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A new method of level-2 uncertainty analysis in risk assessment based on uncertainty theory

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Abstract

The objective of this study is to present a novel method of level-2 uncertainty analysis in risk assessment by means of uncertainty theory. In the proposed method, aleatory uncertainty is characterized by probability distributions, whose parameters are affected by epistemic uncertainty. These parameters are described as uncertain variables. For monotone risk models, such as fault trees or event trees, the uncertainty is propagated analytically based on the operational rules of uncertain variables. For non-monotone risk models, we propose a simulation-based method for uncertainty propagation. Three indexes, i.e., average risk, value-at-risk and bounded value-at-risk, are defined for risk-informed decision making in the level-2 uncertainty setting. Two numerical studies and an application on a real example from literature are worked out to illustrate the developed method. A comparison is made to some commonly used uncertainty analysis methods, e.g., the ones based on probability theory and evidence theory.

Keywords Uncertainty theory · Uncertainty analysis · Epistemic uncertainty

1 Introduction

Uncertainty modeling and analysis is an essential part of probabilistic risk assessment (PRA) and has drawn numerous attentions since 1980s (Apostolakis 1990; Parry and Winter 1981). Two types of uncertainty are usually distinguished: aleatory uncertainty, which refers to the uncertainty inherent in the physical behavior of a system, and epistemic uncertainty, which refers to the uncertainty in the modeling caused by lack of knowledge on the system behavior (Kiureghian and Ditlevsen 2009). In practice, uncertainty modeling and analysis involving both aleatory and epistemic uncertainty is often formulated in a level-2 setting: aleatory uncertainty is considered by developing probabilistic models for risk assessment, while the parameters in the probabilistic models might subject to epistemic uncertainty (Aven et al. 2014).

In general, it has been well acknowledged that aleatory uncertainty should be modeled using probability theory. However, there appears to be no consensus on which mathematical framework should be used to describe epistemic uncertainty, since its modeling usually involves subjective information from human judgements. Indeed, various mathematical frameworks have been proposed in the literature to model the epistemically uncertain variables, e.g., probability theory (subjective interpretation), evidence theory, possibility theory (Aven 2013; Aven and Zio 2011; Helton et al. 2010). As a result, different methods for level-2 uncertainty analysis are developed. Aven et al. (2014) systematically elaborate on level-2 uncertainty analysis methods and developed a purely probabilistic for level-2 uncertainty analysis. Limbourg and Rocquigny (2010) apply evidence theory to both level-1 and level-2 uncertainty modeling and analysis, and the two settings were compared through a benchmark problem. Some explanations of the results are discussed in the context of evidence theory. Considering the large calculation cost for level-2 uncertainty analysis, Limbourg et al. (2010) develop an accelerated method for monotonous problems using the monotonous reliability method (MRM). Pedroni et al. (2013) and Pedroni and Zio (2012) model the epis-

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temic uncertainty using possibility distributions and develop a level-2 Monte Carlo simulation for uncertainty analysis, which is then compared to a purely probabilistic approach and an evidence theory-based (ETB) approach. Pasanisi et al. (2012) reinterpret the level-2 purely probabilistic frameworks in the light of Bayesian decision theory and apply the approach to risk analysis. Hybrid methods based on probability theory and evidence theory are also presented (Aven et al. 2014). Baraldi et al. (2013) introduce the hybrid level-2 uncertainty models to consider maintenance policy performance assessment.

In this paper, we enrich the research of level-2 uncertainty analysis by introducing a new mathematical framework, the uncertainty theory, to model the epistemically uncertain variables. Uncertainty theory has been founded in 2007 by Liu (2007) as an axiomatic mathematical framework to model subjective belief degrees. It is viewed as a reasonable and effective approach to describe epistemic uncertainty (Kang et al. 2016). To simulate the evolution of an uncertain phenomenon with time, concepts of uncertain process (Liu 2015) and uncertain random process (Gao and Yao 2015) are proposed. The uncertain differential equation is also developed as an effective tool to model events affected by epistemic uncertainty (Yang and Yao 2016). After these years of development, uncertainty theory has been applied in various areas, including finance (Chen and Gao 2013; Guo and Gao 2017), decision making under uncertain environment (Wen et al. 2015a, b), game theory (Yang and Gao 2013, 2016; Gao et al. 2017; Yang and Gao 2014). There are also considerable real applications in reliability analysis and risk assessment considering epistemic uncertainties. For example, Zeng et al. (2013) propose a new concept of belief reliability based on uncertainty theory accounting for both aleatory and epistemic uncertainties. Wen et al. (2017) develop an uncertain optimization model of spare parts inventory for equipment system, where the subjective belief degree is adopted to compensate the data deficiency. Ke and Yao (2016) apply uncertainty theory to optimize scheduled replacement time under block replacement policy considering human uncertainty. Wen and Kang (2016) model the reliability of systems with both random components and uncertain components. Wang et al. (2017) develop a new structural reliability index based on uncertainty theory.

To the best of our knowledge, in this paper, it is the first time that uncertainty theory is applied to level-2 uncertainty analysis. Through comparisons to some commonly used level-2 uncertainty analysis methods, new insights are brought with respect to strength and limitations of the developed method.

The remainder of the paper is structured as follows. Section 2 recalls some basic concepts of uncertainty theory. Level-2 uncertainty analysis method is developed in Sect. 3, for monotone and non-monotone risk models. Numerical case studies and applications are presented in Sect. 4. The paper is concluded in Sect. 5.

