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METHODOLOGIES AND APPLICATION

Trapezoidal cubic fuzzy number Einstein hybrid weighted averaging operators and its application to decision making

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Abstract

In this paper, we define some Einstein operations on trapezoidal cubic fuzzy set and develop three arithmetic averaging operators, that is trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating trapezoidal cubic fuzzy information. The TrCFEHWA operator generalizes both the TrCFEWA and TrCFEOWA operators. Furthermore, we establish various properties of these operators and derive the relationship between the proposed operators and the exiting aggregation operators. We apply on the TrCFEHWA operator to multiple attribute decision making with trapezoidal cubic fuzzy information. Finally, a numerical example is providing to demonstrate the submission of the established approach.

Keywords Trapezoidal cubic fuzzy set (TrCFS) · Einstein *t*-norm · Arithmetic averaging operator · Multi-attribute decision making (MADM)

1 Introduction

Fuzzy set was first introduced by Zadeh (1965). Atanassov's intuitionistic fuzzy sets (AIFSs) (Atanassov 1994) and interval-valued fuzzy sets (IVFSs) (Atanassov 1986, 1994) are two reasonably basic developments, which were introduced to help a segment of the drawbacks of fuzzy set theory. As observed in Zadeh (1973) and recent made in Bustince and Burillo (1996); Deschrijver and Kerre (2003), the AIFS is numerically equivalent to the IVFSs. The AIFSs address one of the main ideological headways, which consult not solely to what degree a part has a place with a particular set (non-membership); moreover, to what degree this segment does not have a place with the set (non-membership function). Throughout in the last decade, the AIFS theory has been associated with aggregate intuitionistic fuzzy information (Deschrijver and Kerre 2007; Liu and Wang 2007; Turksen 1986; Xu and Cai 2010). Recently, the possibility of AIFSs is summed up by allowing the membership and non-membership capacity to expect interval values along these lines displaying the possibility of interval-valued intuitionistic fuzzy sets (IVIFSs) (Liu and Wang 2007). Wang and Liu (2013), proposed Einstein intersection, Einstein product, Einstein scalar multiplication and Einstein exponentiation and then defined new concentration and dilation of AIFSs. Wang and Zhang (2009), introduced the concept of intuitionistic trapezoidal fuzzy numbers. In Wu and Cao (2013), developed some aggregation operators based on intuitionistic trapezoidal fuzzy numbers, namely intuitionistic trapezoidal fuzzy weighted geometric (ITFWG) operator, the intuitionistic trapezoidal fuzzy ordered weighted geometric (ITFOWG) operator, the induced intuitionistic trapezoidal fuzzy ordered weighted geometric (I-ITFOWG) operator and the intuitionistic trapezoidal fuzzy hybrid geometric (ITFHG) operator. Zhang et al. (2013) developed gray relational projection method, combined gray relational analysis method and projection method under intuitionistic trapezoidal fuzzy number. Liu et al. (2017) developed the trapezoidal intuitionistic fuzzy Einstein weighted averaging (TIFEWA) operator, the trapezoidal intuitionistic fuzzy Einstein ordered weighted averaging (TIFEOWA) operator and the trapezoidal intuitionistic fuzzy Einstein hybrid aver-

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aging (TIFEHA) operator. Zhang et al. (2014) developed interval-valued dual hesitant fuzzy Einstein weighted averaging (IVDHFEWA) operator, interval-valued dual hesitant fuzzy Einstein ordered weighted averaging (IVDHFEOWA) operator, interval-valued dual hesitant fuzzy Einstein hybrid averaging (IVDHFEHA) operator, interval-valued dual hesitant fuzzy Einstein weighted geometric (IVDHFEWG) operator, interval-valued dual hesitant fuzzy Einstein ordered weighted geometric (IVDHFEOWG) operator, and interval-valued dual hesitant fuzzy Einstein hybrid geometric (IVDHFEHG) operator.

Cubic set was introduced by Jun et al. (2011), as a generalization of fuzzy set and intuitionistic fuzzy set and is characterized by membership degree and non-membership degree. The membership function is an interval fuzzy number, while non-membership function is a fuzzy set. Fahmi et al. (2018) defined the triangular cubic fuzzy number and operational laws. We developed the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structure provide by the decision makers into the aggregate cubic fuzzy decision matrix. Fahmi et al. (2017) proposed the cubic TOPSIS method and cubic gray relation analysis (GRA) method. Finally, the proposed method is used for selection in sol–gel synthesis of titanium carbide nano-powders. Fahmi et al. (2017) defined weighted average operator of triangular cubic fuzzy numbers and hamming distance of the TCFN. We develop an MCDM method approach based on an extended VIKOR method using triangular cubic fuzzy numbers (TCFNS) and multi-criteria decision-making (MCDM) method using triangular cubic fuzzy numbers (TCFNs).

Based on the above analysis in this paper, we develop trapezoidal cubic fuzzy numbers (TrCFNs), which is the generalization of trapezoidal intuitionistic fuzzy number and trapezoidal interval fuzzy number. We propose some operations based on Einstein t-norm and Einstein T-conorm for TrCFNs. We also develop score and accuracy function to compare two TrCFNs. Due to the developed operation, we propose trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating a collection of trapezoidal cubic fuzzy numbers (TrCFNs).

This paper is organized as follows: In Sect. 2, we discuss some definitions and properties. In Sect. 3, we discuss the trapezoidal cubic fuzzy number (TrCFN) and operational laws. In Sect. 4, we present some Einstein operations on trapezoidal cubic fuzzy sets (TrCFSs) and analysis some desirable properties of the proposed operations. In Sect. 5, we first develop some novel arithmetic averaging operators, such as the trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating a collection of trapezoidal cubic fuzzy numbers (TrCFNs). In Sect. 6, we apply the TrCFEHWA operator to multiple attribute decision making (MADM) with trapezoidal cubic fuzzy material. Section 7 gives a numerical example according to our approach. In Sect. 8, we discuss comparison analysis. In Sect. 9, we consume a conclusion.

2 Preliminaries

In this section, we give a brief review of some preliminaries.

Definition 1 (Zadeh 1965) H be a universe of discourse. Then, the fuzzy set can be defined as: $J = \{h, \mu_J(h)|h \in H\}$. A fuzzy set in a set H is denoted by $\mu_J : H \to I$. The function $\mu_J(h)$ denoted the degree of membership of the element h to the set H, where $I = [0, 1]$. The collection of all fuzzy subsets of H is denoted by I^H . Define a relation on I^H as follows: $(\forall\mu, \eta \in I^H)(\mu \leq \eta \Leftrightarrow (\forall h \in H)(\mu(h) \leq \eta(h)))$.

Definition 2 (Atanassov 1986) Let H is a non-empty set. A function $I : H \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in H . Let $[I]^H$ stand for the set of all IVF sets in H . For every $I \in [I]^H$ and $h \in H$, $I(h) = [I^-(h), I^+(h)]$ is called the degree of membership of an element h to I, where $I^- : H \to I$ and $I^+ : H \to I$ are fuzzy sets in H which are called a lower fuzzy set and an upper fuzzy set in H, respectively. For simplexes, we denote $I = [I^-, I^+]$. For every $I_1, I_2 \in [I]^H$, we define $I_1 \subseteq I_2 \Leftrightarrow I_1(h) \leq I_2(h)$ for all $h \in H$ and $I_1 = I_2 \Leftrightarrow I_1(h) = I_2(h) \forall h \in H$.

Definition 3 (Jun et al. 2011) Let H is a non-empty set. A cubic set in H is a structure of the form $F = \{\langle h, I(h), \mu(h)\rangle : h \in H\}$ in which I is an IVF set in H and μ is a fuzzy set in H. A cubic set $F = \{\langle h, I(h), \mu(h)\rangle : h \in H\}$ is simply denoted by $F = \langle I, \mu\rangle$. Denote by C^H the collection of all cubic sets in H. A cubic set $F = \langle I, \mu\rangle$ in which $I(h) = 0$ and $\mu(h) = 1$ (resp. $I(h) = 1$ and $\mu(h) = 0$ for all $h \in H$ is denoted by 0 (resp. 1). A cubic set $D = \langle J, \lambda\rangle$ in which $J(h) = 0$ and $\lambda(h) = 0$ (resp. $J(h) = 1$ and $\lambda(h) = 1$) for all $h \in H$ is denoted by (resp. $\hat{1}$).

Definition 4 (Jun et al. 2011) Let H is a non-empty set. A cubic set $F = \langle I, \mu\rangle$ in H is said to be an internal cubic set if $I^-(h) \leq \mu(h) \leq I^+(h) \quad \forall h \in H$.

Definition 5 (Jun et al. 2011) Let H is a non-empty set. A cubic set $F = \langle I, \mu\rangle$ in H is said to be an external cubic set if $\mu(h) \notin [I^-(h), I^+(h)] \quad \forall h \in H$.

3 Trapezoidal cubic fuzzy numbers

In this section, we introduce the concept of trapezoidal cubic fuzzy numbers, which is the generalization of trapezoidal intuitionistic fuzzy numbers and trapezoidal interval-valued fuzzy numbers.

Definition 6 Let Abe the trapezoidal cubic fuzzy number on the set of real numbers, its IVTrFN is defined as:

$$\left\{\begin{array}{l} \lambda_A^-(h)=\left\{\begin{array}{ll} \left\{\frac{(h-p^-)}{(q^- - p^-)}I_A^-\right\} & p^- \le h < q^- \\ \left\{I_A^-\right\} & q^- \le h < r^- \\ \left\{\frac{(r^- - h)}{(s^- - r^-)}I_A^-\right\} & r^- \le h < s^- \\ 0 & h < p^- \text{ or } h < s^- \end{array}\right. , \\ \lambda_A^+(h)=\left\{\begin{array}{ll} \left\{\frac{(h-p^+)}{(q^+ - p^+)}I_A^+\right\} & p^+ \le h < q^+ \\ \left\{I_A^+\right\} & q^+ \le h < r^+ \\ \left\{\frac{(r^+ - h)}{(s^+ - r^+)}I_A^+\right\} & r^+ \le h < s^+ \\ 0 & h > p^+ \text{ or } h > s^+ \end{array}\right. \end{array}\right\}$$

and its TrFN is $\Gamma_A(h) = \left\{\begin{array}{ll} \frac{q-h+(h-p)\mu_A}{(p-q)}, & p \le h < q \\ \mu_A & q \le h \le r \\ \frac{h-r+(s-h)\mu_A}{(s-r)} & r < h \le s \\ 0 & h < p \text{ or } h > s) \end{array}\right.$. Then the trapezoidal cubic fuzzy number A basically denoted by $A = \left\{\begin{array}{l} \langle[(p^-, q^-, \\ r^-, s^-), (I_A^-)], \\ [(p^+, q^+, \\ r^+, s^+), (I_A^+)], \\ [(p, q, r, s), \\ (\mu_A)]\rangle \end{array}\right\}$. Then A is called trapezoidal cubic fuzzy number (TrCFN).

Definition 7 Let $A_1 = \left\{\begin{array}{l} \langle[\{p_1^-(h), q_1^-(h), \\ r_1^-(h), s_1^-(h)\}; I_{A_1}^-], \\ [\{p_1^+(h), q_1^+(h), \\ r_1^+(h), s_1^+(h)\}; I_{A_1}^+], \\ [\{p_1(h), q_1(h), \\ r_1(h), s_1(h)\}; \mu_A]\rangle \\ |h \in H \end{array}\right\}$ and $A_2 = \left\{\begin{array}{l} \langle[\{p_2^-(h), q_2^-(h), \\ r_2^-(h), s_2^-(h)\}; I_A^-], \\ [\{p_2^+(h), q_2^+(h), \\ r_2^+(h), s_2^+(h)\}; I_{A_2}^+], \\ [\{p_2(h), q_2(h), \\ r_2(h), s_1(h)\}; \mu_A\rangle \\ |h \in H \end{array}\right\}$ be two trapezoidal cubic fuzzy sets; some operational laws on trapezoidal cubic fuzzy sets are defined as follows:

$$\text{(a)} \left\{\begin{array}{l} A_1 \subseteq A_2 \text{ iff } \forall h \in H, [p_1^-(h) \ge p_2^-(h), \\ \quad q_1^-(h) \ge q_2^-(h), \\ r_1^-(h) \ge r_1^-(h), s_1^-(h) \ge s_2^-(h), I_{A_1}^- \ge I_{A_2}^-], \\ [p_1^+(h) \ge p_2^+(h), q_1^+(h) \ge q_2^+(h), \\ r_1^+(h) \ge r_1^+(h), s_1^+(h) \ge s_2^+(h), I_{A_1}^+ \ge I_{A_2}^+] \text{ and} \\ [p_1(h) \le p_2(h), q_1(h) \le q_2(h), \\ r_1(h) \le r_1(h), s_1(h) \le s_2(h), \mu_{A_1} \le \mu_{A_2}] \end{array}\right\};$$

$$\text{(b)} \left\{\begin{array}{l} A_1 \cap_{T,S} A_2 = \langle T[p_1^-(h), p_2^-(h)], \\ T[q_1^-(h), q_2^-(h)], T[r_1^-(h), r_2^-(h)], \\ T[s_1^-(h), s_2^-(h)], T[I_{A_1}^-, I_{A_2}^-], \\ T[p_1^+(h), p_2^+(h)], T[q_1^+(h), q_2^+(h)], \\ T[r_1^+(h), r_2^+(h)], T[s_1^+(h), s_2^+(h)], \\ T[I_{A_1}^+, I_{A_2}^+], S[p_1(h), p_2(h)], \\ S[q_1(h), q_2(h)], S[r_1(h), r_2(h)], \\ S[s_1(h), s_2(h)], S[\mu_{A_1}, \mu_{A_2}]\rangle \end{array}\right\};$$

$$\text{(c)} \left\{\begin{array}{l} A_1 \cup_{T,S} A_2 = \langle S[p_1^-(h), p_2^-(h)], \\ S[q_1^-(h), q_2^-(h)], S[r_1^-(h), r_2^-(h)], \\ S[s_1^-(h), s_2^-(h)], S[I_{A_1}^-, I_{A_2}^-], \\ S[p_1^+(h), p_2^+(h)], \\ S[q_1^+(h), q_2^+(h)], S[r_1^+(h), r_2^+(h)], \\ S[s_1^+(h), s_2^+(h)], \\ S[I_{A_1}^+, I_{A_2}^+], T[p_1(h), p_2(h)], \\ T[q_1(h), q_2(h)], \\ T[r_1(h), r_2(h)], \\ T[s_1(h), s_2(h)], T[\mu_{A_1}, \mu_{A_2}]\rangle \end{array}\right\};$$

where any pair (T, S) can be used. T denotes a ***t***-norm and S a so-called t-conorm dual to the t-norm T, defined by $S(h, \tau) = 1 - T(1-h, 1-\tau)$.

Example 8 Let $A_1 = \left\{\begin{array}{l} \langle[(\{0.2, 0.4, \\ 0.6, 0.8\}; 0.10], \\ [\{0.4, 0.6, \\ 0.8, 0.10\}; 0.12)]; \\ [\{0.3, 0.5, \\ 0.7, 0.9\}; 0.11]\rangle \end{array}\right\}$ and $A_2 = \left\{\begin{array}{l} \langle[\{0.10, 0.12, \\ 0.14, 0.16\}; 0.20], \\ [\{0.12, 0.14, \\ 0.16, 0.18\}; 0.24], \\ [\{0.11, 0.13, \\ 0.15, 0.17\}; 0.22]\rangle \end{array}\right\}$ be two trapezoidal cubic fuzzy sets (TrCFSs)

$$\text{(a)} \left\{\begin{array}{l} A_1 \subseteq A_2 \text{ iff } \forall h \in H, 0.2 \ge 0.10, \\ 0.4 \ge 0.12, 0.6 \ge 0.14, 0.8 \ge 0.16, \\ 0.10 \ge 0.20, 0.4 \ge 0.12, 0.6 \ge 0.14, \\ 0.8 \ge 0.16, 0.10 \ge 0.18, 0.12 \ge 0.24 \\ \text{and } 0.3 \le 0.11, 0.5 \le 0.13, \\ 0.7 \le 0.15, 0.9 \le 0.17, 0.11 \le 0.22 \end{array}\right\};$$

(b) $A_1 \cap_{T,S} A_2 = \left\{\begin{array}{l} \langle T[0.2, 0.10], T[0.4, 0.12], \\ T[0.6, 0.14], T[0.8, 0.16], \\ T[0.10, 0.20], T[0.4, 0.12], \\ T[0.6, 0.14], T[0.8, 0.16], \\ T[0.10, 0.18], T[0.12, 0.24] \text{ and} \\ S[0.3, 0.11], S[0.5, 0.13], \\ \quad S[0.7, 0.15], \\ S[0.9, 0.17], S[0.11, 0.22]\rangle \end{array}\right\}$

(c) $A_1 \cup_{T,S} A_2 = \left\{\begin{array}{l} \langle S[0.2, 0.10], S[0.4, 0.12], \\ S[0.6, 0.14], S[0.8, 0.16], \\ S[0.10, 0.20], S[0.4, 0.12], \\ S[0.6, 0.14], S[0.8, 0.16], \\ S[0.10, 0.18], S[0.12, 0.24] \text{ and} \\ T[0.3, 0.11], T[0.5, 0.13], \\ T[0.7, 0.15], T[0.9, 0.17], \\ \quad T[0.11, 0.22]\rangle \end{array}\right\}$

Definition 9 Let $A = \left\{\begin{array}{l} \langle\{[p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)], I_A^-\}, \\ \{[p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)], I_A^+\}, \\ \{[p_A(h), q_A(h), \\ r_A(h), s_A(h)], \mu_A\}\rangle \\ |h \in H \end{array}\right\}$ be the trapezoidal cubic fuzzy number (TrCFN). Then, the score function $S(A)$, accuracy function $H(A)$, membership uncertainty index $T(A)$ and hesitation uncertainty index $G(A)$of the trapezoidal cubic fuzzy number (TrCFN) A are defined by

$$S(A) = \left\{ \frac{\begin{array}{l} \langle[(I_A^-)[p_A^-(h) + q_A^-(h) + r_A^-(h) + s_A^-(h)]+ \\ (I_A^+)[p_A^+(h) + q_A^+(h) + r_A^+(h) + s_A^+(h)]] \\ -[p_A(h) + q_A(h) + r_A(h) + s_A(h)](\mu_A)\rangle \end{array}}{12} \right\};$$

$$H(A) = \left\{ \frac{\begin{array}{l} \langle[(I_A^-)[p_A^-(h) + q_A^-(h) + r_A^-(h) + s_A^-(h)]+ \\ [p_A^+(h) + q_A^+(h) + r_A^+(h) + s_A^+(h)](I_A^+)] \\ +[p_A(h) + q_A(h) + r_A(h) + s_A(h)](\mu_A)\rangle \end{array}}{12} \right\};$$

$$T(A) = \left\{\begin{array}{l} \langle[(I_A^+)[p_A^+(h) + q_A^+(h) + r_A^+(h) + s_A^+(h)]+ \\ [p_A(h) + q_A(h) + r_A(h) + s_A(h)](\mu_A)- \\ [p_A^-(h) + q_A^-(h) + r_A^-(h) + s_A^-(h)](I_A^-)\rangle \end{array}\right\};$$

$$G(A) = \left\{\begin{array}{l} \langle[(I_A^+)[p_A^+(h) + q_A^+(h) + r_A^+(h) + s_A^+(h)]+ \\ [p_A^-(h) + q_A^-(h) + r_A^-(h) + s_A^-(h)](I_A^-) \\ -[p_A(h) + q_A(h) + r_A(h) + s_A(h)](\mu_A)\rangle \end{array}\right\}.$$

