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METHODOLOGIES AND APPLICATION

An improved algorithm based on deviation of the error estimation for first-order integro-differential equations

R. Parvaz[1] · M. Zarebnia[1] · A. Saboor Bagherzadeh[2]

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Abstract

In this paper, we study efficient asymptotically correct a posteriori error estimates for the numerical approximation of first-order Fredholm–Volterra integro-differential equations. In the first step, we find the deviation of the error for Fredholm–Volterra integro-differential equations by using defect correction principle. Then we show that for m degree piecewise polynomial collocation method, our method provides order $\mathcal{O}(h^{m+1})$ for the deviation of the error. Also we improve the piecewise polynomial collocation method by using the deviation of the error estimation. Numerical results in the last section are included to confirm the theoretical results.

Keywords Integro-differential equations · Defect correction principle · Collocation · Finite difference · Error analysis

1 Introduction

In this work, we study the deviation of the error estimation for the linear and nonlinear Fredholm–Volterra integro-differential equations. The first-order Fredholm–Volterra integro-differential (FVID) equation is given by the following form

$$y'(t) = F\big(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)\big), \quad t \in I := [a, b], \tag{1.1}$$

$$\alpha y(a) + \beta y(b) = r, \tag{1.2}$$

with

$$z_{\mathbf{f}}[y](t) := \int_a^b K_{\mathbf{f}}\big(t, s, y(s), y'(s)\big)\mathrm{d}s, \tag{1.3}$$

$$z_{\mathbf{v}}[y](t) := \int_a^t K_{\mathbf{v}}\big(t, s, y(s), y'(s)\big)\mathrm{d}s, \tag{1.4}$$

where $a, b, \alpha, \beta, r \in R = (-\infty, \infty)$, $\alpha + \beta \neq 0$ and $b > a$.

Integro-differential equations can be found in many branches of science and engineering, for example, in the electrical circuit analysis (Moura and Darwazeh 2005) and mechanical engineering (Yogi Goswami 2004; Fidlin 2005). These equations appear in the computer vision and image processing. In particular, these equations are used in the image deblurring, denoising and its regularization (Chen et al. 2016; Huang et al. 2009; Athavale and Tadmor 2011). In addition, in the pattern recognition and machine intelligence, we can see the application of these equations (Doroshenko et al. 2011). Therefore, the numerical studies for the integro-differential equations have important role in sciences and computer vision. The numerical solution based on the piecewise polynomial collocation method is studied in Hangelbroek et al. (1977), Brunner (2004), Boor and Swartz (1973). Also other methods can be found in Volk (1988), Daşcioğlu and Sezer (2005), Reutskiy (2016). In this work, improved piecewise polynomial collocation method is introduced. In the previous work (Parvaz et al. 2016), the deviation of the error estimation analysis is given for the second-order

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✉ M. Zarebnia
zarebnia@uma.ac.ir

R. Parvaz
rparvaz@uma.ac.ir

A. Saboor Bagherzadeh
saboorbagherzadeh.a@gmail.com

[1] Department of Mathematics, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

[2] Institute of Structural Mechanics, Bauhaus Universität-Weimar, Marienstrasse 15, 99423 Weimar, Germany

Fredholm–Volterra integro-differential equations. It is shown that for m degree piecewise polynomial collocation method, the order of deviation of the error is at least $\mathcal{O}(h^{m+1})$. In this study, the first-order Fredholm–Volterra integro-differential equations are studied. We prove that for m degree piecewise polynomial, collocation method provides $\mathcal{O}(h^{m+1})$ as the order of the deviation of the error. Then according to the defect correction and the deviation of the error, piecewise polynomial collocation method can be improved. The general studies on the structure of the defect correction principle can be found in Stetter (1978), Bohmer et al. (1984). The deviation of the error estimation analysis for boundary value problems has been given in Saboor Bagherzadeh (2011). Also the error estimation based on locally weighted defect for linear and nonlinear second-order boundary value problems can be found in Saboor Bagherzadeh (2011), Auzinger et al. (2014).

This article is organized as follows: The method is presented in Sect. 2. A complete analysis of the deviation of the error for linear and nonlinear cases is given in Sect. 3. In Sect. 4, numerical results are presented. Finally, we give a summary of the main conclusions in Sect. 5.

2 Description of the method

We define W and S as follows

$$W := \{(t, y, z_{\mathbf{f}}, z_{\mathbf{v}}) \,|\, t \in I \;\&\; y, z_{\mathbf{f}}, z_{\mathbf{v}} \in R\}, \tag{2.1}$$

$$S := \{(t, s, y, y') \,|\, t, s \in I \;\&\; y, y' \in R\}. \tag{2.2}$$

In this paper, we shall assume that F and K_l, $(l = \mathbf{f}, \mathbf{v})$ are uniformly continuous in W and S, respectively. We say that $z_{\mathbf{f}}[y](t)$ and $z_{\mathbf{v}}[y](t)$ are linear if we can write

$$z_{\mathbf{f}}[y](t) = \sum_{l=0}^{1} \int_a^b \Lambda_{l,\mathbf{f}}(t, s) y^{(l)}(s) \mathrm{d}s, \tag{2.3}$$

$$z_{\mathbf{v}}[y](t) = \sum_{l=0}^{1} \int_a^t \Lambda_{l,\mathbf{v}}(t, s) y^{(l)}(s) \mathrm{d}s. \tag{2.4}$$

In this paper, when we study linear case we assume that $\Lambda_{l,\mathbf{f}}(t, s)$, $\Lambda_{l,\mathbf{v}}(t, s)$ $(l = 0, 1)$ are sufficiently smooth in $J_{\mathbf{f}} := \{(t, s) \,|\, t, s \in I\}$ and $J_{\mathbf{v}} := \{(t, s) \,|\, a \le s \le t \le b\}$, respectively. Also we say that F is semilinear if we can write $F(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t))$ as

$$F(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)) = a_1(t) y(t) + a_2(t) + z_{\mathbf{f}}[y](t) + z_{\mathbf{v}}[y](t). \tag{2.5}$$

In the nonlinear case, we assume that $F(t, y, z_{\mathbf{f}}, z_{\mathbf{v}})$, $F_l(t, y, z_{\mathbf{f}}, z_{\mathbf{v}})$ $(l = t, y, z_{\mathbf{f}}, z_{\mathbf{v}})$ are Lipschitz-continuous. Also when $z_l[y](t)$ $(l = \mathbf{f}, \mathbf{v})$ are nonlinear we assume that $K_l(t, s, y, y')$ and $(K_l)_j(t, s, y, y')$ $(l = \mathbf{f}, \mathbf{v}$ & $j = y, y')$ are Lipschitz-continuous. We say FVID equation with boundary condition (1.2) is linear if we can write (1.1) as follows

$$y'(t) = a_1(t) y(t) + a_2(t) + z_{\mathbf{f}}[y](t) + z_{\mathbf{v}}[y](t), \quad t \in I, \tag{2.6}$$

with linear $z_l[y](t) (l = \mathbf{f}, \mathbf{v})$. Also in the linear case we assume that $a_1(t), a_2(t)$ are sufficiently smooth in I.

2.1 Collocation method

In this subsection, we introduce the piecewise polynomial collocation method for solution of the FVID problem (1.1), (1.2). Let

$$a = \tau_0 < \tau_1 < \cdots < \tau_n = b, \;\; (n \ge 1), \tag{2.7}$$

$$0 = \rho_0 < \rho_1 < \cdots < \rho_m < \rho_{m+1} = 1, \tag{2.8}$$

and $h_i := \tau_{i+1} - \tau_i$. We define X_n, Z_n and $S_m^{(0)}(Z_n)$ as follows

$$X_i := \{t_{i,j} := \tau_i + \rho_j h_i;\; j = 1, \ldots, m\}, \tag{2.9}$$

$$Z_n := \{t_{i,0} := \tau_i;\; i = 0, \ldots, n\}, \tag{2.10}$$

$$S_m^{(0)}(Z_n) := \{p \in \mathcal{C}(I);\; p \restriction [\tau_i, \tau_{i+1}] \in \Pi_m([\tau_i, \tau_{i+1}])\; (i = 0, \ldots, n-1)\}, \tag{2.11}$$

where $\Pi_m([\tau_i, \tau_{i+1}])$ is space of real polynomial functions on $[\tau_i, \tau_{i+1}]$ of degree $\leqslant m$. Also we define the set of collocation points as

$$X(n) := \bigcup_{i=0}^{n-1} X_i. \tag{2.12}$$

We define h (the diameter of gird Z_n) and h' as

$$h := \max\{h_i;\; i = 0, \ldots, n-1\},$$

$$h' := \min\{h_i;\; i = 0, \ldots, n-1\}. \tag{2.13}$$

In this paper, the set $X(n) := \bigcup_{i=0}^{n-1} X_i$ is called the set of collocation points. According to the piecewise polynomial collocation method, we are looking to find a $p \in S_m^{(0)}(Z_n)$ so that (1.1), (1.2) hold for all $t \in X(n)$. In the collocation method, since we can not determine exact value for $z_{\mathbf{f}}[\cdot](t)$ and $z_{\mathbf{v}}[\cdot](t)$, we use the following quadrature method to determine $z_{\mathbf{f}}[\cdot](t)$ and $z_{\mathbf{v}}[\cdot](t)$.