2 Preliminaries

In this section, we briefly review some basic knowledge on uncertainty theory. Uncertainty theory is a new branch of axiomatic mathematics built on four axioms, i.e., normality, duality, subadditivity and product axioms. Founded by Liu (2007) and refined by Liu (2010) , uncertainty theory has been widely applied as a new tool for modeling subjective (especially human) uncertainties. In uncertainty theory, belief degrees of events are quantified by defining uncertain measures:

Definition 1 (*Uncertain measure* Liu 2007) Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following three axioms,

Axiom 1 (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (*Duality Axiom*) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.

Axiom 3 (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

Uncertain measures of product events are calculated following the product axiom (Liu 2009):

Axiom 4 (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where $\mathcal{L}_k$ are σ-algebras over Γ_k, and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

In uncertainty theory, if an uncertain measure of one event can take multiple reasonable values, a value as close to 0.5 as possible is assigned to the event so as to maximize the uncertainty (maximum uncertainty principle) (Liu 2009). Hence, the uncertain measure of an arbitrary event in the product σ-algebra $\mathcal{L}$ is calculated by

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k \le \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \quad \text{if} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k \le \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k \le \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \quad \text{if} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k \le \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \quad \text{otherwise.} \end{cases} \tag{1}$$

Definition 2 (*Uncertain variable* Liu 2007) An uncertain variable is a function ξ from an uncertainty space $(\Lambda, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$ of real numbers.

Definition 3 (*Uncertainty distribution* Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x.

For example, a linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an uncertainty distribution

$$\Phi_1(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases} \tag{2}$$

and a normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an uncertainty distribution

$$\Phi_2(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in \Re \tag{3}$$

An uncertainty distribution Φ is said to be regular if it is a continuous and strictly increasing with respect to x, with $0 < \Phi(x) < 1$, and $\lim\limits_{x \to -\infty} \Phi(x) = 0$, $\lim\limits_{x \to +\infty} \Phi(x) = 1$. A regular uncertainty distribution has an inverse function, and this inverse function is defined as the inverse uncertainty distribution, denoted by $\Phi^{-1}(\alpha)$, $\alpha \in (0, 1)$. It is clear that linear uncertain variables and normal uncertain variables are regular, and their inverse uncertainty distributions are written as:

$$\Phi_1^{-1}(\alpha) = (1-\alpha)a + \alpha b, \tag{4}$$

$$\Phi_2^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{5}$$

Inverse uncertainty distributions play a central role in uncertainty theory, since the uncertainty distribution of a function of uncertain variables is calculated using the inverse uncertainty distributions:

Theorem 1 *(Operational law* Liu 2010 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_n^{-1}(1-\alpha)\right). \tag{6}$$

Definition 4 (*Expected value* Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\} \mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\} \mathrm{d}x. \tag{7}$$

It is clear that, if ξ has an uncertainty distribution $\Phi(x)$, the expected value of ξ can be calculated by (Liu 2015):

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) \, \mathrm{d}x - \int_{-\infty}^0 \Phi(x) \mathrm{d}x. \tag{8}$$

For ξ with a regular uncertainty distribution, the expected value $E[\xi]$ is given by (Liu 2015)

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) \mathrm{d}\alpha. \tag{9}$$

3 Level-2 uncertainty analysis based on uncertainty theory

In this section, a new method for level-2 uncertainty analysis is presented based on uncertainty theory. Sect. 3.1 formally defines the problem of level-2 uncertainty analysis. Then, the uncertainty analysis method is introduced for monotone and non-monotone models in Sects. 3.2 and 3.3, respectively.

3.1 Problem definition

Conceptually, uncertainty analysis of a risk model can be represented as:

$$\begin{aligned} & z = g(\mathbf{x}), \\ & p = h(g(\mathbf{x}), z_{th}), \end{aligned} \tag{10}$$

where z is the safety variable of the system of interest, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of input parameters, p is the risk indicator expressed in probabilistic terms and calculated by a distance function $h(\cdot)$ between the value of z and safety threshold z_{th}:

$$p = \Pr\{z > z_{th}\} \text{ or } p = \Pr\{z < z_{th}\}. \tag{11}$$

In practice, $g(\cdot)$ could be logical models, e.g., fault trees, event trees, Bayesian networks, or physical models of failure dynamics, e.g., see Baraldi and Zio (2008) and Ripamonti et al. (2013).

Uncertainty in (10) is assumed to come from the input parameters $\mathbf{x}$, i.e., model uncertainty (e.g., see Nilsen and Aven (2003)) is not considered in the present paper. Aleatory and epistemic uncertainty are considered separately. Depending on the ways the uncertainty in the model parameters is handled, level-1 and level-2 uncertainty models are distinguished.

Level-1 uncertainty models separate the input vector into $\mathbf{x} = (\mathbf{a}, \mathbf{e})$, where $\mathbf{a} = (x_1, x_2, \ldots, x_m)$ represents the parameters affected by aleatory uncertainty while $\mathbf{e} = (x_{m+1}, x_{m+2}, \ldots, x_n)$ represents the parameters that are affected by epistemic uncertainty (Limbourg and Rocquigny 2010). In level-1 uncertainty models, probability theory is used to model the aleatory uncertainty in $\mathbf{a} = (x_1, x_2, \ldots, x_m)$ by identifying their probability density functions (PDF) $f(x_i|\theta_i)$. These PDFs are assumed to be known, i.e., the parameters in the PDFs, denoted by $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, are assumed to have precise values. In practice, however, $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, are subject to epistemic uncertainty, and the corresponding uncertainty model is called level-2 uncertainty model.

