Soft Computing (2019) 23:6231–6247
https://doi.org/10.1007/s00500-018-3281-z

METHODOLOGIES AND APPLICATION

Multi-period mean–semivariance portfolio optimization based on uncertain measure

Wei Chen[1] · Dandan Li[1] · Shan Lu[1] · Weiyi Liu[2]

Published online: 2 June 2018
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Abstract
In this paper, we discuss a multi-period portfolio selection problem when security returns are given by experts' estimations. By considering the security returns as uncertain variables, we propose a multi-period mean–semivariance portfolio optimization model with real-world constraints, in which transaction costs, cardinality and bounding constraints are considered. Furthermore, we provide an equivalent deterministic form of mean–semivariance model under the assumption that the security returns are zigzag uncertain variables. After that, a modified imperialist competitive algorithm is developed to solve the corresponding optimization problem. Finally, a numerical example is given to illustrate the effectiveness of the proposed model and the corresponding algorithm.

Keywords Multi-period portfolio optimization · Uncertain variable · Semivariance · Cardinality constraint · Imperialist competitive algorithm

1 Introduction

Portfolio selection discusses the problem of how to allocate a certain amount of investor's wealth among different assets and from a satisfying portfolio. The mean–variance (M–V) model formulated by Markowitz (1952) lays the basis for single-period portfolio selection. It combines probability theory with optimization techniques to model the investment behavior with some uncertainties. Quantifying investment return as the mean of returns, and investment risk as the variance from the mean, Markowitz formulated his models mathematically in two ways: minimizing variance for a given expected value, or maximizing expected value for a given variance. Since then, variance is widely used as a risk measure, and the mean–variance models are well investigated such as Yoshimoto (1996), Best and Hlouskova (2000), Liu et al. (2003), Corazza and Favaretto (2007), etc. However, as pointed out by Grootveld and Hallerbach (1999), the distinguished drawback is that variance treats high returns as equally undesirable as low returns because high returns will also contribute to the extreme of variance. In particular, when probability distributions of security returns are asymmetric, variance becomes a deficient measure of investment risk because it may have a potential danger to sacrifice too much expected return in eliminating both low and high return extremes. To overcome this disadvantage, semivariance was proposed to replace variance in portfolio selection. Semivariance only measures the variability of returns below the mean and gauges no variability of returns above the mean and thus better matches investors' intuition of risk than the variance. Afterward, several researchers have employed the semivariance as risk measure to study portfolio selection problems in random environment, such as Markowitz (1959, 1993), Hogan and Warren (1974), Choobineh and Branting (1986), Ballestero (2005).

Note that all the models mentioned above are single-period portfolio selection models, which provide an one-off decision at the beginning of the investment period and suggest to

Communicated by V. Loia.

✉ Wei Chen
chenwei@cueb.edu.cn

Dandan Li
ldd@cueb.edu.cn

Shan Lu
lushan@cueb.edu.cn

Weiyi Liu
liuweiyi@cueb.edu.cn

[1] School of Information, Capital University of Economics and Business, Beijing, China

[2] School of Finance, Capital University of Economics and Business, Beijing, China

hold it until the end of the investment period. But, in the real world, investors often adjust their wealth from time to time by taking into consideration the volatile market conditions. Thus, a lot of work has been done to extend portfolio selection from single-period case to multi-period case by using different approaches. Mossin (1968) presented an optimal multi-period portfolio polices by using dynamic programming approach. Dantzig and Infanger (1993) demonstrated how multi-period portfolio optimization problems can be efficiently solved as multistage stochastic linear programs. Li and Ng (2000) presented a multi-period mean–variance model and derived an analytical optimal solution. Zhu et al. (2004) developed a generalized mean–variance multi-period portfolio model incorporating bankruptcy. Wei and Ye (2007) developed a multi-period mean–variance model for portfolio selection under bankruptcy risk control. Geyer et al. (2009) used stochastic linear programming approach to study the multi-period portfolio optimization. Yan and Li (2009) proposed a class of multi-period semivariance portfolio selection with a four-factor futures price model. Brieca and Kerstens (2009) presented a general approach for multi-horizon mean–variance portfolio analysis. Fu et al. (2010) considered a continuous-time mean–variance model with borrowing constraint in portfolio selection. Zeng et al. (2013) studied a multi-period investment-consumption problem with state-dependent utility functions and uncertain time-horizon in a regime-switching market. Sun et al. (2016) developed a minimax model for a multi-period portfolio selection problem. Yao et al. (2016) studied multi-period mean–variance portfolio selection with stochastic interest rate and uncontrollable liability.

All the above literature assumes that the security returns are random variables with probability distributions. As we know, a premise of applying probability theory is that the obtained probability distribution is close enough to the true frequency. That is to say, we should have enough samples to estimate probability distributions of the security returns. However, in real financial markets, there are situations where people have none or not enough historical data. Thus, in such situations, the predictions of the security returns have to rely on experts' estimations rather than historical data. With the widely use of fuzzy set theory, more and more researchers have studied fuzzy portfolio selection problems, see, for example, Carlsson et al. (2002), Abiyev and Menekay (2007), Zhang et al. (2009), Chen (2015), Zhang (2016), Liu et al. (2016a), Chen et al. (2016), Mehlawat (2016), Wang et al. (2017). These researches broadened the way to handle portfolio selection problems with returns given by experts' estimations. However, deeper researches find that paradoxes will appear when fuzzy variable is used to describe the security returns (see Liu 2012). To better deal with this subjective uncertainty, an uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010) in 2010. By using uncertainty theory, no counterintuitive results occur. So far, the uncertainty theory has been used to handle many optimization problems, such as the shortest path (Gao 2011), single-period inventory (Qin and Kar 2013), facility location-allocation (Wen et al. 2014), and option pricing (Shen and Yao 2016; Sun et al. 2017). In particular, some researchers have applied the uncertainty theory for portfolio selection problems. In an early study, Huang (2012) proposed a risk index model for uncertain portfolio selection. Li and Qin (2014) formulated a mean-semiabsolute deviation model for portfolio selection under assumption that security returns are uncertain interval variables. Zhang et al. (2015) proposed two uncertain portfolio selection models, that is, an expected-variance-chance model and a chance-expected-variance model. Qin et al. (2016) formulated an uncertain portfolio rebalancing model with transaction costs by using semiabsolute deviation to measure the portfolio risk. Chen et al. (2017a) proposed two uncertain diversified mean–semivariance models for portfolio selection. Chen et al. (2017b) formulated an uncertain multi-objective mean–variance–skewness–kurtosis portfolio optimization model. Recently, Zhai and Bai (2018) presented an uncertain mean-risk model with background risk for portfolio selection. Chen et al. (2018) proposed an uncertain portfolio selection model under consideration of the skewness, transaction costs, cardinality and minimum transaction lots constraints. Apart from the above-mentioned studies, there exist few studies on multi-period uncertain portfolio selection problems. Huang and Qiao (2012) developed a multi-period uncertain mean-risk index model. Li et al. (2018) proposed an uncertain multi-period portfolio selection model, in which transaction cost and bankruptcy of investor are considered.

Imperialist competitive algorithm (ICA) is a new population-based meta-heuristic algorithm proposed by Atashpaz-Gargari and Lucas (2007) in 2007. This novel optimization method was developed based on a socio-politically motivated strategy. The ICA uses an initial population that consists of colonies and imperialists that are assigned to several empires. The empires then compete with each other, which cause the weak empires to collapse and the powerful empires to dominate and overtake their colonies. The ICA has shown potential superiority to other well-known optimization methods; for example, Atashpaz-Gargari and Lucas (2007) showed the ICA has potential superiority to other well-known optimization techniques, such as the genetic algorithm (GA) and particle swarm optimization (PSO) in terms of its convergence rate and global optima achievement; Towsyfyan et al. (2013) also demonstrated that ICA has considerable improvement over GA in finding the optimum solution in less computational time with the same population size and iterations; Niknam et al. (2011) mixed ICA with a K-means algorithm to show its advantages over GA, PSO, ant colony optimization (ACO), simulated annealing

(SA), Tabu search (TS), and honeybee mating optimization (HBMO) algorithms. Therefore, ICA has aroused much interest and has been successfully used in different fields, such as traveling salesman (Xu et al. 2014), job-shop scheduling (Zandieha et al. 2017), economic load dispatch (Morshed and Asgharpour 2014), feature selection (MousaviRad et al. 2012), distribution network design (Ghorbani and Akbari Jokar 2016) problems. A comprehensive survey of ICA is presented in the studies of Hosseini and Al Khaled (2014), Sharafi et al. (2016).