$$z_{\mathbf{f}}[p](t_{i,j}) \approx \sum_{k=0}^{n-1}\sum_{z=1}^{m+1} \alpha_{k,z} K_{\mathbf{f}}\big(t_{i,j}, t_{k,z}, p(t_{k,z}),$$
$$p'(t_{k,z})\big) =: \widetilde{z}_{\mathbf{f}}[p](t_{i,j}), \tag{2.14}$$
$$z_{\mathbf{v}}[p](t_{i,j}) \approx \sum_{k=0}^{i-1}\sum_{z=1}^{m+1} \alpha_{k,z} K_{\mathbf{v}}\big(t_{i,j}, t_{k,z}, p_k(t_{k,z}), p_k'(t_{k,z})\big)$$
$$+ (t_{i,j}-\tau_i)\sum_{z=1}^{m+1} \beta_z K_{\mathbf{v}}\Big(t_{i,j}, \tilde{t}_{i,j}^{z}, p_i(\tilde{t}_{i,j}^{z}), p_i'(\tilde{t}_{i,j}^{z})\Big)$$
$$=: \widetilde{z}_{\mathbf{v}}[p](t_{i,j}), \tag{2.15}$$

where $\tilde{t}_{i,j}^{z} := \tau_i + \rho_z(t_{i,j} - \tau_i)$ and the quadrature weights are given by

$$\alpha_{k,z} := \int_{\tau_k}^{\tau_{k+1}} L_z^{[\tau_k,\tau_{k+1}]}(s)\mathrm{d}s, \quad \beta_z := \int_0^1 L_z(s)\mathrm{d}s, \tag{2.16}$$

where

$$L_j(\rho) := \prod_{\substack{i=1\\ i\neq j}}^{m+1} \frac{\rho-\rho_i}{\rho_j-\rho_i}, \quad L_j^{[a',b']}(\rho) := L_j\left(\frac{\rho-a'}{b'-a'}\right),$$
$$a \le a' < b' \le b. \tag{2.17}$$

In summary, collocation method can be written as Algorithm 1.

Algorithm 1: Collocation method

Input : $a, b, r, \alpha, \beta, m, n, \{\tau_i\}_{i=0}^{n}, \{t_{i,j}\}_{(i,j)=(0,0)}^{(n,m+1)}$.
Output: p.

1. Consider unknown parameters $\{c_{(i+1)m+j}\}_{(i,j)=(0,0)}^{(n-1,m)}$.
2. Define $p_i = \sum_{j=0}^{m} c_{(i+1)m+j}x^j, \quad i = 0,\dots,n-1$.
3. Find $\{c_{(i+1)m+j}\}_{(i,j)=(0,0)}^{(n-1,m)}$ by using following system, $p_0(a) - r = 0$, $p_{i-1}(\tau_i) - p_i(\tau_i) = 0, \ i = 1,\dots,n-1$, $p_i' - F\big(t_{i,j}, p_i(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p_i](t_{i,j})\big), \widetilde{z}_{\mathbf{v}}[p_i](t_{i,j})\big) = 0, \quad i = 0,\dots,n-1, j = 1,\dots,m$.
4. Consider collocation solution for each interval $[\tau_i, \tau_{i+1}]$ as $p_i \ (i = 0,\dots,n-1)$.

By using the interpolation error theorem (see Stoer and Bulirsch 2002, Section 2.1), we can find the following lemma.

Lemma 2.1 *For sufficiently smooth f, the following estimates hold*

$$|z_l[f](t_{i,j}) - \widetilde{z}_l[f](t_{i,j})| = \mathcal{O}(h^{m+1}), \quad l = \mathbf{f}, \mathbf{v}. \tag{2.18}$$

In a similar way to Brunner (2004), for the piecewise polynomial collocation method, we can find the following theorem.

Theorem 2.2 *Assume that the FVID problem (1.1), (1.2) has a unique and sufficiently smooth solution $y(t)$. Also assume that $p(t)$ is a piecewise polynomial collocation solution of degree $\le m$. Then for sufficiently small h, the collocation solution $p(t)$ is well-defined and the following uniform estimates at least hold:*

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^m), \quad j = 0, 1. \tag{2.19}$$

Remark 2.3 In special case, we can see that for equidistant collocation gird points with odd m the following uniform estimates hold

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^{m+1}), \quad j = 0, 1. \tag{2.20}$$

By using Theorem 2.2 and Lemma 2.1, we have the following lemma.

Lemma 2.4 *For linear and nonlinear $z_l[\cdot](t)$ $(l = \mathbf{f}, \mathbf{v})$, we have*

$$|\widetilde{z}_l[p](t_{i,j}) - \widetilde{z}_l[y](t_{i,j})| = \mathcal{O}(h^m), \quad l = \mathbf{f}, \mathbf{v}. \tag{2.21}$$

2.2 Finite difference scheme

In this section, we define $\Delta_{i,j}$, $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\Delta_{i,j} := \{(l,k);\ l = 0,\dots,i-1 \,\&\, k = 0,\dots,m\}$$
$$\cup \{(i,k);\ k = 0,\dots,j-1\}, \tag{2.22}$$
$$\mathcal{A} := \{(i,j);\ t_{i,j} \in X(n) \cup Z_n\}, \tag{2.23}$$
$$\mathcal{B} := \mathcal{A} - \{(n,0)\}. \tag{2.24}$$

Also we define

$$\left(L_{\mathcal{A}}^{(1)}\eta\right)_{i,j} := \frac{\eta_{i,j+1} - \eta_{i,j}}{\delta_{i,j}}, \tag{2.25}$$
$$\chi^{\mathbf{f}}[\eta]_{i,j} := \sum_{(l,v)\in\mathcal{B}} \delta_{l,v} K_{\mathbf{f}}\left(t_{i,j}, t_{l,v}, \eta_{l,v}, \left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}\right), \tag{2.26}$$
$$\chi^{\mathbf{v}}[\eta]_{i,j} := \sum_{(l,v)\in\Delta_{i,j}} \delta_{l,v} K_{\mathbf{v}}\left(t_{i,j}, t_{l,v}, \eta_{l,v}, \left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}\right), \tag{2.27}$$

where $\delta_{i,j} := t_{i,j+1} - t_{i,j}$.

We write a general one-step finite difference scheme as

$$\left(L_{\mathcal{A}}^{(1)}\eta\right)_{i,j} = F(t_{i,j}, \eta_{i,j}, \chi^{\mathbf{f}}[\eta]_{i,j}, \chi^{\mathbf{v}}[\eta]_{i,j}), \quad (i,j) \in \mathcal{B}, \tag{2.28}$$

$$\alpha\,\eta_{0,0} + \beta\,\eta_{n,0} = r. \tag{2.29}$$

Definition 2.5 For any function u, we define

$$\mathcal{R}(u) := \{u(t_{i,j})\, ;\ (i,j) \in \mathcal{A}\},$$

also we define

$$\eta := \{\eta_{i,j}\, ;\ (i,j) \in \mathcal{A}\}, \quad L_{\mathcal{A}}^{(1)}\eta := \left\{\left(L_{\mathcal{A}}^{(1)}\eta\right)_{i,j}\, ; (i,j) \in \mathcal{A}\right\}.$$

By using Taylor expansions, the following lemma is obtained easily.

Lemma 2.6 *For sufficiently smooth f, the following estimates hold*

$$|\chi^{l}[f]_{i,j} - z_l[f](t_{i,j})| = \mathcal{O}(h), \quad l = \mathbf{f}, \mathbf{v}. \tag{2.30}$$

By using Taylor expansion and Lemma 2.6, we can find the following estimates

$$\|\eta - \mathcal{R}(y)\|_{\infty} = \mathcal{O}(h), \tag{2.31}$$

$$\|L_{\mathcal{A}}^{(1)}\eta - \mathcal{R}(y')\|_{\infty} = \mathcal{O}(h), \tag{2.32}$$

where η and $L_{\mathcal{A}}^{(1)}\eta$ is defined in the Definition 2.5.