In this paper, we consider the generic model in (10) and develop a new method for level-2 uncertainty analysis, based on uncertainty theory. More specifically, it is assumed that:

(1) The aleatory uncertainty in the input parameters is described by the PDFs $f(x_i|\theta_i)$, $i = 1, 2, \ldots, n$.
(2) $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ are modeled as independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$.

The uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$ describe the epistemic uncertainty in the parameter values of $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ and can be determined based on expert knowledge, using uncertain statistical methods such as interpolation (Liu 2015), optimization (Hou 2014) and the method of moments (Wang and Peng 2014). The problem, then, becomes: given $\Phi_1, \Phi_2, \ldots, \Phi_n$, how to assess the epistemic uncertainty in the risk index of interest p. In the following sections, we first develop the uncertainty analysis method for monotone models in Sect. 3.2, where p is a monotone function of the parameters $\mathbf{\Theta}$, and, then, discuss a more general case in Sect. 3.3, where there are no requirements on the monotony of the risk model.

3.2 Monotone risk model

3.2.1 Uncertainty analysis using operational laws

In monotone uncertainty models, the risk index of interest can be explicitly expressed as:

$$p = h(\mathbf{\Theta}), \tag{12}$$

where $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ is the vector of the parameters in the PDFs whose values are subject to epistemic uncertainty and h is a strictly monotone function with respect to $\mathbf{\Theta}$. According to Assumption (2) in Sect. 3.1, the risk index of interest p is also an uncertain variable. Given regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$ for $\theta_1, \theta_2, \ldots, \theta_n$, the epistemic uncertainty in p can be represented by an uncertainty distribution $\Psi(p)$. Without loss of generality, we assume h is strictly increasing with respect to $\theta_1, \theta_2, \ldots, \theta_m$, and strictly decreasing with respect to $\theta_{m+1}, \theta_{m+2}, \ldots, \theta_n$. Then, the inverse uncertainty distribution of p can be calculated based on Theorem 1, i.e.,

$$\begin{aligned}\Psi_p^{-1}(\alpha) = & h(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \\ & \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)), \quad 0 \le \alpha \le 1.\end{aligned} \tag{13}$$

The uncertainty distribution $\Psi(p)$ can be obtained from the inverse function $\Psi_p^{-1}(\alpha)$.

Two risk indexes are defined for risk-informed decision making, considering the level-2 uncertainty settings presented.

Definition 5 Let p represent a probabilistic risk index and $\Psi(p)$ be the uncertainty distribution of p. Then

$$\bar{p} = \int_0^{+\infty} [1 - \Psi(p)] \, \mathrm{d}p \tag{14}$$

is defined as the average risk, and

$$\mathrm{VaR}(\gamma) = \sup \{p | \Psi(p) \le \gamma\} \tag{15}$$

is defined as the value-at-risk.

It should be noted that the average risk can be also calculated using the inverse distribution of p:

$$\overline{p} = \int_0^1 \Psi^{-1}(\alpha) \mathrm{d}\alpha, \tag{16}$$

and the value-at-risk can also be calculated by

$$\mathrm{VaR}(\gamma) = \Psi^{-1}(\gamma). \tag{17}$$

According to Definition 5, the average risk is the expected value of the uncertain variable p, which reflects our average belief degree of the risk index p. A greater value of the average risk indicates that we believe the risk is more severe. The physical meaning of value-at-risk is that, with belief degree γ, we believe that the value of the risk index is p. It is clear that, for a fixed value of γ, a greater $\mathrm{VaR}(\gamma)$ means that the risk is more severe.

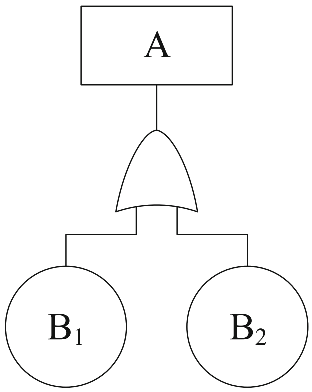

Fig. 1 Simple fault tree for the case study

3.2.2 Numerical case study

We take a simple fault tree (shown in Fig. 1) as a numerical case study to demonstrate the application of the developed method. The fault tree represents a top event A as the union (logic gate OR) of the two basic events B_1 and B_2. The risk index of interest is the probability that event A occurs before time t_0, determined by

$$p = \Pr\{t_A < t_0\}, \tag{18}$$

where t_A denotes the occurrence time of A. Let t_{B1} and t_{B2} be the occurrence times of events B_1 and B_2, respectively. Then, $t_A = \min(t_{B1}, t_{B2})$. Assume that t_{B1} and t_{B2} follow exponential distributions with parameters λ_1 and λ_2, respectively. Thus, (5) can be further expressed as:

$$\begin{aligned} p &= \Pr\{t_{B1} < t_0, t_{B2} < t_0\} \\ &= p_{B1} + p_{B2} - p_{B1} \cdot p_{B2} \\ &= 1 - e^{-(\lambda_1+\lambda_2)t_0} \end{aligned} \tag{19}$$

