)=0.52,$$

$I([x_5]_{R^{\uparrow}_{EC_2}}, D_2) = 0.55$, $I([x_6]_{R^{\uparrow}_{EC_2}}, D_2) = 0.48$,
$I([x_7]_{R^{\uparrow}_{EC_2}}, D_2) = 0.56$, $I([x_8]_{R^{\uparrow}_{EC_2}}, D_2) = 0.57$,
$I([x_9]_{R^{\uparrow}_{EC_2}}, D_2) = 0.54$.

Moreover, based on the inclusion degree from Eq. (20), we have that

$I([x_1]_{R^{\uparrow}_{EC_3}}, D_1) = 0.58$, $I([x_2]_{R^{\uparrow}_{EC_3}}, D_1) = 0.67$,
$I([x_3]_{R^{\uparrow}_{EC_3}}, D_1) = 0.45$, $I([x_4]_{R^{\uparrow}_{EC_3}}, D_1) = 0.48$,
$I([x_5]_{R^{\uparrow}_{EC_3}}, D_1) = 0.52$, $I([x_6]_{R^{\uparrow}_{EC_3}}, D_1) = 0.49$,
$I([x_7]_{R^{\uparrow}_{EC_3}}, D_1) = 0.55$, $I([x_8]_{R^{\uparrow}_{EC_3}}, D_1) = 0.48$,
$I([x_9]_{R^{\uparrow}_{EC_3}}, D_1) = 0.52. And$
$I([x_1]_{R^{\uparrow}_{EC_3}}, D_2) = 0.42$, $I([x_2]_{R^{\uparrow}_{EC_3}}, D_2) = 0.33$,
$I([x_3]_{R^{\uparrow}_{EC_3}}, D_2) = 0.55$, $I([x_4]_{R^{\uparrow}_{EC_3}}, D_2) = 0.52$,
$I([x_5]_{R^{\uparrow}_{EC_3}}, D_2) = 0.48$, $I([x_6]_{R^{\uparrow}_{EC_3}}, D_2) = 0.51$,
$I([x_7]_{R^{\uparrow}_{EC_3}}, D_2) = 0.45$, $I([x_8]_{R^{\uparrow}_{EC_3}}, D_2) = 0.52$,
$I([x_9]_{R^{\uparrow}_{EC_3}}, D_2) = 0.48$.

Based on the above three fuzzy preference granular structures and the inclusion degree between fuzzy preference granules $[x_i]_{R^{\uparrow}_{EC_l}}$ and $[x_j]_{R^{\uparrow}_{EC_l}}$ and assume that $\alpha = 0.60$, $\beta = 0.35$, one can get the upward optimistic/pessimistic fuzzy multi-granulation lower and upper approximations of the decision concepts D_1 and D_2, respectively.

A upward optimistic fuzzy multi-granulation lower approximation of D_1:

$$\underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_1) = \{x_1, x_2\};$$

A upward optimistic fuzzy multi-granulation upper approximation of D_1:

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(D_1) = \{x_1, x_2, x_4\};$$

A upward pessimistic fuzzy multi-granulation lower approximation of D_1:

$$\underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(P)}(D_1) = \emptyset;$$

A upward pessimistic fuzzy multi-granulation upper approximation of D_1:

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(D_1) = U;$$

$$R^{\uparrow}_{EC_2}(x_i, x_j) = \begin{array}{c} \\ x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array} \\ \begin{pmatrix}
0.5000 & 0.1667 & 0.0000 & 0.3333 & 0.0000 & 0.1667 & 0.0000 & 0.3000 & 0.0000\\
0.7000 & 0.5000 & 0.2000 & 0.6000 & 0.3333 & 0.5000 & 0.3333 & 0.5000 & 0.3333\\
0.2000 & 0.0000 & 0.5000 & 0.5000 & 0.3333 & 0.1000 & 0.3333 & 0.5000 & 0.3000\\
0.2000 & 0.0000 & 0.1000 & 0.5000 & 0.1667 & 0.1000 & 0.1667 & 0.4000 & 0.1667\\
0.3000 & 0.1000 & 0.2000 & 0.6000 & 0.5000 & 0.2000 & 0.5000 & 0.5000 & 0.4000\\
0.6000 & 0.4000 & 0.1000 & 0.5000 & 0.3333 & 0.5000 & 0.3333 & 0.4000 & 0.3333\\
0.2000 & 0.0000 & 0.2000 & 0.5000 & 0.4000 & 0.1000 & 0.5000 & 0.5000 & 0.3000\\
0.2000 & 0.0000 & 0.2000 & 0.5000 & 0.1667 & 0.1000 & 0.1667 & 0.5000 & 0.1667\\
0.4000 & 0.2000 & 0.2000 & 0.6000 & 0.5000 & 0.3000 & 0.5000 & 0.5000 & 0.5000
\end{pmatrix} \end{array} \quad (19)$$