Though there are some researches about portfolio selection problems under the framework of uncertainty theory, most of them are mainly focused on single-period portfolio selections. To the best of the author's knowledge, there is few research on constructing multi-period uncertain portfolio selection model by using semivariance as risk measurement under the real-world constraints, and then solving the corresponding model by ICA algorithm. Therefore, the purpose of this paper is to present a multi-period mean–semivariance model for uncertain portfolio selection with real-world constraints and to develop an efficient heuristic approach based on ICA algorithm for solving proposed model. Our contributions can be summarized under two aspects as follows: (1) We formulate a new multi-period mean–semivariance portfolio optimization model including transaction costs, cardinality and bounding constraints, in which the security returns are the uncertain variables whose values are obtained from experts' evaluations rather than historical data. (2) Considering the complexity of the proposed model, we design a MICA algorithm for the solution, in which the GA's crossover and mutation operators are integrated into the ICA by introducing a new type of country called 'independent' country, and a dynamic switching revolution strategy is also proposed.

The rest of the paper is organized as follows. In Sect. 2, we review some basic concepts and results of the uncertainty theory. In Sect. 3, we present a multi-period mean–semivariance model for uncertain portfolio selection and then give an equivalent of the model when security returns are zigzag uncertain variables. In Sect. 4, we design a MICA algorithm to solve the proposed model. After that, an example is given to illustrate the effectiveness of the proposed model and algorithm in Sect. 5. Finally, we present conclusions and future research directions in Sect. 6.

2 Preliminaries

Uncertainty theory is developed based on the following four axioms.

Definition 1 (Liu 2007) Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra on Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2: (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\}=1$ for any event $\Lambda \in \mathcal{L}$;
Axiom 3: (Subadditivity) For every sequence $\{\Lambda_i\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Moreover, a product axiom was given by Liu (2009) for the operation of uncertain variables.
Axiom 4: (Product) Let $(\Gamma_k, \mathcal{L}, \mathcal{M})$ be the uncertainty spaces for $k = 1, 2, \ldots$. Now, the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}$ for $k = 1, 2, \ldots$, respectively.

Definition 2 (Liu 2007) Let Γ be a nonempty set, $\mathcal{L}$ the $\sigma-$algebra over Γ, and $\mathcal{M}$ be an uncertain measure. Then, the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space.

Definition 3 (Liu 2007) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain variable ξ is a measurable function from Γ to a set of real numbers; that is, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in \mathrm{B}\}$$

defines an event.

Definition 4 (Liu 2007) The uncertainty distribution $\Phi : R \to [0, 1]$ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \le x\}.$$

An uncertainty distribution Φ is called regular if it has an inverse function Φ^{-1}. In this case, the inverse function Φ^{-1} is called an inverse uncertainty distribution.

Definition 5 (Liu 2007) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n} \xi_i\right\} = \min_{1\le i\le n} M\{\xi_i \in B_i\},$$

for Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 1 (Liu 2007) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be the independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly* *decreasing with respect to* $x_{m+1}, x_{m+2}, \ldots, x_n$ *then* $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ *is an uncertain variable with an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)). \quad (1)$$

Definition 6 (Liu 2007) Let ξ be an uncertain variable; then the expected value of ξ is given by

$$E(\xi) = \int_0^{+\infty} M\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} M\{\xi \leq r\}\mathrm{d}r \quad (2)$$

provided that at least one of the two integrals is finite.

Since the uncertain variables operate mainly in the form of inverse uncertainty distributions, Liu (2010) presented the following formulas to calculate the expected value of an uncertain variable via inverse uncertainty distributions.

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \quad (3)$$

Definition 7 (Liu 2007) Let ξ be an uncertain variable with finite expected value e. Then, the variance of ξ is, respectively, given by

$$V[\xi] = E\{(\xi - e)^2\}. \quad (4)$$

Theorem 2 (Liu 2010) *Let* ξ *and* η *be independent uncertain variables. Then, for any real numbers a and b we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \quad (5)$$

3 The multi-period uncertain portfolio optimization model

As discussed earlier, in many situations, historical data are insufficient for the estimation of security returns. In such cases, we have to rely on the domain experts' subjective evaluations for estimating the security returns. In this section, by regarding security returns as the uncertain variables, we will propose a multi-period mean–semivariance model for portfolio selection with real-world constraints.

3.1 Mathematical modeling

Assume that an investor allocates his initial wealth W_0 among the n securities at the beginning of period 1, and obtain the terminal wealth at the end of period T. The investor could readjust his wealth at the every beginning of the following $T-1$ consecutive time periods. For the better understanding of our paper, we first introduce all the notations that will be used in the following sections. For $i = 1, 2, \ldots, n$, and $t = 1, 2, \ldots, T$, we let

$r_{t,i}$	The uncertain return rate of ith security at period t;
r_f	The risk-free interest rate;
$d_{t,i}$	The transaction cost on security i at period t;
W_t	The available wealth at the end of period t;
ΔW_t	The increment wealth between the period t and $t-1$;
$\varepsilon_{t,i}$	The minimum proportion allocated to security i at period t, if security i is held;
$\delta_{t,i}$	The maximum proportion allocated to security i at period t, if security i is held;
m_t	The number of securities that investors wish to hold in the portfolio at period t, $1 \leq m_t \leq n$;
$z_{t,i}$	The binary variable, $z_{t,i} \in \{0, 1\}$. If security i is included in the portfolio at period t, $z_{t,i} = 1$, and $z_{t,i} = 0$ otherwise.

Transaction cost is an important factor for an investor to take into consideration in portfolio selection. It is not trivial enough to be neglected, and the optimal portfolio depends on the total costs of transaction. Similar to the researches in Chen (2015); Chen et al. (2016) and others, we assume that the transaction cost is a V-shaped function of the differences between portfolio x_t at period t and the portfolio x_{t-1} at period $t-1$. Hence, the total transaction cost of the portfolio x_t at period t can be represented by

$$D_t = \sum_{i=1}^{n} d_{t,i}|x_{t,i} - x_{t-1,i}|, \quad t = 1, 2, \ldots, T. \quad (6)$$

Thus, the net return rate of the portfolio x_t at period $t (t = 1, 2, \ldots, T)$ after paying transaction costs is obtained as

$$R_{t,N} = \sum_{i=1}^{n} \left(r_{t,i}x_{t,i} - d_{t,i}|x_{t,i} - x_{t-1,i}| \right), \quad t = 1, 2, \ldots, T. \quad (7)$$

Furthermore, the wealth at the end of period $t+1$ can be expressed as

$$W_{t+1} = W_t(1 + R_{t,N}) = W_t\left[1 + \sum_{i=1}^{n} \left(r_{t,i}x_{t,i} - d_{t,i}|x_{t,i} - x_{t-1,i}| \right)\right]. \quad (8)$$

Solving Eq. (8), recursively, the terminal wealth obtained at the end of period T can be expressed by

$$W_T = W_0 \prod_{t=1}^{\mathrm{T}} \left[1 + \sum_{i=1}^{n} \left(r_{t,i} x_{t,i} - d_{t,i} |x_{t,i} - x_{t-1,i}| \right) \right]. \quad (9)$$

Besides the transaction costs, investors commonly face other real-world constraints such as cardinality and bounding constraints. The cardinality constraint imposes a limit on the number of assets in the portfolio either to simplify the management of the portfolio or to reduce transaction costs. The bounding constraint restricts the proportion of each asset in the portfolio to lie between the lower and upper bounds in order to avoid very small (or large) and unrealistic holdings. Specifically, a minimum $\varepsilon_{t,i}$ and a maximum $\delta_{t,i}$ for each asset i at period t are given, and we impose that either $x_{t,i} = 0$ or $\varepsilon_{t,i} \leq x_{t,i} \leq \delta_{t,i}$. In addition, the cardinality constraints about the portfolio at period t can be expressed by

$$\sum_{i=1}^{n} z_{t,i} = m_t, \quad t = 1, 2, \ldots, T, \quad (10)$$

where $z_{t,i} \in \{0, 1\}$ is a binary variable that controls whether asset i at period t should be selected in the portfolio or not.

Taking above facts into consideration, we assume that the investor wants to maximize his/her terminal wealth over the T period. At the same time, the cumulative investment risk of the portfolios must be smaller than the given maximum risk tolerance level. Thus, the multi-period uncertain portfolio selection problem with real-world constraints can be formulated as the follows:

$$\begin{aligned} \max \quad & E[W_T] \\ \text{s.t.} \quad & \sum_{t=1}^{\mathrm{T}} V[W_t] \leq \sigma, \\ & \sum_{i=1}^{n} x_{t,i} = 1, \\ & \sum_{i=1}^{n} z_{t,i} = m_t, \\ & \varepsilon_{t,i} z_{t,i} \leq x_{t,i} \leq \delta_{t,i} z_{t,i}, \\ & z_{t,i} \in \{0, 1\}, \\ & x_{t,i} \geq 0, \\ & i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T, \end{aligned} \quad (11)$$

where σ is the preset maximum risk tolerance level.