It is assumed that λ_1 and λ_2 are subject to epistemic uncertainty. The developed methods in Sect. 3.2.1 are used for level-2 uncertainty analysis based on uncertainty theory. In accordance with expert experience, linear uncertainty distributions are used to model the epistemic uncertainty in λ_1 and λ_2, i.e., $\lambda_1 \sim \mathcal{L}(a_1, b_1)$ and $\lambda_2 \sim \mathcal{L}(a_2, b_2)$. From (13), the inverse uncertainty distribution of the risk index p is calculated as

$$\Psi_p^{-1}(\alpha) = 1 - \exp\left[-(1-\alpha)(a_1+a_2)t_0 - \alpha(b_1+b_2)t_0\right], \quad 0 \le \alpha \le 1 \tag{20}$$

and the uncertainty distribution of p is

$$\Psi(p) = \begin{cases} 0, & \text{if } p \le \Psi_p^{-1}(0) \\ \dfrac{-\frac{1}{t_0}\ln(1-p) - (a_1+a_2)}{(b_1+b_2)-(a_1+a_2)}, & \text{if } \Psi_p^{-1}(0) \le p \le \Psi_p^{-1}(1) \\ 1, & \text{if } p \ge \Psi_p^{-1}(0). \end{cases} \tag{21}$$

Table 1 Time threshold and distributions for level-2 uncertain parameters

	$\lambda_1(10^{-5}/h^{-1})$	$\lambda_2(10^{-5}/h^{-1})$	t_0	γ
UTB method	$\mathcal{L}(0.8, 1.2)$	$\mathcal{L}(0.5, 0.8)$	$10^4 h$	0.9
PB method	$U(0.8, 1.2)$	$U(0.5, 0.8)$		

According to (16) and (17), $\overline{p}$ and VaR can be calculated by

$$\begin{aligned} \overline{p} &= \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 1\mathrm{d}\alpha - \int_0^1 \exp\left[(a_1+a_2-b_1-b_2)t_0\alpha - (a_1+a_2)t_0\right]\mathrm{d}\alpha \\ &= 1 - \frac{1}{(a_1+a_2-b_1-b_2)t_0}\left[e^{-(b_1+b_2)t_0} - e^{-(a_1+a_2)t_0}\right], \end{aligned} \tag{22}$$

$$\begin{aligned} \mathrm{VaR}(\gamma) &= \Psi_p^{-1}(\gamma) \\ &= 1 - \exp\left[-(1-\gamma)(a_1+a_2)t_0 - \gamma(b_1+b_2)t_0\right]. \end{aligned} \tag{23}$$

Assuming the parameter values in Table 1, we have $\overline{p} = 0.1519$ and $\mathrm{VaR}(0.9) = 0.1755$. The results are compared to those from a similar method based on probability theory, hereafter indicated as probability-based (PB) method, whereby the belief degrees on λ_1, λ_2 and p are modeled by random variables. In this paper, we assume that λ_1 and λ_2 follow uniform distributions whose parameter values are given in Table 1. Monte Carlo (MC) sampling is used to generate samples from the probability distribution of p. Average risk and value-at-risk can, then, be calculated using the MC samples:

$$\overline{p} = \frac{1}{n}\sum_{i=1}^{n} p_i, \tag{24}$$

$$\mathrm{VaR}(\gamma) = \sup\{p_i | p_i \le \gamma, i = 1, 2, \ldots, n\} \tag{25}$$

where $p_i, i = 1, 2, \ldots, n$ are the samples obtained by MC simulation.

Figure 2 compares the distributions of the risk indexes obtained from the two methods. Both distributions have the same supports, but the uncertainty distribution has more weights on high values of the risk index than the probability theory. This means that the uncertainty theory-based (UTB) method is more conservative than the PB method, since it tends to evaluate a higher risk. This is obtained also from the values in Table 2: although both methods have roughly the same $\overline{p}$, the UTB method yields a higher VaR(0.9), which indicates a high risk.

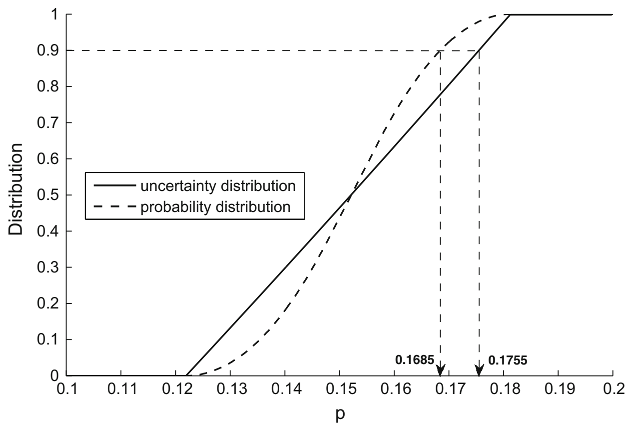

Fig. 2 Level-2 propagation results from uncertainty theory-based (UTB, *solid line*) and probability-based (PB, *dashed line*) methods

Table 2 Risk indexes of the monotone risk model

Method	$\overline{p}$	VaR(0.9)
UTB method	0.1519	0.1755
PB method	0.1520	0.1685

3.3 Non-monotone risk model

3.3.1 Uncertainty analysis using uncertain simulation

In many practical situations, the risk index of interest cannot be expressed as a strictly monotone function of the level-2 uncertain parameters. For such cases, we cannot obtain the exact uncertainty distributions for p by directly applying the operational laws. Rather, the maximum uncertainty principle needs to be used to derive the upper and lower bounds for the uncertainty distribution based on an uncertain simulation method developed by (Zhu 2012). The uncertain simulation can provide a reasonable uncertainty distribution of a function of uncertain variables and does not require the monotonicity of the function with respect to the variables. In this section, the method is extended to calculate the upper and lower bounds of an uncertainty distribution for risk assessment.