$$R^{\uparrow}_{EC_3}(x_i, x_j) = \begin{array}{c} \\ x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array} \\ \begin{pmatrix}
0.5000 & 0.0000 & 0.0000 & 0.1250 & 0.1667 & 0.2500 & 0.1667 & 0.1667 & 0.0000\\
0.2500 & 0.5000 & 0.0000 & 0.1250 & 0.1250 & 0.2500 & 0.3750 & 0.0000 & 0.2500\\
0.3750 & 0.1667 & 0.5000 & 0.3750 & 0.2500 & 0.5000 & 0.3333 & 0.1250 & 0.1667\\
0.5000 & 0.1667 & 0.3750 & 0.5000 & 0.3333 & 0.5000 & 0.3333 & 0.2500 & 0.1667\\
0.5000 & 0.3333 & 0.0000 & 0.1250 & 0.5000 & 0.2500 & 0.3750 & 0.2500 & 0.2500\\
0.3750 & 0.1667 & 0.2500 & 0.3750 & 0.2500 & 0.5000 & 0.3333 & 0.1250 & 0.1667\\
0.3750 & 0.3333 & 0.1250 & 0.2500 & 0.2500 & 0.3750 & 0.5000 & 0.1250 & 0.3333\\
0.7500 & 0.3333 & 0.2500 & 0.3750 & 0.5000 & 0.5000 & 0.5000 & 0.5000 & 0.3333\\
0.3750 & 0.5000 & 0.2500 & 0.3750 & 0.2500 & 0.5000 & 0.5000 & 0.1250 & 0.5000
\end{pmatrix} \end{array} \quad (20)$$

A upward optimistic fuzzy multi-granulation lower approximation of D_2:

$$\underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_2) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\};$$

A upward optimistic fuzzy multi-granulation upper approximation of D_2:

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(O)}(D_2) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\};$$

A upward pessimistic fuzzy multi-granulation lower approximation of D_2:

$$\underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(P)}(D_2) = \emptyset;$$

A upward pessimistic fuzzy multi-granulation upper approximation of D_2:

$$\overline{\sum_{l=1}^{m} R_l^{\uparrow}}^{\beta(P)}(D_2) = U.$$

(3) Decision rules

(a) Upward optimistic decision rules:

(UOP1) if $x \in \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_1)$ or $x \in U - \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_2)$, then decide *Accept*;

(UON1) if $x \in U - \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_1)$ or $x \in \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_2)$, then *Decline*;

(UOB1) otherwise, decide *Retard*.

(b) Upward pessimistic decision rules:

(UPP1) if $x \in \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(P)}(D_1)$ or $x \in U - \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(P)}(D_2)$, then decide *Accept*;

(UPN1) if $x \in U - \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(O)}(D_1)$ or $x \in \underline{\sum_{l=1}^{3} R_l^{\uparrow}}^{\alpha(P)}(D_2)$, then *Decline*;

(UPB1) otherwise, decide *Retard*.

By using the above same method, we can compute the downward optimistic/pessimistic fuzzy multi-granulation lower and upper approximations of the decision concepts D_1 and D_2, respectively.

A downward optimistic fuzzy multi-granulation lower approximation of D_1:

$$\underline{\sum_{l=1}^{3} R_l^{\downarrow}}^{\alpha(O)}(D_1) = \{x_1\};$$

A downward optimistic fuzzy multi-granulation upper approximation of D_1:

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(D_1) = \{x_1, x_2, x_4, x_6, x_8\};$$

A downward pessimistic fuzzy multi-granulation lower approximation of D_1:

$$\underline{\sum_{l=1}^{3} R_l^{\downarrow}}^{\alpha(P)}(D_1) = \emptyset;$$

A downward pessimistic fuzzy multi-granulation upper approximation of D_1:

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(D_1) = U;$$

A downward optimistic fuzzy multi-granulation lower approximation of D_2:

$$\underline{\sum_{l=1}^{3} R_l^{\downarrow}}^{\alpha(O)}(D_2) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\};$$

A downward optimistic fuzzy multi-granulation upper approximation of D_2:

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(O)}(D_2) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\};$$