In the model (11), variance is used as risk measure. However, as discussed in introduction section, we know that when return distributions of securities are asymmetric, using variance as risk measure leads to an unsatisfactory prediction of portfolio behavior. Thus, some researchers began to employ semivariance as an alternative risk measure to qualify the risk of portfolio, see, for example, Markowitz (1959), Liu and Zhang (2015). Especially, Huang (2012) introduced a new easier-to-use risk measure for single-period portfolio selection, where risk-free interest rate is set as a base target and any portfolio returns below the risk-free interest rate are regarded as losses. In this paper, the semivariance of the return rate on the portfolio at period t can be defined as follows:

Definition 8 Let W_0 be the initial wealth, and r_{f} the risk-free interest rate. Then, the semivariance of the return rate on the portfolio at period t is defined as

$$\mathrm{SV}_t = E\left[\left[(\Delta W_t - W_0 \cdot r_{\mathrm{f}})^- \right]^2 \right], \quad (12)$$

where

$$(\Delta W_t - W_0 \cdot r_{\mathrm{f}})^- = \begin{cases} \Delta W_t - W_0 \cdot r_{\mathrm{f}}, & \text{if } \Delta W_t \leq W_0 \cdot r_{\mathrm{f}}, \\ 0. & \text{if } \Delta W_t > W_0 \cdot r_{\mathrm{f}}. \end{cases}$$

Now, we assume that the investor adopts semivariance as risk measure. Then, the multi-period mean–semivariance model for uncertain portfolio selection can be formulated as follows:

$$\begin{aligned} \max \quad & E[W_T] \\ \text{s.t.} \quad & \sum_{t=1}^{\mathrm{T}} \mathrm{SV}[W_t] \leq \sigma, \\ & \sum_{i=1}^{n} x_{t,i} = 1, \\ & \sum_{i=1}^{n} z_{t,i} = m_t, \\ & \varepsilon_{t,i} z_{t,i} \leq x_{t,i} \leq \delta_{t,i} z_{t,i}, \\ & z_{t,i} \in \{0, 1\}, \\ & x_{t,i} \geq 0, \\ & i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T. \end{aligned} \quad (13)$$

3.2 Equivalent of the proposed model

In order to solve the proposed model (13), we need to convert it into its equivalent. Before giving the crisp form, we first give a theorem for calculating the semivariance.

Theorem 3 *Let ξ be an uncertain variable with continuous uncertainty distribution Φ whose inverse function $\Phi^{-1}(\alpha)$*

exists and is unique for each $\alpha \in (0,1)$*, and semivariance defined as*

$$\mathrm{SV}(\xi) = E\left[[(\xi - W_0 \cdot r_{\mathrm{f}})^-]^2\right].$$

Then

$$\mathrm{SV}(\xi) = \int_0^\beta (\Phi^{-1}(\alpha) - W_0 \cdot r_{\mathrm{f}})^2 \mathrm{d}\alpha, \tag{14}$$

where $\Phi^{-1}(\beta) = W_0 \cdot r_{\mathrm{f}}$.

Proof According to Definitions 6 and 8, we have

$$\begin{aligned}\mathrm{SV}(\xi) &= \int_0^{+\infty} M\{(\xi - W_0 \cdot r_{\mathrm{f}})^2 \geq r\}\mathrm{d}r \\ &= \int_0^{+\infty} M\{\xi \leq W_0 \cdot r_{\mathrm{f}} - \sqrt{r}\}\mathrm{d}r \\ &= \int_{-\infty}^{W_0 \cdot r_{\mathrm{f}}} 2(W_0 \cdot r_{\mathrm{f}} - r)\Phi(r)\mathrm{d}r.\end{aligned}$$

□

The theorem is proved.

In the following section, we will convert the proposed model (13) into its equivalent. With the same consideration as Chen et al. (2018) and Qin et al. (2016), we assume that the returns of security $r_{t,i}$ are zigzag uncertain variables for all $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$, denoted by $r_{t,i} = \mathcal{Z}(a_{t,i}, b_{t,i}, c_{t,i})$ whose uncertainty distribution can be described by

$$\Phi(r_{t,i}) = \begin{cases} 0, & r_{t,i} \leq a_{t,i}, \\ (r_{t,i} - a_{t,i})/2(b_{t,i} - a_{t,i}), & a_{t,i} \leq r_{t,i} \leq b_{t,i}, \\ (r_{t,i} + c_{t,i} - 2b_{t,i})/2(c_{t,i} - b_{t,i}), & b_{t,i} \leq r_{t,i} \leq c_{t,i}, \\ 1, & r_i \geq c_{t,i}, \end{cases}$$

where $a_{t,i} < b_{t,i} < c_{t,i}$.

Furthermore, the inverse uncertainty distributions of zigzag uncertain variables $r_{t,i} (i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T)$ are

$$\Phi^{-1}(\alpha_{t,i}) = \begin{cases} (1 - 2\alpha_{t,i})a_{t,i} + 2\alpha_{t,i} b_{t,i}, & 0 < \alpha_{t,i} < 0.5, \\ (2 - 2\alpha_{t,i})b_{t,i} + (2\alpha_{t,i} - 1)c_{t,i}, & 0.5 \leq \alpha_{t,i} < 1. \end{cases} \tag{15}$$

Then, from Eq. (3), we have

$$E[r_{t,i}] = \frac{a_{t,i} + 2b_{t,i} + c_{t,i}}{4}. \tag{16}$$

According to the results in Liu (2007), the sum of two independent zigzag uncertain variables $\xi_1 = \mathcal{Z}(a_1, b_1, c_1)$ and $\xi_2 = \mathcal{Z}(a_2, b_2, c_2)$ is also zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, i.e., $\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. In addition, the product of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ and a real number $k > 0$ is also a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$, i.e., $k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc)$.

Therefore, for any real numbers $x_{t,i} \geq 0\ (i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T)$, the total uncertain return at period t is still a zigzag uncertain variable with the following form:

$$\begin{aligned}\sum_{i=1}^n r_{t,i} x_{t,i} &= \sum_{i=1}^n (a_{t,i} x_{t,i}, b_{t,i} x_{t,i}, c_{t,i} x_{t,}) \\ &= \mathcal{Z}\left(\sum_{i=1}^n a_{t,i} x_{t,i}, \sum_{i=1}^n b_{t,i} x_{t,i}, \sum_{i=1}^n c_{t,i} x_{t,i}\right).\end{aligned} \tag{17}$$

Thus, we easily obtain the expected total return at period t as follows,

$$E\left(\sum_{i=1}^n r_{t,i} x_{t,i}\right) = \frac{1}{4}\left(\sum_{i=1}^n a_{t,i} x_{t,i} + 2\sum_{i=1}^n b_{t,i} x_{t,i} + \sum_{i=1}^n c_{t,i} x_{t,i}\right). \tag{18}$$

Furthermore, according to Eq. (9), we can obtain the final expected wealth at the end of period T as follows:

$$\begin{aligned}E[W_T] &= W_0 \prod_{t=1}^{\mathrm{T}} \left[1 + \sum_{i=1}^n \left(x_{t,i} E(r_{t,i}) - d_{t,i}|x_{t,i} - x_{(t-1),i}|\right)\right] \\ &= W_0 \prod_{t=1}^{\mathrm{T}} \left[1 + \frac{1}{4}\left(\sum_{i=1}^n a_{t,i} x_{t,i} + 2\sum_{i=1}^n b_{t,i} x_{t,i} + \sum_{i=1}^n c_{t,i} x_{t,i}\right) \right. \\ &\quad \left. - d_{t,i}|x_{t,i} - x_{(t-1),i}|\Big)\right].\end{aligned} \tag{19}$$

Moreover, for the zigzag uncertain variables $r_{t,i} = \mathcal{Z}(a_{t,i}, b_{t,i}, c_{t,i})$, according to Eqs. (14) and (15), we have following results.