Definition 6 (Zhu 2012) An uncertain variable ξ is common if it is from the uncertain space $(\Re, \mathcal{B}, \mathcal{M})$ to $\Re$ defined by $\xi(\gamma) = \gamma$, where $\mathcal{B}$ is the Borel algebra over $\Re$. An uncertain vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is common if all the elements of $\boldsymbol{\xi}$ are common.

Theorem 2 (Zhu 2012) *Let $f : \Re^n \to \Re$ be a Borel function, and $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a common uncertain vector. Then the uncertainty distribution of f is:*

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \le x\}$$
$$= \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, \\ \quad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, \\ \quad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \quad \text{otherwise.} \end{cases} \tag{26}$$

In (26), $\Lambda = f^{-1}(-\infty, x)$, $\{A_i\}$ denotes a collection of all intervals of the form $(-\infty, a]$, $[b, +\infty)$, $\varnothing$ and $\Re$, and each $\mathcal{M}_k\{\Lambda_k\}$ is derived based on (27):

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B\subset\bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, \\ \quad \text{if} \inf\limits_{B\subset\bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c\subset\bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, \\ \quad \text{if} \inf\limits_{B^c\subset\bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, \quad \text{otherwise}, \end{cases} \tag{27}$$

where $B \in \mathcal{B}$, and $B \subset \bigcup_{i=1}^{\infty} A_i$.

From Theorem 2, it can be seen that (27) gives a theoretical bound of each $\mathcal{M}_k\{\Lambda_k\}$ in (26). Let $m = \inf_{B\subset\bigcup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}$, $n = \inf_{B^c\subset\bigcup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}$. It is clear that any values within m and $1 - n$ is a reasonable value for $\mathcal{M}\{B\}$. Hence, we use m as the upper bound and $1-n$ as the lower bound of $\mathcal{M}_k\{\Lambda_k\}$ and develop a numerical algorithm for level-2 uncertainty analysis.

Algorithm 1. (Level-2 uncertainty analysis for non-monotone models)

step 1 Set $m_1(i) = 0$ and $m_2(i) = 0$, $i = 1, 2, \ldots, n$.

step 2 Randomly generate $u_k = \left(\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(n)}\right)$ with $0 < \Phi_i\left(\gamma_k^{(i)}\right) < 1$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, N$.

step 3 From $k = 1$ to $k = N$, if $f(u_k) \le c$, $m_1(i) = m_1(i) + 1$, denote $x_{m_1(i)}^{(i)} = \gamma_k^{(i)}$; otherwise, $m_2(i) = m_2(i) + 1$, denote $y_{m_2(i)}^{(i)} = \gamma_k^{(i)}$, $i = 1, 2, \ldots, n$.

step 4 Rank $x_{m_1}^{(i)}$ and $y_{m_2}^{(i)}$ from small to large, respectively.

step 5 Set

$$a^{(i)} = \Phi\left(x_{m_1(i)}^{(i)}\right) \wedge \left(1 - \Phi\left(x_1^{(i)}\right)\right) \wedge$$

$$\left(\Phi\left(x_1^{(i)}\right)+1-\Phi\left(x_2^{(i)}\right)\right)\wedge$$
$$\left(\Phi\left(x_{m_1(i)-1}^{(i)}\right)+1-\Phi\left(x_{m_1(i)}^{(i)}\right)\right);$$
$$b^{(i)}=\Phi\left(y_{m2(i)}^{(i)}\right)\wedge\left(1-\Phi\left(y_1^{(i)}\right)\right)\wedge$$
$$\left(\Phi\left(y_1^{(i)}\right)+1-\Phi\left(y_2^{(i)}\right)\right)\wedge$$
$$\left(\Phi\left(y_{m_2(i)-1}^{(i)}\right)+1-\Phi\left(y_{m_2(i)}^{(i)}\right)\right).$$

step 6 $L_{1U}^{(i)}=a^{(i)}, L_{1L}^{(i)}=1-b^{(i)}, L_{2U}^{(i)}=b^{(i)}, L_{2L}^{(i)}=1-a^{(i)}$.

step 7 If $a_U=L_{1U}^{(1)}\wedge L_{1U}^{(2)}\wedge\cdots\wedge L_{1U}^{(n)}>0.5, L_U=a_U$; if $b_U=L_{2L}^{(1)}\wedge L_{2L}^{(2)}\wedge\cdots\wedge L_{2L}^{(n)}>0.5$, $L_U=1-b_U$; otherwise, $L_U=0.5$.
If $a_L=L_{1L}^{(1)}\wedge L_{1L}^{(2)}\wedge\cdots\wedge L_{1L}^{(n)}>0.5, L_L=a_L$; if $b_L=L_{2U}^{(1)}\wedge L_{2U}^{(2)}\wedge\cdots\wedge L_{2U}^{(n)}>0.5, L_L=1-b_L$; otherwise, $L_L=0.5$.

Through this algorithm, the upper and lower bounds for the uncertainty distribution of p can be constructed, denoted by $[\Psi_L(p),\Psi_U(p)]$. Similar to the monotone case, we define two risk indexes considering the level-2 uncertainty:

Definition 7 Let p described by (11) be the probability that a hazardous event will happen, and let $\Psi_L(p)$ and $\Psi_U(p)$ be the lower bound and upper bound of the uncertainty distribution of p, respectively. Then

$$\overline{p}=\int_0^{+\infty}\left[1-\frac{\Psi_L(p)+\Psi_U(p)}{2}\right]dp \tag{28}$$

is defined as average risk, and

$$[\mathrm{VaR}_L,\mathrm{VaR}_U](\gamma)=\left[\sup\{p|\Psi_L(p)\le\gamma\},\sup\{p|\Psi_U(p)\le\gamma\}\right] \tag{29}$$

is defined as bounded value-at-risk.