A downward pessimistic fuzzy multi-granulation lower approximation of D_2:

$$\underline{\sum_{l=1}^{3} R_l^{\downarrow}}^{\alpha(P)}(D_2) = \{x_3\};$$

A downward pessimistic fuzzy multi-granulation upper approximation of D_2:

$$\overline{\sum_{l=1}^{m} R_l^{\downarrow}}^{\beta(P)}(D_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}.$$

Decision rules

(a) Downward optimistic decision rules:

(DOP1) if $x \in \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(O)}}(D_1)$ or $x \in U - \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(O)}}(D_2)$, then decide *Accept*;

(DON1) if $x \in U - \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(O)}}(D_1)$ or $x \in \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(O)}}(D_2)$, then decide *Decline*;

(DOB1) otherwise, decide *Retard*.

(b) Downward pessimistic decision rules:

(DPP1) if $x \in \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(P)}}(D_1)$ or $x \in U - \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(P)}}(D_2)$, then decide *Accept*;

(DPN1) if $x \in U - \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(O)}}(D_1)$ or $x \in \underline{\sum_{l=1}^{3} R_l^{\downarrow\alpha(P)}}(D_2)$, then decide *Decline*;

(DPB1) otherwise, decide *Retard*.

If we defined the accuracies degrees of X in terms of optimistic fuzzy multi-granulation decision-theoretic rough set (OF-MG-DTRS) (Definition 6, Lin et al. 2016) and pessimistic fuzzy multi-granulation decision-theoretic rough set (PF-MG-DTRS) (Definition 7, Lin et al. 2016) are, respectively, as

$$\rho^{O,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\alpha}(X)\right|}{\left|\overline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\beta}(X)\right|}, \tag{21}$$

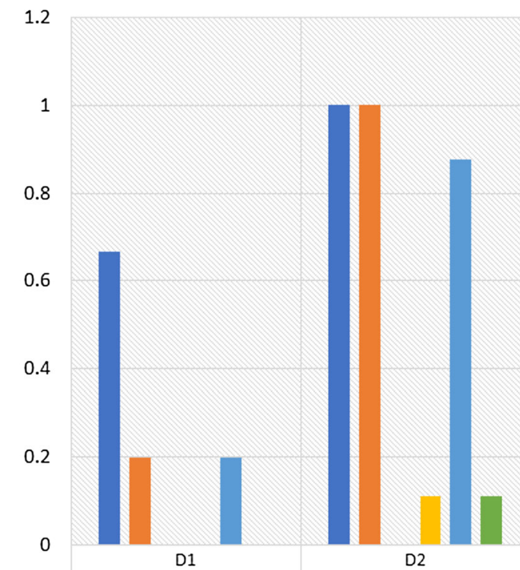
$$\rho^{P,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\alpha}(X)\right|}{\left|\overline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\beta}(X)\right|}, \tag{22}$$

then from Eqs. (3)–(6) and (21)–(22), the comparison of our proposed models and the model given in (Lin et al. 2016) according to accuracies degrees of D_1 and D_2 are shown in Fig. 1, where $\underline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\alpha}(D_1)$, $\overline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\beta}(D_1)$, $\underline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\alpha}(D_2)$, $\overline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\beta}(D_2)$, $\underline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\alpha}(D_1)$, $\overline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\beta}(D_1)$, $\underline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\alpha}(D_2)$ and $\overline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\beta}(D_2)$ computed in (Lin et al. 2016).

Similarly, if we defined the approximated degrees of X in terms of OF-MG-DTRS and PF-MG-DTRS are, respectively, as

$$\rho^{O,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\alpha}(X)\right|}{|X|}, \tag{23}$$

$$\rho^{P,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\alpha}(X)\right|}{|X|}, \tag{24}$$



	D1	D2
UOFPR-MG-DTRS	0.6667	1
DOFPR-MG-DTRS	0.2	1
UPFPR-MG-DTRS	0	0
DPFPR-MG-DTRS	0	0.1111
OF-MG-DTRS	0.2	0.875
PF-MG-DTRS	0	0.1111

Fig. 1 A comparison of the accuracy

then from Eqs. (7)–(10) and (23)–(24), the comparison of our proposed models and the model given in (Lin et al. 2016) according to approximated degrees of D_1 and D_2 are shown in Fig. 2.

