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METHODOLOGIES AND APPLICATION

Handling group decision-making model with incomplete hesitant fuzzy preference relations and its application in medical decision

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Abstract
On account of uncertainty and complexity of environments, it is more suitable to express their assessed value by means of hesitant fuzzy information for decision makers. In this paper, we establish a new group decision-making (GDM) model with incomplete hesitant fuzzy preference relations (HFPRs) based on mathematical programming approach. Firstly, based on the multiplicative consistency of incomplete HFPR, a mathematical programming model is established to obtain multiplicative consistent fuzzy preference relation (FPR) from a given incomplete HFPR. Following this, experts are assigned with weights according to their consistency degree. Subsequently, a group consensus reaching process algorithm is constructed based on the obtained multiplicative consistent FPRs. Correspondingly, a GDM model is further established. Finally, a medical decision application is studied to present the practicability and effectiveness of the proposed method.

Keywords Incomplete hesitant fuzzy preference relation · Multiplicative consistency · Mathematical programming · Consensus · Group decision making

1 Introduction

As an extension of Zadeh's fuzzy sets (Zadeh 1965), all kinds of fuzzy sets were established and have been developed in recent years. Subsequently, a variety of preference relations were formed, for example fuzzy preference relation (FPR) (Orlovsky 1978; Herrera-Viedma et al. 2004) and linguistic preference relation (Xu 2005). Besides, some extended fuzzy numbers were proposed and applied to decision problems (Li et al. 2015a, b). Because of the uncertainty and complexity of decision-making environment, it is not easy to just afford a single term to evaluate two objects for decision makers (DMs) in the actual decision-making process. In order to manage this issue, Torra (2010) developed hesitant fuzzy set, which allows DMs to take into account simultaneously several possible values to evaluate two objects. Complying with the cognitive characteristics of DMs, the hesitant fuzzy set includes more influential information of DMs. Moreover, Rodríguez et al. (2012) proposed hesitant fuzzy linguistic term sets (HFLTSs) based on the hesitant fuzzy sets. Afterward, the GDM problems based on preference relations with HFLTSs have received widespread attention (Rodríguez et al. 2013; Song and Hu 2017; Xu and Wang 2017).

On the basis of the hesitant fuzzy set, Xia and Xu (2013) put forward hesitant fuzzy preference relation (HFPR). The GDM models based on HFPRs have been extensively studied (Xia and Xu 2013; Liao et al. 2014; Zhu et al. 2014; Zhang et al. 2015b; Xu et al. 2017). Xia and Xu (2013) introduced the concept of HFPR and applied four operators to obtain the collective matrices, respectively. Recommending the concept of multiplicative consistency of HFPR, Liao et al. (2014) developed two algorithms to improve consistency and consensus level, respectively. Zhang et al. (2015b) constructed a GDM model, simultaneously considering consensus reaching process. Zhu et al. (2014) proposed two methods to obtain the ranking results of alternatives based on the α-normalization and the β-normalization, respectively. Xu et al. (2017) developed firstly a normalization method to obtain the normalized HFPRs based on additive consistency. And then, a group consensus model was established based on two feedback mechanisms, namely interactive mechanism and automatic mechanism.

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Meanwhile, the incomplete evaluations frequently exit owed to varied reasons, for example lack of background knowledge or time pressure for DMs. Various incomplete preference relations acquired development constantly in recent years (Xu and Wang 2013; Xu et al. 2013a, 2018; Ureña et al. 2015; Zhang and Guo 2016a). Furthermore, the research of incomplete HFPRs in group decision making has received attention quickly (Zhang et al. 2015a; Xu et al. 2016). Zhang et al. (2015a) defined the additive consistent incomplete HFPR. First, the normalized incomplete HFPR was obtained to ensure the same number for every known elements. Then, two estimation procedures were constructed to estimate the missing values with incomplete HFPR based on the additive consistency. Finally, the collective HFPR was obtained by means of WA operator. Xu et al. (2016) developed two goal programming models to get the priority weights from an incomplete HFPR based on additive consistency and multiplicative consistency, respectively. In addition to these two goal programming models were extended to deal with GDM problems. However, the above existing GDM models to deal with incomplete HFPRs exhibit some drawbacks as the mentioned methods do not consider group consensus reaching process, and Zhang et al. (2015a)'s model carries out normalization process which biases the original information of DMs (Rodríguez et al. 2016). On account of the uncertainty for hesitant information, the handling method with HFPR should draw the reasonable information from the HFPR rather than endeavor to satisfy that all the preference information should be perfectly consistent because the normalization process will bias the original information (Rodríguez et al. 2016). In addition, Zhu and Xu (2013) established a regression algorithm to obtain the highest consistent FPR within all possible FPRs from a given HFPR.

Based on the above motivations, we develop a novel group decision-making model with incomplete HFPRs by means of mathematical programming approach, considering simultaneously group consensus reaching process. The multiplicative consistent FPR obtained by mathematical programming may be interpreted as the most reasonable information of given incomplete HFPR, which means a process of regression.

The following contents of this paper are structured as below: In Sect. 2, the relevant knowledge for FPR, HFPR, and incomplete HFPR is reviewed. In Sect. 3, an addressing method for incomplete HFPR via mathematical programming is proposed. In Sect. 4, a GDM model with incomplete HFPRs is established. In Sect. 5, a medical decision problem is resolved by the proposed model, and comparison between our method and other relevant approaches is fulfilled. Some concluding remarks are presented in Sect. 6.

2 Preliminaries

In this section, we review the relevant knowledge for FPR, HFPR, and incomplete HFPR.