The defined average risk is a reflection of the average belief degree of the risk index p, and a greater value of $\overline{p}$ means more severe risk that we believe we will suffer. The meaning of the bounded value-at-risk is that, with belief degree γ, we believe that the value of risk index is within the interval $[\mathrm{VaR}_L,\mathrm{VaR}_U](\gamma)$. Obviously, if we fix the value of γ, a wider bounded value-at-risk means a more conservative assessment result. Meanwhile, we believe a greater $\mathrm{VaR}_U(\gamma)$ reflects that the risk is more severe.

Table 3 Distributions of level-1 and level-2 parameters

Parameter	Level-1	Level-2	
		UTB method	ETB method
x_1	$N(\mu_1,5)$	$\mu_1\sim\mathcal{L}(9,11)$	$\mu_1\sim U(9,11)$
x_1	$N(\mu_2,5)$	$\mu2\sim\mathcal{N}(10,0.3)$	$\mu_2\sim N(10,0.3)$

3.3.2 Numerical case study

We consider a problem of structural reliability in Choi et al. (2007) to further elaborate on the developed method. Let the limit-state function of a structure be

$$g(x_1,x_2)=x_1^4+2x_2^4-20. \tag{30}$$

where x_1 and x_2 are random variables, and the risk index of interest is the probability that the structure fails, which can be written as

$$p_f=\Pr\{g(x_1,x_2)<0\}. \tag{31}$$

Assume that x_1 and x_2 follow normal distributions with parameters (μ_1,σ_1) and (μ_2,σ_2), respectively. The parameters μ_1 and μ_2 are not precisely known due to the epistemic uncertainty, whereas σ_1 and σ_2 are known as crisp values. Based on experts knowledge, the belief degree of μ_1 is modeled by a linear uncertainty distribution and μ_2 is described by a normal uncertainty distribution (see Table 3). The bounded uncertainty distribution can, then, be obtained through Algorithm 1.

The solid line and dashed line in Fig. 3 show the upper and lower uncertainty distributions of the risk index p_f, respectively. Average risk and bounded value-at-risk are calculated using the numerical method based on (28) and (29), i.e., $\overline{p}=0.001980$ and $[\mathrm{VaR}_L,\mathrm{VaR}_U](0.9)=[0.001689,0.003548]$.

Since the developed method offers a bounded uncertainty distribution of p_f, it is then compared with an evidence theory-based (ETB) method, in which the belief degree of p_f is also given as upper and lower distributions called plausibility (Pl) and belief (Bel) function, respectively. In this paper, the ETB method models the belief degrees of μ_1 and μ_2 using probability distributions (see Table 3). A double loop Monte Carlo simulation combined with a discretization method for getting basic probability assignments (BPAs) is used to obtain $Bel\left(p_f\right)$ and $Pl\left(p_f\right)$ (Limbourg and Rocquigny 2010; Tonon 2004). In Fig. 3, the dotted line and dot–dash line represent Bel and Pl, respectively. It should be noted that although we use Bel and Pl as mathematical constructs, they are not strictly the concepts of belief and plausibility defined by Shafer (i.e., the degree of truth of a proposition Shafer 1976). The two functions only represent

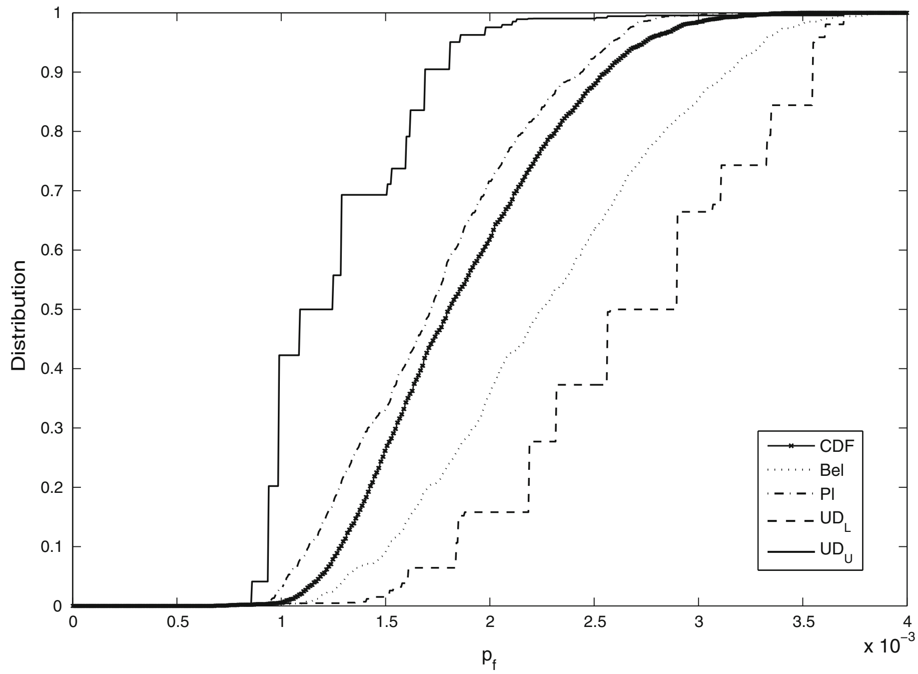

Fig. 3 Results of level-2 uncertainty analysis (*CDF* cumulative distribution function, *Bel* belief function, *Pl* plausibility function, UD_L lower uncertainty distribution, UD_U upper uncertainty distribution)