(i) If $\beta < 0.5$, then

$$\begin{aligned}\mathrm{SV}(r_{t,i}) &= \int_0^\beta (\Phi^{-1}(\alpha) - W_0 \cdot r_{\mathrm{f}})^2 \mathrm{d}\alpha \\ &= \int_0^\beta [(1 - 2\alpha)a_{t,i} + 2\alpha b_{t,i} - W_0 \cdot r_{\mathrm{f}}]^2 \mathrm{d}\alpha \\ &= \int_0^\beta [(2b_{t,i} - 2a_{t,i})^2\alpha^2 + 2(2b_{t,i} - 2a_{t,i})(a_{t,i} \\ &\quad - W_0 \cdot r_{\mathrm{f}})\alpha + (a_{t,i} - W_0 \cdot r_{\mathrm{f}})^2]\mathrm{d}\alpha \\ &= \frac{1}{3}\Big[4(b_{t,i} - a_{t,i})^2\beta^3 + 6(b_{t,i} - a_{t,i})(a_{t,i} - W_0 \cdot r_{\mathrm{f}})\beta^2 \\ &\quad + 3(a_{t,i} - W_0 \cdot r_{\mathrm{f}})^2\beta\Big]\end{aligned} \tag{20}$$

(ii) If $\beta \geq 0.5$, then

$$
\begin{aligned}
\mathrm{SV}(r_{t,i}) &= \int_0^{\beta} (\Phi^{-1}(\alpha) - W_0 \cdot r_{\mathrm{f}})^2 \mathrm{d}\alpha \\
&= \int_0^{0.5} [(1-2\alpha)a + 2\alpha b - W_0 \cdot r_{\mathrm{f}}]^2 \mathrm{d}\alpha \\
&+ \int_{0.5}^{\beta} [(2-2\alpha)b + (2\alpha - 1)c - W_0 \cdot r_{\mathrm{f}}]^2 \mathrm{d}\alpha \\
&= \frac{4}{3}(c_{t,i} - b_{t,i})^2 \beta^3 + 2(c_{t,i} - b_{t,i})(2b_{t,i} \\
&- c_{t,i} - W_0 \cdot r_{\mathrm{f}})\beta^2 + (2b_{t,i} - c_{t,i} - W_0 \cdot r_{\mathrm{f}})^2 \beta \\
&- \frac{1}{6}[(c_{t,i} - b_{t,i})^2 + 3(c_{t,i} - b_{t,i})(2b_{t,i} \\
&-_{t,i} c - W_0 \cdot r_{\mathrm{f}}) + 3(2b_{t,i} - c_{t,i} - W_0 \cdot r_{\mathrm{f}})^2]
\end{aligned}
\tag{21}
$$

Furthermore, the semivariance about the return rate of the portfolio at period t can be expressed by

(i) If $\beta < 0.5$, then

$$
\begin{aligned}
\mathrm{SV}(W_t) = \frac{1}{3}\Big[& 4(b_{t,i} W_{t-1} - a_{t,i} W_{t-1})^2 \beta_t^3 \\
& - 6(a_{t,i} W_{t-1} - b_{t,i} W_{t-1})(a_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})\beta_t^2 \\
& + 3(a_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})^2 \beta_t \Big].
\end{aligned}
\tag{22}
$$

(ii) If $\beta \geq 0.5$, then

$$
\begin{aligned}
\mathrm{SV}(W_t) = \Big\{ & \frac{4}{3}(c_{t,i} W_{t-1} - b_{t,i} W_{t-1})^2 \beta_t^3 + 2(c_{t,i} W_{t-1} \\
& - b_{t,i} W_{t-1})(2b_{t,i} W_{t-1} - c_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})\beta_t^2 \\
& + (2b_{t,i} W_{t-1} - c_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})^2 \beta_t \\
& + \frac{1}{6}\big[(a_{t,i} W_{t-1})^2 + a_{t,i} b_{t,i} W_{t-1}^2 + 6b_{t,i} W_{t-1} W_0 \cdot r_{\mathrm{f}} \\
& - 3(a_{t,i} W_{t-1} + c_{t,i} W_{t-1}) W_0 \cdot r_{\mathrm{f}} \\
& - (c_{t,i} W_{t-1})^2 + 5b_{t,i} c_{t,i} W_{t-1}^2 - 6b_{t,i}^2\big] \Big\}.
\end{aligned}
\tag{23}
$$

Based on the above analysis, the uncertain multi-period portfolio selection model (13) can be transformed into the following optimization problem:

(i) If $\beta < 0.5$, the model (13) equals

$$
\begin{aligned}
\max \quad & W_0 \prod_{t=1}^{\mathrm{T}} \Big[1 + \sum_{i=1}^{n} \big(\frac{1}{4}(a_{t,i} + 2b_{t,i} + c_{t,i}) x_{t,i} \\
& - d_{t,i} |x_{t,i} - x_{(t-1),i}| \big) \Big] \\
\text{s.t.} \quad & \sum_{t=1}^{\mathrm{T}} \frac{1}{3} \Big[4(b_{t,i} W_{t-1} - a_{t,i} W_{t-1})^2 \beta_t^3 \\
& - 6(a_{t,i} W_{t-1} - b_{t,i} W_{t-1})(a_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})\beta_t^2 \\
& + 3(a_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})^2 \beta_t \Big] \leq \sigma, \\
& \sum_{i=1}^{n} x_{t,i} = 1, \\
& \sum_{i=1}^{n} z_{t,i} = m_t, \\
& \varepsilon_{t,i} z_{t,i} \leq x_{t,i} \leq \delta_{t,i} z_{t,i}, \\
& z_{t,i} \in \{0, 1\}, \\
& x_{t,i} \geq 0, \\
& i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T.
\end{aligned}
\tag{24}
$$

(ii) If $\beta \geq 0.5$, the model (11) equals

$$
\begin{aligned}
\max \quad & W_0 \prod_{t=1}^{\mathrm{T}} \Big[1 + \sum_{i=1}^{n} \big(\frac{1}{4}(a_{t,i} + 2b_{t,i} + c_{t,i}) x_{t,i} \\
& - d_{t,i} |x_{t,i} - x_{(t-1),i}| \big) \Big] \\
\text{s.t.} \quad & \sum_{t=1}^{\mathrm{T}} \Big\{ \frac{4}{3}(c_{t,i} W_{t-1} - b_{t,i} W_{t-1})^2 \beta_t^3 + 2(c_{t,i} W_{t-1} \\
& - b_{t,i} W_{t-1})(2b_{t,i} W_{t-1} - c_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})\beta_t^2 \\
& + (2b_{t,i} W_{t-1} - c_{t,i} W_{t-1} - W_0 \cdot r_{\mathrm{f}})^2 \beta_t \\
& + \frac{1}{6}\big[(a_{t,i} W_{t-1})^2 + a_{t,i} b_{t,i} W_{t-1}^2 + 6b_{t,i} W_{t-1} W_0 \cdot r_{\mathrm{f}} \\
& - 3(a_{t,i} W_{t-1} + c_{t,i} W_{t-1}) W_0 \cdot r_{\mathrm{f}} - (c_{t,i} W_{t-1})^2 \\
& + 5b_{t,i} c_{t,i} W_{t-1}^2 - 6b_{t,i}^2\big] \Big\} \leq \sigma, \\
& \sum_{i=1}^{n} x_{t,i} = 1, \\
& \sum_{i=1}^{n} z_{t,i} = m_t, \\
& \varepsilon_{t,i} z_{t,i} \leq x_{t,i} \leq \delta_{t,i} z_{t,i}, \\
& z_{t,i} \in \{0, 1\}, \\
& x_{t,i} \geq 0, \\
& i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T.
\end{aligned}
\tag{25}
$$

4 Hybrid of imperialist competitive and genetic algorithm

The proposed models (24) and (25) are classified as a mixed integer nonlinear programming model necessitating the use of efficient heuristic algorithms to find the solution. In this paper, we propose a modified imperialist competitive algorithm (MICA) for the solution. In the following, basic ICA is first reviewed and then, MICA algorithm to solve the proposed models is presented.

4.1 Basic ICA

Imperialist competitive algorithm (ICA) (Atashpaz-Gargari and Lucas 2007) is an evolutionary meta-heuristic algorithm that mimics socio-political imperialist competition to search for a global optimal solution. It is a population-based algorithm in which every element of the population is called a country. Each country denotes an encoded solution to the problem. The better countries are chosen to become the imperialists. The other countries are divided among the imperialists as colonies. The whole set of an imperialist and its colonies is called an empire. After seizing the colonies, the imperialists try to penetrate into their colonies by making them similar to themselves. Then, the imperialists compete with each other to seize more colonies and gain more power. This process makes some imperialists more powerful and some weaker. The weak empires eventually collapse. An overview of the main steps of the ICA is presented next.