2.3 Deviation of the error estimation

In this subsection, we study the deviation of the error estimation for (1.1), (1.2) by using the defect correction principle. In the first step, we consider $y'(t) = f(t)$, $a \le t \le b$, where $f(t)$ is permitted to have jump discontinuities in the points belonging to Z_n. Using the Taylor expansion, we can find "exact finite difference scheme" for $y'(t) = f(t)$, which is satisfied by the exact solution.

$$\left(L_{\mathcal{A}}^{(1)}y\right)_{i,j} = \int_0^1 f(t_{i,j} + \xi\delta_{i,j})\mathrm{d}\xi := \mathcal{I}_{\mathcal{A}}(f, t_{i,j}). \tag{2.33}$$

Therefore, we can say that a solution of problem (1.1), (1.2) satisfies in the following exact finite difference scheme

$$\left(L_{\mathcal{A}}^{(1)}y\right)_{i,j} = \mathcal{I}_{\mathcal{A}}\big(F(\cdot, y, z_{\mathbf{f}}[y], z_{\mathbf{v}}[y]), t_{i,j}\big). \tag{2.34}$$

According to the collocation method, we have

$$p'(t_{i,j}) - F\big(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j})\big) \equiv 0, \ (i,j) \in X(n). \tag{2.35}$$

Now in this step we define defect at $t_{i,j}$ as

$$D_{i,j} := \left(L_{\mathcal{A}}^{(1)}p\right)_{i,j} - \mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, z_{\mathbf{f}}[p], z_{\mathbf{v}}[p]), t_{i,j}\big), \ (i,j) \in \mathcal{B}. \tag{2.36}$$

We use quadrature formula to compute integral in (2.36)

$$\begin{aligned}\mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, z_{\mathbf{f}}[p], z_{\mathbf{v}}[p]), t_{i,j}\big) &\approx \mathcal{Q}_{\mathcal{A}}\big(F(\cdot, p, \widetilde{z}_{\mathbf{f}}[p], \widetilde{z}_{\mathbf{v}}[p]), t_{i,j}\big)\\ &:= \sum_{k=1}^{m+1} \gamma_{i,j}^{k} F\big(t_{i,k}, p(t_{i,k}), \widetilde{z}_{\mathbf{f}}[p](t_{i,k}), \widetilde{z}_{\mathbf{v}}[p](t_{i,k})\big),\end{aligned} \tag{2.37}$$

where

$$\gamma_{i,j}^{k} := \int_0^1 L_k\left(\rho_j + \frac{\xi\delta_{i,j}}{h_i}\right)\mathrm{d}\xi. \tag{2.38}$$

For sufficiently smooth f, the following error holds

$$\mathcal{I}_{\mathcal{A}}\big(f, t_{i,j}\big) - \mathcal{Q}_{\mathcal{A}}\big(f, t_{i,j}\big) = \mathcal{O}(h^{m+1}). \tag{2.39}$$

Also when m is odd and the nodes ρ_i are symmetrically, we can find the following relation.

$$\mathcal{I}_{\mathcal{A}}\big(f, t_{i,j}\big) - \mathcal{Q}_{\mathcal{A}}\big(f, t_{i,j}\big) = \mathcal{O}(h^{m+2}). \tag{2.40}$$

Then we consider defect at $t_{i,j}$ as follows

$$D_{i,j} \approx \left(L_{\mathcal{A}}^{(1)}p\right)_{i,j} - \mathcal{Q}_{\mathcal{A}}\big(F(\cdot, p, z_{\mathbf{f}}[p], z_{\mathbf{v}}[p]), t_{i,j}\big), \ (i,j) \in \mathcal{B}. \tag{2.41}$$

In this step, we define $\pi = \{\pi_{i,j}\, ;\ (i,j) \in \mathcal{A}\}$ as the solution of the following finite difference scheme

$$\left(L_{\mathcal{A}}^{(1)}\pi\right)_{i,j} = F(t_{i,j}, \pi_{i,j}, \chi^{\mathbf{f}}[\pi]_{i,j}, \chi^{\mathbf{v}}[\pi]_{i,j}) + D_{i,j}, \ (i,j) \in \mathcal{B}, \tag{2.42}$$

$$\alpha\,\pi_{0,0} + \beta\,\pi_{n,0} = r. \tag{2.43}$$

We define $\mathbf{D} := \{D_{i,j}\, ;\ (i,j) \in \mathcal{B}\}$. For small value $\mathbf{D}$, we can say that

$$\pi - \mathcal{R}(p) \approx -\mathcal{R}(y), \tag{2.44}$$

where η can be found in (2.28), (2.29). We define ε and e as follows

$$\varepsilon := \pi - \eta \approx \mathcal{R}(p) - \mathcal{R}(y) := e. \tag{2.45}$$

An estimate for the error e can be found in Theorem 2.2. We consider the deviation of the error in the following form

$$\theta := e - \varepsilon. \tag{2.46}$$

By using above discussion, improved collocation method can be written as Algorithm 2.

Algorithm 2: Improved collocation method

1 Find collocation solution by using Algorithm 1.
2 Find defect by using (2.41).
3 Solve finite difference scheme (2.28), (2.29).
4 Solve finite difference scheme (2.42), (2.43).
5 Consider $\varepsilon := \pi - \eta$.
6 Improve collocation solution by using $\mathcal{R}(p) - \varepsilon$.

In the next section, we will study the order of the deviation of the error estimate for FVID equation. We can easily find the following lemmas.

Lemma 2.7 *The defined defect in (2.41) has order $\mathcal{O}(h^m)$.*

Lemma 2.8 *The $\pi - \eta$ has order $\mathcal{O}(h^m)$.*

3 Analysis of the deviation of the error

Definition 3.1 In this section, we define $\overline{\varepsilon}$ and $\widehat{\varepsilon}$ as follows

$$\overline{\varepsilon} := \pi - \mathcal{R}(p), \tag{3.1}$$

$$\widehat{\varepsilon} := \eta - \mathcal{R}(y). \tag{3.2}$$

Definition 3.2 We define

$$\overline{\chi}^{\mathbf{f}}[\overline{\varepsilon}]_{i,j} := \sum_{(l,v)\in\mathcal{B}} \delta_{l,v}\Big(\overline{\Upsilon}_0^{\mathbf{f}}(t_{i,j},t_{l,v})\overline{\varepsilon}_{l,v} + \overline{\Upsilon}_1^{\mathbf{f}}(t_{i,j},t_{l,v})\Big(L_{\mathcal{A}}^{(1)}\overline{\varepsilon}\Big)_{l,v}\Big), \tag{3.3}$$

$$\overline{\chi}^{\mathbf{v}}[\overline{\varepsilon}]_{i,j} := \sum_{(l,v)\in\Delta_{i,j}} \delta_{l,v}\Big(\overline{\Upsilon}_0^{\mathbf{v}}(t_{i,j},t_{l,v})\overline{\varepsilon}_{l,v} + \overline{\Upsilon}_1^{\mathbf{v}}(t_{i,j},t_{l,v})\Big(L_{\mathcal{A}}^{(1)}\overline{\varepsilon}\Big)_{l,v}\Big), \tag{3.4}$$

where ($l = \mathbf{f}, \mathbf{v}$)

$$\overline{\Upsilon}_0^{l}(t_{i,j},t_{l,v}) = \begin{cases} \Lambda_{0,l}(t_{i,j},t_{l,v}), & when\ z_l\ is\ linear,\\ \int_0^1 (K_l)_y\Big(t_{i,j},t_{l,v},p(t_{l,v})+\nu\overline{\varepsilon}_{l,v},\Big(L_{\mathcal{A}}^{(1)}\pi\Big)_{l,v}\Big)\mathrm{d}\nu, & when\ z_l\ is\ nonlinear,\end{cases} \tag{3.5}$$

$$\overline{\Upsilon}_1^{l}(t_{i,j},t_{l,v}) = \begin{cases} \Lambda_{1,l}(t_{i,j},t_{l,v}), & when\ z_l\ is\ linear,\\ \int_0^1 (K_l)_{y'}\Big(t_{i,j},t_{l,v},p(t_{l,v}),\Big(L_{\mathcal{A}}^{(1)}p\Big)_{l,v}+\nu\Big(L_{\mathcal{A}}^{(1)}\overline{\varepsilon}\Big)_{l,v}\Big)\mathrm{d}\nu, & when\ z_l\ is\ nonlinear.\end{cases} \tag{3.6}$$