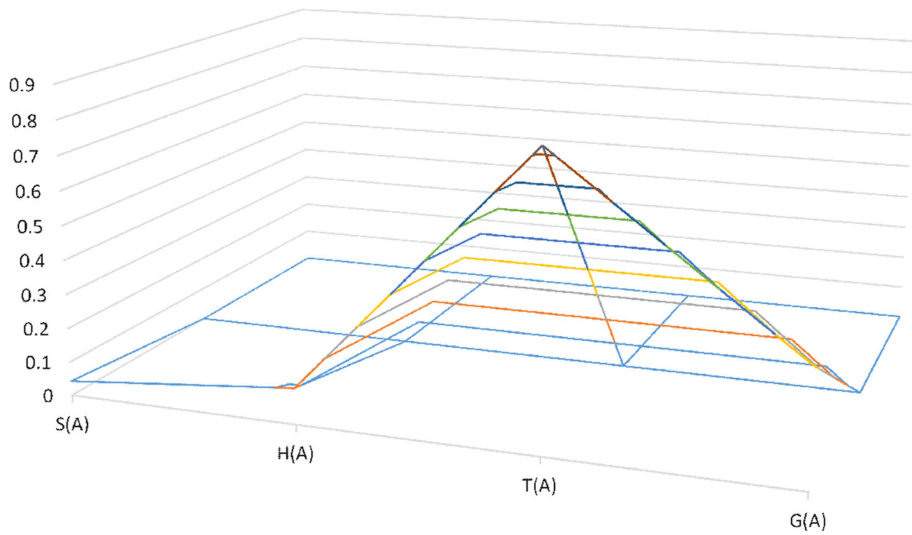
Example 10 Let $A = \left\{\begin{array}{l} \langle[\{0.6, 0.8, \\ 0.10, 0.12\}; 0.3]; \\ [\{0.8, 0.10, \\ 0.12, 0.14\}; 0.5], \\ [\{0.7, 0.9, \\ 0.11, 0.13\} \\ ; 0.4]\rangle \end{array}\right\}$ be the trapezoidal cubic fuzzy number (TrCFN). Then the score function $S(A)$, accuracy function $H(A)$, membership uncertainty index $T(A)$ and hesitation uncertainty index $G(A)$ of the trapezoidal cubic fuzzy number (TrCFN) A are defined by

$$S(A) = \left\{\begin{array}{l} \frac{\begin{array}{c} [0.6 + 0.8 + 0.10 + 0.12]\,(0.3)+ \\ [0.8 + 0.10 + 0.12 + 0.14]\,(0.5) \\ [0.7 + 0.9 + 0.11 + 0.13]\,(0.4)\rangle \end{array}}{12} \\ = \frac{0.306+0.58-0.376}{12} = \frac{0.886-0.376}{12} \\ = \frac{0.51}{12} = 0.0425 \end{array}\right\};$$

$$H(A) = \left\{\begin{array}{l} \frac{\begin{array}{c} [0.6 + 0.8 + 0.10 + 0.12]\,(0.3)+ \\ [0.8 + 0.10 + 0.12 + 0.14]\,(0.5) \\ +[0.7 + 0.9 + 0.11 + 0.13]\,(0.4)\rangle \end{array}}{12} \\ = \frac{0.306+0.58+0.376}{12} = \frac{1.262}{12} = 0.1051 \end{array}\right\};$$

$$T(A) = \left\{\begin{array}{l} [0.8 + 0.10 + 0.12 + 0.14](0.5)+ \\ [0.7 + 0.9 + 0.11 + 0.13](0.4) \\ -[0.6 + 0.8 + 0.10 + 0.12](0.3)\rangle \\ = 0.58 + 0.736 - 0.486 = 0.83 \end{array}\right\};$$

$$G(A) = \left\{\begin{array}{l} \langle[[0.8 + 0.10 + 0.12 + 0.14](0.5)+ \\ [0.6 + 0.8 + 0.10 + 0.12](0.3)- \\ [0.7 + 0.9 + 0.11 + 0.13](0.4)\rangle \\ = 0.58 + 0.486 - 0.736 = 0.33 \end{array}\right\}.$$



4 Some Einstein operations on trapezoidal cubic fuzzy sets

In this section, we introduce the Einstein t-norm

$$T\left\{T(h,\tau)=\frac{h\tau}{1+(1-h)(1-\tau)}\right\}$$

and its dual t-conorm

$$S\left\{(S(h,\tau)=\frac{h+\tau}{1+h\tau})\right\},$$

(then the generalized union d) on TrCFSs A_1 and A_2 becomes to the Einstein sum (denoted by A_1+A_2) on A_1 and A_2 as follows:

$$A_1+A_2=\left\{\left\langle\begin{array}{l}\left[\begin{array}{l}\max(I^-_{A_1},I^-_{A_2})\left[\frac{p_1^-(h)+p_2^-(h)}{1+p_1^-(h)p_2^-(h)},\right.\\ \frac{q_1^-(h)+q_2^-(h)}{1+q_1^-(h)q_2^-(h)},\\ \left.\frac{r_1^-(h)+r_2^-(h)}{1+r_1^-(h)r_2^-(h)},\frac{s_1^-(h)+s_2^-(h)}{1+s_1^-(h)s_2^-(h)}\right],\\ \max(I^+_{A_1},I^+_{A_2})\\ \left[\frac{p_1^+(h)+p_2^+(h)}{1+p_1^+(h)p_2^+(h)},\frac{q_1^+(h)+q_2^+(h)}{1+q_1^+(h)q_2^+(h)},\right.\\ \frac{r_1^+(h)+r_2^+(h)}{1+r_1^+(h)r_2^+(h)},\\ \left.\frac{s_1^+(h)+s_2^+(h)}{1+s_1^+(h)s_2^+(h)}\right],\end{array}\right],\\ \left[\begin{array}{l}\min(\mu_{A_1},\mu_{A_2})\\ \left[\frac{p_1\cdot p_2}{1+((1-p_1(h))(1-p_2(h)))},\right.\\ \frac{q_1\cdot q_2}{1+((1-q_1(h))(1-q_2(h)))},\\ \frac{r_1\cdot r_2}{1+((1-r_1(h))(1-r_2(h)))},\\ \left.\frac{s_1\cdot s_2}{1+((1-s_1(h))(1-s_2(h)))}\right]\end{array}\right]\end{array}\right\rangle\right\}.$$

By mathematical induction, we can derive the multiplication operation on trapezoidal cubic fuzzy sets (TrCFSs) as follows:

$$\lambda A=\left\{\left\langle\begin{array}{l}\left[\begin{array}{l}\max(I^-_A)\left[\frac{[1+p_A^-(h)]^\lambda-[1-p_A^-(h)]^\lambda}{[1+p_A^-(h)]^\lambda+[1-p_A^-(h)]^\lambda},\right.\\ \frac{[1+q_A^-(h)]^\lambda-[1-q_A^-(h)]^\lambda}{[1+q_A^-(h)]^\lambda+[1-q_A^-(h)]^\lambda},\\ \frac{[1+r_A^-(h)]^\lambda-[1-r_A^-(h)]^\lambda}{[1+r_A^-(h)]^\lambda+[1-r_A^-(h)]^\lambda},\\ \left.\frac{[1+s_A^-(h)]^\lambda-[1-s_A^-(h)]^\lambda}{[1+s_A^-(h)]^\lambda+[1-s_A^-(h)]^\lambda}\right]\\ \max(I^+_A)\left[\frac{[1+p_A^+(h)]^\lambda-[1-p_A^+(h)]^\lambda}{[1+p_A^+(h)]^\lambda+[1-p_A^+(h)]^\lambda},\right.\\ \frac{[1+q_A^+(h)]^\lambda-[1-q_A^+(h)]^\lambda}{[1+q_A^+(h)]^\lambda+[1-q_A^+(h)]^\lambda},\\ \frac{[1+r_A^+(h)]^\lambda-[1-r_A^+(h)]^\lambda}{[1+r_A^+(h)]^\lambda+[1-r_A^+(h)]^\lambda},\\ \left.\frac{[1+s_A^+(h)]^\lambda-[1-s_A^+(h)]^\lambda}{[1+s_A^+(h)]^\lambda+[1-s_A^+(h)]^\lambda}\right]\end{array}\right],\\ \left[\begin{array}{l}\min(\mu_A)\left[\frac{2[p_A(h)]^\lambda}{[(2-p_A(h)]^\lambda+[p_A(h)]^\lambda},\right.\\ \frac{2[p_A(h)]^\lambda}{[(2-p_A(h)]^\lambda+[p_A(h)]^\lambda},\\ \frac{2[r_A(h)]^\lambda}{[(2-r_A(h)]^\lambda+[r_A(h)]^\lambda},\\ \left.\frac{2[s_A(h)]^\lambda}{[(2-s_A(h)]^\lambda+[s_A(h)]^\lambda}\right]\end{array}\right]\end{array}\right\rangle\right\}.$$

where λ is any positive real number.

Definition 11 Let $A_1 = \left\{ \begin{array}{l} \langle[\{p_1^-(h), q_1^-(h), \\ r_1^-(h), s_1^-(h)\}; \\ I_{A_1}^-], \\ [\{p_1^+(h), q_1^+(h), \\ r_1^+(h), s_1^+(h)\}; \\ I_{A_1}^+], \\ [\{p_1(h), q_1(h), \\ r_1(h), s_1(h)\}; \\ \mu_{A_1}]\rangle | h \in H \end{array} \right\}$ and $A_2 = \left\{ \begin{array}{l} \langle[\{p_2^-(h), q_2^-(h), \\ r_2^-(h), s_2^-(h)\}; \\ I_{A_2}^-], \\ [\{p_2^+(h), q_2^+(h), \\ r_2^+(h), s_2^+(h)\}; \\ I_{A_2}^+], \\ [\{p_2(h), q_2(h), \\ r_2(h), s_2(h)\}; \\ \mu_{A_2}]\rangle | h \in H \end{array} \right\}$ be two trapezoidal cubic fuzzy numbers (TrCFNs). Then some Einstein operations of A_1 and A_2 can be defined as:

$$A_1 + A_2 = \left\{ \begin{array}{l} \left\langle \left[\begin{array}{c} \max(I_{A_1}^-, I_{A_2}^-) \\ \left[\frac{p_1^-(h)+p_2^-(h)}{1+p_1^-(h)p_2^-(h)}, \right. \\ \frac{q_1^-(h)+q_2^-(h)}{1+q_1^-(h)q_2^-(h)}, \\ \frac{r_1^-(h)+r_2^-(h)}{1+r_1^-(h)r_2^-(h)}, \\ \left. \frac{s_1^-(h)+s_2^-(h)}{1+s_1^-(h)s_2^-(h)} \right]; \\ \max(I_{A_1}^+, I_{A_2}^+) \left[\frac{p_1^+(h)+p_2^+(h)}{1+p_1^+(h)p_2^+(h))}, \right. \\ \frac{q_1^+(h)+q_1^+(h)}{1+q_1^+(h)q_2^+(h))}, \\ \frac{r_1^+(h)+r_2^+(h)}{1+r_1^+(h)r_2^+(h))}, \\ \left. \frac{s_1^+(h)+s_2^+(h)}{1+s_1^+(h)s_2^+(h))} \right] \end{array} \right], \right. \\ \left. \left[\begin{array}{l} \min(\mu_{A_1}, \mu_{A_2}) \\ \left[\frac{p_1(h)\cdot p_2(h)}{(1+(1-p_1(h))(1-p_2(h)))}, \right. \\ \frac{q_1(h)\cdot q_2(h)}{(1+((1-q_1(h))(1-q_2(h)))}, \\ \frac{r_1(h)\cdot r_2(h)}{(1+(1-r_1(h)(1-r_2(h)))}, \\ \left. \frac{s_1(h)\cdot s_2(h)}{(1+((1-s_1(h))(1-s_2(h)))} \right] \end{array} \right] \right\rangle \end{array} \right\};$$

$$\lambda A = \left\{ \begin{array}{l} \left\langle \left[\begin{array}{l} \max(I_A^-) \left[\frac{[1+p_A^-(h)]^\lambda-[1-p_A^-(h)]^\lambda}{[1+p_A^-(h)]^\lambda+[1-p_A^-(h)]^\lambda}, \right. \\ \frac{[1+q_A^-(h)]^\lambda-[1-q_A^-(h)]^\lambda}{[1+q_A^-(h)]^\lambda+[1-q_A^-(h)]^\lambda}, \\ \frac{[1+r_A^-(h)]^\lambda-[1-r_A^-(h)]^\lambda}{[1+r_A^-(h)]^\lambda+[1-r_A^-(h)]^\lambda}, \\ \frac{[1+s_A^-(h)]^\lambda-[1-s_A^-(h)]^\lambda}{[1+s_A^-(h)]^\lambda+[1-s_A^-(h)]^\lambda} \\ \max(I_A^+) \left[\frac{[1+p_A^+(h)]^\lambda-[1-p_A^+(h)]^\lambda}{[1+p_A^+(h)]^\lambda+[1-p_A^+(h)]^\lambda}, \right. \\ \frac{[1+q_A^+(h)]^\lambda-[1-q_A^+(h)]^\lambda}{[1+q_A^+(h)]^\lambda+[1-q_A^+(h)]^\lambda}, \\ \frac{[1+r_A^+(h)]^\lambda-[1-r_A^+(h)]^\lambda}{[1+r_A^+(h)]^\lambda+[1-r_A^+(h)]^\lambda}, \\ \left. \frac{[1+s_A^+(h)]^\lambda-[1-s_A^+(h)]^\lambda}{[1+s_A^+(h)]^\lambda+[1-s_A^+(h)]^\lambda} \right]; \end{array} \right], \right. \\ \left. \left[\begin{array}{l} \min(\mu_A) \left[\frac{2[p_A(h)]^\lambda}{[(2-p_A(h)]^\lambda+[p_A(h)]^\lambda}, \right. \\ \frac{2[q_A(h)]^\lambda}{[(2-q_A(h)]^\lambda+[q_A(h)]^\lambda}, \\ \frac{2[r_A(h)]^\lambda}{[(2-r_A(h)]^\lambda+[r_A(h)]^\lambda}, \\ \left. \frac{2[s_A(h)]^\lambda}{[(2-s_A(h)]^\lambda+[s_A(h)]^\lambda} \right] \end{array} \right] \right\rangle \end{array} \right\}.$$

Proposition 12 *Let A, A_1 and A_2 be three trapezoidal cubic fuzzy numbers and $\lambda, \lambda_1, \lambda_2 > 0$, then we have:*

(1) $A_1 + A_2 = A_2 + A_1$,
(2) $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$,
(3) $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$.

Proof The proof of this proposition is provided in Appendix A. □

Remark 13 If $\alpha_1 \leq_{L_{\text{TrCFN}}} \alpha_2$, then $\alpha_1 \leq \alpha_2$, that is the total order contains the usual partial order on L_{TrCFN}.

5 Trapezoidal cubic fuzzy arithmetic averaging operators based on Einstein operations

In this section, we introduce the concept of trapezoidal cubic fuzzy Einstein weighted averaging (TCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TCFEOWA) operator and discuss some of its properties.

Definition 14 (Beliakov et al. 2007; Grabisch et al. 2009) An aggregation function $f : [0, 1]^n \rightarrow [0, 1]$ is a function non-decreasing in each argument, that is $A_j \leq B_j$ for all $j \in \{1, 2, ..., n\}$ implies $f(A_1, A_2, ..., A_n) \leq f(B_1, B_2, ..., B_n)$ and satisfying $f(0, 0, ..., 0) = 0$ and $f(1, 1, ..., 1) = 1$.

Definition 15 $f_{L_{\text{TrCFN}}} : L^n_{\text{TrCFN}} \rightarrow L_{\text{TrCFN}}$ is an aggregation function if it is monotone with respect to $\leq$ L_{TrCFN} and satisfies $f_{L_{\text{TrCFN}}} \left(0_{L_{\text{TrCFN}}}, \ldots, 0_{L_{\text{TrCFN}}}\right) = 0_{L_{\text{TrCFN}}}$ and $f_{L_{\text{TrCFN}}}(1_{L_{\text{TrCFN}}}, \ldots, 1_{L_{\text{TrCFN}}}) = 1_{L_{\text{TrCFN}}}$.

Place the file in any of the directories where MS Word looks for templates. These directories are defined within MS Word under *Tools/Options/File Locations*.