2.1 Fuzzy preference relation

Definition 2.1 (Tanino 1984) Suppose P be FPR about the alternatives $A = \{A_1, A_2, \ldots, A_n\}$, presented as hereunder:

$$P = (p_{ij})_{n\times n} = \begin{bmatrix} 0.5 & p_{12} & \cdots & p_{1n} \\ p_{21} & 0.5 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{bmatrix},$$

where p_{ij} suggests the degree of preference for alternative A_i over A_j, $p_{ij} \in [0, 1]$, $p_{ij} = 0.5$ denotes indifference between A_i and A_j, $p_{ij} = 1$ denotes that A_i is entirely preferred to A_j, and $p_{ij} > 0.5$ denotes A_i is preferred to A_j, where $1 \le i, j \le n$.

Definition 2.2 (Tanino 1984) Suppose $P = (p_{ij})_{n\times n}$ be FPR. If P satisfies the following equation:

$$p_{ij} = \frac{\omega_i}{\omega_i + \omega_j}, \quad i, j = 1, 2, \ldots, n, \tag{1}$$

then P is referred to as a multiplicative consistent FPR.

2.2 Hesitant fuzzy preference relation (HFPR)

Xia and Xu (2013) firstly proposed the definition of HFPR. However, it needs the values of HFPR to be arranged in ascending sequence, i.e., $h_{ij}^{\beta} < h_{ij}^{\beta+1}$ $(i < j)$, which will distort the expert's original information. To overcome the above weaknesses, Xu et al. (2017) developed a new definition of HFPR that need not arrange the elements in ascending or descending sequence. Therefore, this paper uses the definition proposed by Xu et al. (2017) as follows:

Definition 2.3 (Xu et al. 2017) A hesitant fuzzy preference relation H about the alternatives $X = \{x_1, x_2, \ldots, x_n\}$ is presented by $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \left\{h_{ij}^{\beta} \,\middle|\, \beta = 1, 2, \ldots, \#h_{ij}\right\}$ ($\#h_{ij}$ is the number of values in h_{ij}) is a hesitant fuzzy element, which indicates all the possible preference degree(s) of the objective x_i over x_j. Moreover, h_{ij} should satisfy the following conditions:

$$h_{ij}^{\beta} + h_{ji}^{\beta} = 1,\ h_{ii} = \{0.5\},\ \#h_{ij} = \#h_{ji}, \\ i, j = 1, 2, \ldots, n.$$

where h_{ij}^{β} refers to the βth element in h_{ij}.

2.3 Incomplete hesitant fuzzy preference relation

As we have already said in introduction, incomplete preference relations usually exist on account of DMs' limited expertise related to the problem domain or time pressure. In the following, we review the definition of incomplete HFPR.

Definition 2.4 (Xu et al. 2016) Suppose $X = \{x_1, x_2, \ldots, x_n\}$ is a fixed set, then an incomplete HFPR H on X is indicated by $H = \left(h_{ij}\right)_{n\times n} \subset X \times X$ where all known hesitant fuzzy elements (HFEs) $h_{ij} = \left\{h_{ij}^{\beta} \left|\beta = 1, 2, \ldots, \#h_{ij}\right.\right\}$ ($\#h_{ij}$ is the number of values in h_{ij}) which indicates all the possible preference degree(s) of the objective x_i over x_j. Moreover, h_{ij} should satisfy the following conditions:

$$h_{ij}^{\beta} + h_{ji}^{\beta} = 1, h_{ii} = \{0.5\}, \#h_{ij} = \#h_{ji}, \quad i, j = 1, 2, \ldots, n.$$

where h_{ij}^{β} refers to the βth element in h_{ij}.

Definition 2.5 (Xu et al. 2016) Let $H = \left(h_{ij}\right)_{n\times n}$ be an incomplete HFPR. If H satisfies the following conditions:

$$\frac{\omega_i}{\omega_i + \omega_j} = h_{ij}^{1} \; or \; h_{ij}^{2} \; or \ldots or \; h_{ij}^{\#h_{ij}}, \tag{2}$$

then H is referred to as a multiplicative consistent incomplete HFPR, where $i, \; j = 1, 2, \ldots, n, , h_{ij}^{\beta}$ being the βth element in h_{ij} and the $\#h_{ij}$ being the number of h_{ij}.

3 An addressing method for incomplete HFPR via mathematical programming

In this section, the multiplicative consistent FPR is extracted from incomplete HFPR by means of mathematical programming.

For convenience of calculation, an indication matrix $\Delta = \left(\lambda_{ij}\right)_{n\times n}$ of the incomplete HFPR $H = \left(h_{ij}\right)_{n\times n}$ is constructed (Xu et al. 2016) as follows:

$$\lambda_{ij} = \begin{cases} 0, \; h_{ij} = x \\ 1, \; h_{ij} \neq x \end{cases} \text{ and } h_{ij} = x \text{ represents a missing HFE } h_{ij}.$$

Given an incomplete HFPR $H = \left(h_{ij}\right)_{n\times n}$, let $\delta\left(h_{ij}\right) = h_{ij}^{1} \; or \; h_{ij}^{2} \; or \ldots or \; h_{ij}^{\#h_{ij}}$, then Eq. (2) is rewritten as follows:

$$\lambda_{ij} \cdot \frac{\omega_i}{\omega_i + \omega_j} = \lambda_{ij}\delta\left(h_{ij}\right) \Leftrightarrow \lambda_{ij}\omega_i = \lambda_{ij}\delta\left(h_{ij}\right)\left(\omega_i + \omega_j\right)$$
$$\Leftrightarrow \lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j = 0, \tag{3}$$