Table 4 Risk indexes of the non-monotone risk model

Method	$\overline{p}$	$[\mathrm{VaR}_L, \mathrm{VaR}_U]\,(0.9)$
UTB method	0.001980	[0.001689, 0.003548]
ETB method	0.002012	[0.002440, 0.003140]

bounds on a true quantity. To illustrate this, a cumulative density function (CDF) of p_f is calculated via a double loop MC simulation method, shown as the crossed line in Fig. 3. It is seen that the CDF is covered by the area enclosed by Bel and Pl. In this sense, the CDF obtained in PB method is a special case of the ETB model, and the $Bel\left(p_f\right)$ and $Pl\left(p_f\right)$ give a reasonable bound of the probability distribution of p_f.

Given $Bel\left(p_f\right)$ and $Pl\left(p_f\right)$, the two risk indexes can be calculated by

$$\overline{p} = \int_0^{\infty} \left(1 - \frac{Bel(p_f) + Pl(p_f)}{2}\right) \mathrm{d}p, \tag{32}$$

and

$$[\mathrm{VaR}_L, \mathrm{VaR}_U]\,(\gamma) = \left[\sup\left\{p_f | Bel(p_f) \le \gamma\right\}, \sup\left\{p_f | Pl(p_f) \le \gamma\right\}\right], \tag{33}$$

and the results are tabulated in Table 4.

Figure 3 shows a comparison of the distributions of belief degrees on p_f in UTB method and ETB method. The distributions have the same supports, whereas the upper and lower uncertainty distributions fully cover the CDF and the area enclosed by Bel and Pl, which indicates that the developed method is more conservative. This is because the subjective belief described by uncertainty distributions usually tends to be more conservative and is more easily affected by epistemic uncertainty. This phenomenon is also reflected by the two defined risk indexes: the average risk $\overline{p}$s are nearly the same on different theory basis, while the bounded value-at-risk of ETB method is within that of the UTB method.

We also find that the bounded value-at-risk obtained by the developed UTB method may be too wide for some decision makers. This may be a shortcoming of the proposed method. Therefore, when choosing a method for risk analysis from the PB method, ETB method and UTB method, we need to consider the attitude of decision maker. For a conservative decision maker, the bounded uncertainty distribution is an alternative choice.

4 Application

In this section, the developed level-2 uncertainty analysis method is applied to a real application of flood risk assessment. In Sect. 4.1, we briefly introduce the system of interest. Sections 4.2 and 4.3 show the process of level-2 uncertainty analysis based on uncertainty theory, to illustrate the effectiveness of the method.

4.1 System description

In this case study, we consider a residential area located near a river, which is subject to potential risks of floods, as shown in Fig. 4. As a mitigation and prevention measure, a dike is

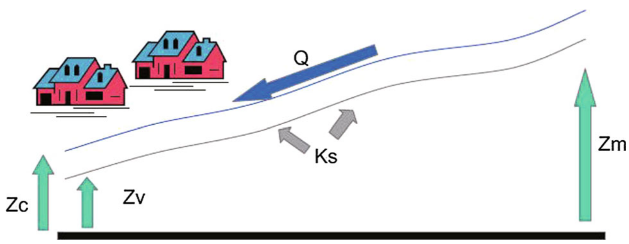

Fig. 4 Flooding risk system (Limbourg and Rocquigny 2010)

constructed to protect this area. The final goal is to calculate the risk of floods to determine whether the dike needs to be heightened. A mathematical model is develop in (Limbourg and Rocquigny 2010) which calculates the maximal water level Z_c:

$$Z_c = g(Q, K_s, Z_m, Z_v, l, b)$$

$$= Z_v + \left(\frac{Q}{K_s \cdot b \cdot \sqrt{(Z_m - Z_v)/l}} \right)^{3/5}, \tag{34}$$

where Z_m denotes the riverbed level at the upstream part of the river, Z_v denotes the riverbed level at the downstream part of the river, K_s denotes the friction coefficient of the riverbed, Q denotes the yearly maximal water flow, l denotes the length of river, and b denotes the width of river (Limbourg et al. 2010). The risk of floods can, then, be calculated as the probability that the annual maximum water level exceeds the dike height:

$$p_{\text{flood}} = \Pr\{Z_c > H\}. \tag{35}$$

4.2 Parameter setting

The input variables in 34 are assumed to be random variables and the form of their PDFs are assumed to be known, as

Table 5 Uncertainty description of level-1 and level-2 parameters

Parameter	Probability distribution	Level-2 uncertainty distribution		Theoretical bounds
Q	$Gum(\alpha, \beta)$	α	$\mathcal{N}_\alpha(1013, 48)$	[10, 10000]
		β	$\mathcal{N}_\beta(558, 36)$	
K_s	$N(\mu_{K_s}, \sigma^2_{K_s})$	μ_{K_s}	$\mathcal{L}(22.3, 33.3)$	[5, 60]
		σ_{K_s}	$\mathcal{L}(2.5, 3.5)$	
Z_m	$N(\mu_{Z_m}, \sigma^2_{Z_m})$	μ_{Z_m}	$\mathcal{L}(54.87, 55.19)$	[53.5, 57]
		σ_{Z_m}	$\mathcal{L}(0.33, 0.57)$	
Z_v	$N(\mu_{Z_v}, \sigma^2_{Z_v})$	μ_{Z_v}	$\mathcal{L}(50.05, 50.33)$	[48, 51]
		σ_{Z_v}	$\mathcal{L}(0.28, 0.48)$	
l	5000 (constant)			–
b	30 (constant)			–