5.1 Trapezoidal cubic fuzzy Einstein weighted averaging operator

Definition 16 Let $A = \left\{ \begin{array}{l} \langle[\{p^-_A(h), q^-_A(h), \\ r^-_A(h), s^-_A(h)\}; I^-_A] \\ , [\{p^+_A(h), q^+_A(h), \\ r^+_A(h), s^+_A(h)\}; I^+_A] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \mu_A]\rangle | \\ h \in H \end{array} \right\}$ be a collection of TrCFNs in L_{TrCFN} and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $A_j (j = 1, 2, ..., n)$ such that $\varpi_j \in [0, 1]$ and $\sum_{j=1}^{n} \varpi_j = 1$. Then trapezoidal cubic fuzzy Einstein weighted averaging operator of dimension n is a mapping TrCFEWA : $L^n_{\text{TrCFN}} \rightarrow L_{\text{TrCFN}}$ and TrCFEWA $(A_1, A_2, ..., A_n) = \varpi_1 A_1 + \varpi_2 A_2, ..., \varpi_n A_n$ If $\varpi = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$. Then the TrCFEWA operator is reduced to trapezoidal cubic fuzzy Einstein averaging operator of dimension n, which is defined as follows: $\text{TrCFEA}(A_1, A_2, ..., A_n) = \frac{1}{n}(A_1 + A_2 + ... + A_n)$

Example 17 Let $A_1 = \left\{ \begin{array}{l} \langle[0.9, 0.11, \\ 0.13, 0.15], (0.2) \\ [0.11, 0.13, \\ 0.15, 0.17], (0.4) \\ [0.10, 0.12, \\ 0.14, 0.16](0.3)\rangle \end{array} \right\}$ and $A_2 = \left\{ \begin{array}{l} \langle[0.2, 0.4, \\ 0.6, 0.8], (0.1) \\ [0.4, 0.6, \\ 0.8, 0.10], (0.3) \\ [0.3, 0.5, \end{array} \right\}$ be a collection of TrCFNs in L_{TrCFN}.
Then their aggregated value by using the TrCFEWA operator is also the TrCFN and $\varpi = 0.3, 0.4, 0.3$

TrCFEWA $(A_1, A_2, ..., A_n) =$

$$\langle 0.2 \begin{bmatrix} \frac{\prod_{j=1}^{2}[1+0.9]^{0.3}[1+0.2]^{0.3} - \prod_{j=1}^{2}[1-0.9]^{0.3}[1-0.2]^{0.3}}{\prod_{j=1}^{2}[1+0.9]^{0.3}[1+0.2]^{0.3} + \prod_{j=1}^{2}[1-0.9]^{0.3}[1-0.s]^{0.3}}, \\ \frac{\prod_{j=1}^{2}[1+0.11]^{0.4}[1+0.4]^{0.4} - \prod_{j=1}^{2}[1-0.11]^{0.4}[1-0.4]^{0.4}}{\prod_{j=1}^{2}[1+0.11]^{0.4}[1+0.4]^{0.4} + \prod_{j=1}^{2}[1-0.11]^{0.4}[1-0.4]^{0.4}}, \\ \frac{\prod_{j=1}^{2}[1+0.13]^{0.4}[1+0.6]^{0.4} - \prod_{j=1}^{2}[1-0.13]^{0.4}[1-0.6]^{0.4}}{\prod_{j=1}^{2}[1+0.13]^{0.4}[1+0.6]^{0.4} + \prod_{j=1}^{2}[1-0.13]^{0.4}[1-0.6]^{0.4}}, \\ \frac{\prod_{j=1}^{2}[1+0.15]^{0.4}[1+0.8]^{0.4} - \prod_{j=1}^{2}[1-0.15]^{0.4}[1-0.8]^{0.4}}{\prod_{j=1}^{2}[1+0.15]^{0.4}[1+0.8]^{0.4} + \prod_{j=1}^{2}[1-0.15]^{0.4}[1-0.8]^{0.4}} \end{bmatrix},$$

$$0.4 \begin{bmatrix} \frac{\prod_{j=1}^{2}[1+0.11]^{0.4}[1+0.4]^{0.4} - \prod_{j=1}^{2}[1-0.11]^{0.4}[1-0.4]^{0.4}}{\prod_{j=1}^{2}[1+0.11]^{0.4}[1+0.4]^{0.4} + \prod_{j=1}^{2}[1-0.11]^{0.4}[1-0.4]^{0.4}}, \\ \frac{\prod_{j=1}^{2}[1+0.13]^{0.4}[1+0.6]^{0.4} - \prod_{j=1}^{2}[1-0.13]^{0.4}[1-0.6]^{0.4}}{\prod_{j=1}^{2}[1+0.13]^{0.4}[1+0.6]^{0.4} + \prod_{j=1}^{2}[1-0.13]^{0.4}[1-0.6]^{0.4}}, \\ \frac{\prod_{j=1}^{2}[1+0.15]^{0.4}[1+0.8]^{0.4} - \prod_{j=1}^{2}[1-0.15]^{0.4}[1-0.8]^{0.4}}{\prod_{j=1}^{2}[1+0.15]^{0.4}[1+0.8]^{0.4} + \prod_{j=1}^{2}[1-0.15]^{0.4}[1-0.8]^{0.4}}, \\ \frac{\prod_{j=1}^{2}[1+0.17]^{0.4}[1+0.10]^{0.4} - \prod_{j=1}^{2}[1-0.17]^{0.4}[1-0.10]^{0.4}}{\prod_{j=1}^{2}[1+0.17]^{0.4}[1+0.10]^{0.4} + \prod_{j=1}^{2}[1-0.17]^{0.4}[1-0.10]^{0.4}} \end{bmatrix}$$

$$(0.2) \begin{bmatrix} \frac{2\prod_{j=1}^{2}[0.10]^{0.3}[0.3]^{0.3}}{\prod_{j=1}^{2}[(2-0.10]^{0.3}[(2-0.3]^{0.3} + \prod_{j=1}^{2}[0.10]^{0.3}[0.3]^{0.3}}, \\ \frac{2\prod_{j=1}^{2}[0.12]^{0.3}[0.5]^{0.3}}{\prod_{j=1}^{2}[(2-0.12]^{0.3}[(2-0.5]^{0.3} + \prod_{j=1}^{2}[0.12]^{0.3}[0.5]^{0.3}}, \\ \frac{2\prod_{j=1}^{2}[0.14]^{0.3}[0.7]^{0.3}}{\prod_{j=1}^{2}[(2-0.14]^{0.3}[(2-0.7]^{0.3} + \prod_{j=1}^{2}[0.14]^{0.3}[0.7]^{0.3}}, \\ \frac{2\prod_{j=1}^{2}[0.16]^{0.3}[0.9]^{0.3}}{\prod_{j=1}^{2}[(2-0.16]^{0.3}[(2-0.9]^{0.3} + \prod_{j=1}^{2}[0.16]^{0.3}[0.9]^{0.3}} \end{bmatrix} \rangle$$

$$= \left\{ \begin{array}{l} \langle[0.2\frac{1.1382-0.8312}{1.1382+0.8312}, 0.4\frac{1.3493-0.6009}{1.3493+0.6009}], \\ (0.2)[\frac{1.0023}{1.3471+0.5012}]\rangle \\ = \langle[(0.2\frac{0.307}{1.9694}, 0.4\frac{0.7484}{1.9502}], \\ (0.2)[\frac{1.0023}{1.8486}]\rangle \\ = \langle[0.2(0.1558); 0.4(0.3837)]; \\ (0.2)[0.5421]\rangle \\ = \langle[0.0311; 0.1534]; 0.1084\rangle \end{array} \right\}.$$

Theorem 18 *Let* $A = \left\{ \begin{array}{l} \langle [\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; I_A^-], \\ [\{p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)\}; I_A^+], \\ [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \mu_A]\rangle \\ | h \in H \end{array} \right\}$ *be a collection of TrCFNs in* L_{TrCFN} *. Then their aggregated value by using the TrCFEWA operator is also the TrCFN and TrCFEWA* $(A_1, A_2, ..., A_n) = \langle \max(I_A^-)$

$$\begin{bmatrix} \frac{[\prod_{j=1}^n [1+p_1^-(h)]^{\varpi} - \prod_{j=1}^n [1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^n [1+p_1^-(h)]^{\varpi} + \prod_{j=1}^n [1-p_1^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+q_1^-(h)]^{\varpi} - \prod_{j=1}^n [1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^n [1+q_1^-(h)]^{\varpi} + \prod_{j=1}^n [1-q_1^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+r_1^-(h)]^{\varpi} - \prod_{j=1}^n [1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^n [1+r_1^-(h)]^{\varpi} + \prod_{j=1}^n [1-r_1^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+s_1^-(h)]^{\varpi} - \prod_{j=1}^n [1-s_1^-(h)]^{\varpi}}{\prod_{j=1}^n [1+s_1^-(h)]^{\varpi} + \prod_{j=1}^n [1-s_1^-(h)]^{\varpi}} \end{bmatrix},$$

$$\max(I_A^+) \begin{bmatrix} \frac{\prod_{j=1}^n [1+p_1^+(h)]^{\varpi} - \prod_{j=1}^n [1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^n [1+p_1^+(h)]^{\varpi} + \prod_{j=1}^n [1-p_1^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+q_1^+(h)]^{\varpi} - \prod_{j=1}^n [1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^n [1+q_1^+(h)]^{\varpi} + \prod_{j=1}^n [1-q_1^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+r_1^+(h)]^{\varpi} - \prod_{j=1}^n [1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^n [1+r_1^+(h)]^{\varpi} + \prod_{j=1}^n [1-r_1^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+s_1^+(h)]^{\varpi} - \prod_{j=1}^n [1-s_1^+(h)]^{\varpi}}{\prod_{j=1}^n [1+s_1^+(h)]^{\varpi} + \prod_{j=1}^n [1-s_1^+(h)]^{\varpi}} \end{bmatrix},$$

$$\min(\mu_A) \begin{bmatrix} \frac{2\prod_{j=1}^n [p_1(h)]^{\varpi}}{\prod_{j=1}^n [(2-p_1(h)]^{\varpi} + \prod_{j=1}^n [p_1(h)]^{\varpi}}, \\ \frac{2\prod_{j=1}^n [q_1(h)]^{\varpi}}{\prod_{j=1}^n [(2-q_1(h)]^{\varpi} + \prod_{j=1}^n [q_1(h)]^{\varpi}}, \\ \frac{2\prod_{j=1}^n [r_1(h)]^{\varpi}}{\prod_{j=1}^n [(2-r_1(h)]^{\varpi} + \prod_{j=1}^n [r_1(h)]^{\varpi}}, \\ \frac{2\prod_{j=1}^n [s_1(h)]^{\varpi}}{\prod_{j=1}^n [(2-s_1(h)]^{\varpi} + \prod_{j=1}^n [s_1(h)]^{\varpi}} \end{bmatrix} \rangle$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *is the weight vector of* $A_j (j = 1, 2, ..., n)$ *such that* $\varpi_j \in [0, 1]$ *and* $\sum_{j=1}^n \varpi_j = 1$.

Proof The proof of this theorem is provided in "Appendix B." □

Proposition 19 *Let* $A = \left\{ \begin{array}{l} \langle [\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; I_A^-] \\ [\{p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)\}; I_A^+] \\ [\{p_A(h), q_A(h), \\ r_A(h), s_A(h)\}, \mu_A]\rangle | \\ h \in H \end{array} \right\}$ *be a collection of TrCFNs in* L_{TrCFN} *and where* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *is the weight vector of* $A_j (j = 1, 2, ..., n)$ *such that* $\varpi_j \in [0, 1]$ *and* $\sum_{j=1}^n \varpi_j = 1$ *. Then*

(1) *(Idempotency): If all* A_j, $j = 1, 2, ..., n$ *are equal, i.e.,* $A_j = A$, *for all* $j = 1, 2, ..., n$, *then TrCFEWA* $(A_1, A_2, ..., A_n) = A$.

(2) *(Boundary): If*

$$p_{\min}^- = \min_{1\le j\le n} p_j^-,$$
$$q_{\min}^- = \min_{1\le j\le n} q_j^-, \quad r_{\min}^- = \min_{1\le j\le n} r_j^-,$$
$$s_{\min}^- = \min_{1\le j\le n} s_j^-, \quad I_{\min}^- = \min_{1\le j\le n} I_j^-,$$
$$p_{\min}^+ = \min_{1\le j\le n} p_j^+, \quad q_{\min}^+ = \min_{1\le j\le n} q_j^+,$$
$$r_{\min}^+ = \min_{1\le j\le n} r_j^+, \quad s_{\min}^+ = \min_{1\le j\le n} s_j^+,$$
$$I_{\min}^+ = \min_{1\le j\le n} I_j^+, \quad p_{\max} = \max_{1\le j\le n} p_j,$$
$$q_{\max} = \max_{1\le j\le n} q_j, \quad r_{\max} = \max_{1\le j\le n} r_j,$$
$$s_{\max} = \max_{1\le j\le n} s_j, \quad \mu_{\max} = \max_{1\le j\le n} \mu_j,$$
$$p_{\max}^- = \max_{1\le j\le n} p_j^-, \quad q_{\max}^- = \max_{1\le j\le n} q_j^-,$$
$$r_{\max}^- = \max_{1\le j\le n} r_j^-, \quad s_{\max}^- = \max_{1\le j\le n} s_j^-$$
$$\mu_{\max}^- = \max_{1\le j\le n} \mu_j^-, \quad p_{\max}^+ = \max_{1\le j\le n} p_j^+,$$
$$q_{\max}^+ = \max_{1\le j\le n} q_j^+, \quad r_{\max}^+ = \max_{1\le j\le n} r_j^+,$$
$$s_{\max}^+ = \max_{1\le j\le n} s_j^+, \quad I_{\max}^+ = \max_{1\le j\le n} I_j^+,$$
$$p_{\min} = \min_{1\le j\le n} p_j, \quad q_{\min} = \min_{1\le j\le n} q_j,$$
$$r_{\min} = \min_{1\le j\le n} r_j, \quad s_{\min} = \min_{1\le j\le n} s_j,$$
$$\mu_{\min} = \min_{1\le j\le n} \mu_j$$

for all $j = 1, 2, .., n$, *we can obtain that* $\left\{ \begin{array}{l} \langle [p_{\min}^-(h), q_{\min}^-(h), \\ r_{\min}^-(h), s_{\min}^-(h)], (I_{A_j}^-) \\ [p_{\min}^+(h), q_{\min}^+(h), \\ r_{\min}^+(h), s_{\min}^+(h)], (I_{A_j}^+) \\ [p_{\max}(h), q_{\max}(h), \\ r_{\max}(h), s_{\max}(h)], (\mu_{A_j})\rangle | \\ h \in H \end{array} \right\} \le$ *TrCFEWA* $(A_1, A_2, ..., A_n)$

$$\le \left\{ \begin{array}{l} \langle [p_{\max}^-(h), q_{\max}^-(h), \\ r_{\max}^-(h), s_{\max}^-(h)], (I_{A_j}^-) \\ [p_{\max}^+(h), q_{\max}^+(h), \\ r_{\max}^+(h), s_{\max}^+(h)], (I_{A_j}^+) \\ [p_{\min}(h), q_{\min}(h), \\ r_{\min}(h), s_{\min}(h)], (\mu_{A_j})\rangle | \\ h \in H \end{array} \right\}.$$

(3) *(Monotonicity):* $A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; \\ I_A^-], [\{p_A^+(h), \\ q_A^+(h), r_A^+(h), \\ s_A^+(h)\}; I_A^+] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \\ \mu_A]\rangle | h \in H \end{array}\right\}$ *and* $B = \left\{\begin{array}{c} \langle[p_B^-(h), q_B^-(h), \\ r_B^-(h), s_B^-(h)], \\ (I_B^-), [p_B^+(h), \\ q_B^+(h), r_B^+(h), \\ s_B^+(h)], (I_B^+) \\ [p_B(h), q_B(h), \\ r_B(h), s_B(h)], \\ (\mu_B)\rangle | h \in H \end{array}\right\}$ *be two collection of TrCFNs in* L_{TrCFN} *and* $A_j \leq L_{\text{TrCFN}} B_j$*, i.e.,*

$$\begin{array}{ll} p_A^-(h) \leq p_B^-(h), & q_A^-(h) \leq q_B^-(h), \\ r_A^-(h) \leq r_B^-(h), & s_A^-(h) \leq s_B^-(h), \\ p_A^+(h) \leq p_B^+(h), & q_A^+(h) \leq q_B^+(h), \\ r_A^+(h) \leq r_B^+(h), & s_A^+(h) \leq s_B^+(h) \text{ and } p_A(h) \leq p_B(h), \\ q_A(h) \leq q_B(h), & r_A(h) \leq r_B(h), \\ s_A(h) \leq s_B(h) & \end{array}$$

then $TrCFEWA(A_1, A_2, ..., A_n) \leq TrCFEWA(B_1, B_2, ..., B_n)$.

Proof The proof of these propositions 1, 2, 3 is provided in "Appendix C." □

Corollary 20 *Let* $A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; \\ I_A^-], [\{p_A^+(h), \\ q_A^+(h), r_A^+(h), \\ s_A^+(h)\}; I_A^+] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \\ \mu_A]\rangle | h \in H \end{array}\right\}$ *be a collection of TrCFNs in* L_{TrCFN} *and where* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *is the weight vector of* $A_j (j = 1, 2, ..., n)$ *such that* $\varpi_j \in [0, 1]$ *and* $\sum_{j=1}^n \varpi_j = 1$. *Then*
TrCFWA $(A_1, A_2, ., A_n) \leq$ *TrCFWA* $(A_1, A_2, ., A_n)$.

Proof Omitted. □

5.2 Trapezoidal cubic fuzzy Einstein ordered weighted averaging operator

We also develop a type of trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator.

Definition 21 Let $A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}, \\ I_A^-], [\{p_A^+(h), \\ q_A^+(h), r_A^+(h), \\ s_A^+(h)\}, I_A^+] \\ [\{p_A(h), q_A(h), \\ r_A(h), s_A(h)\}, \\ \mu_A]\rangle | h \in H \end{array}\right\}$ be a collection of TrCFNs in L_{TrCFN}, is also the TrCFEOWA operator of dimension n, is a mapping TrCFEOWA : $L_{\text{TCFN}}^n \to L_{\text{TCFN}}$, that is an associated vector $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ such that $\varpi_j \in [0, 1]$ and $\sum_{j=1}^n \varpi_j = 1$. TrCFEOWA $(A_1, A_2, ..., A_n) = \varpi_1 A_{(\sigma)1} + \varpi_2 A_{(\sigma)2}, ..., \varpi_n A_{(\sigma)n}$, where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $A_{\sigma(1)} \leq A_{\sigma(j-1)}$ for all $j = 2, 3, ..., n (i.e., A_{\sigma(j)}$ is the j th largest value in the collection $(A_1, A_2, ..., A_n)$. If $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the TrCFEOWA operator is reduced to the TrCFA operator (2) of dimension n.