If an HFPR H is not consistent, then there is no one vector satisfying Eq. (3). In real world, it is not easy to meet the perfect consistency like Eq. (3). Therefore, "soft" consistent concept was proposed to express approximate consistency (Kacprzyk and Fedrizzi 1988; Herrera-Viedma et al. 2014). It is generally known that the most deviation element with consistent level plays a central role in deriving the priority weight of alternatives, based on which, in order to obtain the best weight for the standards, a solution should be found so as to minimize the maximum absolute differences $\left|\lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j\right|$ for all j based on Eq. (3). Taking the sum and nonnegativity conditions for the weights, the below model is established:

$$\begin{aligned} &\min \; \max \left\{\left|\lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j\right|\right\} \\ &s.t. \begin{cases} \sum\limits_{i=1}^{n} \omega_i = 1 \\ \omega_i \geq 0 \\ i, j = 1, 2, \ldots, n \end{cases}, \end{aligned} \tag{4}$$

Also, because the following equation holds

$$\begin{aligned} &\left|\lambda_{ji}\left(\delta\left(h_{ji}\right) - 1\right)\omega_j + \lambda_{ji}\delta\left(h_{ji}\right)\omega_i\right| \\ &\quad = \left|\lambda_{ij}\left(1 - \delta\left(h_{ij}\right) - 1\right)\omega_j + \lambda_{ij}\left(1 - \delta\left(h_{ij}\right)\right)\omega_i\right| \\ &\quad = \left|\lambda_{ij}\left(-\delta\left(h_{ij}\right)\right)\omega_j + \lambda_{ij}\left(1 - \delta\left(h_{ij}\right)\right)\omega_i\right| \\ &\quad = \left|\lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j\right| \end{aligned}$$

Therefore, model (4) is equivalent to the following model (5)

$$\begin{aligned} &\min \; \max \left\{\left|\lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j\right|\right\} \\ &s.t. \begin{cases} \sum\limits_{i=1}^{n} \omega_i = 1 \\ \omega_i \geq 0 \\ i, j = 1, 2, \ldots, n, i < j \end{cases}, \end{aligned} \tag{5}$$

Model (5) can be transferred to the following model:

$$\begin{aligned} &\min \; \xi \\ &s.t. \begin{cases} \left|\lambda_{ij}\left(\delta\left(h_{ij}\right) - 1\right)\omega_i + \lambda_{ij}\delta\left(h_{ij}\right)\omega_j\right| \leq \xi \\ \sum\limits_{i=1}^{n} \omega_i = 1 \\ \omega_i \geq 0 \\ i, j = 1, 2, \ldots, n, i < j \end{cases}, \end{aligned} \tag{6}$$

Moreover, model (6) can be reduced to the following model (7)

$$
\min \xi \\
s.t. \begin{cases} \lambda_{ij}\left(\delta\left(h_{ij}\right)-1\right)\omega_i+\lambda_{ij}\delta\left(h_{ij}\right)\omega_j-\xi\leq 0 \\ \lambda_{ij}\left(1-\delta\left(h_{ij}\right)\right)\omega_i-\lambda_{ij}\delta\left(h_{ij}\right)\omega_j-\xi\leq 0 \\ \sum_{i=1}^{n}\omega_i=1 \\ \omega_i\geq 0 \\ i,j=1,2,\ldots,n, i<j \end{cases}, \tag{7}
$$

Finally, model (7) can be translated into the following model (8)

$$
\min \xi \\
s.t. \begin{cases} \lambda_{ij}\left(\left(\sum_{\beta=1}^{\#h_{ij}} z_{ij}^{\beta}\cdot h_{ij}^{\beta}\right)-1\right)\omega_i \\ \quad +\lambda_{ij}\left(\sum_{\beta=1}^{\#h_{ij}} z_{ij}^{\beta}\cdot h_{ij}^{\beta}\right)\omega_j-\xi\leq 0, \\ \lambda_{ij}\left(1-\left(\sum_{\beta=1}^{\#h_{ij}} z_{ij}^{\beta}\cdot h_{ij}^{\beta}\right)\right)\omega_i \\ \quad -\lambda_{ij}\left(\sum_{\beta=1}^{\#h_{ij}} z_{ij}^{\beta}\cdot h_{ij}^{\beta}\right)\omega_j-\xi\leq 0, \\ \sum_{i=1}^{n}\omega_i=1, \\ \omega_i\geq 0, \\ \sum_{\beta=1}^{\#h_{ij}} z_{ij}^{\beta}=1, \\ z_{ij}^{\beta}=0\, or\, 1, \\ i,j=1,2,\ldots,n,\ i<j. \end{cases}, \tag{8}
$$

Remark 1 Since $\xi_{\min}$ denotes the deviation of consistency level, the $1-\xi_{\min}$ may indicate the consistent level for decision maker. It becomes clear that the $1-\xi_{\min}$ larger, the higher the consistent level for decision maker.

Example 1 Assume an incomplete HFPR H as follows:

$$
H=\begin{bmatrix} \{0.5\} & x & \{0.6\} & \{0.7\} \\ x & \{0.5\} & x & \{0.7,0.8\} \\ \{0.4\} & x & \{0.5\} & \{0.6,0.7\} \\ \{0.3\} & \{0.3,0.2\} & \{0.4,0.3\} & \{0.5\} \end{bmatrix}.
$$