Also we consider $\widehat{\chi}^{l}[\widehat{\varepsilon}]_{i,j} (l = \mathbf{f}, \mathbf{v})$ as follows

$$\widehat{\chi}^{\mathbf{f}}[\widehat{\varepsilon}]_{i,j} := \sum_{(l,v)\in\mathcal{B}} \delta_{l,v}\Big(\widehat{\Upsilon}_0^{\mathbf{f}}(t_{i,j},t_{l,v})\widehat{\varepsilon}_{l,v} + \widehat{\Upsilon}_1^{\mathbf{f}}(t_{i,j},t_{l,v})\Big(L_{\mathcal{A}}^{(1)}\widehat{\varepsilon}\Big)_{l,v}\Big), \tag{3.7}$$

$$\widehat{\chi}^{\mathbf{v}}[\widehat{\varepsilon}]_{i,j} := \sum_{(l,v)\in\Delta_{i,j}} \delta_{l,v}\Big(\widehat{\Upsilon}_0^{\mathbf{v}}(t_{i,j},t_{l,v})\widehat{\varepsilon}_{l,v} + \widehat{\Upsilon}_1^{\mathbf{v}}(t_{i,j},t_{l,v})\Big(L_{\mathcal{A}}^{(1)}\widehat{\varepsilon}\Big)_{l,v}\Big), \tag{3.8}$$

where ($l = \mathbf{f}, \mathbf{v}$)

$$\widehat{\Upsilon}_0^{l}(t_{i,j},t_{l,v}) = \begin{cases} \Lambda_{0,l}(t_{i,j},t_{l,v}), & when\ z_l\ is\ linear,\\ \int_0^1 (K_l)_y\Big(t_{i,j},t_{l,v},y(t_{l,v})+\nu\widehat{\varepsilon}_{l,v},\Big(L_{\mathcal{A}}^{(1)}\eta\Big)_{l,v}\Big)\mathrm{d}\nu, & when\ z_l\ is\ nonlinear,\end{cases} \tag{3.9}$$

$$\widehat{\Upsilon}_1^{l}(t_{i,j},t_{l,v}) = \begin{cases} \Lambda_{1,l}(t_{i,j},t_{l,v}), & when\ z_l\ is\ linear,\\ \int_0^1 (K_l)_{y'}\Big(t_{i,j},t_{l,v},y(t_{l,v}),\Big(L_{\mathcal{A}}^{(1)}y\Big)_{l,v}+\nu\Big(L_{\mathcal{A}}^{(1)}\widehat{\varepsilon}\Big)_{l,v}\Big)\mathrm{d}\nu, & when\ z_l\ is\ nonlinear.\end{cases} \tag{3.10}$$

By using Theorem 2.2, (2.31), (2.32) and Lemma 2.8, we can find the following lemma.

Lemma 3.3 *The* $\overline{\varepsilon}$ *and the* $\widehat{\varepsilon}$ *have order* $\mathcal{O}(h)$.

Lemma 3.4 *We have*

$$||\overline{\varepsilon}-\widehat{\varepsilon}||_\infty=\mathcal{O}(h^m). \tag{3.11}$$

Proof By using Lemma 2.8 and Theorem 2.2, we can write

$$||\overline{\varepsilon}-\widehat{\varepsilon}||_\infty\le\underbrace{||\pi-\eta||_\infty}_{\mathcal{O}(h^m)}+\underbrace{||\mathcal{R}(p)-\mathcal{R}(y)||_\infty}_{\mathcal{O}(h^m)}=\mathcal{O}(h^m). \tag{3.12}$$

□

By using Definition 3.2, we can say that

$$\chi^l[\pi]-\chi^l[p]=\overline{\chi}^l[\overline{\varepsilon}],\quad (l=\mathbf{f},\mathbf{v}), \tag{3.13}$$
$$\chi^l[\eta]-\chi^l[y]=\widehat{\chi}^l[\widehat{\varepsilon}],\quad (l=\mathbf{f},\mathbf{v}). \tag{3.14}$$

Lemma 3.5 *For linear and nonlinear* $z_l[\cdot](t)\quad (l=\mathbf{f},\mathbf{v})$, *we have*

$$|\chi^l[y]_{i,j}-\widetilde{z}_l[y](t_{i,j})|=\mathcal{O}(h), \tag{3.15}$$
$$|\chi^l[p]_{i,j}-\widetilde{z}_l[p](t_{i,j})|=\mathcal{O}(h), \tag{3.16}$$
$$|\chi^l[\eta]_{i,j}-\chi^l[\pi]_{i,j}|=\mathcal{O}(h^m), \tag{3.17}$$
$$|\chi^l[y]_{i,j}-\chi^l[p]_{i,j}|=\mathcal{O}(h^m), \tag{3.18}$$
$$|\chi^l[\widehat{\varepsilon}]_{i,j}-\chi^l[\overline{\varepsilon}]_{i,j}|=\mathcal{O}(h^m), \tag{3.19}$$
$$|\chi^l[\widehat{\varepsilon}]_{i,j}|=\mathcal{O}(h). \tag{3.20}$$

Proof For (3.15) by using Lemmas 2.1 and 2.6 we have

$$|\chi^l[y]_{i,j}-\widetilde{z}_l[y](t_{i,j})|=|\underbrace{\chi^l[y]_{i,j}-z_l[y](t_{i,j})}_{\mathcal{O}(h)}+\underbrace{z_l[y](t_{i,j})-\widetilde{z}_l[y](t_{i,j})}_{\mathcal{O}(h^{m+1})}|=\mathcal{O}(h). \tag{3.21}$$

Similarly, we can prove (3.16). Now we prove (3.17) for $l=\mathbf{f}$. When $z_l[\cdot](t)$ is linear, we find

$$\begin{aligned}
&\chi^{\mathbf{f}}[\eta]_{i,j}-\chi^{\mathbf{f}}[\pi]_{i,j}=\chi^{\mathbf{f}}[\eta-\pi]_{i,j}\\
&=\sum_{(l,v)\in\mathcal{B}}\delta_{l,v}\Lambda_{0,\mathbf{f}}(t_{i,j},t_{l,v})\varepsilon_{l,v}\\
&\quad+\sum_{(l,v)\in\mathcal{B}}\delta_{l,v}\Lambda_{1,\mathbf{f}}(t_{i,j},t_{l,v})\left(L_{\mathcal{A}}^{(1)}\varepsilon\right)_{l,v}\\
&\le(b-a)(m+1)\frac{h}{h'}\Big(\max_{(l,v)\in\mathcal{B}}\Lambda_{0,\mathbf{f}}(t_{i,j},t_{l,v})\underbrace{\varepsilon_{l,v}}_{\mathcal{O}(h^m)}\\
&\quad+\max_{(l,v)\in\mathcal{B}}\Lambda_{1,\mathbf{f}}(t_{i,j},t_{l,v})\underbrace{\left(L_{\mathcal{A}}^{(1)}\varepsilon\right)_{l,v}}_{\mathcal{O}(h^m)}\Big)=\mathcal{O}(h^m).
\end{aligned} \tag{3.22}$$

Also for nonlinear case we can get

$$\begin{aligned}
&\chi^{\mathbf{f}}[\pi]_{i,j}-\chi^{\mathbf{f}}[\eta]_{i,j}\\
&=\sum_{(l,v)\in\mathcal{B}}\delta_{l,v}\Big(K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\pi_{l,v},\left(L_{\mathcal{A}}^{(1)}\pi\right)_{l,v}\Big)\\
&\quad-K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\eta_{l,v},\left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}\Big)\Big)\\
&\le(b-a)(m+1)\frac{h}{h'}\max_{(l,v)\in\mathcal{B}}\Big(K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\pi_{l,v},\left(L_{\mathcal{A}}^{(1)}\pi\right)_{l,v}\Big)\\
&\quad-K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\eta_{l,v},\left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}\Big)\Big),
\end{aligned} \tag{3.23}$$

from the Lipschitz condition for $K_{\mathbf{f}}$ and Lemma 2.8, we get

$$\begin{aligned}
&|K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\pi_{l,v},\left(L_{\mathcal{A}}^{(1)}\pi\right)_{l,v}\Big)\\
&\quad-K_{\mathbf{f}}\Big(t_{i,j},t_{l,v},\eta_{l,v},\left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}\Big)|\\
&\le C|\pi_{l,v}-\eta_{l,v}|+|\left(L_{\mathcal{A}}^{(1)}\pi\right)_{l,v}-\left(L_{\mathcal{A}}^{(1)}\eta\right)_{l,v}|=\mathcal{O}(h^m).
\end{aligned} \tag{3.24}$$

In the same way, we can find (3.17) for $l=\mathbf{v}$. Similarly, we can prove (3.18),(3.19) and (3.20). □

In this step, we study linear case.