Theorem 22 *Let* $A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}, \\ I_A^-][\{p_A^+(h), \\ q_A^+(h), r_A^+(h), \\ s_A^+(h)\}, I_A^+] \\ [\{p_A(h), q_A(h), \\ r_A(h), s_A(h)\}, \\ \mu_A]\rangle | h \in H \end{array}\right\}$ *be a collection of trapezoidal cubic fuzzy numbers (TrCFNs) in* L_{TrCFN}. *Then their aggregated value by using the TCFEOWA operator is also the TrCFN and*
TrCFEOWA $(A_1, A_2, ..., A_n) =$

$$\langle \max[I_A^-] \left[\begin{array}{c} \frac{\prod_{j=1}^n [1+p_{\sigma(j)}^-(h)]^{\varpi} - \prod_{j=1}^n [1-p_{\sigma(j)}^-(h)]^{\varpi}}{\prod_{j=1}^n [1+p_{\sigma(j)}^-(h)]^{\varpi} + \prod_{j=1}^n [1-p_{\sigma(j)}^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+q_{\sigma(j)}^-(h)]^{\varpi} - \prod_{j=1}^n [1-q_{\sigma(j)}^-(h)]^{\varpi}}{\prod_{j=1}^n [1+q_{\sigma(j)}^-(h)]^{\varpi} + \prod_{j=1}^n [1-q_{\sigma(j)}^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+r_{\sigma(j)}^-(h)]^{\varpi} - \prod_{j=1}^n [1-r_{\sigma(j)}^-(h)]^{\varpi}}{\prod_{j=1}^n [1+r_{\sigma(j)}^-(h)]^{\varpi} + \prod_{j=1}^n [1-r_{\sigma(j)}^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+s_{\sigma(j)}^-(h)]^{\varpi} - \prod_{j=1}^n [1-s_{\sigma(j)}^-(h)]^{\varpi}}{\prod_{j=1}^n [1+s_{\sigma(j)}^-(h)]^{\varpi} + \prod_{j=1}^n [1-s_{\sigma(j)}^-(h)]^{\varpi}} \end{array}\right],$$

$$\max[I_A^+] \left[\begin{array}{c} \frac{\prod_{j=1}^n [1+p_{\sigma(j)}^+(h)]^{\varpi} - \prod_{j=1}^n [1-p_{\sigma(j)}^+(h)]^{\varpi}}{\prod_{j=1}^n [1+p_{\sigma(j)}^+(h)]^{\varpi} + \prod_{j=1}^n [1-p_{\sigma(j)}^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+q_{\sigma(j)}^+(h)]^{\varpi} - \prod_{j=1}^n [1-q_{\sigma(j)}^+(h)]^{\varpi}}{\prod_{j=1}^n [1+q_{\sigma(j)}^+(h)]^{\varpi} + \prod_{j=1}^n [1-q_{\sigma(j)}^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+r_{\sigma(j)}^+(h)]^{\varpi} - \prod_{j=1}^n [1-r_{\sigma(j)}^+(h)]^{\varpi}}{\prod_{j=1}^n [1+r_{\sigma(j)}^+(h)]^{\varpi} + \prod_{j=1}^n [1-r_{\sigma(j)}^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^n [1+s_{\sigma(j)}^+(h)]^{\varpi} - \prod_{j=1}^n [1-s_{\sigma(j)}^+(h)]^{\varpi}}{\prod_{j=1}^n [1+s_{\sigma(j)}^+(h)]^{\varpi} + \prod_{j=1}^n [1-s_{\sigma(j)}^+(h)]^{\varpi}} \end{array}\right],$$

$$\min[\mu_A]\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}(h)]^{\mu}+\prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}(h)]^{\varpi}+\prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}(h)]^{\varpi}+\prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[s_{\sigma(j)}(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-s_{\sigma(j)}(h)]^{\varpi}+\prod_{j=1}^{n}[s_{\sigma(j)}(h)]^{\varpi}}\end{array}\right]\rangle$$

where $(\sigma(1),\sigma(2),...,\sigma(n))$ *is a permutation of* $(1,2,...,n)$ *such that* $A_{\sigma(1)}\le A_{\sigma(j-1)}$ *for all* $j=2,3,...,n$, $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ *is the weight vector of* $A_j(j=1,2,...,n)$ *such that* $\varpi_j\in[0,1]$, *and* $\sum_{j=1}^{n}\varpi_j=1$.

Proof The process of this proof is the same as theorem 1. □

Proposition 23 *Let* $A=\left\{\begin{array}{l}\langle[\{p_A^-(h),q_A^-(h),\\ r_A^-(h),s_A^-(h)\},I_A^-]\\ [\{p_A^+(h),q_A^+(h),\\ r_A^+(h),s_A^+(h)\},I_A^+]\\ [\{p_A(h),q_A(h),\\ r_A(h),s_A(h)\},\\ \mu_A]\rangle|\\ h\in H\end{array}\right\}$ *be a collection of TrCFNs in* L_{TrCFN} *and where* $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ *be the weighting vector of the TrCFEOWA operator, such that* $\varpi_j\in[0,1]$ *and* $\sum_{j=1}^{n}\varpi_j=1$ *. Then TrCFEOWA* $(A_1,A_2,...,A_n)$ $\le$ *TrCFEOWA* $(B_1,B_2,...,B_n)$ *, where* $(A_1,A_2,...,A_n)$ *is any permutation of* $(B_1,B_2,...,B_n)$*.*

Based on algebraic operations on trapezoidal cubic fuzzy sets (TrCFSs), defined the TrCFEOWA operator as follows:

TrCFEOWA $(A_1,A_2,...,A_n)=$

$$\left\{\begin{array}{l}\langle\{\max(I_A^-)[1-\prod_{j=1}^{n}[1-p_{\sigma(j)}^-],\\ [1-\prod_{j=1}^{n}[1-q_{\sigma(j)}^-]],[1-\prod_{j=1}^{n}[1-r_{\sigma(j)}^-]]\\ ,[1-\prod_{j=1}^{n}[1-s_{\sigma(j)}^-]]\},\{\max(I_A^+)\\ [1-\prod_{j=1}^{n}[1-p_{\sigma(j)}^+]],[1-\prod_{j=1}^{n}[1-q_{\sigma(j)}^+]],\\ [1-\prod_{j=1}^{n}[1-r_{\sigma(j)}^+]],[1-\prod_{j=1}^{n}[1-s_{\sigma(j)}^+]]\},\\ \min(\mu_A)\{\prod_{j=1}^{n}p_{\sigma(j)},\prod_{j=1}^{n}q_{\sigma(j)},\prod_{j=1}^{n}r_{\sigma(j)},\\ \prod_{j=1}^{n}s_{\sigma(j)}\}\rangle\end{array}\right\}$$

and proposed an approach to solving group decision-making problems, where $A=\left\{\begin{array}{l}\langle[\{p_A^-(h),q_A^-(h),\\ r_A^-(h),s_A^-(h)\};I_A^-]\\ ,[\{p_A^+(h),q_A^+(h),\\ r_A^+(h),s_A^+(h)\};I_A^+]\\ ,[\{(p_A(h),q_A(h),\\ r_A(h),s_A(h)\};\mu_A]\rangle|\\ h\in H\end{array}\right\}$ *be a collection of TrCFNs in* L_{TrCFN}, $(\sigma(1),\sigma(2),...,\sigma(n))$ *is a permutation of* $(1,2,...,n)$ *such that* $A_{\sigma(j)}\le A_{\sigma(j-1)}$ *for all* $j=2,3,...,n$ *and where* $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ *is the weight vector of the TrCFEOWA operator* $A_j(j=1,2,...,n)$ *such that* $\varpi_j\in[0,1]$ *and* $\sum_{j=1}^{n}\varpi_j=1$.

Proof Omitted. □

Corollary 24 *Let* $A=\left\{\begin{array}{l}\langle[\{p_A^-(h),q_A^-(h),\\ r_A^-(h),s_A^-(h)\},I_A^-]\\ [\{p_A^+(h),q_A^+(h),\\ r_A^+(h),s_A^+(h)\},I_A^+]\\ [\{p_A(h),q_A(h),\\ r_A(h),s_A(h)\},\mu_A]\rangle|\\ h\in H\end{array}\right\}$ *be a collection of TrCFNs in* L_{TrCFN} *and where* $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ *is the weight vector of TrCFEOWA such that* $\varpi_j\in[0,1]$ *and* $\sum_{j=1}^{n}\varpi_j=1$ *. Then TrCFEOWA* $(A_1,A_2,...,A_n)\le$ *TrCFEOWA* $(A_1,A_2,...,A_n)$*.*

Proof Omitted □

5.3 Trapezoidal cubic fuzzy Einstein hybrid weighted averaging operator

The TrCFEWA operator weights individual the TrCFNs and the TrCFEOWA operator weights individual the ordered positions of the TrCFNs. We develop trapezoidal cubic fuzzy hybrid averaging (TrCFEHWA) operator, which weights together the given TrCFN and its well-ordered position.

Definition 25 Let $A=\left\{\begin{array}{l}\langle[\{p_A^-(h),q_A^-(h),\\ r_A^-(h),s_A^-(h)\},I_A^-]\\ [\{p_A^+(h),q_A^+(h),\\ r_A^+(h),s_A^+(h)\},I_A^+]\\ [\{p_A(h),q_A(h),\\ r_A(h),s_A(h)\},\mu_A]\rangle|\\ h\in H\end{array}\right\}$ be a collection of TrCFNs in L_{TrCFN} and $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ is the weight vector of $A_j(j=1,2,...,n)$ such that $\varpi_j\in[0,1]$ and $\sum_{j=1}^{n}\varpi_j=1$. Then trapezoidal cubic fuzzy Einstein hybrid weighted averaging operator of dimension n is a mapping TrCFEHWA : $L_{\text{TCFN}}^n\to L_{\text{TCFN}}$, that is an associated vector $w=(w_1,w_2,...,w_n)^T$ such that $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$.

TrCFEHWA $(A_1,A_2,...,A_n)=p_1A_{\sigma(1)}+p_2A_{\sigma(1)},...,p_nA_{\sigma(1)}$. If $p=\theta\varpi_{\sigma(j)}+(1-\theta)w_{\sigma(j)}$ with a balancing coefficient $\theta\in[0,1]$, $(\sigma(1),\sigma(2),...,\sigma(n))$ is a permutation of $(1,2,...,n)$ such that $A_{\sigma(j)}\le A_{\sigma(j-1)}$ for all $j=2,3,...,n(i.e.,A_{\sigma(j)}$ is the j th largest value in the collection $(A_1,A_2,...,A_n)$.

Theorem 26 *Let* $A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; I_A^-] \\ , [\{p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)\}; I_A^+] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \mu_A]\rangle| \\ h \in H \end{array}\right\}$ *be a collection of trapezoidal cubic fuzzy numbers (TrCFNs) in* L_{TrCFN} *and* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *is the weight vector of* $A_j (j = 1, 2, ..., n)$ *such that* $\varpi_j \in [0, 1]$ *and* $\sum_{j=1}^n \varpi_j = 1$ *. Then their aggregated value by using the TrCFEHWA operator, which is an associated vector* $w = (w_1, w_2, ..., w_n)^T$*, is the weight vector of* $A_j (j = 1, 2, ..., n)$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^n w_j = 1$ *, is also the TrCFN and TrCFEWA* $(A_1, A_2, ..., A_n) = \langle \max(I_A^-)$

$$\left[\begin{array}{c} \frac{\prod_{j=1}^n[1+p^-_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-p^-_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+p^-_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-p^-_{\sigma(j)}]^\alpha}, \\ \frac{\prod_{j=1}^n[1+q^-_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-q^-_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+q^-_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-q^-_{\sigma(j)}]^\alpha} \\ , \frac{\prod_{j=1}^n[1+r^-_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-r^-_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+r^-_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-r^-_{\sigma(j)}]^\alpha}, \\ \frac{\prod_{j=1}^n[1+s^-_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-s^-_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+s^-_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-s^-_{\sigma(j)}]^\alpha} \end{array}\right],$$

$$\max(I_A^+)\left[\begin{array}{c} \frac{\prod_{j=1}^n[1+p^+_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-p^+_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+p^+_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-p^+_{\sigma(j)}]^\alpha}, \\ \frac{\prod_{j=1}^n[1+q^+_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-q^+_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+q^+_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-q^+_{\sigma(j)}]^\alpha}, \\ \frac{\prod_{j=1}^n[1+r^+_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-r^+_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+r^+_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-r^+_{\sigma(j)}]^\alpha}, \\ \frac{\prod_{j=1}^n[1+s^+_{\sigma(j)}]^\alpha-\prod_{j=1}^n[1-s^+_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[1+s^+_{\sigma(j)}]^\alpha+\prod_{j=1}^n[1-s^+_{\sigma(j)}]^\alpha} \end{array}\right],$$

$$\min(\mu_A)\left[\begin{array}{c} \frac{2\prod_{j=1}^n[p_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[(2-p_{\sigma(j)}]^\alpha+\prod_{j=1}^n[p_{\sigma(j)}]^\alpha}, \\ \frac{2\prod_{j=1}^n[q_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[(2-q_{\sigma(j)}]^\alpha+\prod_{j=1}^n[q_{\sigma(j)}]^\alpha}, \\ \frac{2\prod_{j=1}^n[r_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[(2-r_{\sigma(j)}]^\alpha+\prod_{j=1}^n[r_{\sigma(j)}]^\alpha}, \\ \frac{2\prod_{j=1}^n[s_{\sigma(j)}]^\alpha}{\prod_{j=1}^n[(2-s_{\sigma(j)}]^\alpha+\prod_{j=1}^n[s_{\sigma(j)}]^\alpha} \end{array}\right]\rangle.$$

If $p = \theta\varpi_{\sigma(j)} + (1-\theta)w_{\sigma(j)}$ *with a balancing coefficient* $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $A_{\sigma(j)} \leq A_{\sigma(j-1)}$ *for all* $j = 2, 3, ..., n$ $(i.e., A_{\sigma(j)}$ *is the* j *th largest value in the collection* $(A_1, A_2, ..., A_n)$.

6 An approach to multiple attribute decision making with trapezoidal cubic fuzzy information

A multiple attribute decision-making (MADM) problem is to find a best compromise solution from all feasible alternatives assessed on multiple attributes. Let $h = \{h_1, h_2, ..., h_n\}$ be a discrete set of alternatives and $G = \{g_1, g_2, ..., g_n\}$ be the set of attributes. Suppose the rating of alternatives $h_i \quad (i = 1, 2, ..., n)$ on attributes $g_j \quad (j = 1, 2, ..., m)$ given by decision maker are trapezoidal cubic fuzzy numbers (TrCFNs) in L_{TrCFN} :

$$A = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; I_A^-] \\ , [\{p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)\}; I_A^+] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \mu_A]\rangle| \\ h \in H \end{array}\right\}, \quad \text{where}$$

$\left\{\begin{array}{l} [p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)](I_A^-), \\ [p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)](I_A^+) \end{array}\right\}$ indicates the interval-valued trapezoidal fuzzy set that the alternative h_i satisfies the attribute g_j and $\left\{\begin{array}{l} [p_A(h), q_A(h), \\ r_A(h), s_A(h)](\mu_A) \end{array}\right\}$ indicates the trapezoidal fuzzy set that the alternative h_i does not satisfy the attribute g_j . Hence, a multiple attribute decision-making (MADM) problem can be concisely expressed in the trapezoidal cubic fuzzy decision matrix $D = (A_{ij})_{m\times n} = \left\{\begin{array}{l} \langle[\{p_A^-(h), q_A^-(h), \\ r_A^-(h), s_A^-(h)\}; I_A^-] \\ , [\{p_A^+(h), q_A^+(h), \\ r_A^+(h), s_A^+(h)\}; I_A^+] \\ , [\{(p_A(h), q_A(h), \\ r_A(h), s_A(h)\}; \mu_A]\rangle| \\ h \in H \end{array}\right\}$.

Next, we shall apply the TrCFEHWA operator to deal with the multiple attribute decision-making (MADM) problem, which involves the following steps.

Step 1: Get the normalized trapezoidal cubic fuzzy decision matrix. The following normalization formula $E^k = (\beta_{ij}^k)_{n\times 1}$. Accordingly, we attain the normalized trapezoidal cubic fuzzy decision matrix $E = (\beta_{ij})_{n\times m}$.

Step 2. Utilize the TrCFEHWA operator to aggregate all the rating values $\beta_{ij} \quad (j = 1, 2, ..., m)$ of the i th line and get the overall rating value β_{ij} corresponding to the alternative $h_i \quad (i = 1, 2, ..., n)$, i.e., $\beta_{ij} =$ TrCFHWA $(\beta_{i1}, \beta_{i2}, ..., \beta_{im})$, $(i = 1, 2, ..., n)$, where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the attribute weight vector of $g_j \quad (j = 1, 2, ..., m)$ such that $\varpi_j \in [0, 1]$, $(j = 1, 2, ..., m)$ and $\sum_{j=1}^n \varpi_j = 1$. $w = (w_1, w_2, ..., w_n)^T$ is the associated vector of the TrCFEHWA operator, such that $w_j \in [0, 1]$, $j = 1, 2, ..., n$ and $\sum_{j=1}^n w_j = 1$.

Step 3. Rank the order of all alternatives. Utilize the method in definition 8 to rank the overall rating values $\beta_i \quad (i = 1, 2, ..., n)$ and rank all the alternatives $h_i \quad (i = 1, 2, ..., n)$ in accordance with $\beta_i \quad (i = 1, 2, ..., n)$ in ascending order.

Step 4: Lastly, we choice the most appropriate alternative(s) with the smallest largely rating value.

7 Illustrative example

In this section, a multiple attribute decision-making (MADM) problem involves the prioritization of a set of propulsion systems used to illustrate the developed operator.

The propulsion system selection is based on the study that has been conducted for the selection of propulsion system of a double-ended passenger ferry to operate across the Lahore in Karachi with the aim of reducing the journey time in highly congested seaway traffic. Propulsion system alternatives are given the set of alternatives $A = \{A_1, A_2, A_3, A_4\}$

A_1 : Conventional propeller and high lift rudder;
A_2 : Get-up-and-go,
A_3 : Cycloidal propeller,
A_4 : Outmoded

The selection decision is made on the basis of one objective and four subjective attributes, which are the following:

C_1 : Investment cost (IC);
C_2 : Reparation and maintenance expenditures;
C_3 : Maneuverability (MV);
C_4 : Vibration and noise (VN).

where the attribute weight vector is $\mu = (0.24, 0.35, 0.41)$. Therefore, trapezoidal cubic fuzzy multiple attribute decision making (MADM) problem is to choose the appropriate propulsion system from among three alternatives.

Assume that the decision maker uses the linguistic terms to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown in Table 1. (Here the relationship between the linguistic terms and the corresponding TrCFNs in L_{TrCFN} as shown in Tables 2, 3, 4.

Step 1: Get the normalized trapezoidal cubic fuzzy decision matrix. The following normalization formula $E^k = (\beta_{ij}^k)_{n\times 1}$.

Henceforth, we acquire the standardized trapezoidal cubic fuzzy decision matrix $E = (\beta_{ij})_{n\times m}$.

Step 2. Utilize the TrCFEHWA operator to total all the rating values $\beta_{ij}(j = 1, 2, ..., m)$ of the i th line and get the general rating value β_{ij} comparing to the alternative h_i

$$w = (0.25, 0.25, 0.25, 0.25)$$

Step 3. Rank the request of all choices. Use the technique in definition 8 to rank the general rating values $\beta_i \quad (i = 1, 2, ..., n)$ and rank every one of the alternatives $h_i \quad (i = 1, 2, ..., n)$ in accordance with $\beta_i \quad (i = 1, 2, ..., n)$ in ascending order. At long last, we select the most alluring alternative(s) with the littlest general rating esteem.