Moreover, if we defined the degree of dependency of X in terms of OF-MG-DTRS and PF-MG-DTRS are, respectively, as

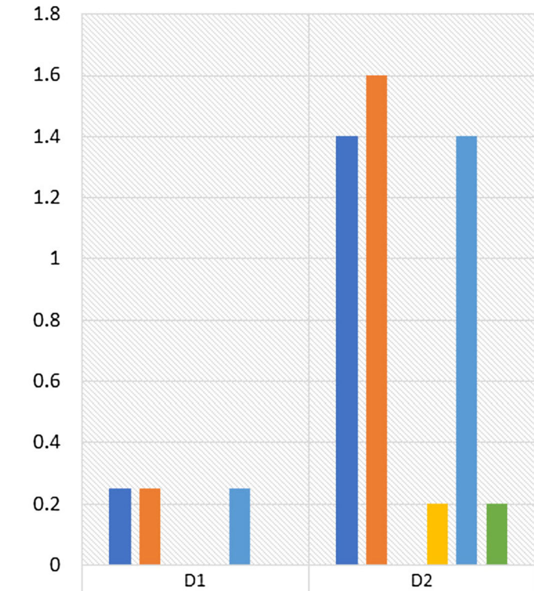
$$\rho^{O,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{O,\alpha}(X)\right|}{|U|}, \tag{25}$$

$$\rho^{P,\alpha,\beta}_{\sum_{i=1}^{m}\tilde{R}_i}(X) = \frac{\left|\underline{\sum_{i=1}^{m}\tilde{R}_i}^{P,\alpha}(X)\right|}{|U|}, \tag{26}$$

then from Eqs. (11)–(14) and (25)–(26), the comparison of our proposed models and the model given in (Lin et al. 2016) according to degree of dependency of D_1 and D_2 are shown in Fig. 3.

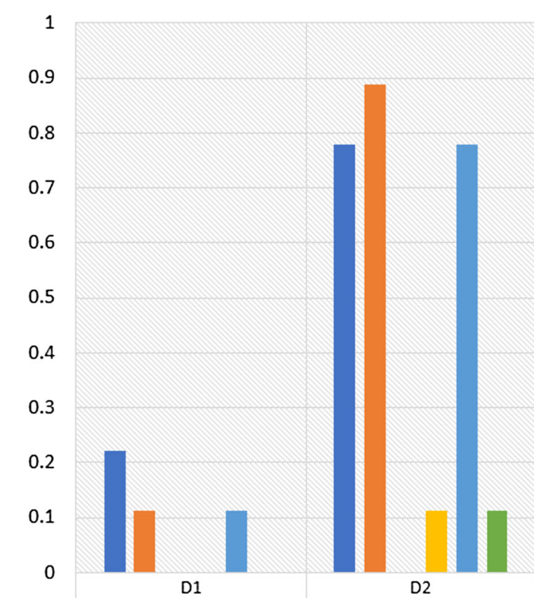
6 Conclusions

In this paper, we combined MG-DTRS with FPR in order to dealing the problems of uncertainty and imprecision easily. The FPRs can be used to represent the fuzzy and uncertain preferences of the experts in the process of group decision making, while MG-DTRSs can be used to multi-source data



	D1	D2
UOFPR-MG-DTRS	0.25	1.4
DOFPR-MG-DTRS	0.25	1.6
UPFPR-MG-DTRS	0	0
DPFPR-MG-DTRS	0	0.2
OF-MG-DTRS	0.25	1.4
PF-MG-DTRS	0	0.2

Fig. 2 A comparison of the approximation degree



	D1	D2
UOFPR-MG-DTRS	0.2222	0.7778
DOFPR-MG-DTRS	0.1111	0.8889
UPFPR-MG-DTRS	0	0
DPFPR-MG-DTRS	0	0.1111
OF-MG-DTRS	0.1111	0.7778
PF-MG-DTRS	0	0.1111

Fig. 3 A comparison of the degree of dependency

analysis, knowledge discovery from data with high dimensions and distributive information systems. We are combined use of two ideas, this paper proposed FPR-MG-DTRS, which can be used to solve multi-criteria preference analysis problems, where data come from the multi-source fuzzy information system. The contribution of this paper has constructed two different types of FPR-MG-DTRS associated with granular computing, in which approximation operators are defined based on multiple additive FPRs. We also discuss the uncertainty measure of proposed model by using the concept of the granularity of additive FPR. Finally, we use a example to illustrate our methods effectiveness in real applications and compare the our proposed and existing models. It shows that the propose approach will be helpful for dealing with multi-criteria preference analysis problems, where data come from multi-source information system. In future, we will apply our proposed model for ordinal decision system.

Acknowledgements The authors would like to thank the Associate Editor and reviewers for their thoughtful comments and valuable suggestions.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any study performed on humans or animals by the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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