Theorem 3.6 *Assume that the FVID problem (2.6) with boundary conditions (1.2) has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty=||e-\varepsilon||_\infty=\mathcal{O}(h^{m+1}), \tag{3.25}$$

where e *is error,* ε *is the error estimate and* θ *is the deviation of the error estimate.*

Proof Since F is linear then by using (2.28) and (2.42) we get

$$\left(L_{\mathcal{A}}^{(1)}\varepsilon\right)_{i,j}=a_1(t_{i,j})\varepsilon_{i,j}+\chi^{\mathbf{f}}[\varepsilon]_{i,j}+\chi^{\mathbf{v}}[\varepsilon]_{i,j}+D_{i,j}. \tag{3.26}$$

Also we can write

$$\begin{aligned}
\left(L_{\mathcal{A}}^{(1)}e\right)_{i,j}&=(L_{\mathcal{A}}^{(1)}p)_{i,j}-\left(L_{\mathcal{A}}^{(1)}y\right)_{i,j}\\
&=D_{i,j}+Q_{\mathcal{A}}(a_1e+\widetilde{z}_{\mathbf{f}}[e]+\widetilde{z}_{\mathbf{v}}[e],t_{i,j})\\
&\quad+\mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we have

$$\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} = (L_{\mathcal{A}}^{(1)}e)_{i,j} - \left(L_{\mathcal{A}}^{(1)}\varepsilon\right)_{i,j}$$
$$= a_1(t_{i,j})\theta_{i,j} + \sum_{l=\mathbf{f}\&\mathbf{v}} \chi^l[\theta]_{i,j}$$
$$+ \underbrace{\mathcal{Q}_{\mathcal{A}}(a_1 e, t_{i,j}) - a_1(t_{i,j})e(t_{i,j})}_{S_1}$$
$$+ \underbrace{\sum_{l=\mathbf{f}\&\mathbf{v}} \left(\mathcal{Q}_{\mathcal{A}}(\widetilde{z}_l[e], t_{i,j}) - \chi^l[e]_{i,j}\right)}_{S_2}$$
$$+ \mathcal{O}(h^{m+1}). \tag{3.28}$$

Since $\sum_{k=1}^{m+1} \gamma_{i,j}^k = 1$, then we rewrite S_1 as

$$S_1 = \sum_{k=1}^{m+1} \gamma_{i,j}^k \left(a_1(t_{i,k})e(t_{i,k}) - a_1(t_{i,j})e(t_{i,j})\right), \tag{3.29}$$

and by using Taylor expansion, we have

$$a_1(t_{i,k})e(t_{i,k}) - a_1(t_{i,j})e(t_{i,j})$$
$$= \underbrace{(t_{i,k} - t_{i,j})}_{\mathcal{O}(h)} \left(a_1'(\xi_i)\underbrace{e(\xi_i)}_{\mathcal{O}(h^m)} + a_1(\xi_i)\underbrace{e'(\xi_i)}_{\mathcal{O}(h^m)}\right)$$
$$= \mathcal{O}(h^{m+1}), \tag{3.30}$$

where $\xi_i \in [\tau_i, \tau_{i+1}]$. By using Theorem 2.2 and (3.30), we can get $S_1 = \mathcal{O}(h^{m+1})$. For S_2 by using the Lemma 2.1, we get

$$\sum_{r=0}^{1} \int_a^b \Lambda_{r,\mathbf{f}}(t_{i,j}, s)e^{(r)}(s)\mathrm{d}s - \chi^{\mathbf{f}}[e]_{i,j} + \mathcal{O}(h^{m+1})$$
$$\leq \mathcal{O}(h^m)(b-a)(m+1)\frac{h}{2h'}h$$
$$\sum_{r=0}^{1} \max_{s\in[a,b]} \left(\frac{\partial \Lambda_{r,\mathbf{f}}(t_{i,j}, s)}{\partial s}\right) + \mathcal{O}(h^{m+1})$$
$$= \mathcal{O}(h^{m+1}). \tag{3.31}$$

In a similar way, we can find

$$\sum_{r=0}^{1} \int_a^{t_{i,j}} \Lambda_{r,\mathbf{v}}(t_{i,j}, s)e^{(l)}(s)\mathrm{d}s - \chi^{\mathbf{v}}[e]_{i,j} = \mathcal{O}(h^{m+1}). \tag{3.32}$$

By using (3.31), (3.32), we obtain

$$S_2 = \sum_{w=\mathbf{f}\&\mathbf{v}} \left(\mathcal{Q}_{\mathcal{A}}(\widetilde{z}_w[e], t_{i,j}) - \chi^w[e]_{i,j}\right)$$
$$= \mathcal{Q}_{\mathcal{A}}(\widetilde{z}_{\mathbf{f}}[e], t_{i,j}) - \chi^{\mathbf{f}}[e]_{i,j} + \mathcal{Q}_{\mathcal{A}}(\widetilde{z}_{\mathbf{v}}[e], t_{i,j}) - \chi^{\mathbf{v}}[e]_{i,j}$$
$$= \sum_{r=0}^{1}\sum_{k=1}^{m+1} \gamma_{i,j}^k \int_a^b \left(\Lambda_{r,\mathbf{f}}(t_{i,k}, s) - \Lambda_{r,\mathbf{f}}(t_{i,j}, s)\right)e^{(r)}(s)\mathrm{d}s$$
$$+ \sum_{r=0}^{1}\sum_{k=1}^{m+1} \gamma_{i,j}^k \Big(\int_a^{t_{i,k}} \Lambda_{r,\mathbf{v}}(t_{i,k}, s)e^{(r)}(s)\mathrm{d}s$$
$$- \int_a^{t_{i,j}} \Lambda_{r,\mathbf{v}}(t_{i,j}, s)e^{(r)}(s)\mathrm{d}s\Big)$$
$$= (b-a)\sum_{r=0}^{1}\sum_{k=1}^{m+1} \gamma_{i,j}^k (t_{i,k} - t_{i,j})\frac{\partial \Lambda_{r,\mathbf{f}}}{\partial t}(\zeta_{k,r}^i, \zeta_{k,r}')e^{(r)}(\zeta_{k,r}')$$
$$+ \sum_{r=0}^{1}\sum_{k=1}^{m+1} \gamma_{i,j}^k \Big(\int_{t_{i,j}}^{t_{i,k}} \Lambda_{r,\mathbf{v}}(t_{i,j}, s)e^{(r)}(s)\mathrm{d}s$$
$$+ (t_{i,k} - t_{i,j})\int_a^{t_{i,k}} \frac{\partial \Lambda_{r,\mathbf{v}}(\xi_{i,j}^k, s)}{\partial t}e^{(r)}(s)\mathrm{d}s\Big)$$
$$= \mathcal{O}(h^{m+1}), \tag{3.33}$$

where $\xi_{i,j}^k,\ \zeta_{k,r}^i,\ \zeta_{k,r}' \in [a, b]$. Therefore, we can rewrite (3.28) as

$$\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} = a_1(t_{i,j})\theta_{i,j} + \sum_{l=\mathbf{f}\&\mathbf{v}} \chi^l[\theta]_{i,j} + \mathcal{O}(h^{m+1}). \tag{3.34}$$

By using stability of forward Euler scheme, we find

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}). \tag{3.35}$$

□

For nonlinear case we have the following theorem.