Table 1 Linguistic terms

Linguistic terms	TrCFVs
Very high (VH)	{⟨[0.4,0.6,0.8,0.10],0.2[0.6,0.8,0.10,0.12], 0.4[0.5,0.7,0.9,0.11],0.3⟩}
Very low (VL)	{⟨[0.6,0.8,0.10,0.12],0.2[0.8,0.10,0.12,0.14],0.4 [0.7,0.9,0.11,0.13],0.3⟩}
Low (L)	{⟨[0.16,0.18,0.20,0.22],0.2[0.18,0.20,0.22,0.24],0.4 [0.17,0.19,0.21,0.23],0.3⟩}
Medium low (ML)	{⟨[0.5,0.7,0.9,0.11],0.2[0.7,0.9,0.11,0.13],0.4 [0.6,0.8,0.10,0.12],0.3⟩}
Medium (M)	{⟨[0.9,0.11,0.13,0.15],0.2[0.11,0.13,0.15,0.17],0.4 [0.10,0.12,0.14,0.16],0.3⟩}
Medium high (MH)	{⟨[0.12,0.14,0.16,0.18],0.2[0.14,0.16,0.18,0.20],0.4 [0.13,0.15,0.17,0.19],0.3⟩}
High	{⟨[0.25,0.27,0.29,0.31],0.2[0.27,0.29,0.31,0.33],0.4 [0.26,0.28,0.30,0.32],0.3⟩}

Table 2 Decision matrix 2

	C_1	C_2	C_3	C_4
A_1	Very low	Very high	Very high	Very low
A_2	Very low	Very high	Very low	low
A_3	low	Very low	Medium high	Very high
A_4	Very high	Medium low	Medium low	Very high

Table 3 Decision matrix 3

	C_1	C_2	C_3	C_4
A_1	High	Low	Very high	Medium high
A_2	Medium high	High	Very low	Low
A_3	Low	Medium high	High	Medium low
A_4	Very high	Medium low	Medium low	High

Table 4 Decision matrix 4

	C_1	C_2	C_3	C_4
A_1	Medium low	Low	Very high	Medium low
A_2	Medium high	Very low	Very low	Medium
A_3	Low	Medium	Medium	Low
A_4	Medium	Medium low	Medium low	Very high

Table 5 Expert decision matrix 5

	C_1	C_2	C_3	C_4
A_1	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2 \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4 \\ [0.7, 0.9, \\ 0.11, 0.13], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, 0 \\ .8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11]; 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2; \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4; \\ [0.7, 0.9, \\ 0.11, 0.13]0.3 > \end{array}\right\}$
A_2	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2; \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2; \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.16, 0.18, \\ 0.20, 0.22], 0.2; \\ [0.18, 0.20, \\ 0.22, 0.24], 0.4; \\ [0.17, 0.19, \\ 0.21, 0.23], 0.3 > \end{array}\right\}$
A_3	$\left\{\begin{array}{l} < [0.16, 0.18, \\ 0.20, 0.22], 0.2; \\ [0.18, 0.20, \\ 0.22, 0.24], 0.4; \\ [0.17, 0.19, \\ 0.21, 0.23], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2; \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4 \\ [0.7, 0.9, \\ 0.11, 0.13], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.12, 0.14, \\ 0.16, 0.18], 0.2; \\ [0.14, 0.16, \\ 0.18, 0.20], 0.4 \\ [0.13, 0.15, \\ 0.17, 0.19], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11]; 0.3 > \end{array}\right\}$
A_4	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2 \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.5, 0.7, \\ 0.9, 0.11], 0.2; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.4; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.5, 0.7, \\ 0.9, 0.11], 0.2; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.4; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$

Table 6 Expert decision matrix 6

	C_1	C_2	C_3	C_4
A_1	$\left\{\begin{array}{l} < [0.25, 0.27, \\ 0.29, 0.31], 0.2; \\ [0.27, 0.29, \\ 0.31, 0.33], 0.4; \\ [0.26, 0.28, \\ 0.30, 0.32], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.16, 0.18, \\ 0.20, 0.22], 0.2; \\ [0.18, 0.20, \\ 0.22, 0.24], 0.4; \\ [0.17, 0.19, \\ 0.21, 0.23], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.12, 0.14, \\ 0.16, 0.18], 0.2; \\ [0.14, 0.16, \\ 0.18, 0.20], 0.4; \\ [0.13, 0.15, \\ 0.17, 0.19], 0.3 > \end{array}\right\}$
A_2	$\left\{\begin{array}{l} < [0.12, 0.14, \\ 0.16, 0.18], 0.2; \\ [0.14, 0.16, \\ 0.18, 0.20], 0.4; \\ [0.13, 0.15, \\ 0.17, 0.19]; 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.25, 0.27, \\ 0.29, 0.31], 0.2; \\ [0.27, 0.29, \\ 0.31, 0.33], 0.4; \\ [0.26, 0.28, \\ 0.30, 0.32], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.6, 0.8, \\ 0.10, 0.12], 0.2; \\ [0.8, 0.10, \\ 0.12, 0.14], 0.4; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.16, 0.18, \\ 0.20, 0.22], 0.2; \\ [0.18, 0.20, \\ 0.22, 0.24], 0.4; \\ [0.17, 0.19, \\ 0.21, 0.23], 0.3 > \end{array}\right\}$
A_3	$\left\{\begin{array}{l} < [0.16, 0.18, \\ 0.20, 0.22], 0.2; \\ [0.18, 0.20, \\ 0.22, 0.24], 0.4; \\ [0.17, 0.19, \\ 0.21, 0.23], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.12, 0.14, \\ 0.16, 0.18], 0.2; \\ [0.14, 0.16, \\ 0.18, 0.20], 0.4; \\ [0.13, 0.15, \\ 0.17, 0.19], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.25, 0.27, \\ 0.29, 0.31], 0.2; \\ [0.27, 0.29, \\ 0.31, 0.33], 0.4; \\ [0.26, 0.28, \\ 0.30, 0.32], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.5, 0.7, \\ 0.9, 0.11], 0.2; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.4; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.3 > \end{array}\right\}$
A_4	$\left\{\begin{array}{l} < [0.4, 0.6, \\ 0.8, 0.10], 0.2; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.4; \\ [0.5, 0.7, \\ 0.9, 0.11], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.5, 0.7, \\ 0.9, 0.11], 0.2; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.4; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.5, 0.7, \\ 0.9, 0.11], 0.2; \\ [0.7, 0.9, \\ 0.11, 0.13], 0.4; \\ [0.6, 0.8, \\ 0.10, 0.12], 0.3 > \end{array}\right\}$	$\left\{\begin{array}{l} < [0.25, 0.27, \\ 0.29, 0.31], 0.2; \\ [0.27, 0.29, \\ 0.31, 0.33], 0.4; \\ [0.26, 0.28, \\ 0.30, 0.32], 0.3 > \end{array}\right\}$

Table 7 Expert decision matrix 7

	C_1	C_2	C_3	C_4
A_1	{< [0.5, 0.7, 0.9, 0.11], 0.2; [0.7, 0.9, 0.11, 0.13], 0.4; [0.6, 0.8, 0.10, 0.12], 0.3 >}	{< [0.6, 0.8, 0.10, 0.12], 0.2; [0.8, 0.10, 0.12, 0.14], 0.4; [0.7, 0.9, 0.11, 0.13], 0.3 >}	{< [0.4, 0.6, 0.8, 0.10], 0.2; [0.6, 0.8, 0.10, 0.12], 0.4; [0.5, 0.7, 0.9, 0.11], 0.3 >}	{< [0.5, 0.7, 0.9, 0.11], 0.2; [0.7, 0.9, 0.11, 0.13], 0.4; [0.6, 0.8, 0.10, 0.12], 0.3 >}
A_2	{< [0.5, 0.7, 0.9, 0.11], 0.2; [0.7, 0.9, 0.11, 0.13], 0.4; [0.6, 0.8, 0.10, 0.12], 0.3 >}	{< [0.6, 0.8, 0.10, 0.12], 0.2; [0.8, 0.10, 0.12, 0.14], 0.4; [0.7, 0.9, 0.11, 0.13], 0.3 >}	{< [0.6, 0.8, 0.10, 0.12], 0.2; [0.8, 0.10, 0.12, 0.14], 0.4; [0.7, 0.9, 0.11, 0.13], 0.3 >}	{< [0.9, 0.11, 0.13, 0.15], 0.2; [0.11, 0.13, 0.15, 0.17], 0.4 [0.10, 0.12, 0.14, 0.16], 0.3 >}
A_3	{< [0.6, 0.8, 0.10, 0.12], 0.2; [0.8, 0.10, 0.12, 0.14], 0.4; [0.7, 0.9, 0.11, 0.13], 0.3 >}	{< [0.9, 0.11, 0.13, 0.15], 0.2; [0.11, 0.13, 0.15, 0.17], 0.4; [0.10, 0.12, 0.14, 0.16], 0.3 >}	{< [0.9, 0.11, 0.13, 0.15], 0.2; [0.11, 0.13, 0.15, 0.17], 0.4; [0.10, 0.12, 0.14, 0.16], 0.3 >}	{< [0.6, 0.8, 0.10, 0.12], 0.2; [0.8, 0.10, 0.12, 0.14], 0.4; [0.7, 0.9, 0.11, 0.13], 0.3 >}
A_4	{< [0.9, 0.11, 0.13, 0.15], 0.2; [0.11, 0.13, 0.15, 0.17], 0.4; [0.10, 0.12, 0.14, 0.16], 0.3 >}	{< [0.5, 0.7, 0.9, 0.11], 0.2; [0.7, 0.9, 0.11, 0.13], 0.4; [0.6, 0.8, 0.10, 0.12], 0.3 >}	{< [0.5, 0.7, 0.9, 0.11], 0.2; [0.7, 0.9, 0.11, 0.13], 0.4; [0.6, 0.8, 0.10, 0.12], 0.3 >}	{< [0.4, 0.6, 0.8, 0.10], 0.2; [0.6, 0.8, 0.10, 0.12], 0.4; ; 0.5, 0.7, 0.9, 0.11], 0.3 >}

Table 8 Normalized trapezoidal cubic fuzzy decision matrix 8

	C_1	C_2	C_3	C_4
A_1	{< [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6 [0.3, 0.1, 0.89, 0.87], 0.7 >}	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}	{< [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6; [0.3, 0.1, 0.89, 0.87], 0.7 >}
A_2	{< [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6; [0.3, 0.1, 0.89, 0.87], 0.7 >}	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}	{< [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6; [0.3, 0.1, 0.89, 0.87], 0.7 >}	{< [0.84, 0.82, 0.8, 0.78], 0.8; [0.82, 0.8, 0.78, 0.76], 0.6; [0.83, 0.81, 0.79, 0.77], 0.7 >}
A_3	{< [0.84, 0.82, 0.8, 0.78], 0.8; [0.82, 0.8, 0.78, 0.76], 0.6; [0.83, 0.81, 0.79, 0.77], 0.7 >}	{< [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6; [0.3, 0.1, 0.89, 0.87], 0.7 >}	{< [0.88, 0.86, 0.84, 0.82], 0.8; [0.86, 0.84, 0.82, 0.8], 0.6 [0.87, 0.85, 0.83, 0.81], 0.7 >}	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}
A_4	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}	{< [0.5, 0.3, 0.1, 0.89], 0.8; [0.3, 0.1, 0.89, 0.87], 0.6; [0.4, 0.2, 0.9, 0.88], 0.7 >}	{< [0.5, 0.3, 0.1, 0.89], 0.8; [0.3, 0.1, 0.89, 0.87], 0.6; [0.4, 0.2, 0.9, 0.88], 0.7 >}	{< [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89]; 0.7 >}

Table 9 Normalized trapezoidal cubic fuzzy decision matrix 9

	C_1	C_2	C_3	C_4
A_1	$\left\{ < [0.75, 0.73, 0.71, 0.69], 0.8; [0.73, 0.71, 0.69, 0.67], 0.6; [0.74, 0.72, 0.70, 0.68], 0.7 > \right\}$	$\left\{ < [0.84, 0.82, 0.80, 0.78], 0.8; [0.82, 0.80, 0.78, 0.76], 0.6; [0.83, 0.81, 0.79, 0.77], 0.7 > \right\}$	$\left\{ < [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89], 0.7 > \right\}$	$\left\{ < [0.88, 0.86, 0.84, 0.82], 0.8; [0.86, 0.84, 0.82, 0.80], 0.6; [0.87, 0.85, 0.83, 0.81]; 0.7 > \right\}$
A_2	$\left\{ < [0.88, 0.86, 0.84, 0.82], 0.8; [0.86, 0.84, 0.82, 0.80], 0.6; [0.87, 0.85, 0.83, 0.81]; 0.7 > \right\}$	$\left\{ < [0.75, 0.73, 0.71, 0.69], 0.8; [0.73, 0.71, 0.69, 0.67], 0.6; [0.74, 0.72, 0.70, 0.68], 0.7 > \right\}$	$\left\{ < [0.4, 0.2, 0.9, 0.88], 0.8; [0.2, 0.9, 0.88, 0.86], 0.6; [0.3, 0.1, 0.89, 0.87], 0.7 > \right\}$	$\left\{ < [0.84, 0.82, 0.80, 0.78], 0.8; [0.82, 0.80, 0.78, 0.76], 0.6; [0.83, 0.81, 0.79, 0.77], 0.7 > \right\}$
A_3	$\left\{ < [0.84, 0.82, 0.80, 0.78], 0.8; [0.82, 0.80, 0.78, 0.76], 0.6; [0.83, 0.81, 0.79, 0.77], 0.7 > \right\}$	$\left\{ < [0.88, 0.86, 0.84, 0.82], 0.8; [0.86, 0.84, 0.82, 0.80], 0.6; [0.87, 0.85, 0.83, 0.81]; 0.7 > \right\}$	$\left\{ < [0.75, 0.73, 0.71, 0.69], 0.8; [0.73, 0.71, 0.69, 0.67], 0.6; [0.74, 0.72, 0.70, 0.68], 0.7 > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], 0.8; [0.3, 0.1, 0.89, 0.87], 0.6; [0.4, 0.2, 0.9, 0.88], 0.7 > \right\}$
A_4	$\left\{ < [0.6, 0.4, 0.2, 0.9], 0.8; [0.4, 0.2, 0.9, 0.88], 0.6; [0.5, 0.3, 0.1, 0.89], 0.7 > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], 0.8; [0.3, 0.1, 0.89, 0.87], 0.6; [0.4, 0.2, 0.9, 0.88], 0.7 > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], 0.8; [0.3, 0.1, 0.89, 0.87], 0.6; [0.4, 0.2, 0.9, 0.88], 0.7 > \right\}$	$\left\{ < [0.75, 0.73, 0.71, 0.69], 0.8; [0.73, 0.71, 0.69, 0.67], 0.6; [0.74, 0.72, 0.70, 0.68], 0.7 > \right\}$

Table 10 Normalized trapezoidal cubic fuzzy decision matrix 10

	C_1	C_2	C_3	C_4
A_1	$\left\{ < [0.5, 0.3, 0.1, 0.89], (0.8) [0.3, 0.1, 0.89, 0.87], (0.6) [0.4, 0.2, 0.9, 0.88], (0.7) > \right\}$	$\left\{ < [0.84, 0.82, 0.80, 0.78], (0.8) [0.82, 0.80, 0.78, 0.76], (0.6) [0.83, 0.81, 0.79, 0.77], (0.7) > \right\}$	$\left\{ < [0.6, 0.4, 0.2, 0.9], (0.8) [0.4, 0.2, 0.9, 0.88], (0.6) [0.5, 0.3, 0.1, 0.89], (0.7) > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], (0.8) [0.3, 0.1, 0.89, 0.87], (0.6) [0.4, 0.2, 0.9, 0.88], (0.7) > \right\}$
A_2	$\left\{ < [0.5, 0.3, 0.1, 0.89], (0.8) [0.3, 0.1, 0.89, 0.87], (0.6) [0.4, 0.2, 0.9, 0.88], (0.7) > \right\}$	$\left\{ < [0.4, 0.2, 0.9, 0.88], (0.8) [0.2, 0.9, 0.88, 0.86], (0.6) [0.3, 0.1, 0.89, 0.87], (0.7) > \right\}$	$\left\{ < [0.4, 0.2, 0.9, 0.88], (0.8) [0.2, 0.9, 0.88, 0.86], (0.6) [0.3, 0.1, 0.89, 0.87], (0.7) > \right\}$	$\left\{ < [0.1, 0.89, 0.87, 0.85], (0.8) [0.89, 0.87, 0.85, 0.83], (0.6) [0.9, 0.88, 0.86, 0.84], (0.7) > \right\}$
A_3	$\left\{ < [0.84, 0.82, 0.80, 0.78], (0.8); [0.82, 0.80, 0.78, 0.76], (0.6); [0.83, 0.81, 0.79, 0.77], (0.7) > \right\}$	$\left\{ < [0.1, 0.89, 0.87, 0.85], (0.8) [0.89, 0.87, 0.85, 0.83], (0.6) [0.9, 0.88, 0.86, 0.84], (0.7) > \right\}$	$\left\{ < [0.1, 0.89, 0.87, 0.85], (0.8) [0.89, 0.87, 0.85, 0.83], (0.6) [0.9, 0.88, 0.86, 0.84], (0.7) > \right\}$	$\left\{ < [0.84, 0.82, 0.80, 0.78], (0.8) [0.82, 0.80, 0.78, 0.76], (0.6) [0.83, 0.81, 0.79, 0.77], (0.7) > \right\}$
A_4	$\left\{ < [0.1, 0.89, 0.87, 0.85], (0.8) [0.89, 0.87, 0.85, 0.83], (0.6) [0.9, 0.88, 0.86, 0.84], (0.7) > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], (0.8) [0.3, 0.1, 0.89, 0.87], (0.6) [0.4, 0.2, 0.9, 0.88], (0.7) > \right\}$	$\left\{ < [0.5, 0.3, 0.1, 0.89], (0.8) [0.3, 0.1, 0.89, 0.87], (0.6) [0.4, 0.2, 0.9, 0.88], (0.7) > \right\}$	$\left\{ < [0.6, 0.4, 0.2, 0.9], (0.8) [0.4, 0.2, 0.9, 0.88], (0.6) [0.5, 0.3, 0.1, 0.89], (0.7) > \right\}$