4.1.1 Initialization

Similar to other population-based meta-heuristic algorithms, ICA also begins with a randomly generated population which contains N_{pop} initial solutions. In ICA, each individual is called a 'country.' For an N_{var}-dimensional optimization problem, a country is a $1 * N_{\text{var}}$ array whose elements are randomly generated in the allowable range of the corresponding parameters as:

$$\text{Country} = [p_1, p_2, \ldots, p_{N_{\text{var}}}]. \tag{26}$$

The power of country is inversely proportional to the value of the cost function. The country with smaller cost has greater power. Thus, the cost of each country is evaluated with the cost function f at variables $(p_1, p_2, \ldots, p_{N_{\text{var}}})$ as the follows:

$$\text{Cost} = f(\text{country}) = f(p_1, p_2, \ldots, p_{N_{\text{var}}}). \tag{27}$$

In the initialization step, we need to generate an initial population with the size of N_{pop}. Next, we have to select N_{imp} of the most powerful countries to construct the empires. The remaining $N_{\text{col}} (N_{\text{col}} = N_{\text{pop}} - N_{\text{imp}})$ will be the colonies such that each colony belongs to an empire. So, an empire consists of an imperialist and set of colonies.

To form the initial empires, the colonies are divided among the imperialist countries according to the power of the imperialists. The normalized fitness of each imperialist is determined by,

$$C_n = \max_i \{c_i\} - c_n, \quad i = 1, 2, \ldots, N_{\text{imp}} \tag{28}$$

where c_n and C_n are the cost and normalized cost of nth imperialist, respectively. An imperialist with larger cost (i.e., a weaker imperialist country) has smaller normalized cost. When the normalized costs of all imperialists are gathered, the normalized power of each imperialist can be evaluated as follows:

$$p_n = \left| \frac{C_n}{\sum_{i=1}^{N_{\text{imp}}} c_i} \right|. \tag{29}$$

The initial colonies are distributed among empires based on their power; therefore, the initial number of colonies for nth empire can be defined as follows:

$$NC_n = round(p_n \cdot N_{\text{col}}), \tag{30}$$

where NC_n is the number of initial colonies possessed by the nth imperialist, N_{col} is the total number of colonies in the initial population, and $round$ is a function that gives the nearest integer of a fractional number.

4.1.2 Assimilation process

After forming initial empires, the colonies in each empire start to improve their power by moving toward their relevant imperialist. This process is known as the assimilation process, illustrated in Fig. 1.

As shown in Fig. 1, the colony moves x distance along with d direction toward its imperialist. The moving distance x is a random number generated by random distribution within interval $[0, \beta * d]$. The value of β falls between 1 and 2. Setting $\beta > 1$ causes the colony to move toward the imperialist direction. Meanwhile, a random amount of deviation (the θ in Fig. 1) is added to the movement direction, to enhance the exploration ability of the algorithm. The θ is defined as follows:

$$\theta \sim \cup(-\varphi, \varphi), \tag{31}$$

where φ is an arbitrary parameter, where a larger value will facilitate a global search and a smaller value will encourage a local search. Setting φ to $\pi/4$ (radian) could be a good choice.

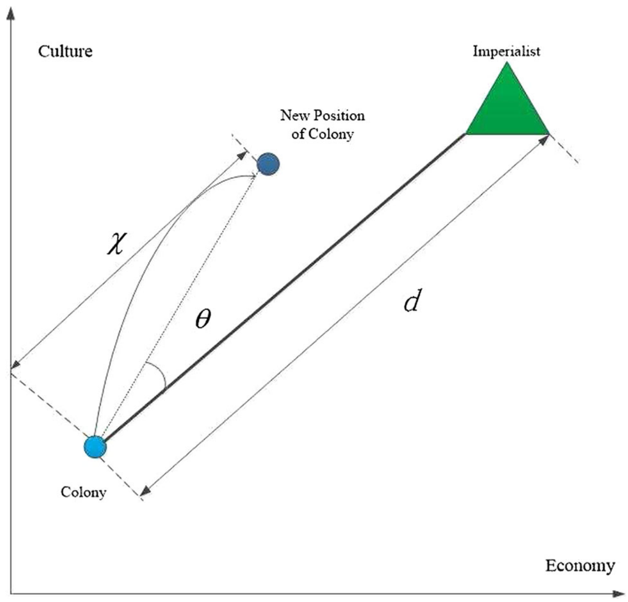

Fig. 1 Assimilation process

4.1.3 Revolution operator

Similar to the mutation process in GA, ICA allows a sudden change in socio-political characteristic of a country, known as revolution. During revolution, the positions of some countries suddenly change which facilitate diversification. The revolution rate decides the percentage of countries inside an empire which will randomly change. This process avoids the algorithm to converge to local optimal solution in the early iterations. The revolution operator must be precisely determined since a very high value of it decreases the intensification power of algorithm while a very low value of it decreases diversification.

4.1.4 Exchanging the position of a colony and an imperialist

After assimilation and revolution processes, it is possible that a colony reaches a position with lower cost than its imperialist. If there exists any colony with lower cost, then the position of this colony exchanges with the position of its imperialist. The algorithm then continues with imperialist in a new position, and new imperialist attempts to assimilate its colonies.

4.1.5 Total power of an empire

The power of an empire is computed based on the power of its imperialist and a fraction of the power of its colonies.

$$\begin{aligned} TC_n = & \text{Cost}(\text{imperialist}_n) \\ & + \xi * \text{mean}\{\text{Cost}(\text{colonies of empire}_n)\}, \end{aligned} \tag{32}$$

where TC_n is the total cost of nth empire, and ξ is a positive number between 0 and 1, usually close to zero. A small value of ξ emphasizes a greater influence of imperialist power in determining the total power of empire, while a large value of ξ indicates the influence of the mean power of colonies in determining the total power of the empire. And all the empires except the most powerful one will collapse and all colonies will be under the control of this unique empire.

4.1.6 Imperialist competition

Any empire that is unable to gain power will finally be eliminated, and more powerful empires will seize their colonies. In an iteration of the algorithm, a set of the weakest empires are considered and the more powerful empires try to conquer colonies from the weakest empires. The competition for colonies is based on the power of empires. In fact the more powerful an empire is, the better chance it has to win the competition for taking a colony. To model this competition, first the probability of every empire for winning the competition is calculated based on the normalized total cost of every empire. Normalized total cost of an empire is given by

$$NTC_n = \max_i\{TC_i\} - TC_n, \tag{33}$$

where TC_n is the total cost of nth empire and NTC_n is the normalized total cost of corresponding nth empire. After calculating the normalized cost (power) of every empire, the probability with which each empire might win the competition for conquering colonies is given by

$$p_n = \left| \frac{NTC_n}{\sum_{i=1}^{N_{\text{imp}}} NTC_i} \right|. \tag{34}$$

In the imperialist competition step, if the weakest imperialist loses all of its colonies, then this imperialist is collapsed. A collapsed imperialist is possessed by other imperialists as a colony.

4.1.7 Convergence

After a certain amount of iterations, there will be only one empire left, as that empire is the most powerful empire among the other empires. The other empires have collapsed during imperialistic competition. Also, the colonies in the most powerful empire will have the same position and same fitness as the imperialist. The program will terminate when the number of imperialist, $N_{\text{imp}} = 1$ or the maximum number of iterations has reached.

4.2 The proposed MICA algorithm

4.2.1 Fitness function

The fitness function is a factor for measuring the quality of a solution. In an optimization problem, fitness is often determined by the value of its objective function. Thus, for the maximization problems, the individual with higher fitness will have more chance to generate offspring. In this paper, we take the objective functions of the models (24) and (25) as the fitness functions.

In addition, we know that, in the basic ICA, the power of country is inversely proportional to the value of the cost function. So, for the proposed MIC algorithm, the cost function is defined as the reciprocal of the fitness function:

$$\text{Cost} = \frac{1}{\text{fit(country)}}. \tag{35}$$

where fit(country) refers to the fitness of the country.

4.2.2 Hybrid ICA-GA

ICA has attracted much attention and wide applications for solving optimization problems; however, this algorithm is easily to be trapped into a local optimum. Thus, inspired by the ideas in Mehdinejad et al. (2016), in this paper, a new type of country called 'independent' country is introduced, and the GA's crossover and mutation operators are applied for these independent countries. The independent countries are anti-imperialism and are in competition with imperialist countries. Independent countries' aim is to free the colonies from the empires and let them join independent countries to cause the collapse of all empires.