Theorem 3.7 *Consider the FVID equation (1.1) with boundary conditions (1.2), where $F(t, y, z_{\mathbf{f}}, z_{\mathbf{v}})$, $F_l(t, y, z_{\mathbf{f}}, z_{\mathbf{v}})$ ($l = t, y, z_{\mathbf{f}}, z_{\mathbf{v}}$) are Lipschitz-continuous. Also when $z_{\mathbf{f}}$ and $z_{\mathbf{v}}$ are nonlinear, we assume that $K_l(t, s, y, y')$ and $(K_l)_j(t, s, y, y')$ ($l = \mathbf{f}, \mathbf{v}\ \&\ j = y, y'$) are Lipschitz-continuous. Assume that the FVID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}), \tag{3.36}$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

Proof For nonlinear case, we have

$$\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} = -\Big(\underbrace{F(t_{i,j},\pi_{i,j},\chi^{\mathbf{f}}[\pi]_{i,j},\chi^{\mathbf{v}}[\pi]_{i,j}) - F(t_{i,j},p(t_{i,j}),\chi^{\mathbf{f}}[p]_{i,j},\chi^{\mathbf{v}}[p]_{i,j})}_{I_1}\Big)$$
$$- \Big(\underbrace{F(t_{i,j},p(t_{i,j}),\chi^{\mathbf{f}}[p]_{i,j},\chi^{\mathbf{v}}[p]_{i,j}) - F(t_{i,j},p(t_{i,j}),\widetilde{z}_{\mathbf{f}}[p](t_{i,j}),\widetilde{z}_{\mathbf{v}}[p](t_{i,j}))}_{I_2}\Big)$$
$$- F(t_{i,j},p(t_{i,j}),\widetilde{z}_{\mathbf{f}}[p](t_{i,j}),\widetilde{z}_{\mathbf{v}}[p](t_{i,j}))$$
$$+ Q_{\mathcal{A}}\Big(F(t_{i,j},p(t_{i,j}),\widetilde{z}_{\mathbf{f}}[p](t_{i,j}),\widetilde{z}_{\mathbf{v}}[p](t_{i,j}))\Big)$$
$$+ \underbrace{F(t_{i,j},\eta_{i,j},\chi^{\mathbf{f}}[\eta]_{i,j},\chi^{\mathbf{v}}[\eta]_{i,j}) - F(t_{i,j},y(t_{i,j}),\chi^{\mathbf{f}}[y]_{i,j},\chi^{\mathbf{v}}[y]_{i,j})}_{I_3}$$
$$+ \underbrace{F(t_{i,j},y(t_{i,j}),\chi^{\mathbf{f}}[y]_{i,j},\chi^{\mathbf{v}}[y]_{i,j}) - F(t_{i,j},y(t_{i,j}),\widetilde{z}_{\mathbf{f}}[y](t_{i,j}),\widetilde{z}_{\mathbf{v}}[y](t_{i,j}))}_{I_4}$$
$$+ F(t_{i,j},y(t_{i,j}),\widetilde{z}_{\mathbf{f}}[y](t_{i,j}),\widetilde{z}_{\mathbf{v}}[y](t_{i,j}))$$
$$- Q_{\mathcal{A}}\Big(F(t_{i,j},y(t_{i,j}),\widetilde{z}_{\mathbf{f}}[y](t_{i,j}),\widetilde{z}_{\mathbf{v}}[y](t_{i,j}))\Big)$$
$$+ \mathcal{O}(h^{m+1}). \tag{3.37}$$

We can rewrite I_1, I_2, I_3 and I_4 as

$$I_1 = \Psi^1_{i,j}\overline{\varepsilon}_{i,j} + \Psi^{1,\mathbf{f}}_{i,j}\overline{\chi}^{\mathbf{f}}[\overline{\varepsilon}]_{i,j} + \Psi^{1,\mathbf{v}}_{i,j}\overline{\chi}^{\mathbf{v}}[\overline{\varepsilon}]_{i,j}, \tag{3.38}$$
$$I_3 = \Psi^2_{i,j}\widehat{\varepsilon}_{i,j} + \Psi^{2,\mathbf{f}}_{i,j}\widehat{\chi}^{\mathbf{f}}[\widehat{\varepsilon}]_{i,j} + \Psi^{2,\mathbf{v}}_{i,j}\widehat{\chi}^{\mathbf{v}}[\widehat{\varepsilon}]_{i,j}, \tag{3.39}$$
$$I_2 = R^{1,\mathbf{f}}_{i,j}(\chi^{\mathbf{f}}[p]_{i,j} - \widetilde{z}_{\mathbf{f}}[p](t_{i,j})) + R^{1,\mathbf{v}}_{i,j}(\chi^{\mathbf{v}}[p]_{i,j} - \widetilde{z}_{\mathbf{v}}[p](t_{i,j})), \tag{3.40}$$
$$I_4 = R^{2,\mathbf{f}}_{i,j}(\chi^{\mathbf{f}}[y]_{i,j} - \widetilde{z}_{\mathbf{f}}[y](t_{i,j})) + R^{2,\mathbf{v}}_{i,j}(\chi^{\mathbf{v}}[y]_{i,j} - \widetilde{z}_{\mathbf{v}}[y](t_{i,j})), \tag{3.41}$$

with

$$\Psi^1_{i,j} := \int_0^1 F_y(t_{i,j}, p(t_{i,j}) + \overline{\varepsilon}\nu, \chi^{\mathbf{f}}[\pi]_{i,j}, \chi^{\mathbf{v}}[\pi]_{i,j})\mathrm{d}\nu, \tag{3.42}$$
$$\Psi^{1,\mathbf{f}}_{i,j} := \int_0^1 F_{z_{\mathbf{f}}}(t_{i,j}, p(t_{i,j}), \chi^{\mathbf{f}}[p]_{i,j} + \overline{\chi}^{\mathbf{f}}[\overline{\varepsilon}]_{i,j}\nu, \chi^{\mathbf{v}}[\pi]_{i,j})\mathrm{d}\nu, \tag{3.43}$$
$$\Psi^{1,\mathbf{v}}_{i,j} := \int_0^1 F_{z_{\mathbf{v}}}(t_{i,j}, p(t_{i,j}), \chi^{\mathbf{f}}[p]_{i,j}, \chi^{\mathbf{v}}[p]_{i,j} + \overline{\chi}^{\mathbf{v}}[\overline{\varepsilon}]_{i,j}\nu)\mathrm{d}\nu, \tag{3.44}$$
$$\Psi^2_{i,j} := \int_0^1 F_y(t_{i,j}, y(t_{i,j}) + \widehat{\varepsilon}\nu, \chi^{\mathbf{f}}[\eta]_{i,j}, \chi^{\mathbf{v}}[\eta]_{i,j})\mathrm{d}\nu, \tag{3.45}$$
$$\Psi^{2,\mathbf{f}}_{i,j} := \int_0^1 F_{z_{\mathbf{f}}}(t_{i,j}, y(t_{i,j}), \chi^{\mathbf{f}}[y]_{i,j} + \overline{\chi}^{\mathbf{f}}[\widehat{\varepsilon}]_{i,j}\nu, \chi^{\mathbf{v}}[\eta]_{i,j})\mathrm{d}\nu, \tag{3.46}$$
$$\Psi^{2,\mathbf{v}}_{i,j} := \int_0^1 F_{z_{\mathbf{v}}}(t_{i,j}, y(t_{i,j}), \chi^{\mathbf{f}}[y]_{i,j}, \chi^{\mathbf{v}}[y]_{i,j} + \widehat{\chi}^{\mathbf{v}}[\widehat{\varepsilon}]_{i,j}\nu)\mathrm{d}\nu, \tag{3.47}$$
$$R^{1,\mathbf{f}}_{i,j} := \int_0^1 F_{z_{\mathbf{f}}}(t_{i,j}, p(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p](t_{i,j}) + (\chi^{\mathbf{f}}[p]_{i,j} - \widetilde{z}_{\mathbf{f}}[p](t_{i,j}))\nu, \chi^{\mathbf{v}}[p]_{i,j})\mathrm{d}\nu, \tag{3.48}$$
$$R^{1,\mathbf{v}}_{i,j} := \int_0^1 F_{z_{\mathbf{v}}}(t_{i,j}, p(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p](t_{i,j}), \widetilde{z}_{\mathbf{v}}[p](t_{i,j}) + (\chi^{\mathbf{v}}[p]_{i,j} - \widetilde{z}_{\mathbf{v}}[p](t_{i,j}))\nu)\mathrm{d}\nu. \tag{3.49}$$

By using the Lipschitz condition for F_y, $F_{z_{\mathbf{f}}}$, $F_{z_{\mathbf{v}}}$, Lemma 3.4 and Lemma 2.8, we get

$$|\Psi^1_{i,j} - \Psi^2_{i,j}| = \mathcal{O}(h^m), \tag{3.50}$$
$$|\Psi^{1,\mathbf{l}}_{i,j} - \Psi^{2,\mathbf{l}}_{i,j}| = \mathcal{O}(h^m), \quad l = \mathbf{f}, \mathbf{v}, \tag{3.51}$$
$$|R^{1,\mathbf{l}}_{i,j} - R^{2,\mathbf{l}}_{i,j}| = \mathcal{O}(h^m), \quad l = \mathbf{f}, \mathbf{v}. \tag{3.52}$$

We can get

$$\Psi^2_{i,j}\widehat{\varepsilon}_{i,j} = \Psi^1_{i,j}\widehat{\varepsilon}_{i,j} + (\Psi^2_{i,j} - \Psi^1_{i,j})\widehat{\varepsilon}_{i,j} = \Psi^1_{i,j}\widehat{\varepsilon}_{i,j} + \mathcal{O}(h^{m+1}). \tag{3.53}$$