Table 11 Utilize the TrCFEHWA operator

	C_1	C_2	C_3	C_4
A_1	{< [0.6229, 0.1806, 0.6841, 0.8087], (0.8); [0.1806, 0.6841, 0.8087, 0.7850], (0.6), [0.5267, 0.4850, 0.7954, 0.7753], (0.7) >}	{< [0.7966, 0.7502, 0.7037, 0.0007], (0.8), [0.7502, 0.7037, 0.0007, 0.7972], (0.6); [0.7400, 0.6538, 0.5162, 0.7995], (0.7) >}	{< [0.5068, 0.3033, 0.6844, 0.8904], (0.8); [0.3033, 0.6844, 0.8904, 0.8703], (0.6); [0.3903, 0.1758, 0.3408, 0.8799], (0.7) >}	{< [0.7311, 0.6497, 0.7632, 0.8559], (0.8); [0.6497, 0.7632, 0.8559, 0.8355], (0.6); [0.5736, 0.3841, 0.8623, 0.8423], (0.7) >}
A_2	{< [0.7311, 0.6497, 0.7949, 0.8559], (0.8)[0.6497, 0.7949, 0.8559, 0.8355], (0.6); [0.5736, 0.3841, 0.8623, 0.8423], (0.7) >}	{< [0.6444, 0.5491, 0.6975, 0.8131], (0.8); [0.5491, 0.6975, 0.8131, 0.7904], (0.6); [0.5485, 0.3776, 0.4991, 0.7776], (0.7) >}	{< [0.4, 0.2, 0.9, 0.88], (0.8); [0.2, 0.9, 0.88, 0.86], (0.6); , [0.3, 0.1, 0.89, 0.87], (0.7) >}	{< [0.7356, 0.8406, 0.8202, 0.7998], (0.8); [0.8406, 0.8202, 0.7998, 0.7796], (0.6); [0.8474, 0.8269, 0.8073, 0.7832], (0.7) >}
A_3	{< [0.84, 0.82, 0.80, 0.78], (0.8); [0.82, 0.80, 0.78, 0.76], (0.6); , [0.83, 0.81, 0.79, 0.77], (0.7) >}	{< [0.6745, 0.7829, 0.8648, 0.8444], (0.8); [0.7829, 0.8648, 0.8444, 0.5514], (0.6); [0.7026, 0.5587, 0.8523, 0.8323], (0.7) >}	{< [0.6939, 0.8155, 0.7939, 0.7728], (0.8); [0.8155, 0.7939, 0.7728, 0.7518], (0.6); [0.8114, 0.7913, 0.7711, 0.7510], (0.7) >}	{< [0.6371, 0.5004, 0.3587, 0.8717], (0.8); [0.5004, 0.3587, 0.8717, 0.8509], (0.6); [0.5170, 0.3280, 0.5606, 0.8446], (0.7) >}
A_4	{< [0.3886, 0.5100, 0.8076, 0.8787], (0.8); [0.5100, 0.8076, 0.8787, 0.8585], (0.6); [0.4657, 0.2455, 0.5711, 0.8674], (0.7) >}	{< [0.5, 0.3, 0.1, 0.89], (0.8); [0.3, 0.1, 0.89, 0.87], (0.6); , [0.4, 0.2, 0.9, 0.88], (0.7) >}	{< [0.5, 0.3, 0.1, 0.89], (0.8); [0.3, 0.1, 0.89, 0.87], (0.6); [0.4, 0.2, 0.9, 0.88], (0.7) >}	{< [0.6821, 0.5890, 0.4967, 0.8211], (0.8); [0.5890, 0.4967, 0.8211, 0.7981], (0.6); [0.6134, 0.4791, 0.2881, 0.7824], (0.7) >}

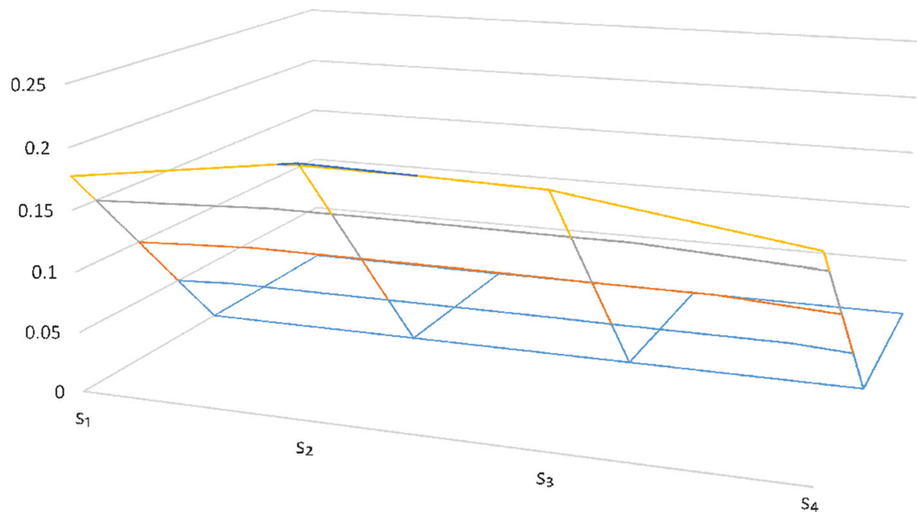
Table 12 Aggregation operator

A_1	{⟨[0.6789, 0.5088, 0.7104, 0.7426], (0.8); [0.5088, 0.7104, 0.7426, 0.8250], (0.6)], [0.5489, 0.3926, 0.6061, 0.8239], (0.7)⟩}
A_2	{⟨[[0.6442, 0.6071, 0.8158, 0.8403], (0.8)[0.6071, 0.8158, 0.8403, 0.8190], (0.6) [0.5451, 0.3509, 0.7832, 0.8179], (0.7)⟩}
A_3	{⟨[0.7222, 0.7512, 0.7473, 0.8218], (0.8); [0.7512, 0.7473, 0.8218, 0.7457], (0.6); [0.7096, 0.6004, 0.7395, 0.6654], (0.7)⟩}
A_4	{⟨[0.5265, 0.4337, 0.4353, 0.8726], (0.8); [0.4337, 0.4353, 0.8726, 0.8216], (0.6) [0.4542, 0.2649, 0.6307, 0.8521], (0.7)⟩}

$A_1 = 0.1771, A_2 = 0.2022,$

$A_3 = 0.1977, A_4 = 0.1709.$

Step 4: Ranking $A_2 > A_3 > A_1 > A_4$ and A_2 is the best solution.



8 Comparison analyses

In way to verify the sagacity and efficiency of the proposed approach, a comparative study is driven overshadowing the methods of interval-valued intuitionistic trapezoidal fuzzy number (Wu and Liu 2013), intuitionistic trapezoidal fuzzy number (Liu et al. 2017) and triangular cubic fuzzy number (Fahmi et al. 2018), which are special cases of trapezoidal cubic fuzzy numbers (TrCFNs), to the related expressive example.

8.1 A comparison analysis with the existing MCDM interval-valued intuitionistic trapezoidal fuzzy number

(Wu and Liu 2013) Step 1: According to the decision information given in the interval-valued intuitionistic trapezoidal fuzzy number decision matrix $R^k = (r_{ij}^k)$

Step 2. Utilize the IVTrFHWA operator to total all the rating values $\beta_{ij}(j = 1, 2, ..., m)$ of the i th line and get the general rating value β_{ij} comparing to the alternative h_i. Utilize the decision information given in matrix R, and we get: (0.2, 0.3, 0.2, 0.3)

Step 3. Find the score value

$$S(r_1) = -0.0406, \quad S(r_2) = -0.0864,$$
$$S(r_3) = 0.0585, \quad S(r_4) = -0.0412.$$

Step 4. Rank all the alternatives $A_i(i = 1, 2, 3, 4)$ in accordance with the scores $S(r_i)$ of the overall preference values $r_i : A_3 > A_2 > A_4 > A_1$, and thus, the most desirable alternative is A_3.

Table 13 Interval valued intuitionistic trapezoidal fuzzy number

	C_1	C_2	C_3	C_4
A_1	$\left\{\begin{array}{l}\langle[0.6.0.8,\\ 0.10, 0.12]\\ [0.8, 0.10]\\ [0.7, 0.9]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10],\\ [0.6, 0.8],\\ [0.5, 0.7]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10]\\ [0.6, 0.8]\\ [0.5, 0.7]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.6, 0.8,\\ 0.10, 0.12],\\ [0.8, 0.10]\\ [0.7, 0.9]\rangle\end{array}\right\}$
A_2	$\left\{\begin{array}{l}\langle[0.6, 0.8,\\ 0.10, 0.12],\\ [0.8, 0.10]\\ [0.7, 0.11]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10],\\ [0.6, 0.8],\\ [0.5, 0.6]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.6, 0.8,\\ 0.10, 0.12],\\ [0.8, 0.10];\\ [0.7, 0.9]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.16, 0.18,\\ 0.20, 0.22];\\ [0.18, 0.20],\\ [0.17, 0.19]\rangle\end{array}\right\}$
A_3	$\left\{\begin{array}{l}\langle[0.16, 0.18,\\ 0.20, 0.22],\\ [0.17, 0.21]\\ [0.15, 0.19]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.6, 0.8,\\ 0.10, 0.12],\\ [0.8, 0.10]\\ [0.3, 0.4]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.12, 0.14,\\ 0.16, 0.18]\\ [0.14, 0.16],\\ [0.13, 0.15]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10]\\ [0.5, 0.9]\\ [0.14, 0.16]\rangle\end{array}\right\}$
A_4	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10],\\ [0.6, 0.8],\\ [0.5, 0.7]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.5, 0.7,\\ 0.9, 0.11],\\ [0.7, 0.9],\\ [0.6, 0.8]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.5, 0.7,\\ 0.9, 0.11],\\ [0.7, 0.9],\\ [0.6, 0.8]\rangle\end{array}\right\}$	$\left\{\begin{array}{l}\langle[0.4, 0.6,\\ 0.8, 0.10],\\ [0.6, 0.8],\\ [0.5, 0.7]\rangle\end{array}\right\}$

Table 14 Utilize the IVTrFHWA operator

A_1	{[0.5650, 0.7455, 0.3641, 0.1704]; [0.7455, 0.3641]; [0.5317, 0.7541]}
A_2	{[0.3226, 0.4486, 0.1449, 0.0891], [0.4486, 0.1449]; [0.6271, 0.6548]}
A_3	{[0.3409, 0.4135, 0.3031, 0.2164], [0.3942, 0.3134]; [0.1494, 0.1907]}
A_4	{[0.3807, 0.5942, 0.8211, 0.0668], [0.5942, 0.8211], [0.6192, 0.9731]}

Table 15 Intuitionistic trapezoidal fuzzy number

A_1	{[0.6, 0.8, 0.10, 0.12] [0.8, 0.10]}	{[0.4, 0.6, 0.8, 0.10], [0.6, 0.8]}	{[0.4, 0.6, 0.8, 0.10], [0.6, 0.8]}	{[0.6, 0.8, 0.10, 0.12], [0.8, 0.10]}
A_2	{[0.6, 0.8, 0.10, 0.12], [0.8, 0.10]}	{[0.4, 0.6, 0.8, 0.10], [0.6, 0.8]}	{[0.6, 0.8, 0.10, 0.12], [0.8, 0.10]}	{[0.16, 0.18, 0.20, 0.22]; [0.18, 0.20]}
A_3	{[0.16, 0.18, 0.20, 0.22], [0.18, 0.20]}	{[0.6, 0.8, 0.10, 0.12], [0.8, 0.10]}	{[0.12, 0.14, 0.16, 0.18] [0.14, 0.16]}	{[0.4, 0.6, 0.8, 0.10] [0.6, 0.8]}
A_4	{[0.4, 0.6, 0.8, 0.10], [0.6, 0.8]}	{[0.5, 0.7, 0.9, 0.11], [0.7, 0.9]}	{[0.5, 0.7, 0.9, 0.11], [0.7, 0.9]}	{[0.4, 0.6, 0.8, 0.10], [0.6, 0.8]}

Table 16 Overall preference value

A_1	{[0.5068, 0.7143, 0.5367, 0.1100]; [0.7143, 0.3155]}
A_2	{[0.4239, 0.6056, 0.4058, 0.2876], [0.4168, 0.2472]}
A_3	{[0.3725, 0.5385, 0.4074, 0.1463], [0.5385, 0.2528]}
A_4	{[0.4586, 0.6528, 0.8579, 0.1051], [0.6528, 0.5751]}

8.2 A comparison analysis with the existing MCDM intuitionistic trapezoidal fuzzy number

(Liu et al. 2017) Step 1. According to the decision information given in the trapezoidal intuitionistic fuzzy decision

Table 17 Triangular cubic fuzzy number

	C_1	C_2	C_3	C_4
A_1	{[0.6, 0.8, 0.10] < [0.12, 0.14], 0.13 >}	{[0.4, 0.6, 0.8]; < [0.10, 0.12], 0.11 >}	{[0.4, 0.6, 0.8]; < [0.10, 0.12], 0.11 >}	{[0.6, 0.8, 0.10]; < [0.12, 0.14], 0.13 >}
A_2	{[0.6, 0.8, 0.10] < [0.12, 0.14], 0.13 >}	{[0.4, 0.6, 0.8]; < [0.10, 0.12], 0.11 >}	{[0.8, 0.10, 0.12], < [0.7, 0.11], 0.9 >}	{[0.16, 0.18, 0.20]; < [0.17, 0.19], 0.21 >}
A_3	{[0.16, 0.18, 0.20], < [0.17, 0.19], 0.21 >}	{[0.6, 0.8, 0.10]; < [0.12, 0.14], 0.13 >}	{[0.14, 0.16, 0.18], < [0.15, 0.19], 0.17 >}	{[0.4, 0.6, 0.8]; < [0.10, 0.12], 0.11 >}
A_4	{[0.4, 0.6, 0.8]; < [0.10, 0.12], 0.11 >}	{[0.7, 0.9, 0.11]; [0.12, 0.14], 0.13 >}	{[0.5, 0.7, 0.9], < [0.10, 0.12], 0.11 >}	{[0.4, 0.6, 0.8], < [0.10, 0.12], 0.11 >}

Table 18 Using the TCFHA operator

A_1	$\{[0.5651, 0.7455, 0.3641]; <[0.0891, 0.1054], 0.1828>\}$
A_2	$\{[0.3517, 0.2404, 0.1531]; <[0.3855, 0.1662], 0.1696>\}$
A_3	$\{[0.3516, 0.4247, 0.3103]; <[0.1098, 0.1306], 0.2195>\}$
A_4	$\{[0.4211, 0.6407, 0.4371]; <[0.1246, 0.1481], 0.0743>\}$

Table 19 Comparison analysis with existing methods

Method	Ranking
Trapezoidal cubic fuzzy information	$A_2 > A_3 > A_1 > A_4$
Interval-valued intuitionistic trapezoidal fuzzy number (Wu and Liu 2013)	$A_3 > A_2 > A_4 > A_1$
Intuitionstic trapezoidal fuzzy number (Liu et al. 2017)	$A_1 > A_3 > A_2 > A_4$
Triangular cubic fuzzy number (Fahmi et al. 2018)	$A_4 > A_2 > A_3 > A_1$

matrix $R^k = (r_{ij}^k)$, and the TIFEWA operator to derive the individual overall preference trapezoidal intuitionistic fuzzy values r_i^k of the alternative A_i, we get:

Step 2. Utilize the individual overall preference trapezoidal intuitionistic fuzzy values r_i^k of the alternative A_i and we get

Step 3. Calculate the scores $S(r_i)$ of the overall trapezoidal intuitionistic fuzzy preference values r_i

$$S(r_1) = 0.0932, \quad S(r_2) = 0.0365,$$
$$S(r_3) = 0.0523, \quad S(r_4) = 0.0201.$$

Step 4. Rank all the alternatives $A_i (i = 1, ..., n)$ in accordance with the scores $S(r_i)$ of the overall preference values r_i:

$A_1 > A_3 > A_2 > A_4$, and thus the most desirable alternative is A_1.

8.3 A comparison analysis with the existing MCDM method triangular cubic fuzzy number

(Fahmi et al. 2018) Step 1: According to the decision information given in the triangular cubic fuzzy number decision matrix $R^k = (r_{ij}^k)$,

By step 2 using the TCFHA Operator to aggregate all the decision matrices into single collective decision matrix with triangular cubic fuzzy ratings. Consider

Step 3. To find the ranking order of the alternatives, use the score function

$$S(r_1) = 0.0021, \quad S(r_2) = 0.0316,$$
$$S(r_3) = 0.0025, \quad S(r_4) = 0.0331.$$

Step 4. Rank all the alternatives $A_i (i = 1, 2, 3, 4)$ in accordance with the scores $S(r_i)$ of the overall preference values $r_i : A_4 > A_2 > A_3 > A_1$, and thus the most desirable alternative is A_2 (Tables 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19).