More specifically, in the initial population, N_{imp} of the most powerful countries from initial countries N_{pop} are selected to be imperialists, N_{ind} of the most powerful countries from remaining countries ($N_{\text{pop}} - N_{\text{imp}}$) are selected to be independent countries, and the remaining N_{col} ($N_{\text{col}} = N_{\text{pop}} - N_{\text{imp}} - N_{\text{ind}}$) countries form the colonies. Then, in each iteration, three operators of GA are added as follows:

(1) **Mutation Operator**. In the evolutionary progress of ICA, population diversity may be lost and premature convergence always happens. Here, we design a new mutation strategy to perform mutation operator. Specifically, a new independent country Ind_i' is produced according to the following equation:

$$\text{Ind}_i' = \text{Ind}_i + \varphi_i \times \text{rand}(N_{\text{var}}, N_{\text{var}}) \times (\text{Cou}_{\text{best}} - \text{Ind}_i), \tag{36}$$

$$\varphi_i = 1 - \frac{\text{fit}(\text{Ind}_i)}{\sum_{i=1}^{N_{\text{pop}}} \text{fit}(\text{cou}_i)}, \tag{37}$$

where Ind_i refers to the ith independent country, $\text{rand}(N_{\text{var}}, N_{\text{var}})$ is a $N_{\text{var}} \times N_{\text{var}}$ matrix, whose elements are random numbers between 0 and 1. Cou_{best} refers to the country with the best fitness, and φ_i is the mutation probability of ith independent country. $\text{fit}(\text{Ind}_i)$ refers to the fitness of the ith independent country. The countries with greater fitness have less probability to mutation. It should be noted that, after mutation operator, if the new country is not better than old one, we let $\text{Ind}_i' = \text{Ind}_i$.

(2) **Crossover Operator**. This paper adopts the single-point crossover operation method. In single-point crossover, one crossover position k is selected uniformly at random in the interval $[1, 2, \ldots, N_{\text{var}}]$, and then the variables exchanged between the 'independent countries' in this point. Thus, two new independent countries are produced.

(3) **Selection Operator**. We select some of the best countries with the size of N_{imp} from the new countries including imperialists, independent countries and colonies to be new imperialists, and the rest of the countries are to be the new colonies.

4.2.3 Dynamic switching revolution strategy

Like many meta-heuristic methods, parameters selection has great effects on the algorithm performance. Revolution operator is the counterpart of mutation in GA. A large value of revolution rate reinforces the exploration, while a small value of it may encourage exploitation. Hosseini and Al Khaled (2014) showed that the value of revolution rate is highly dependent on the magnitude of solution spaces, and the value of 0.2 could be an appropriate choice in general. Besides, Sadeghi et al. (2016) suggested that 0.1 is the better set for revolution rate. However, for any generalized algorithm, we know that more global search should be done at the beginning as it will tend to explore the search space better and when enough exploration has been done, the process should move on to local search that is exploitation. Thus, in this paper, dynamic switching revolution strategy is employed to make the algorithm more reliable in exploring and exploiting the search space. The revolution operator is performed as follows:

$$P_{r,i} = p_0\left(1 - \frac{\text{fit}(\text{col}_i)}{\text{fit}(\text{cou}_{\text{best}})} \cdot e^{\lambda}\right), i = 1, 2, \ldots, N_{\text{col}}, \tag{38}$$

$$\lambda = \frac{\text{rand}\,[0, N_{\text{var}}]}{N_{\text{var}}}, \tag{39}$$

where $\text{fit}(\text{col}_i)$ is the fitness for the ith colony, cou_{best} is the country with the best fitness, $\text{rand}\,[0, N_{\text{var}}]$ is a random number between 0 and N_{var}.

From Eq. (38), we can see that the bigger the fitness value, the smaller the probability of performing revolution operation. Moreover, the revolution rate is also associated with the dimension of the optimization problem. As the dimension N_{var} increases, the probability of performing revolution operation also increases.

4.2.4 Constraint satisfaction

We use hybrid representation to define a portfolio, in which two vectors are defined as following forms:

$$\Delta = \{z_{1,1}, z_{1,2}, \ldots, z_{1,n}; \ldots; z_{T,1}, z_{T,2}, \ldots, z_{T,n}\},$$
$$X = \{x_{1,1}, x_{1,2}, \ldots, x_{1,n}; \ldots; x_{T,1}, x_{T,2}, \ldots, x_{T,n}\}.$$

Here, $z_{t,i}$ is a binary vector that specifies whether a particular asset participate in the portfolio at period t, where $z_{t,i} \in \{0, 1\} (i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T)$. $x_{t,i}$ is a real-valued vector used to compute the investment proportions of the budget invested on the portfolio at period t, where $x_{t,i} \in [0, 1] (i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T)$.

Similar to the studies in Chang et al. (2000), Liu et al. (2016b), Mishra et al. (2014), we performed the following operators to find the portfolio x associated with the above encoding. First, if the number of assets in the portfolio at each period (i.e., the number of 1's in z_t) exceeds the maximum allowed number m, we deleted (by changing its value from 1 to 0 in z_t) those assets with the $(n - m)$ smallest weights in x_t. In this way, we can keep the portfolio at each period satisfy the cardinality constraints.

To meet the budget constraint, we perform the following normalization operation

$$x'_{t,i} = \frac{x_{t,i} z_{t,i}}{\sum_{j=1}^{n} x_{t,i} z_{t,i}}, \quad i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T \tag{40}$$

Since the normalize investment proportion may not satisfy the bound constraints, the following three cases are discussed.

Case 1: If both lower and upper constraints are present, then the investment proportions are computed by

$$x'_{t,i} = \varepsilon_{t,i} z_{t,i} + \frac{x_{t,i} z_{t,i}}{\sum_{i=1}^{n} x_{t,i} z_{t,i}} \left(\delta_{t,i} z_{t,i} - \sum_{i=1}^{n} \varepsilon_{t,i} z_{t,i} \right), \quad t = 1, 2, \ldots, T. \tag{41}$$

Case2: If the investment proportion has to be adjusted only for the lower constraint, and there is no restriction on the upper limit, then the adjusted investment proportions are computed as:

$$x'_{t,i} = \varepsilon_{t,i} z_{t,i} + \frac{x_{t,i} z_{t,i}}{\sum_{i=1}^{n} x_{t,i} z_{t,i}} \left(1 - \sum_{i=1}^{n} \varepsilon_{t,i} z_{t,i} \right), \quad t = 1, 2, \ldots, T. \tag{42}$$

Case 3: If the investment proportion has to be adjusted for the upper constraint, and there is no restriction on the lower limit, then the adjusted investment proportions are calculated by

$$x'_{t,i} = \delta_{t,i} z_{t,i} - \frac{x_{t,i} z_{t,i}}{\sum_{i=1}^{n} x_{t,i} z_{t,i}} \left(\sum_{i=1}^{n} \delta_{t,i} z_{t,i} - 1 \right), \quad t = 1, 2, \ldots, T. \tag{43}$$

4.2.5 Stop criteria

We consider two stopping criteria. The program will terminate when the number of imperialist $N_{\text{imp}} = 1$ or the maximum number of iterations has reached.

The pseudo-code about the MICA algorithm is showed in Algorithm 1.

5 Numerical example

In this section, we give a numerical example to illustrate the applications of the proposed models and demonstrate the validity of the designed algorithm. In this example, the investor intends to make four consecutive periods investment among the 10 securities with initial wealth 10,000 RMB. In addition, we assume that security returns cannot be well reflected by the historical data and are given by experts' evaluations. Here, the return rates are characterized by zigzag uncertain variables $r_{t,i}$ $(i = 1, 2, \ldots, 10; \ t = 1, 2, 3, 4)$, which are shown in Table 1.

In this example, we assume that the initial portfolio at the beginning of period is $x_{0,i} = 0.1$ for $i = 1, 2, \ldots, 10$. The risk-free interest rate is 0.02. The transaction costs of securities at the four periods are set as $d_{t,i} = 0.0015$ $(i = 1, 2, \ldots, 10; \ t = 1, 2, 3, 4)$. The maximum number of securities held in the portfolio in each period is $m_t = 5$ $(t = 1, 2, 3, 4)$. The lower bound constraint $\varepsilon_{t,i} = 0.1$, and upper bound constraint $\delta_{t,i} = 0.6$ for all $i = 1, 2, \ldots, 10$ and $t = 1, 2, 3, 4$.