Analogously we can get

$$\Psi^{2,l}_{i,j}\widehat{\chi}^l[\widehat{\varepsilon}]_{i,j} = \Psi^{1,l}_{i,j}\widehat{\chi}^l[\widehat{\varepsilon}]_{i,j} + \mathcal{O}(h^{m+1}), \quad l = \mathbf{f}, \mathbf{v}, \tag{3.54}$$

$$R^{2,l}_{i,j}(\chi^l[y]_{i,j} - \widetilde{z}_l[y](t_{i,j})) = R^{1,l}_{i,j}(\chi^l[p]_{i,j}$$

$$- \widetilde{z}_l[p](t_{i,j})) + \mathcal{O}(h^{m+1}), \ l = \mathbf{f}, \mathbf{v}. \tag{3.55}$$

Then based on the above discussion, we rewrite (3.37) in the following form

$$\begin{aligned}
\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} &= \Psi_{i,j}^1\theta_{i,j} + \sum_{l=\mathbf{f},\mathbf{v}} \Psi_{i,j}^{1,l}\chi^l[\theta]_{i,j} \\
&+ F(t_{i,j}, y(t_{i,j}), \widetilde{z}_{\mathbf{f}}[y](t_{i,j}), \widetilde{z}_{\mathbf{v}}[y](t_{i,j})) \\
&- F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j})) \\
&+ F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j})) \\
&- F(t_{i,j}, p(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p](t_{i,j}), \widetilde{z}_{\mathbf{v}}[p](t_{i,j})) \\
&+ F(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j})) \\
&- F(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j})) \\
&+ Q_{\mathcal{A}}\Big(F(t_{i,j}, p(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p](t_{i,j}), \widetilde{z}_{\mathbf{v}}[p](t_{i,j}))\Big) \\
&- Q_{\mathcal{A}}\Big(F(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j}))\Big) \\
&+ Q_{\mathcal{A}}\Big(F(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j}))\Big) \\
&- Q_{\mathcal{A}}\Big(F(t_{i,j}, y(t_{i,j}), \widetilde{z}_{\mathbf{f}}[y](t_{i,j}), \widetilde{z}_{\mathbf{v}}[y](t_{i,j}))\Big) \\
&+ Q_{\mathcal{A}}\Big(F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j}))\Big) \\
&- Q_{\mathcal{A}}\Big(F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j}))\Big) \\
&+ \mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.56}$$

By using the Lipschitz condition for F and lemma 2.1, we have

$$\begin{aligned}
&|F(t_{i,j}, y(t_{i,j}), \widetilde{z}_{\mathbf{f}}[y](t_{i,j}), \widetilde{z}_{\mathbf{v}}[y](t_{i,j})) \\
&\quad - F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j}))| = \mathcal{O}(h^{m+1}),
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
&|F(t_{i,j}, p(t_{i,j}), \widetilde{z}_{\mathbf{f}}[p](t_{i,j}), \widetilde{z}_{\mathbf{v}}[p](t_{i,j})) \\
&\quad - F(t_{i,j}, p(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j}))| = \mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.58}$$

Therefore, we can write (3.56) as

$$\begin{aligned}
\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} &= \Psi_{i,j}^1\theta_{i,j} + \sum_{l=\mathbf{f},\mathbf{v}} \Psi_{i,j}^{1,l}\chi^l[\theta]_{i,j} \\
&\quad + \Theta(t_{i,j}) - Q_{\mathcal{A}}\big(\Theta(t_{i,j})\big) + \mathcal{O}(h^{m+1}),
\end{aligned} \tag{3.59}$$

where

$$\begin{aligned}
\Theta(t_{i,j}) := &F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[y](t_{i,j}), z_{\mathbf{v}}[y](t_{i,j})) \\
&- F(t_{i,j}, y(t_{i,j}), z_{\mathbf{f}}[p](t_{i,j}), z_{\mathbf{v}}[p](t_{i,j})).
\end{aligned} \tag{3.60}$$

By using the Taylor expansion we have

$$|\Theta(t_{i,j}) - Q_{\mathcal{A}}\big(\Theta(t_{i,j})\big)| \leq \ Ch \max|\Theta'(t)|. \tag{3.61}$$

We can find

$$\begin{aligned}
||\Theta'(t)||_\infty = ||&F_t\Big(t, p(t), z_{\mathbf{f}}[p](t), z_{\mathbf{v}}[p](t)\Big) \\
&- F_t\Big(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)\Big) \\
&+ F_y\Big(t, p(t), z_{\mathbf{f}}[p](t), z_{\mathbf{v}}[p](t), \Big)p'(t) \\
&- F_y\Big(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)\Big)y'(t) \\
&+ F_{z_{\mathbf{f}}}\Big(t, p(t), z_{\mathbf{f}}[p](t), z_{\mathbf{v}}[p](t)\Big)z'_{\mathbf{f}}[p](t) \\
&- F_{z_{\mathbf{f}}}\Big(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)\Big)z'_{\mathbf{f}}[y](t) \\
&+ F_{z_{\mathbf{v}}}\Big(t, p(t), z_{\mathbf{f}}[p](t), z_{\mathbf{v}}[p](t)\Big)z'_{\mathbf{v}}[p](t) \\
&- F_{z_{\mathbf{v}}}\Big(t, y(t), z_{\mathbf{f}}[y](t), z_{\mathbf{v}}[y](t)\Big)z'_{\mathbf{v}}[y](t)||_\infty \\
&\leq C_1||p - y||_\infty + C_2||p' - y'||_\infty \\
&\quad + C_3||z'_{\mathbf{f}}[p](t) - z'_{\mathbf{f}}[y](t)||_\infty \\
&\quad + C_4||z'_{\mathbf{v}}[p](t) - z'_{\mathbf{v}}[y](t)||_\infty \\
&= \mathcal{O}(h^m).
\end{aligned} \tag{3.62}$$

In this step based on the above discussion, we have

$$\left(L_{\mathcal{A}}^{(1)}\theta\right)_{i,j} = \Psi_{i,j}^1\theta_{i,j} + \sum_{l=\mathbf{f},\mathbf{v}} \Psi_{i,j}^{1,l}\chi^l[\theta]_{i,j} + \mathcal{O}(h^{m+1}). \tag{3.63}$$

By stability of forward Euler scheme, we can find

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}). \tag{3.64}$$

□

4 Numerical illustration

In this section to illustrate the theoretical results, some numerical results are presented. For all examples, we choose n collocation intervals of length $1/n$. Also we have computed the numerical results by using Mathematica-9.0 programming.

Example 4.1 To check Theorem 3.6, we consider FVID equation as follows

$$\begin{aligned}
y'(t) = t^2y(t) + a(t) + \sum_{l=0}^{1}\Bigg(&\int_0^1 s\cos(t)y^{(l)}(s)\mathrm{d}s \\
&+ \int_0^t t^l\sin(s)y^{(l)}(s)\mathrm{d}s\Bigg).
\end{aligned}$$

In this example, we assume that $\alpha = 1, \beta = 0$ and $I := [0, 1]$. Also $a(t)$ is chosen so that exact solution is $y(t) = \exp(2t)$. In Tables 1 and 2, we choose $m = 2$ and $m = 3$ and assume that ρ_i $(i = 0, \ldots, m+1)$ are equidistant. Also in Table 3, we choose $m = 3$ and $\{\rho_0, \rho_1\ \rho_2, \rho_3\} = \{0, 0.15, 0.80, 1\}$.

Table 1 Numerical results for Example 4.1

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
8	5.18351e−2	–	1.05863e−2	–
16	1.29503e−2	2.00094	1.48574e−3	2.83295
32	3.22797e−3	2.00429	1.97595e−4	2.91056

Table 2 Numerical results for Example 4.1

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
8	1.34073e−4	–	4.02249e−5	–
16	8.90314e−6	3.91256	2.09700e−6	4.26169
32	5.73520e−7	3.95640	1.15804e−7	4.17857

Table 3 Numerical results for Example 4.1

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
8	7.29175e−4	–	1.07691e−4	–
16	9.80916e−5	2.89406	8.18102e−6	3.71848
32	1.27122e−5	2.94792	5.79682e−7	3.81895

Table 4 Numerical results for Example 4.2

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
4	8.93866e−7	–	1.31189e−7	–
8	6.38549e−8	3.80719	1.38742e−9	6.56309
16	4.13234e−9	3.94977	3.13338e−11	5.46854

Example 4.2 Now we consider nonlinear case as follows

$$y'(t) = y^2(t) + a(t) + \int_0^1 st\Big(y(s)y'(s) + 1\Big)ds + \int_0^t s\Big(y^2(s) + t\Big)ds.$$

We assume that $I = [0, 1]$. Also $a(t)$ chosen so that exact solution is $y(t) = \cos(t)$. In Table 4, we choose $\alpha = 1, \beta = 0$ and $m = 4$. In Tables 5 and 6, we choose $m = 2$ and $\alpha = \beta = 1$. Also in Tables 4 and 5, we assume that ρ_i $(i = 0, \ldots, m+1)$ are equidistant, and in Table 6 we choose $\{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.2, 0.7, 1\}$. By using this example, we reveal Theorem 3.7.