9 Conclusion

In this paper, we define the new concept of trapezoidal cubic fuzzy number and Hamming distance. We develop three arithmetic averaging operators, that is trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for gathering cubic fuzzy data. The TrCFEHWA operator simplifies both the TrCFEWA and TrCFEOWA operators. Furthermore, we originate the relationship between the current aggregation operators and suggested operators and establish many properties of these operators. We apply on the TrCFEHWA operator to multiple attribute decision making with fuzzy material. Finally, a numerical example is providing to demonstrate the submission of the established approach. In group decision-making problems, because the experts usually come from different specialty fields and have different backgrounds and levels of knowledge, they usually have diverging opinions. These operators can be applied to many other fields, such as information fusion, data mining, pattern recognition, triangular cubic linguistic fuzzy VIKOR method and trapezoidal cubic linguistic fuzzy VIKOR method, which may be the possible topic for the future research.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A: Proof of Proposition 1

(1) $A_1 + A_2 = A_2 + A_1$;

$$A_1 + A_2 = \left\langle \begin{bmatrix} \max(I_{A_1}^-, I_{A_2}^-) \\ \left[\frac{p_1^-(h)+p_2^-(h)}{1+p_1^-(h)p_2^-(h)}, \frac{q_1^-(h)+q_2^-(h)}{1+q_1^-(h)q_2^-(h)}, \frac{r_1^-(h)+r_2^-(h)}{1+r_1^-(h)r_2^-(h))}, \frac{s_1^-(h)+s_2^-(h)}{1+s_1^-(h)s_2^-(h))} \right], \\ \max(I_{A_1}^+, I_{A_2}^+) \\ \left[\frac{p_1^+(h)+p_2^+(h)}{1+p_1^+(h)p_2^+(h))}, \frac{q_1^+(h)+q_2^+(h)}{1+q_1^+(h)q_2^+(h))}, \frac{r_1^+(h)+r_2^+(h)}{1+r_1^+(h)r_2^+(h))}, \frac{s_1^+(h)+s_2^+(h)}{1+s_1^+(h)s_2^+(h))} \right], \end{bmatrix}, \begin{bmatrix} \min(\mu_{A_1}, \mu_{A_2}) \\ \left[\frac{p_1(h)\cdot p_2(h)}{1+((1-p_1(h))(1-p_2(h)))}, \frac{q_1(h)\cdot q_2(h)}{1+((1-q_1(h))(1-q_2(h)))}, \frac{r_1(h)\cdot r_2(h)}{1+((1-r_1(h))(1-r_2(h)))}, \frac{s_1(h)\cdot s_2(h)}{1+((1-s_1(h))(1-s_2(h)))} \right] \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} \max(I_{A_2}^-, I_{A_1}^-) \\ \left[\frac{p_2^-(h)+p_1^-(h)}{1+p_2^-(h)p_1^-(h)}, \frac{q_2^-(h)+q_1^-(h)}{1+q_2^-(h)q_1^-(h)}, \frac{r_2^-(h)+r_1^-(h)}{1+r_2^-(h)r_1^-(h)}, \frac{s_2^-(h)+s_1^-(h)}{1+s_2^-(h)s_1^-(h)} \right], \\ \max(I_{A_2}^+, I_{A_1}^+) \\ \left[\frac{p_2^+(h)+p_1^+(h)}{1+p_2^+(h)p_1^+(h)}, \frac{q_2^+(h)+q_1^+(h)}{1+q_2^+(h)q_1^+(h)}, \frac{r_2^+(h)+r_1^+(h)}{1+r_2^+(h)r_1^+(h)}, \frac{s_2^+(h)+s_1^+(h)}{1+s_2^+(h)s_1^+(h)} \right], \end{bmatrix}, \begin{bmatrix} \min(\mu_{A_2}, \mu_{A_1}) \\ \left[\frac{p_2(h)\cdot p_1(h)}{1+((1-p_2(h))(1-p_1(h)))}, \frac{q_2(h)\cdot q_1(h)}{1+((1-q_2(h))(1-q_1(h)))}, \frac{r_2(h).r_1(h)}{1+((1-r_2(h))(1-r_1(h)))}, \frac{s_2(h)\cdot s_1(h)}{1+((1-s_2(h))(1-s_1(h)))} \right] \end{bmatrix} \right\rangle$$

$$= A_2 + A_1$$

Hence $A_1 + A_2 = A_2 + A_1$.

(2) $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1 + A_2)$$
$$= \left\langle \begin{bmatrix} \max(I_{A_1}^- I_{A_2}^-) \\ \left[\frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}, \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda}, \frac{[(1+s_1^-(h))(1-s_1^-(h))]^\lambda[(1+s_2^-(h))(1-s_2^-(h))]^\lambda}{[(1+s_1^-(h))(1-s_1^-(h))]^\lambda[(1+s_2^-(h))(1-s_2^-(h))]^\lambda} \right] \end{bmatrix}, \begin{bmatrix} \max(I_{A_1}^+ I_{A_2}^+) \\ \left[\frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+p_{p2}^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+p_2^+(h))(1-p_2^+(h))]^\lambda}, \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda}, \frac{[(1+s_1^+(h))(1-s_1^+(h))]^\lambda[(1+s_2^+(h))(1-s_2^+(h))]^\lambda}{[(1+s_1^+(h))(1-s_1^+(h))]^\lambda[(1+s_2^+(h))(1-s_2^+(h))]^\lambda} \right] \end{bmatrix}, \begin{bmatrix} \min(\mu_{A_1}\mu_{A_2}); \\ \left[\frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^\lambda+[p_1(h)p_2(h)]^\lambda}, \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^\lambda+[q_1(h)q_2(h)]^\lambda}, \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^\lambda+[r_1(h)r_2(h)]^\lambda}, \frac{2[s_1(h)s_2(h)]^\lambda}{[(4-2s_1(h)-2s_2(h)-s_1(h)s_2(h)]^\lambda+[s_1(h)s_2(h)]^\lambda} \right] \end{bmatrix} \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \begin{bmatrix} \max(I_{A_1}^-) \left[\frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]}, \frac{[(1+s_1^-(h))^\lambda-(1-s_1^-(h))^\lambda]}{[(1+s_1^-(h))^\lambda+(1-s_1^-(h))^\lambda]} \right] \\ \max(I_{A_1}^+) \left[\frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]}, \frac{[(1+s_1^+(h))^\lambda-(1-s_1^+(h))^\lambda]}{[(1+s_1^+(h))^\lambda+(1-s_1^+(h))^\lambda]} \right] \end{bmatrix}, \begin{bmatrix} \min(\mu_{A_1}); \left[\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda}, \frac{2s_1^\lambda(h)}{[(2-s_1(h)]^\lambda+[s_1(h)]^\lambda} \right] \end{bmatrix} \right\rangle$$

$$\lambda A_2=\left\langle \begin{bmatrix} \max(I_A^-)\left[\frac{[(1+p_2^-(h))^\lambda-(1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda+(1-p_2^-(h))^\lambda]},\right. \\ \frac{[(1+q_2^-(h))^\lambda-(1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda+(1-q_2^-(h))^\lambda]}, \\ \frac{[(1+r_2^-(h))^\lambda-(1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda+(1-r_2^-(h))^\lambda]}, \\ \left.\frac{[(1+s_2^-(h))^\lambda-(1-s_2^-(h))^\lambda]}{[(1+s_2^-(h))^\lambda+(1-s_2^-(h))^\lambda]}\right]; \\ \max(I_A^-)\left[\frac{[(1+p_2^+(h))^\lambda-(1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda+(1-p_2^+(h))^\lambda]},\right. \\ \frac{[(1+q_2^+(h))^\lambda-(1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda+(1-q_2^+(h))^\lambda]}, \\ \frac{[(1+r_2^+(h))^\lambda-(1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda+(1-r_2^+(h))^\lambda]}, \\ \left.\frac{[(1+s_2^+(h))^\lambda-(1-s_2^+(h))^\lambda]}{[(1+s_2^+(h))^\lambda+(1-s_2^+(h))^\lambda]}\right] \end{bmatrix}, \begin{bmatrix} \min(\mu_{A_2});\left[\frac{2p_2^\lambda(h)}{[(2-p_2(h)]^\lambda+[p_2(h)]^\lambda},\right. \\ \frac{2q_2^\lambda(h)}{[(2-q_2(h)]^\lambda+[q_2(h)]^\lambda}, \\ \frac{2r_2^\lambda(h)}{[(2-r_2(h)]^\lambda+[r_2(h)]^\lambda}, \\ \left.\frac{2s_2^\lambda(h)}{[(2-s_2(h)]^\lambda+[s_2(h)]^\lambda}\right] \end{bmatrix}\right\rangle$$

$\lambda A_2+\lambda A_1$

$$=\left\langle \begin{matrix} \max(I_{A_2}^-,I_{A_1}^-)\begin{bmatrix} \frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}, \\ \frac{[(1+s_2^-(h))(1-s_2^-(h))]^\lambda[(1+s_1^-(h))(1-s_1^-(h))]^\lambda}{[(1+s_2^-(h))(1-s_2^-(h))]^\lambda[(1+s_1^-(h))(1-s_1^-(h))]^\lambda} \end{bmatrix} \\ \max(I_{A_2}^+,I_{A_1}^+)\begin{bmatrix} \frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}, \\ \frac{[(1+s_2^+(h))(1-s_2^+(h))]^\lambda[(1+s_1^+(h))(1-s_1^+(h))]^\lambda}{[(1+s_2^+(h))(1-s_2^+(h))]^\lambda[(1+s_1^+(h))(1-s_1^+(h))]^\lambda} \end{bmatrix}, \end{matrix} \begin{bmatrix} \min(\mu_{A_2}\mu_{A_1}); \\ \left[\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^\lambda+[p_2(h)p_1(h)]^\lambda},\right. \\ \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^\lambda+[q_2(h)q_1(h)]^\lambda}, \\ \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^\lambda+[r_2(h)r_1(h)]^\lambda}, \\ \left.\frac{2[s_2(h)s_1(h)]^\lambda}{[(4-2s_2(h)-2s_1(h)-s_2(h)s_1(h)]^\lambda+[s_2(h)s_1(h)]^\lambda}\right] \end{bmatrix}\right\rangle$$

so, we have $\lambda(A_1+A_2)=\lambda A_2+\lambda A_1$.

(3) $\lambda_1 A+\lambda_2 A=(\lambda_1+\lambda_2)A$

$$\lambda_1 A=\left\langle \begin{bmatrix} \max(I_A^-),\left[\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}},\right. \\ \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}, \\ \frac{[1+s_A^-(h)]^{\lambda_1}-[1-s_A^-(h)]^{\lambda_1}}{[1+s_A^-(h)]^{\lambda_1}+[1-s_A^-(h)]^{\lambda_1}} \\ \max(I_A^+),\left[\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}},\right. \\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \\ \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}, \\ \left.\frac{[1+s_A^+(h)]^{\lambda_1}-[1-s_A^+(h)]^{\lambda_1}}{[1+s_A^+(h)]^{\lambda_1}+[1-s_A^+(h)]^{\lambda_1}}\right] \end{bmatrix} \begin{bmatrix} \min(\mu_A)\left[\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}},\right. \\ \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \\ \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}, \\ \left.\frac{2[s_A(h)]^{\lambda_1}}{[(2-s_A(h)]^{\lambda_1}+[s_A(h)]^{\lambda_1}}\right] \end{bmatrix}\right\rangle$$

and

$$\lambda_2 A=\left\langle \begin{bmatrix} \max(I_A^-),\left[\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}},\right. \\ \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}, \\ \frac{[1+s_A^-(h)]^{\lambda_2}-[1-s_A^-(h)]^{\lambda_2}}{[1+s_A^-(h)]^{\lambda_2}+[1-s_A^-(h)]^{\lambda_2}} \\ \max(I_A^+),\left[\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}},\right. \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \\ \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}, \\ \left.\frac{[1+s_A^+(h)]^{\lambda_2}-[1-s_A^+(h)]^{\lambda_2}}{[1+s_A^+(h)]^{\lambda_2}+[1-s_A^+(h)]^{\lambda_2}}\right] \end{bmatrix} \begin{bmatrix} \min(\mu_A)\left[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}},\right. \\ \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}}, \\ \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}, \\ \left.\frac{2[s_A(h)]^{\lambda_2}}{[(2-s_A(h)]^{\lambda_2}+[s_A(h)]^{\lambda_2}}\right] \end{bmatrix}\right\rangle$$

$$
= \left\langle \begin{bmatrix} \max(I_A^-), \left[\dfrac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \right. \\ \dfrac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \dfrac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \dfrac{[1+s_A^-(h)]^{\lambda_1+\lambda_2}-[1-s_A^-(h)]^{\lambda_1+\lambda_2}}{[1+s_A^-(h)]^{\lambda_2}+[1-s_A^-(h)]^{\lambda_1+\lambda_2}} \\ \max(I_A^+), \left[\dfrac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \right. \\ \dfrac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \dfrac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \left. \dfrac{[1+s_A^+(h)]^{\lambda_1+\lambda_2}-[1-s_A^+(h)]^{\lambda_1+\lambda_2}}{[1+s_A^+(h)]^{\lambda_1+\lambda_2}+[1-s_A^+(h)]^{\lambda_1+\lambda_2}} \right] \end{bmatrix} \right.
$$

$$
\left. \begin{bmatrix} \min(\mu_A) \\ \left[\dfrac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}}, \right. \\ \dfrac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}}, \\ \dfrac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}, \\ \left. \dfrac{2[s_A(h)]^{\lambda_1+\lambda_2}}{[(2-s_A(h)]^{\lambda_1+\lambda_2}+[s_A(h)]^{\lambda_1+\lambda_2}} \right] \end{bmatrix} \right\rangle
$$

$$= (\lambda_1 + \lambda_2)A.$$

Appendix B: Proof of Theorem 1

Assume that $n = 1$, TrCFEWA $(A_1, A_2, ..., A_n) = \oplus_{j=1}^{k} w_1 A_1$

$$\langle \max(\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$$

$$
\lambda(A_1 + A_2) = \left\langle \begin{bmatrix} \max(I_{A_1}^- I_{A_2}^-) \\ \left[\dfrac{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}, \right. \\ \dfrac{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}, \\ \dfrac{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}, \\ \left. \dfrac{[(1+s_1^-(h))(1-s_1^-(h))]^{\lambda}[(1+s_2^-(h))(1-s_2^-(h))]^{\lambda}}{[(1+s_1^-(h))(1-s_1^-(h))]^{\lambda}[(1+s_2^-(h))(1-s_2^-(h))]^{\lambda}} \right] \end{bmatrix}, \right.
$$

$$
\begin{bmatrix} \max(I_{A_1}^+ I_{A_2}^+) \\ \left[\dfrac{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}, \right. \\ \dfrac{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}, \\ \dfrac{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}, \\ \left. \dfrac{[(1+s_1^+(h))(1-s_1^+(h))]^{\lambda}[(1+s_2^+(h))(1-s_2^+(h))]^{\lambda}}{[(1+s_1^+(h))(1-s_1^+(h))]^{\lambda}[(1+s_2^+(h))(1-s_2^+(h))]^{\lambda}} \right] \end{bmatrix},
$$

$$
\left. \begin{bmatrix} \min(\mu_{A_1}\mu_{A_2}); \\ \left[\dfrac{2[p_1(h)p_2(h)]^{\lambda}}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^{\lambda}+[p_1(h)p_2(h)]^{\lambda}}, \right. \\ \dfrac{2[q_1(h)q_2(h)]^{\lambda}}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^{\lambda}+[q_1(h)q_2(h)]^{\lambda}}, \\ \dfrac{2[r_1(h)r_2(h)]^{\lambda}}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^{\lambda}+[r_1(h)r_2(h)]^{\lambda}}, \\ \left. \dfrac{2[s_1(h)s_2(h)]^{\lambda}}{[(4-2s_1(h)-2s_2(h)-s_1(h)s_2(h)]^{\lambda}+[s_1(h)s_2(h)]^{\lambda}} \right] \end{bmatrix} \right\rangle
$$

and we have

$$
\lambda A_1 = \left\langle \begin{bmatrix} \max(I_{A_1}^-) \left[\dfrac{[(1+p_1^-(h))^{\lambda}-(1-p_1^-(h))^{\lambda}]}{[(1+p_1^-(h))^{\lambda}+(1-p_1^-(h))^{\lambda}]}, \right. \\ \dfrac{[(1+q_1^-(h))^{\lambda}-(1-q_1^-(h))^{\lambda}]}{[(1+q_1^-(h))^{\lambda}+(1-q_1^-(h))^{\lambda}]}, \\ \dfrac{[(1+r_1^-(h))^{\lambda}-(1-r_1^-(h))^{\lambda}]}{[(1+r_1^-(h))^{\lambda}+(1-r_1^-(h))^{\lambda}]}, \\ \left. \dfrac{[(1+s_1^-(h))^{\lambda}-(1-s_1^-(h))^{\lambda}]}{[(1+s_1^-(h))^{\lambda}+(1-s_1^-(h))^{\lambda}]} \right] \\ \max(I_{A_1}^+) \left[\dfrac{[(1+p_1^+(h))^{\lambda}-(1-p_1^+(h))^{\lambda}]}{[(1+p_1^+(h))^{\lambda}+(1-p_1^+(h))^{\lambda}]}, \right. \\ \dfrac{[(1+q_1^+(h))^{\lambda}-(1-q_1^+(h))^{\lambda}]}{[(1+q_1^+(h))^{\lambda}+(1-q_1^+(h))^{\lambda}]}, \\ \dfrac{[(1+r_1^+(h))^{\lambda}-(1-r_1^+(h))^{\lambda}]}{[(1+r_1^+(h))^{\lambda}+(1-r_1^+(h))^{\lambda}]}, \\ \left. \dfrac{[(1+s_1^+(h))^{\lambda}-(1-s_1^+(h))^{\lambda}]}{[(1+s_1^+(h))^{\lambda}+(1-s_1^+(h))^{\lambda}]} \right] \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} \min(\mu_{A_1}); \left[\dfrac{2p_1^{\lambda}(h)}{[(2-p_1(h)]^{\lambda}+[p_1(h)]^{\lambda}}, \right. \\ \dfrac{2q_1^{\lambda}(h)}{[(2-q_1(h)]^{\lambda}+[q_1(h)]^{\lambda}}, \\ \dfrac{2r_1^{\lambda}(h)}{[(2-r_1(h)]^{\lambda}+[r_1(h)]^{\lambda}}, \\ \left. \dfrac{2s_1^{\lambda}(h)}{[(2-s_1(h)]^{\lambda}+[s_1(h)]^{\lambda}} \right] \end{bmatrix} \right\rangle
$$

$$
\lambda A_2 = \left\langle \begin{bmatrix} \max(I_A^-) \left[\dfrac{[(1+p_2^-(h))^{\lambda}-(1-p_2^-(h))^{\lambda}]}{[(1+p_2^-(h))^{\lambda}+(1-p_2^-(h))^{\lambda}]}, \right. \\ \dfrac{[(1+q_2^-(h))^{\lambda}-(1-q_2^-(h))^{\lambda}]}{[(1+q_2^-(h))^{\lambda}+(1-q_2^-(h))^{\lambda}]}, \\ \dfrac{[(1+r_2^-(h))^{\lambda}-(1-r_2^-(h))^{\lambda}]}{[(1+r_2^-(h))^{\lambda}+(1-r_2^-(h))^{\lambda}]}, \\ \left. \dfrac{[(1+s_2^-(h))^{\lambda}-(1-s_2^-(h))^{\lambda}]}{[(1+s_2^-(h))^{\lambda}+(1-s_2^-(h))^{\lambda}]} \right], \\ \max(I_A^-) \left[\dfrac{[(1+p_2^+(h))^{\lambda}-(1-p_2^+(h))^{\lambda}]}{[(1+p_2^+(h))^{\lambda}+(1-p_2^+(h))^{\lambda}]}, \right. \\ \dfrac{[(1+q_2^+(h))^{\lambda}-(1-q_2^+(h))^{\lambda}]}{[(1+q_2^+(h))^{\lambda}+(1-q_2^+(h))^{\lambda}]}, \\ \dfrac{[(1+r_2^+(h))^{\lambda}-(1-r_2^+(h))^{\lambda}]}{[(1+r_2^+(h))^{\lambda}+(1-r_2^+(h))^{\lambda}]}, \\ \left. \dfrac{[(1+s_2^+(h))^{\lambda}-(1-s_2^+(h))^{\lambda}]}{[(1+s_2^+(h))^{\lambda}+(1-s_2^+(h))^{\lambda}]} \right] \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} \min(\mu_{A_2}); \left[\dfrac{2p_2^{\lambda}(h)}{[(2-p_2(h)]^{\lambda}+[p_2(h)]^{\lambda}}, \right. \\ \dfrac{2q_2^{\lambda}(h)}{[(2-q_2(h)]^{\lambda}+[q_2(h)]^{\lambda}}, \\ \dfrac{2r_2^{\lambda}(h)}{[(2-r_2(h)]^{\lambda}+[r_2(h)]^{\lambda}}, \\ \left. \dfrac{2s_2^{\lambda}(h)}{[(2-s_2(h)]^{\lambda}+[s_2(h)]^{\lambda}} \right] \end{bmatrix} \right\rangle
$$

$$\lambda A_2+\lambda A_1$$

$$=\left\langle \begin{bmatrix} \max(I^-_{A_2}I^-_{A_1}) \\ \left[\frac{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}}{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}},\right. \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}, \\ \left.\frac{[(1+s_2^-(h))(1-s_2^-(h))]^{\lambda}[(1+s_1^-(h))(1-s_1^-(h))]^{\lambda}}{[(1+s_2^-(h))(1-s_2^-(h))]^{\lambda}[(1+s_1^-(h))(1-s_1^-(h))]^{\lambda}}\right] \end{bmatrix},\right.$$

$$\begin{bmatrix} \max(I^-_{A_2}I^-_{A_1}) \\ \left[\frac{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}}{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}},\right. \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}, \\ \left.\frac{[(1+s_2^+(h))(1-s_2^+(h))]^{\lambda}[(1+s_1^+(h))(1-s_1^+(h))]^{\lambda}}{[(1+s_2^+(h))(1-s_2^+(h))]^{\lambda}[(1+s_1^+(h))(1-s_1^+(h))]^{\lambda}}\right] \end{bmatrix},$$

$$\left.\begin{bmatrix} \min(\mu_{A_2}\mu_{A_1}); \\ \left[\frac{2[p_2(h)p_1(h)]^{\lambda}}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^{\lambda}+[p_2(h)p_1(h)]^{\lambda}},\right. \\ \frac{2[q_2(h)q_1(h)]^{\lambda}}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^{\lambda}+[q_2(h)q_1(h)]^{\lambda}}, \\ \frac{2[r_2(h)r_1(h)]^{\lambda}}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^{\lambda}+[s_2(h)s_1(h)]^{\lambda}}, \\ \left.\frac{2[s_2(h)s_1(h)]^{\lambda}}{[(4-2s_2(h)-2s_1(h)-s_2(h)s_1(h)]^{\lambda}+[s_2(h)s_1(h)]^{\lambda}}\right] \end{bmatrix}\right\rangle$$

so, we have $\lambda(A_1+A_2)=\lambda A_2+\lambda A_1$.