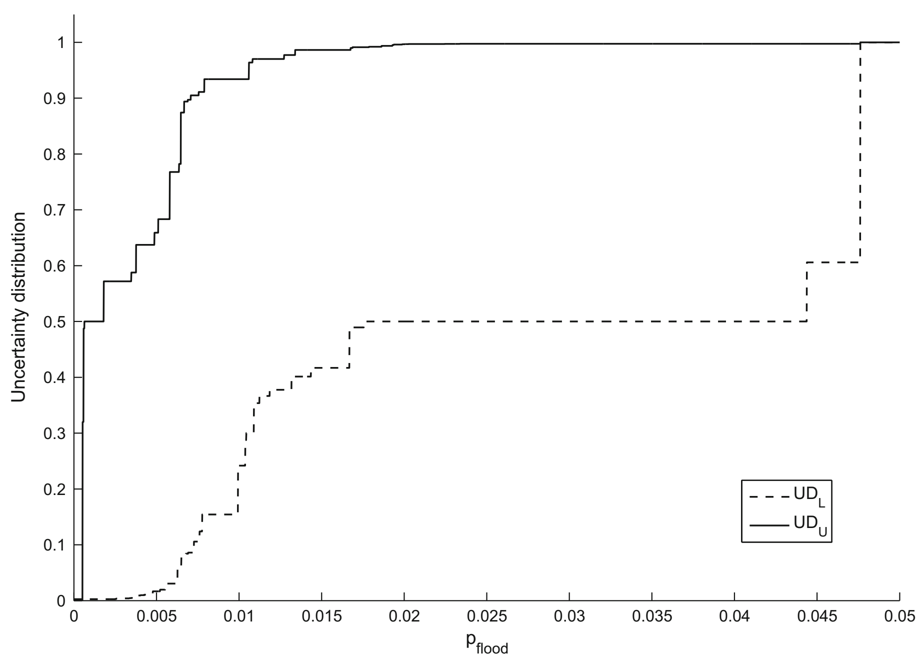

Fig. 5 Result for level-2 uncertainty propagation based on UTB method

Table 6 Risk indexes for the flood system

Index	Value
$\overline{p_{\text{flood}}}$	0.0161
$[\text{VaR}_L, \text{VaR}_U]$ (0.9)	[0.0073, 0.0476]

shown in Table 5 (Limbourg and Rocquigny 2010). However, due to limited statistical data, the distribution parameters of these PDFs cannot be precisely estimated using statistic methods and, therefore, are affected by epistemic uncertainty, which should be evaluated based on experts knowledge. In this paper, the experts knowledge on the distribution of these parameters is obtained by asking the experts to give the uncertainty distributions of the parameters, as shown in Table 5. For example, the yearly maximal water flow, denoted by Q, follows a Gumbel distribution $Gum(\alpha, \beta)$, and according to expert judgements, α and β follow normal uncertainty distributions $\mathcal{N}_\alpha(1013, 48)$ and $\mathcal{N}_\beta(558, 36)$, respectively. In addition, considering some physical constraints, the input quantities also have theoretical bounds, as given in Table 5.

4.3 Results and discussion

Uncertain simulation method is used to propagate the level-2 uncertainty using Algorithm 1. The theoretical bounds in Table 5 are considered by truncating the probability distributions at these bounds. The lower and upper bounds for the uncertainty distributions of p_{flood} are shown in Fig. 5, which represents the belief degrees on p_{flood} considering the level-2 uncertainty. Average risk and bounded value-at-risk are calculated based on (28) and (29) and presented in Table 6.

It follows that the average yearly risk is p_{flood}, which corresponds to an average return period of 62 years. This is unacceptable in practice, because it is too short when compared to a commonly required 100-year-return period. To solve this problem, one measure is to heighten the dike for a more reliable protection. Another solution might be increasing the friction coefficient of the riverbed K_s, noting from 34 that Z_c decreases with K_s.

The bounded value-at-risk is relatively wide, which indicates that due to the presence of epistemic uncertainty, we cannot be too confirmed on the calculated risk index. The same conclusion is also drawn from Fig. 5: the difference between the upper and lower bounds of the uncertainty distributions are large, indicating great epistemic uncertainty. To reduce the effect of epistemic uncertainty, more historical data need to be collected to support a more precise estimation of the distribution parameters in the level-1 probability distributions.

5 Conclusions

In this paper, a new level-2 uncertainty analysis method is developed based on uncertainty theory. The method is discussed in two respects: for monotone risk models, where the risk index of interest is expressed as an explicit monotone function of the uncertain parameters, and level-2 uncertainty analysis is conducted based on operational laws of uncertainty variables; for non-monotone risk models, an uncertain simulation-based method is developed for level-2 uncertainty analysis. Three indexes, i.e., average risk, value-at-risk and bounded value-at-risk, are defined for risk-informed decision making in the level-2 uncertainty setting. Two numerical studies and an application on a real example from literature are worked out to illustrate the developed method. The developed method is also compared to some commonly used level-2 uncertainty analysis methods, e.g., PB method and ETB method. The comparisons show that, in general, the UTB method is more conservative than the methods based on probability theory and evidence theory.

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Compliance with ethical standards

Conflicts of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

Human and animal rights This article does not contain any studies with human participants performed by any of the authors.

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