Table 5 Numerical results for Example 4.2

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
8	8.73973e−5	–	1.65205e−6	–
16	2.16178e−5	2.01537	2.03184e−7	3.02340
32	5.37139e−6	2.00885	2.52772e−8	3.00688

Table 6 Numerical results for Example 4.2

n	$\|\|e\|\|_\infty$	Order	$\|\|\theta\|\|_\infty$	Order
8	4.11799e−5	–	1.04019e−6	–
16	9.58895e−6	2.10250	1.38138e−7	2.91266
32	2.31859e−6	2.04813	1.64964e−8	3.06588

Table 7 Numerical results for Example 4.3

n	$\|\|e\|\|_\infty$	Order	$\|\|e^*\|\|_\infty$	Order
8	3.65601e−4	–	8.76254e−5	–
16	9.44401e−5	1.95280	1.24567e−5	2.81443
32	2.40446e−5	1.97369	1.67629e−6	2.89357

Table 8 Numerical results for Example 4.3

n	$\|\|e\|\|_\infty$	Order	$\|\|e^*\|\|_\infty$	Order
8	2.62394e−6	–	1.91045e−6	–
16	1.72034e−7	3.93097	1.23454e−7	3.95186
32	1.10406e−8	3.96180	8.00212e−9	3.94745

Table 9 Numerical results for Example 4.3

n	$\|\|e\|\|_\infty$	Order	$\|\|e^*\|\|_\infty$	Order
8	2.82533e−8	–	1.36136e−8	–
16	1.38928e−9	4.34601	4.34162e−10	4.97067
32	7.52022e−11	4.20742	1.36821e−11	4.98788

Example 4.3 We consider here the numerical results for Algorithm 2. In this example, Eq. (1.1) is considered as follows

$$y'(t) = y^2(t) + a(t) + \int_0^1 st^2\Big(y(s) + y'(s)\Big)ds + \int_0^t s^3\big(y^2(s) + t\big)ds,$$

in the interval $[0, 1]$. $a(t)$ chosen so that exact solution is $y(t) = \exp(-t)$. Tables 7, 8 and 9 compare our numerical results with collocation method. In the numerical results, e^* denotes the error of the improved collocation method. Table 7 is obtained by using $\alpha = 1,\ \beta = 0,\ m = 2,\ \{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.2, 0.7, 1\}$. In Table 8, we consider $\alpha = 1,\ \beta = 0,\ m = 3,\ \rho_i = i/4\ (i = 0, \ldots, 4)$. Also $\alpha = \beta = 1,\ m = 4,\ \ \rho_i = i/5\ (i = 0, \ldots, 5)$ are considered for Table 9.

Example 4.4 Consider FVID equation as

$$y'(t) + y(t) = a(t) + \frac{1}{4}\int_0^1 ty^3(s)ds - \frac{1}{2}\int_0^t sy^2(s)ds, \quad I = [0, 1],\, y(0) = 0,$$

Table 10 Comparison of point wise absolute errors for Example 4.4

x	n	Present method	Method in Siraj-ul-Islam et al. (2014)	Method in Babolian et al. (2009)	Method in Maleknejad et al. (2011)
0.5	16	1.97715e−6	4.9e−4	1.6e−4	3.4e−3
	32	2.48900e−7	1.2e−4	4.1e−5	1.0e−3
1	16	4.66373e−6	9.1e−4	1.5e−4	1.5e−3
	32	5.78028e−7	2.2e−4	3.7e−5	3.4e−4

where

$$a(t) = \frac{1}{10}t^6 + t^2 + 2t - \frac{1}{32}.$$

The exact solution of this problem is $y(t) = t^2$. We choose $m = 2$ and assume that ρ_i $(i = 0, \ldots, m+1)$ are equidistant. In Table 10, the improved collocation method has been compared with methods in Siraj-ul-Islam et al. (2014), Babolian et al. (2009), Maleknejad et al. (2011). Note that $1/n$ represents the length of the partition interval. By using the results, it can be seen that the results of the proposed method are more accurate than others.

5 Conclusion

In this paper, we study the deviation of the error for the linear and nonlinear first-order FVID equations. It is shown that the order of the deviation of the error estimation is at least $\mathcal{O}(h^{m+1})$, where m is the degree of piecewise polynomial. Also the piecewise polynomial collocation method is improved by using the defect correction principle and the deviation of the error estimation. In numerical section, examples confirming the theoretical results are given. Therefore, based on theoretical results and numerical examples, improved method can be applied to linear and nonlinear first-order Fredholm–Volterra integro- differential equations.

Compliance with ethical standards

Conflict of interest Authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Athavale P, Tadmor E (2011) Integro-differential equations based on (BV, L^1) image decomposition. SIAM J Imaging Sci 4(1):300–312

Auzinger W, Koch O, Saboor Bagherzadeh A (2014) Error estimation based on locally weighted defect for boundary value problems in second order ordinary differential equations. BIT Numer Math 54:873–900

Babolian E, Masouri Z, Hatamzadeh-Varmazyar S (2009) Numerical solution of nonlinear Volterra–Fredholm integro-differential equations via direct method using triangular functions. Comput Math Appl 58:239–247

Bohmer K, Hemker P, Stetter HJ (1984) The defect correction approach. Comput Supply 5:1–32

Boor CD, Swartz B (1973) Collocation at Gaussian points. SIAM J Numer Anal 10:582–606

Brunner H (2004) Collocation methods for volterra integral and related functional differential equations. Cambridge University Press, Cambridge

Chen K, Fairag F, Al-Mahdi A (2016) Preconditioning techniques for an image deblurring problem. Numer Linear Algebr 23(3):570–584

Daşcioğlu AA, Sezer M (2005) Chebyshev polynomial solutions of systems of higher-order linear Fredholm–Volterra integro-differential equations. J Franklin Inst 342:688–701

Doroshenko J, Dulkin L, Salakhutdinov V, Smetanin Y (2011) Principle and method of image recognition under diffusive distortions of image. In: International conference on pattern recognition and machine intelligence, 2011 Jun 27. Springer, Berlin, Heidelberg, pp 130–135

Fidlin A (2005) Nonlinear oscillations in mechanical engineering. Springer, Berlin

Hangelbroek RJ, Kaper HG, Leaf GK (1977) Collocation methods for integro-differential equations. SIAM J Numer Anal 14:377–390

Huang HY, Jia CY, Huan ZD (2009) On weak solutions for an image denoising–deblurring model. Appl Math Ser B 24(3):269–281

Maleknejad K, Basirat B, Hashemizadeh E (2011) Hybrid Legendre polynomials and block-pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations. Comput Math Appl 61:2821–2828

Moura L, Darwazeh I (2005) Introduction to linear circuit analysis and modelling: from DC to RF. Newnes, Oxford

Parvaz R, Zarebnia M, Saboor Bagherzadeh A (2016) Deviation of the error estimation for second order Fredholm–Volterra integro differential equations. Math Model Anal 21(6):719–740

Reutskiy SYu (2016) The backward substitution method for multipoint problems with linear Volterra–Fredholm integro-differential equations of the neutral type. J Comput Appl Math 296:724–738

Saboor Bagherzadeh A (2011) Defect-based error estimation for higher order differential equations. PhD thesis, Vienna University of Technology

Siraj-ul-Islam, Aziz I, Al-Fhaid AS (2014) An improved method based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders. J Comput Appl Math 260:449–469

Stetter HJ (1978) The defect correction principle and discretization methods. Numer Math 29:425–443

Stoer J, Bulirsch R (2002) Introduction to numerical analysis, 3rd edn. Springer, Berlin

Volk W (1988) The iterated Galerkin methods for linear integro differential equations. J Comput Appl Math 21:63–74

Yogi Goswami D (2004) The CRC handbook of mechanical engineering, 2nd edn. CRC Press, Boca Raton