Based on model (8), we establish the following mathematical programming

$$
\min \xi \\
s.t. \begin{cases} -0.4\omega_1+0.6\omega_3-\xi\leq 0, \\ 0.4\omega_1-0.6\omega_3-\xi\leq 0, \\ -0.3\omega_1+0.7\omega_4-\xi\leq 0, \\ 0.3\omega_1-0.7\omega_4-\xi\leq 0, \\ \left[\left(z_{24}^1\times 0.7+z_{24}^2\times 0.8\right)-1\right]\omega_2 \\ \quad +\left(z_{24}^1\times 0.7+z_{24}^2\times 0.8\right)\omega_4-\xi\leq 0, \\ \left[1-\left(z_{24}^1\times 0.7+z_{24}^2\times 0.8\right)\right]\omega_2 \\ \quad -\left(z_{24}^1\times 0.7+z_{24}^2\times 0.8\right)\omega_4-\xi\leq 0, \\ \left[\left(z_{34}^1\times 0.6+z_{34}^2\times 0.7\right)-1\right]\omega_3 \\ \quad +\left(z_{34}^1\times 0.6+z_{34}^2\times 0.7\right)\omega_4-\xi\leq 0, \\ \left[1-\left(z_{34}^1\times 0.6+z_{34}^2\times 0.7\right)\right]\omega_3 \\ \quad -\left(z_{34}^1\times 0.6+z_{34}^2\times 0.7\right)\omega_4-\xi\leq 0, \\ \sum_{i=1}^{4}\omega_i=1, \omega_i\geq 0,\ i=1,2,3,4, \\ z_{24}^1+z_{24}^2=1, \\ z_{34}^1+z_{34}^2=1, \\ z_{24}^1, z_{24}^2, z_{34}^1, z_{34}^2=0\, or\, 1. \end{cases}.
$$

Then, $\xi_{\min}=0.001$ and $\omega^*=(0.2598, 0.4559, 0.1716, 0.1127)^T$ are obtained by MATLAB, and then, multiplicative consistent FPR can be obtained by Eq. (1) as follows:

$$
R=\begin{bmatrix} 0.5 & 0.36 & 0.6 & 0.7 \\ 0.64 & 0.5 & 0.73 & 0.8 \\ 0.4 & 0.27 & 0.5 & 0.6 \\ 0.3 & 0.2 & 0.4 & 0.5 \end{bmatrix}.
$$

For instance, $r_{13}=\frac{\omega_1}{\omega_1+\omega_3}=\frac{0.2598}{0.2598+0.1716}=0.6$.

Remark 2 Based on the above analysis, it should be pointed out that model (8) could play a dual role: It could help to obtain not only the complete FPR but also multiplicative consistent FPR from a given incomplete HFPR. In this paper, the obtained multiplicative consistent FPR by model (8) is called as reduced HFPR.

4 A novel group decision-making model with incomplete HFPRs

In this section, the experts' weight is calculated based on the experts' consistent level $1-\xi_{i,\min}$, where $\xi_{i,\min}$ is obtained by model (8). Then, a group consensus reaching process algorithm is established on the basis of the defined group consensus degree. Finally, a step-by-step procedure of the GDM model with incomplete HFPRs is constructed.

4.1 Determining experts' weight

Once the reduced HFPRs and consistent degree $1-\xi_{i,\min}$ for expert e_i have been obtained by model (8), their corresponding multiplicative consistent FPRs can also be acquired through Eq. (1).

It is quintessential to find out the DMs' weights in the GDM problems, owing to the fact that the DMs typically possess different preferences, capabilities, and hierarchical ranks. It is rational to conclude that the higher the consistent degree of expert's preference relations is, the larger the weights will be allocated to her/he. Therefore, the weights are assigned to experts by means of the following relation:

$$u_i=\frac{1-\xi_{i,\min}}{\sum_{i=1}^{m}\left(1-\xi_{i,\min}\right)}\ (i=1,2,\ldots,m), \tag{9}$$

where m denotes the quantity of experts.

4.2 Consensus reaching process

Before the group consensus is reached, the minimum consensus thresholds $\overline{\mathrm{GCI}}$ should be determined in advance. While there is no specific rule to find out the consensus thresholds $\overline{\mathrm{GCI}}$, it can normally be found by means of a trial-and-error process (Xu et al. 2013b). In addition, Herrera-Viedma et al. (2005) indicated briefly that the determination of $\overline{\mathrm{GCI}}$ should hang on the particular decision-making problem, i.e., if the decision problem is urgent and has to be resolved rapidly, the smaller thresholds can be adopted; otherwise, the minimum group consensus thresholds are required to be as high as possible.

Definition 4.1 Let $P_k=\left(p_{ij,k}\right)_{n\times n}, k=1,2,\ldots,m$ and $P=\left(p_{ij}\right)_{n\times n}$ be m FPRs and the collective FPR, respectively. Then, group consensus index (GCI) of expert e_k is given as follows:

$$\mathrm{GCI}\left(P_k\right)=1-\frac{2}{n\left(n-1\right)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|p_{ij,k}-p_{ij}\right|, \tag{10}$$

From Eq. (10), it can be seen that if $\mathrm{GCI}\left(P_k\right)=1$, then the individual FPR P_k is consistent with the collective FPR P. In addition, if $\mathrm{GCI}\geq\overline{\mathrm{GCI}}$, then the group reaches a satisfied level of consensus, where $\overline{\mathrm{GCI}}$ is a predetermined threshold of group consensus.

Different methods have been proposed to manage the group consensus reaching process (Herrera-Viedma et al. 2005; Pérez et al. 2014; Xu et al. 2015a, b; Li et al. 2016; Zhang and Guo 2016b). Exactly as some research (Herrera-Viedma et al. 2005; Xu et al. 2015a, b), there exists a bounded rational hypothesis in the consensus reaching process to express preferences of their true ideas. Moreover, they agree to resize their preferences by means of a kind of consensus algorithms in the decision process. In what follows, an automatic iterative algorithm is established based on Ref. (Xu et al. 2013b) as follows:

Algorithm 1 Group consensus reaching process

Input: Individual FPRs $P_1,P_2,\cdots,P_m$, the thresholds $\overline{GCI}$, and the maximum iterations $t_{\max}\geq 1$.
Output: Adjusted FPRs $P_1^{(t)},P_2^{(t)},\cdots,P_m^{(t)}$, $GCI\left(P_k^{(t)}\right)$ $(k=1,2,\cdots,m)$, $P_c^{(t)}$ and the iterations t.
Step 1: Set $t=0$ and $P_k^{(0)}=P_k$ $(k=1,2,\cdots,m)$.
Step 2: Using the optimization model (8) to obtain the $\xi_{k,\min}^{(t)}$ for $P_k^{(t)}$ $(k=1,2,\cdots,m)$ and then get the experts' weights $u^{(t)}=\left(u_1^{(t)},u_2^{(t)},\cdots,u_m^{(t)}\right)^T$ according to Eq. (9).
Step 3: Calculate the collective FPR $P_c^{(t)}=\left(p_{ij,c}^{(t)}\right)_{n\times n}$ corresponding to $P_1^{(t)},P_2^{(t)},\cdots,P_m^{(t)}$, where

$$p_{ij,c}^{(t)}=u_1p_{ij,1}^{(t)}+u_2p_{ij,2}^{(t)}+\cdots+u_mp_{ij,m}^{(t)}\quad, \tag{11}$$

Step 4: Count the $GCI\left(P_k^{(t)}\right)$ $(k=1,2,\cdots,m)$ according to Eq. (10). If $GCI\left(P_k^{(t)}\right)\geq\overline{GCI}$ $(k=1,2,\cdots,m)$ or $t\geq t_{\max}$, then go to Step 6; or it, to find out the FPR $P_k^{(t)}$ satisfying $GCI\left(P_k^{(t)}\right)<\overline{GCI}$ and proceed to the next step.
Step 5: Ascertain the position of the elements $d_{i_\tau j_\tau,k}^{(t)}$ for expert e_k satisfying $GCI\left(P_k^{(t)}\right)<\overline{GCI}$, where $d_{i_\tau j_\tau,k}^{(t)}=\max\limits_{(i,j)}\left|p_{ij,k}^{(t)}-p_{ij,c}^{(t)}\right|$ and then adjust expert e_k's FPR according to Eq. (12). Let $P_k^{(t+1)}=\left(p_{ij,k}^{(t+1)}\right)_{n\times n}$, where

$$p_{ij,k}^{(t+1)}=\begin{cases}p_{ij,c}^{(t)},\ i=i_\tau,\ j=j_\tau\\ p_{ij,k}^{(t)},\ otherwise\end{cases}\quad, \tag{12}$$

Set $t=t+1$ and go to Step 2.
Step 6: Let $\overline{P_k}=P_k^{(t)}$, for all $k=1,2,\cdots,m$. Output $\overline{P_1},\overline{P_2},\cdots,\overline{P_m}$, $GCI\left(\overline{P_k}\right)$ for all $k=1,2,\cdots,m$, $\overline{P_c}=P_c^{(t)}$ and the iterations t.

Remark 3 The thresholds $\overline{\mathrm{GCI}}$ provide a flexible option with the group to command the decision process. Once thresholds $\overline{\mathrm{GCI}}$ are specified, Step 4 is implemented to determine which experts need adjust their opinion and then Step 5 gives a specific scheme to make adjustments. After the expert opinion $P_k^{(t)}$ is adjusted, and the optimization model (8) is reused to determine the new experts' weights with those updated information. By iteratively adjusting the experts' opinion and weights, the consensus level gradually is improved.

4.3 Selection process

The objective of this process is to list all alternatives for selecting the optimal one. The score for alternatives $A_i\ (i = 1, 2, \ldots, n)$ is computed by means of the following equation:

$$S(A_i) = \frac{2}{n(n-1)} \sum_{j=1, j \neq i}^{n} \overline{p}_{ij,c}, \tag{13}$$

where $1 \leq i, j \leq n, i \neq j$ and $\sum_{i=1}^{n} S(A_i) = 1$. If $S(A_1) > S(A_2)$, then $A_1 \succ A_2$; if $S(A_1) = S(A_2)$, then $A_1 = A_2$.

4.4 A step-by-step procedure of the GDM model with incomplete HFPRs

In what follows, a GDM model based on mathematical programming is established with incomplete HFPRs. Assume that there are n alternatives $A_1, A_2, \ldots, A_n$ and experts' weights $u^{(t)} = (u_1, u_2, \ldots, u_m)^T$ related to experts $e_1, e_2, \ldots, e_m$ during iteration. The $H_k\ (k = 1, 2, \ldots, m)$ indicates the original HFPR given by expert $e_k\ (k = 1, 2, \ldots, m)$. The flowchart of proposed GDM model is illustrated in Fig. 1, and its procedure is specifically shown as follows:

Step 1 Every expert just need to give the most confident assessed value(s) between two alternatives by pared comparison method based on their experience and expertise knowledge, where missing comparisons are denoted by "x".

Step 2 The complete multiplicative consistent FPRs $P_k\ (k = 1, 2, \ldots, m)$ and $\xi_{k,\min}\ (k = 1, 2, \ldots, m)$ are obtained from the original incomplete HFPRs $H_k\ (k = 1, 2, \ldots, m)$ by means of model (8), respectively.

Step 3 Compute the original experts' weight $u^{(0)} = (u_1, u_2, \ldots, u_m)^T$ on the basis of $\xi_{k,\min}\ (k = 1, 2, \ldots, m)$ by Eq. (9).

Step 4 Put into effect Algorithm 1 and obtain collective FPR that achieves preset group consensus degree.

Step 5 Compute the score value of alternatives by Eq. (13) and rank alternatives $A_i\ (i = 1, 2, \ldots, n)$.

Step 6 End.

5 Case study and contrastive analysis

5.1 Application to medical decision

In dermatology department, it is a tough choice to determine the best treatment choice for a patient with a severe skin lesion since it does not only rely on the disease (Anstey and Edwards 2014). The opinions, preferences, and circumstances of the patient are the most key elements to determine the most suitable treatment. Otherwise, even apparently successful treatments can miss its target completely (Massanet et al. 2016). In order to share decision-making proposal, it is necessary that academic dermatologists, clinical dermatologists, pharmacists, and dermatology nurses to hold a group consultations to decide the best treatment option, namely a group decision making.