The parameters of the MICA algorithm are set as follows: the numbers of countries N_{pop}, imperialists N_{imp}, independent countries N_{ind}, and colonies N_{col} are set to 30, 10, 10,

Table 1 The zigzag uncertain returns of 10 securities

	Security i	1	2	3	4	5	6	7	8	9	10
$t = 1$	$a_{1,i}$	−0.4	−0.8	−0.5	−0.7	−0.6	−0.3	−0.8	−0.5	−0.7	−0.6
	$b_{1,i}$	0.1	−0.2	−0.1	0.1	0.3	−0.1	0.1	0.3	0.2	0.002
	$c_{1,i}$	0.5	0.2	0.3	0.6	0.9	0.4	0.9	0.7	0.6	0.8
$t = 2$	$a_{2,i}$	−0.3	−0.7	−0.5	−0.6	−0.5	−0.5	−0.8	−0.5	−0.7	−0.5
	$b_{2,i}$	0.2	−0.1	−0.2	0.1	0.3	0.1	0.1	0.3	0.2	0.2
	$c_{2,i}$	0.5	0.3	0.5	0.5	0.8	0.4	0.9	0.7	0.6	0.9
$t = 3$	$a_{3,i}$	−0.4	−0.8	−0.5	−0.8	−0.6	−0.6	−0.9	−0.5	−0.6	−0.6
	$b_{3,i}$	0.1	−0.2	−0.1	0.2	0.4	−0.2	−0.1	0.3	0.3	0.003
	$c_{3,i}$	0.5	0.2	0.3	0.7	0.7	0.7	0.8	0.6	0.7	0.8
$t = 4$	$a_{4,i}$	−0.2	−0.6	−0.9	−0.4	−0.6	−0.7	−0.5	−0.8	−0.3	−0.1
	$b_{4,i}$	0.001	−0.2	0.1	0.3	0.2	0.4	−0.3	−0.3	0.2	0.1
	$c_{4,i}$	0.4	0.3	0.6	0.6	0.9	0.5	0.6	0.8	0.7	0.5

Table 2 The optimal strategies with different values of σ

σ	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	Terminal wealth
1500	$t = 1$	0.000	0.264	0.000	0.000	0.240	0.049	0.000	0.354	0.093	0.000	11,380.2209
	$t = 2$	0.265	0.000	0.000	0.000	0.179	0.036	0.000	0.263	0.000	0.256	12,932.0524
	$t = 3$	0.394	0.293	0.156	0.000	0.000	0.054	0.000	0.000	0.103	0.000	14,621.3049
	$t = 4$	0.000	0.000	0.128	0.000	0.218	0.044	0.289	0.321	0.000	0.000	15,548.7712
1600	$t = 1$	0.000	0.331	0.000	0.000	0.000	0.168	0.146	0.000	0.095	0.260	11,439.6252
	$t = 2$	0.245	0.000	0.182	0.316	0.000	0.000	0.000	0.009	0.000	0.247	13,113.5027
	$t = 3$	0.258	0.332	0.000	0.000	0.168	0.000	0.147	0.000	0.095	0.000	14,939.8430
	$t = 4$	0.221	0.000	0.000	0.286	0.000	0.144	0.126	0.000	0.000	0.223	16,966.6278
1700	$t = 1$	0.000	0.145	0.000	0.000	0.368	0.000	0.000	0.349	0.087	0.051	11,681.6879
	$t = 2$	0.113	0.152	0.347	0.336	0.000	0.000	0.000	0.000	0.000	0.053	13,429.2396
	$t = 3$	0.126	0.169	0.000	0.000	0.000	0.000	0.198	0.406	0.101	0.000	15,374.8084
	$t = 4$	0.084	0.000	0.000	0.248	0.285	0.316	0.000	0.000	0.067	0.000	17,487.9872
1800	$t = 1$	0.002	0.000	0.000	0.000	0.128	0.000	0.105	0.255	0.51	0.000	11,876.1322
	$t = 2$	0.003	0.000	0.337	0.000	0.173	0.000	0.143	0.345	0.000	0.000	13,700.0478
	$t = 3$	0.002	0.407	0.242	0.000	0.000	0.000	0.102	0.247	0.000	0.000	15,887.9872
	$t = 4$	0.002	0.321	0.19	0.000	0.098	0.000	0.000	0.000	0.389	0.000	18,016.0462

and 10, respectively. In addition, assimilation coefficient β and crossover rate p_c are set to 0.6 and 0.5, respectively. The maximum generation number is 200.

Using the proposed algorithm, we can obtain some investment strategies by varying the risk tolerate parameter σ. The results are listed in Table 2. We find that the investor will adopt different investment strategies at different investment periods. For example, when $\sigma = 1600$ RMB, the investor adopts the following strategies. At the beginning of period 1, the investor needs to assign his initial wealth among securities 2, 6, 7, 9 and 10 by the investment proportions of 0.331, 0.168, 0.146, 0.095 and 0.260, respectively. At the beginning of period 2, the investor needs to adjust his wealth again. After adjustment, he holds securities 1, 3, 4, 8 and 10 by the investment proportions of 0.245, 0.182, 0.316, 0.009 and 0.247, respectively. Finally, at the beginning of period 4, the investor constructs a portfolio among securities 1, 4, 6, 7 and 10 by the investment proportions of

Algorithm 1 The MICA algorithm

```
Randomly generate N_pop initial countries
Initialize the imperialists, independent countries and colonies
For n = 1 to N_ind   /*the number of the independent countries*/
  For k = 1 to N_var   /* the dimension of the countries*/
  Mutation for independent countries
      Generate a random number r from [0, 1]
      If r < p_c, select the ith and (i + 1)th countries as the parent and children /* p_c is the crossover probability */
      Crossover the parent's and children's gene
  End Loop
  Select the new imperialists and colonies
End Loop
Elimination of empire
  For j = 1 to N_imp   /* the number of the imperialists */
      Move the colony toward the relevant imperialist
      Compute the fitness of assimilated countries
      Imperialist competition
      Perform revolution operator on new colony
      If the fitness of new colony is better than that of imperialist
          Exchange the position of colony and imperialist
      Select the weakest colony from the weakest empire and let it to be the empire
      If there is imperialist with no colonies
          Eliminate the imperialist
  End Loop
Until stopping condition is reached
```

0.221, 0.286, 0.144, 0.126 and 0.223, respectively. In this case, the crisp value of terminal wealth is 16,966.6278 RMB. In addition, Table 2 also indicates that the terminal wealth increases along with the increasement of the risk tolerate value σ. For instance, when $\sigma = 1800$ the best terminal wealth is achieved, while $\sigma = 1500$ leads to the lowest terminal wealth.

To demonstrate the effects of the cardinality constraint on the portfolio selection, given $m_t = 5$ and 6 ($t = 1, 2, 3, 4$), the histograms for the optimal portfolio strategies are depicted in Fig. 2. From Fig. 2, it is clear that with the changes of the m_t, the investors adopt different investment strategies. For example, when $m_t = 5$, securities 1, 6 and 10 are included in the portfolio at period of 4. Nevertheless, when $m_t = 6$, these securities are not selected at the same period. Moreover, taking the security 6 as a example, we can see that when $m_t = 5$, it is selected to invest just at periods 1 and 4 by the investment proportions of 0.168 and 0.144, respectively. However, when $m_t = 6$, the investor holds this security at each period by the investment proportions of 0.114, 0.132, 0.121 and 0.1, respectively.

Furthermore, given $\sigma = 1600$, $r_{\mathrm{f}} = 0.02$, and $m_t = 5$, Table 3 is presented to demonstrate the effect of transaction costs on portfolio selection. It can be seen from Table 3 that when the transaction cost rate increases, the investor gains less expected incremental wealth.

Similarly, Table 4 demonstrates how the risk-free interest rates r_{f} affect the portfolio selection. It can be seen that the obtained terminal wealth as well as the corresponding risk increase at the same time along with the increasing of the r_{f} value. For instance, when $r_{\mathrm{f}} = 0.01$, the terminal wealth is 14,705.8295 at end of period 4, and the cumulative semivariance SV is 899.1219, while $r_{\mathrm{f}} = 0.03$, the corresponding values are 18,280.9852 and 1471.7799, respectively.

Finally, to demonstrate the effectiveness of our designed algorithm, we compare it with the basic ICA and GA by varying the generation numbers. The parameters of the above two algorithms are the same as the proposed MICA. The detailed comparative results are shown in Table 5. It should be noted that Different 1 refers to the difference between the objective values of MICA and ICA, and Different 2 refers to the difference between the objective values of MICA and GA. It can be seen that the objective values obtained by our proposed algorithm are larger than those by the basic ICA and GA algorithms. In addition, we also calculate the maximum deviation of objective for the proposed model with respect to different generations. The maximum deviation generated by the MICA algorithm is smaller than the one generated by the other algorithms. Besides, the convergence characteristic of different algorithms is shown in Fig. 3. It is obvious that the proposed MICA can quickly converge to a better solution much faster than the other algorithms. All these results show that the proposed MICA algorithm has a better performance and is more efficient in finding optimal portfolios.