Suppose that a group of three dermatologists $E = \{e_1, e_2, e_3\}$ hesitate for the best treatment, $A = \{A_1 =$ Photodynamic therapy, $A_2 =$ Isotretinoin, $A_3 =$ Large acne cysts removal, $A_4 = Oral\,antibiotics\}$ are applied to a patient with a severe acne disease. As the evaluation of all these factors is a complex issue, it is more suitable to utilize HFPR for describing their preference for experts, which enhances the preciseness and intelligibility of experts' ideas. Hence, the performance values of alternatives $A_i\ (i = 1, 2, 3, 4)$ are provided by three experts using HFPR.

Step 1 Every expert just need to gives his/her the most confident assessed values between two treatment options by paired comparison method, respectively. Then, three incomplete HFPRs $H_k\ (k = 1, 2, 3)$ are constructed as follows:

$$H_1 = \begin{bmatrix} \{0.5\} & \{0.3\} & \{0.4\} & x \\ \{0.7\} & \{0.5\} & \{0.6\} & x \\ \{0.6\} & \{0.4\} & \{0.5\} & \{0.7, 0.8\} \\ x & x & \{0.3, 0.2\} & \{0.5\} \end{bmatrix},$$

$$H_2 = \begin{bmatrix} \{0.5\} & \{0.7\} & x & \{0.5\} \\ \{0.3\} & \{0.5\} & \{0.6, 0.7\} & x \\ x & \{0.4, 0.3\} & \{0.5\} & \{0.4\} \\ \{0.5\} & x & \{0.6\} & \{0.5\} \end{bmatrix},$$

$$H_3 = \begin{bmatrix} \{0.5\} & \{0.3\} & \{0.5\} & \{0.7\} \\ \{0.7\} & \{0.5\} & x & \{0.8\} \\ \{0.5\} & \{0.4\} & \{0.5\} & \{0.7, 0.8\} \\ \{0.3\} & \{0.2\} & \{0.3, 0.2\} & \{0.5\} \end{bmatrix}.$$

Step 2 Employ model (8), the reduced HFPRs $P_k\ (k = 1, 2, 3)$ and corresponding $\xi_{k,\min}\ (k = 1, 2, 3)$ are obtained as follows:

$$P_1 = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.5 & 0.6 & 0.78 \\ 0.6 & 0.4 & 0.5 & 0.7 \\ 0.4 & 0.22 & 0.3 & 0.5 \end{bmatrix} \left(\xi_{1,\min} = 0.0016\right);$$

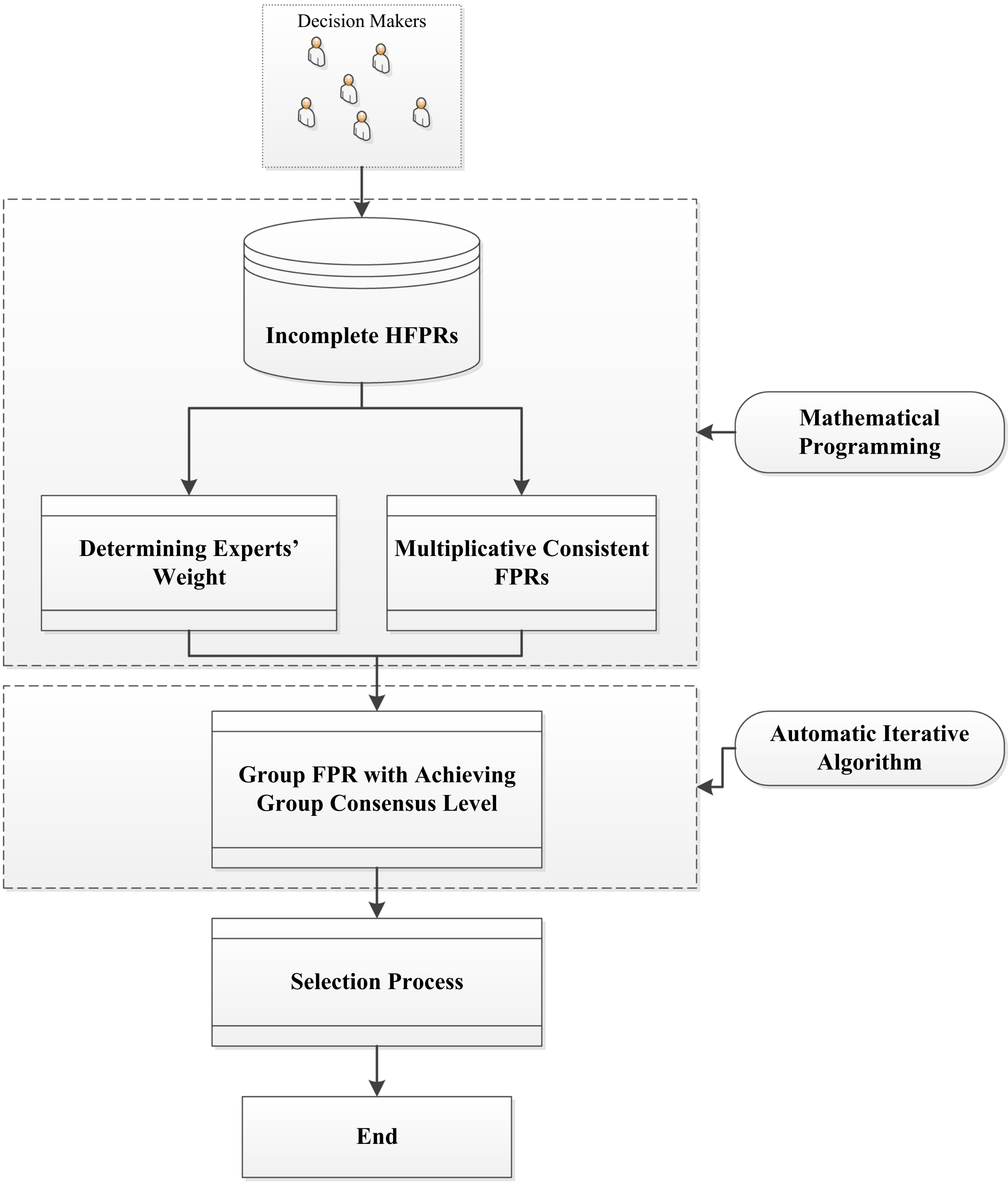

Fig. 1 The framework of GDM with incomplete HFPRs

$$P_2 = \begin{bmatrix} 0.5 & 0.66 & 0.68 & 0.54 \\ 0.34 & 0.5 & 0.53 & 0.38 \\ 0.32 & 0.47 & 0.5 & 0.35 \\ 0.46 & 0.62 & 0.65 & 0.5 \end{bmatrix} \left(\xi_{2,\min} = 0.0238\right);$$

$$P_3 = \begin{bmatrix} 0.5 & 0.31 & 0.48 & 0.67 \\ 0.69 & 0.5 & 0.67 & 0.82 \\ 0.52 & 0.33 & 0.5 & 0.69 \\ 0.33 & 0.18 & 0.31 & 0.5 \end{bmatrix} \left(\xi_{3,\min} = 0.0092\right).$$

Step 3 Based on the obtained $\xi_{k,\min}$ $(k = 1, 2, 3)$ from Step 2, we secure the original experts' weight $u^{(0)} = (0.337, 0.329, 0.334)^T$ by Eq. (9).

Step 4 Reaching the predefined group consensus level.

On the basis of the experts' weight $u^{(0)} = (0.337, 0.329, 0.334)^T$, the original group FPR is shown in Table 1 and original consensus index of each expert is as follows:

$$\mathrm{GCI}\left(P_1^{(0)}\right) = 0.9138, \mathrm{GCI}\left(P_2^{(0)}\right) = 0.8255,$$

Table 1 The original group FPR

	A_1	A_2	A_3	A_4
A_1	0.5	0.422	0.519	0.604
A_2	0.578	0.5	0.6	0.662
A_3	0.481	0.4	0.5	0.582
A_4	0.396	0.338	0.418	0.5

Table 3 The final group FPR

	A_1	A_2	A_3	A_4
A_1	0.5	0.467	0.555	0.603
A_2	0.533	0.5	0.6	0.702
A_3	0.445	0.4	0.5	0.682
A_4	0.397	0.298	0.318	0.5

$$\mathrm{GCI}\left(P_3^{(0)}\right) = 0.9078.$$

According to the practical problems, the experts agree to set up $\overline{\mathrm{GCI}} = 0.95$. Then, Algorithm 1 is applied to adjust the original FPRs. Since $\mathrm{GCI}\left(P_k^{(0)}\right) < \overline{\mathrm{GCI}}\ (k = 1, 2, 3)$, we need to find the position of elements $d_{i_\tau j_\tau,k}^{(0)}\ (k = 1, 2, 3)$, where $d_{i_\tau j_\tau,k}^{(t)} = \max_{(i,j)} \left|p_{ij,k}^{(t)} - p_{ij,c}^{(t)}\right|$. With regard to $P_1^{(0)}$, we obtain $d_{12,1}^{(0)} = d_{21,1}^{(0)} = \max_{(i,j)} \left|p_{ij,1}^{(0)} - p_{ij,c}^{(0)}\right| = 0.122$; hence, according to Eq. (12), $d_{12,1}^{(0)}, d_{21,1}^{(0)}$ need to be adjusted to $d_{12,1}^{(0)} = 0.422, d_{21,1}^{(0)} = 0.578$. Similarly, the same procedure is implemented to adjust the other two experts' FPRs

$$d_{24,2}^{(0)} = 0.662, d_{42,2}^{(0)} = 0.338;$$
$$d_{24,3}^{(0)} = 0.662, d_{42,3}^{(0)} = 0.338.$$

Let $t = 1$, then go to Step 2.

Going through 5 rounds of adjustment, Algorithm 1 terminated. On the whole, e_1, e_2, and e_3 adjust their FPRs 2, 5, and 3 times, respectively. The comparison of experts' FPRs before and after adjustment is shown in Table 2. Furthermore, by means of Eq. (10), the final group consensus index of every expert is:

$$\mathrm{GCI}\left(P_1^{(2)}\right) = 0.97, \mathrm{GCI}\left(P_2^{(5)}\right) = 0.9542,$$
$$\mathrm{GCI}\left(P_3^{(3)}\right) = 0.9673.$$

The final group FPR $\overline{P_c}$ is presented in Table 3.

Step 5 Calculate the score of alternatives $A_i\ (i = 1, 2, 3, 4)$ based on the final group FPR $\bar{P}_c$ by Eq. (13) as follows:

$$S(A_1) = 0.2708;\ S(A_2) = 0.3059;\ S(A_3) = 0.2545;$$
$$S(A_4) = 0.1688.$$

Hence, the ranking of alternatives is $A_2 \succ A_1 \succ A_3 \succ A_4$.

According to this ranking, the isotretinoin (A_2) is recommended as the best treatment plan for this patient among the considered ones.

5.2 Comparison with other relevant approaches

In what follows, we will compare our method with the two previous studies about addressing the GDM problems with incomplete HFPRs.

The comparison results are shown in Table 4 and Fig. 2. As can be seen from Table 4, we can see that the above three methods get the same conclusion that the best alternative is A_2. The same ranking orders are obtained between our method and the Zhang et al. (2015a)'s method, while different ranking orders are obtained between our method and Xu et al. (2016)'s method. Moreover, Fig. 2 intuitively shows the ranking values for different methods.