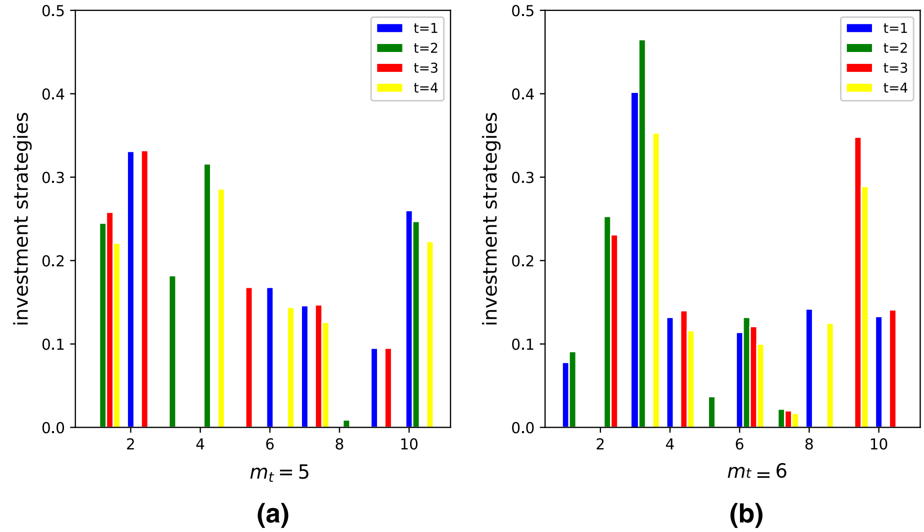

Fig. 2 The optimal investment strategies under different *m*

Table 3 Wealth increment under different transaction costs

d_i	ΔW				Terminal wealth
	$t = 1$	$t = 2$	$t = 3$	$t = 4$	
0.0005	1596.0682	1336.0153	2005.9215	2226.2906	17,137.2956
0.0010	1469.1378	1592.2746	1808.8682	2161.3133	17,031.5939
0.0015	1439.6252	1673.8774	1826.3403	2026.7848	16,966.6278
0.0020	1186.8012	−242.0526	2005.9912	2134.2860	15,085.0258
0.0025	1022.5334	1230.3904	1578.1419	1796.6385	15,627.7042

Table 4 The model results under different risk-free interest rate

r_f	t	β_t	SV	Terminal wealth
$r_f = 0.01$	$t = 1$	0.4120	161.6300	11,507.7000
	$t = 2$	0.4269	107.4156	12,836.2351
	$t = 3$	0.4008	208.7136	14,452.4776
	$t = 4$	0.8257	421.3627	14,705.8295
$r_f = 0.02$	$t = 1$	0.0000	0.0000	11,439.6252
	$t = 2$	0.5745	239.0890	12,673.8774
	$t = 3$	1.0000	310.1759	14,361.8985
	$t = 4$	0.4392	227.9275	16,966.6278
$r_f = 0.03$	$t = 1$	1.0000	185.3339	11,571.6553
	$t = 2$	1.0000	406.0352	13,768.9612
	$t = 3$	0.0000	361.0708	15,965.8657
	$t = 4$	0.4206	519.3400	18,280.9852

6 Conclusion

Since the security market is so complex, sometimes security returns have to be predicted by experts' evaluations rather than historical data. This paper discusses a multi-period portfolio selection problem with returns given by experts' evaluations. By using uncertainty theory, we propose a multi-period mean–semivariance portfolio model with real-world constraints, in which transaction costs, cardinality and bounding constraints are considered simultaneously. We further convert the proposed optimization model into a crisp mathematical programming under the assumption that the security returns are zigzag uncertain variables. After that, we design a MICA algorithm to solve the corresponding optimization model. Finally, a numerical example is given to illustrate the ideas of the proposed model and the validity of the designed algorithm. The experimental results demonstrate that real-world constraints have a great impact on the optimal investment strategy, and the designed MICA algorithm is effective for solving the proposed model.

Table 5 The comparison of the different algorithms

t		ICA	GA	MICA	Different 1	Different 2
$t=1$	$G=100$	10,502.3575	10,458.8725	11,276.8726	774.5151	556.9271
	$G=200$	10,928.1905	11,015.7996	11,439.6252	511.4347	423.8256
	$G=300$	11,028.4114	11,212.6087	11,528.6252	500.2138	316.0165
	$G=400$	11,528.6277	11,528.6283	11,528.6299	0.0022	0.0016
	$G=500$	11,528.6297	11,528.6292	11,528.6299	0.0002	0.0005
	Maximum deviation	1026.2722	839.0744	754.7418		
$t=2$	$G=100$	12,067.8745	12,165.8983	12,448.7221	380.8476	282.8238
	$G=200$	12,861.6152	12,951.9208	13,113.5027	251.8875	161.5819
	$G=300$	13,020.2621	13,065.3628	13,420.3976	400.1355	355.0348
	$G=400$	13,490.0056	13,494.0666	13,496.0866	6.0810	2.0200
	$G=500$	13,496.0864	13,496.0865	13,496.0866	0.0002	0.0001
	Maximum deviation	1428.2119	1330.1882	1047.3645		
$t=3$	$G=100$	13,773.9776	14,048.83393	14,930.2751	1156.2975	881.4412
	$G=200$	13,883.5505	13,979.1193	14,939.8430	1056.2925	960.7237
	$G=300$	14,290.2516	14,304.4206	14,996.8993	706.6477	692.4787
	$G=400$	15,027.4502	15,027.4500	15,027.4505	0.0003	0.0005
	$G=500$	15,028.1928	15,028.1928	15,028.1928	0.0000	0.0000
	Maximum deviation	1154.2152	979.3589	97.9177		
$t=4$	$G=100$	15,987.2108	16,028.8339	16,201.2751	214.0643	172.4412
	$G=200$	16,906.5505	16,875.1193	16,966.6278	60.0773	91.5085
	$G=300$	16,936.2516	16,895.4206	17,041.8993	105.64773	146.4787
	$G=400$	17,306.4553	17,303.4551	173,09.4555	3.0002	6.0004
	$G=500$	17,309.4555	17,309.4555	17,309.4555	0.0000	0.0000
	Maximum deviation	1322.2447	1280.6216	1108.1804		

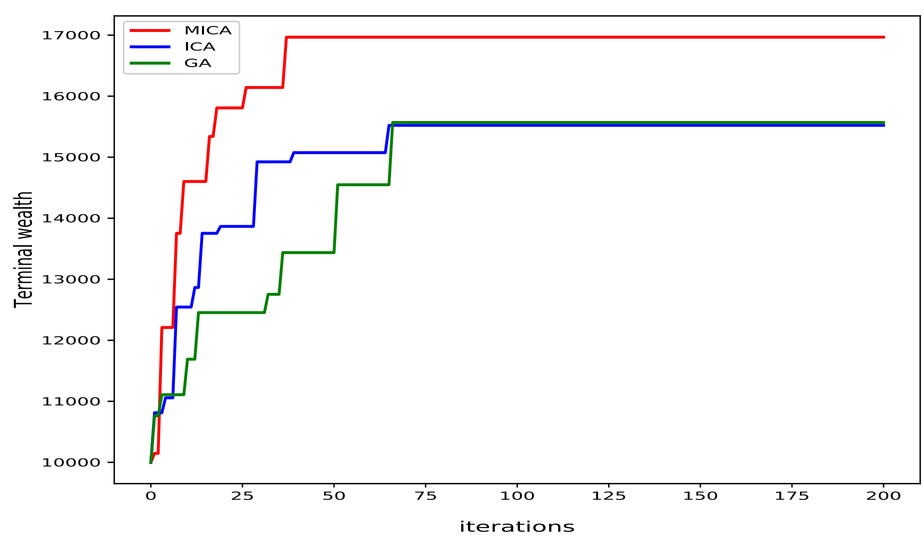

Fig. 3 Convergence through MICA, ICA and GA

Finally, for future researches three areas are proposed: first adding other real-world constraints such as minimum transaction lots and bankruptcy in the investment horizon, second changing the proposed model to a multi-period multi-objective one and third comparing other heuristic algorithms with the designed algorithm.

Acknowledgements This research was supported by the National Natural Science Foundation of China (Nos. 71720107002 and 71601132), the Beijing Municipal Education Commission Foundation of China (Nos. KM201810038001 and SM201810038006) and the Special Research Funds from the Quantitative Finance Research Center of School of Information, Capital University of Economics and Business.

Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

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