In short, the main contributions of the paper can be summed up as follows:

Table 2 The experts' original FPRs and final improved FPRs

Experts	Original FPRs	Final improved FPRs
Expert e_1	$P_1^{(0)} = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.5 & 0.6 & 0.78 \\ 0.6 & 0.4 & 0.5 & 0.7 \\ 0.4 & 0.22 & 0.3 & 0.5 \end{bmatrix}$	$P_1^{(2)} = \begin{bmatrix} 0.5 & 0.422 & 0.519 & 0.6 \\ 0.578 & 0.5 & 0.6 & 0.78 \\ 0.481 & 0.4 & 0.5 & 0.7 \\ 0.4 & 0.22 & 0.3 & 0.5 \end{bmatrix}$
Expert e_2	$P_2^{(0)} = \begin{bmatrix} 0.5 & 0.66 & 0.68 & 0.54 \\ 0.34 & 0.5 & 0.53 & 0.38 \\ 0.32 & 0.47 & 0.5 & 0.35 \\ 0.46 & 0.62 & 0.65 & 0.5 \end{bmatrix}$	$P_2^{(5)} = \begin{bmatrix} 0.5 & 0.516 & 0.583 & 0.54 \\ 0.484 & 0.5 & 0.53 & 0.662 \\ 0.417 & 0.47 & 0.5 & 0.657 \\ 0.46 & 0.338 & 0.343 & 0.5 \end{bmatrix}$
Expert e_3	$P_3^{(0)} = \begin{bmatrix} 0.5 & 0.31 & 0.48 & 0.67 \\ 0.69 & 0.5 & 0.67 & 0.82 \\ 0.52 & 0.33 & 0.5 & 0.69 \\ 0.33 & 0.18 & 0.31 & 0.5 \end{bmatrix}$	$P_3^{(3)} = \begin{bmatrix} 0.5 & 0.463 & 0.562 & 0.67 \\ 0.537 & 0.5 & 0.67 & 0.662 \\ 0.438 & 0.33 & 0.5 & 0.69 \\ 0.33 & 0.338 & 0.31 & 0.5 \end{bmatrix}$

Table 4 Comparison of ranking between the proposed method and previous studies

	Considering how to determine DMs' weights	Considering group consensus	Ranking values for alternatives	Ranking
Our method	Yes	Yes	$(0.2708, 0.3059, 0.2545, 0.1688)^T$	$A_2 \succ A_1 \succ A_3 \succ A_4$
Xu et al. (2016)	No	No	$(0.1854, 0.4172, 0.2781, 0.1192)^T$	$A_2 \succ A_3 \succ A_1 \succ A_4$
Zhang et al. (2015a)	No	No	$(0.2591, 0.2953, 0.2505, 0.1952)^T$	$A_2 \succ A_1 \succ A_3 \succ A_4$

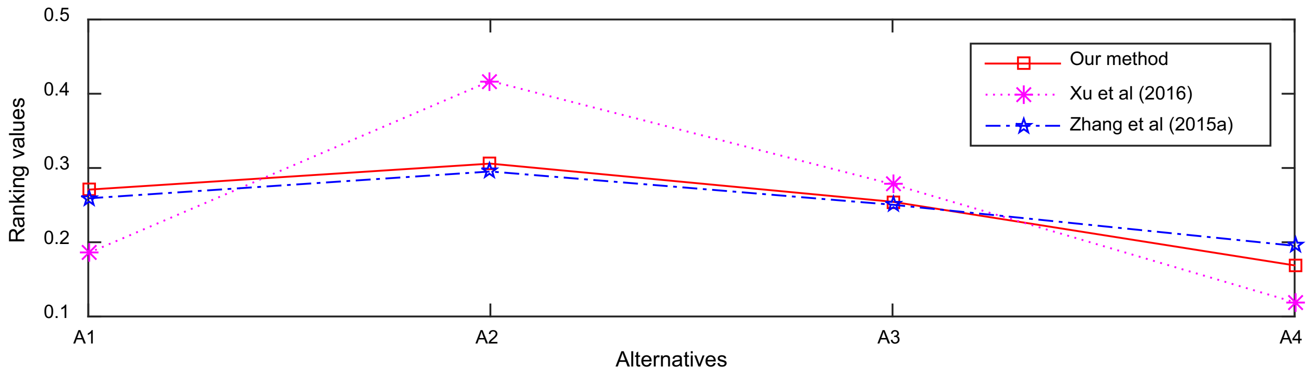

Fig. 2 Ranking values with different methods

- The multiplicative consistent complete FPRs are extracted from given incomplete HFPRs by the proposed mathematical programming approach, in contrast to normalization process used by Ref (Zhang et al. 2015a), which may deviate the original information of experts (Rodríguez et al. 2016).
- Consensus reaching process is considered by the proposed model, while the methods (Zhang et al. 2015a; Xu et al. 2016) do not consider the group consensus reaching process, which makes our model more reliable than the existing methods (Zhang et al. 2015a; Xu et al. 2016).
- The experts' weight is obtained on the basis of consistent level of experts, which is more tallied with the actual situation as different experts have different professional knowledge and background.

6 Conclusions

In this paper, a mathematical programming method has been presented to extract the highest consistent incomplete FPR from all possible FPRs based on a given HFPR and simultaneously improve the highest consistent incomplete FPR to complete multiplicative consistent FPR, namely a process of regression. Moreover, a step-by-step GDM procedure based on the obtained multiplicative consistent FPRs has been concluded. Besides, the weights of the experts are produced on the basis of consistent degree of experts. It is rational to conclude that experts with higher consistency need to be allocated with higher weights, and thus, their corresponding ideas bear more weight during the aggregation process. With the procedure, a medical decision problem is worked out by means of the proposed model.

Furthermore, the proposed model also can be applied to different problems like investment decision making, supplier selection, and decision support system.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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