$$\lambda_1 A+\lambda_2 A=(\lambda_1+\lambda_2)A$$

$$\lambda_1 A=\left\langle \begin{bmatrix} \max(I_A^-),\left[\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}},\right. \\ \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}, \\ \frac{[1+s_A^-(h)]^{\lambda_1}-[1-s_A^-(h)]^{\lambda_1}}{[1+s_A^-(h)]^{\lambda_1}+[1-s_A^-(h)]^{\lambda_1}} \\ \max(I_A^+),\left[\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}},\right. \\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \\ \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}, \\ \left.\frac{[1+s_A^+(h)]^{\lambda_1}-[1-s_A^+(h)]^{\lambda_1}}{[1+s_A^+(h)]^{\lambda_1}+[1-s_A^+(h)]^{\lambda_1}}\right] \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} \min(\mu_A)\left[\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}},\right. \\ \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \\ \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}, \\ \left.\frac{2[s_A(h)]^{\lambda_1}}{[(2-s_A(h)]^{\lambda_1}+[s_A(h)]^{\lambda_1}}\right] \end{bmatrix}\right\rangle$$

and

$$\lambda_2 A=\left\langle \begin{bmatrix} \max(I_A^-),\left[\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}},\right. \\ \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}, \\ \frac{[1+s_A^-(h)]^{\lambda_2}-[1-s_A^-(h)]^{\lambda_2}}{[1+s_A^-(h)]^{\lambda_2}+[1-s_A^-(h)]^{\lambda_2}} \\ \max(I_A^+),\left[\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}},\right. \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \\ \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}, \\ \left.\frac{[1+s_A^+(h)]^{\lambda_2}-[1-s_A^+(h)]^{\lambda_2}}{[1+s_A^+(h)]^{\lambda_2}+[1-s_A^+(h)]^{\lambda_2}}\right] \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} \min(\mu_A)\left[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}},\right. \\ \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}}, \\ \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}, \\ \left.\frac{2[s_A(h)]^{\lambda_2}}{[(2-s_A(h)]^{\lambda_2}+[s_A(h)]^{\lambda_2}}\right] \end{bmatrix}\right\rangle$$

$$=\left\langle \begin{bmatrix} \max(I_A^-),\left[\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}},\right. \\ \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+s_A^-(h)]^{\lambda_1+\lambda_2}-[1-s_A^-(h)]^{\lambda_1+\lambda_2}}{[1+s_A^-(h)]^{\lambda_2}+[1-s_A^-(h)]^{\lambda_1+\lambda_2}} \\ \max(I_A^+),\left[\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}},\right. \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \left.\frac{[1+s_A^+(h)]^{\lambda_1+\lambda_2}-[1-s_A^+(h)]^{\lambda_1+\lambda_2}}{[1+s_A^+(h)]^{\lambda_1+\lambda_2}+[1-s_A^+(h)]^{\lambda_1+\lambda_2}}\right] \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} \min(\mu_A)\left[\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}},\right. \\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}, \\ \left.\frac{2[s_A(h)]^{\lambda_1+\lambda_2}}{[(2-s_A(h)]^{\lambda_1+\lambda_2}+[s_A(h)]^{\lambda_1+\lambda_2}}\right] \end{bmatrix}\right\rangle$$

$$=\max(I^-_{A_1})\begin{bmatrix} \frac{[[1+p_1^-(h)]^{\varpi_1}-[1-p_1^-(h)]^{\varpi_1}}{[1+p_1^-(h)]^{\varpi_1}+[1-p_1^-(h)]^{\varpi_1}}, \\ \frac{[1+q_1^-(h)]^{\varpi_1}-[1-q_1^-(h)]^{\varpi_1}}{[1+q_1^-(h)]^{\varpi_1}+[1-q_1^-(h)]^{\varpi_1}}, \\ \frac{[1+r_1^-(h)]^{\varpi_1}-[1-r_1^-(h)]^{\varpi_1}}{[1+r_1^-(h)]^{\varpi_1}+[1-r_1^-(h)]^{\varpi_1}}, \\ \frac{[1+s_1^-(h)]^{\varpi_1}-[1-s_1^-(h)]^{\varpi_1}}{[1+s_1^-(h)]^{\varpi_1}+[1-s_1^-(h)]^{\varpi_1}} \end{bmatrix};$$

$$\max(I^{+}_{A_1})\left[\begin{array}{c}\frac{[1+p_1^{+}(h)]^{\varpi_1}-[1-p_1^{+}(h)]^{\varpi_1}}{[1+p_1^{+}(h)]^{\varpi_1}+[1-p_1^{+}(h)]^{\varpi_1}},\\ \frac{[1+q_1^{+}(h)]^{\varpi_1}-[1-q_1^{+}(h)]^{\varpi_1}}{[1+q_1^{+}(h)]^{\varpi_1}+[1-q_1^{+}(h)]^{\varpi_1}},\\ \frac{[1+r_1^{+}(h)]^{\varpi_1}-[1-r_1^{+}(h)]^{\varpi_1}}{[1+r_1^{+}(h)]^{\varpi_1}+[1-r_1^{+}(h)]^{\varpi_1}},\\ \frac{[1+s_1^{+}(h)]^{\varpi_1}-[1-s_1^{+}(h)]^{\varpi_1}}{[1+s_1^{+}(h)]^{\varpi_1}+[1-s_1^{+}(h)]^{\varpi_1}}\end{array}\right];$$

$$\min(\mu_{A_1})\left[\begin{array}{c}\frac{2[p_1(h)]^{\varpi_1}}{[(2-p_1(h)]^{\varpi_1}+[p_1(h)]^{\varpi_1}},\\ \frac{2[q_1(h)]^{\varpi_1}}{[(2-q_1(h)]^{\varpi_1}+[q_1(h)]^{\varpi_1}},\\ \frac{2[r_1(h)]^{\varpi_1}}{[(2-r_1(h)]^{\varpi_1}+[r_1(h)]^{\varpi_1}},\\ \frac{2[s_1(h)]^{\varpi_1}}{[(2-s_1(h)]^{\varpi}+[s_1(h)]^{\varpi_1}}\end{array}\right].$$

Assume that $n=k$, TrCFEWA $(A_1, A_2, ..., A_n) = \oplus_{j=1}^{k} w_j A_j$

$$\langle\max(I^{-}_{A})\left[\begin{array}{c}\frac{[\prod_{j=1}^{k}[1+p_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+q_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+s_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+s_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}\end{array}\right],$$

$$\max(I^{+}_{A})\left[\begin{array}{c}\frac{\prod_{j=1}^{k}[1+p_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+q_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+s_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+s_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^{+}(h)]^{\varpi}}\end{array}\right];$$

$$\min(\mu_{A})\left[\begin{array}{c}\frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[s_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-s_1(h)]^{\varpi}+\prod_{j=1}^{k}[s_1(h)]^{\varpi}}\end{array}\right].$$

Then when $n=k+1$, we have

TrCFEWA $(A_1, A_2, ..., A_{k+1})$ = TrCFEWA $(A_1, A_2, ..., A_k) \oplus A_{k+1})$

$$\langle\max(I^{-}_{A})\left[\begin{array}{c}\frac{\prod_{j=1}^{k}[1+p_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+q_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+s_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+s_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}\end{array}\right];$$

$$\langle\max(I^{+}_{A})\left[\begin{array}{c}\frac{\prod_{j=1}^{k}[1+p_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+q_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+s_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k}[1+s_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^{+}(h)]^{\varpi}}\end{array}\right];$$

$$\min(\mu_{A})\left[\begin{array}{c}\frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k}[s_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-s_1(h)]^{\varpi}+\prod_{j=1}^{k}[s_1(h)]^{\varpi}}\end{array}\right]\oplus_{k+1}$$

$$\langle\max(I^{-}_{A})\left[\begin{array}{c}\frac{\prod_{j=1}^{k+1}[1+p_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+q_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+r_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+s_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-s_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+s_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-s_1^{-}(h)]^{\varpi}}\end{array}\right];$$

$$\langle\max(I^{+}_{A})\left[\begin{array}{c}\frac{\prod_{j=1}^{k+1}[1+p_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+q_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+r_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^{+}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+s_1^{+}(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-s_1^{+}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+s_1^{+}(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-s_1^{+}(h)]^{\varpi}}\end{array}\right];$$

$$\min(\mu_{A})\left[\begin{array}{c}\frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[s_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-s_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[s_1(h)]^{\varpi}}\end{array}\right]$$

$$=\max(I^{-}_{A})\left[\begin{array}{c}\frac{[\prod_{j=1}^{k+1}[1+p_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+q_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+r_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^{-}(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+s_1^{-}(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+s_1^{-}(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^{-}(h)]^{\varpi}}\end{array}\right],$$

$$\max(I_A^+)\left[\begin{array}{c}\frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k+1}[1+s_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-s_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+s_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-s_1^+(h)]^{\varpi}}\end{array}\right],$$

$$\min(\mu_A)\left[\begin{array}{c}\frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{k+1}[s_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-s_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[s_1(h)]^{\varpi}}\end{array}\right].$$

In particular, if $w=(\frac{1}{n},\frac{1}{n},....,\frac{1}{n})^T$, then the TrCFEWA operator is reduced to the trapezoidal cubic fuzzy Einstein weighing averaging operator, which is shown as follows:

$$\langle\max(I_A^-)\left[\begin{array}{c}\frac{[\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+s_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-s_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+s_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-s_1^-(h)]^{\frac{1}{n}}}\end{array}\right];$$

$$\max(I_A^+)\left[\begin{array}{c}\frac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}},\\ \frac{\prod_{j=1}^{n}[1+s_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-s_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+s_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-s_1^+(h)]^{\frac{1}{n}}}\end{array}\right];$$

$$\min(\mu_A)\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}},\\ \frac{2\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-q_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}},\\ \frac{2\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-r_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}},\\ \frac{2\prod_{j=1}^{n}[s_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-s_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[s_1(h)]^{\frac{1}{n}}}\end{array}\right].$$

Appendix C: Proof of Proposition 2

(1) (Idempotency) Since $A_j = A$ are equal to

$$\left\{\begin{array}{l}\langle[p^-(h),q^-(h),r^-(h),s^-(h)],(I_A^-)\\ [p^+(h),q^+(h),r^+(h),s^+(h)],(I_A^+)\\ [p(h),q(h),r(h),s(h)],(\mu_A)\rangle|h\in H\end{array}\right\}$$

for $(j=1,2,...,n)$, then TrCFEWA

$$(A_1,A_2,...,A_n)$$

$$=\langle\max[I_A^-]\left[\begin{array}{c}\frac{\prod_{j=1}^{n}[1+p_j^-(h)]^{\varpi j}-\prod_{j=1}^{n}[1-p_j^-(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+p_j^-(h)]^{\varpi j}+\prod_{j=1}^{n}[1-p_j^-(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+q_j^-(h)]^{\varpi j}-\prod_{j=1}^{n}[1-q_j^-(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+q_j^-(h)]^{\varpi j}+\prod_{j=1}^{n}[1-q_j^-(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+r_j^-(h)]^{\varpi j}-\prod_{j=1}^{n}[1-r_j^-(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+r_j^-(h)]^{\varpi j}+\prod_{j=1}^{n}[1-r_j^-(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+s_j^-(h)]^{\varpi j}-\prod_{j=1}^{n}[1-s_j^-(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+s_j^-(h)]^{\varpi j}+\prod_{j=1}^{n}[1-s_j^-(h)]^{\varpi j}}\end{array}\right];$$

$$\max[(I_A^+]\left[\begin{array}{c}\frac{\prod_{j=1}^{n}[1+p_j^+(h)]^{\varpi j}-\prod_{j=1}^{n}[1-p_j^+(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+p_j^+(h)]^{\varpi j}+\prod_{j=1}^{n}[1-p_j^+(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+q_j^+(h)]^{\varpi j}-\prod_{j=1}^{n}[1-q_j^+(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+q_j^+(h)]^{\varpi j}+\prod\limits_{j=1}^{n}[1-q_j^+(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+r_j^+(h)]^{\varpi j}-\prod_{j=1}^{n}[1-r_j^+(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+r_j^+(h)]^{\varpi j}+\prod_{j=1}^{n}[1-r_j^+(h)]^{\varpi j}},\\ \frac{\prod_{j=1}^{n}[1+s_j^+(h)]^{\varpi j}-\prod_{j=1}^{n}[1-s_j^+(h)]^{\varpi j}}{\prod_{j=1}^{n}[1+s_j^+(h)]^{\varpi j}+\prod_{j=1}^{n}[1-s_j^+(h)]^{\varpi j}}\end{array}\right];$$

$$\min[(\mu_A]\left[\begin{array}{c}\frac{2\prod_{j=1}^{n}[p_j(h)]^{\varpi j}}{\prod_{j=1}^{n}[(2-p_j(h)]^{\varpi j}+\prod_{j=1}^{n}[p_j(h)]^{\varpi j}},\\ \frac{2\prod_{j=1}^{n}[q_j(h)]^{\varpi j}}{\prod_{j=1}^{n}[(2-q_j(h)]^{\varpi j}+\prod_{j=1}^{n}[q_j(h)]^{\varpi j}},\\ \frac{2\prod_{j=1}^{n}[r_j(h)]^{\varpi j}}{\prod_{j=1}^{n}[(2-r_j(h)]^{\varpi j}+\prod_{j=1}^{n}[r_j(h)]^{\varpi j}},\\ \frac{2\prod_{j=1}^{n}[s_j(h)]^{\varpi j}}{\prod_{j=1}^{n}[(2-s_j(h)]^{\varpi j}+\prod_{j=1}^{n}[s_j(h)]^{\varpi j}}\end{array}\right]$$

$$=\langle\max[(I_A^-]\left[\begin{array}{c}\frac{\prod_{j=1}^{n}[1+p^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-p^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-p^-(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+q^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-q^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-q^-(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+r^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-r^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-r^-(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+s^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-s^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-s^-(h)]^{\varpi}}\end{array}\right];$$

$$\max[(I_A^+)]\left[\begin{array}{c}\frac{\prod_{j=1}^{n}[1+p^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-p^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-p^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+q^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-q^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-q^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+r^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-r^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-r^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{n}[1+s^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-s^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-s^+(h)]^{\varpi}}\end{array}\right];$$

$$\min[(\mu_A)]\begin{bmatrix}\frac{2\prod_{j=1}^{n}[p(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-p(h)]^{\mu}+\prod_{j=1}^{n}[p(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[q(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-q(h)]^{\varpi}+\prod_{j=1}^{n}[q(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[r(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-r(h)]^{\varpi}+\prod_{j=1}^{n}[r(h)]^{\varpi}},\\ \frac{2\prod_{j=1}^{n}[s(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-s(h)]^{\varpi}+\prod_{j=1}^{n}[s(h)]^{\varpi}}\end{bmatrix}$$

$$=\langle[I_A^-]\begin{bmatrix}\frac{[1+p^-(h)]^{\varpi}-[1-p^-(h)]^{\varpi}}{[1+p^-(h)]^{\varpi}+[1-p^-(h)]^{\varpi}},\\ \frac{[1+q^-(h)]^{\varpi}-[1-q^-(h)]^{\varpi}}{[1+q^-(h)]^{\varpi}+[1-q^-(h)]^{\varpi}},\\ \frac{[1+r^-(h)]^{\varpi}-[1-r^-(h)]^{\varpi}}{[1+r^-(h)]^{\varpi}+[1-r^-(h)]^{\varpi}},\\ \frac{[1+s^-(h)]^{\varpi}-[1-s^-(h)]^{\varpi}}{[1+s^-(h)]^{\varpi}+[1-s^-(h)]^{\varpi}}\end{bmatrix};$$

$$[I_A^+]\begin{bmatrix}\frac{[1+p^+(h)]^{\varpi}-[1-p^+(h)]^{\varpi}}{[1+p^+(h)]^{\varpi}+[1-p^+(h)]^{\varpi}},\\ \frac{[1+q^+(h)]^{\varpi}-[1-q^+(h)]^{\varpi}}{[1+q^+(h)]^{\varpi}+[1-q^+(h)]^{\varpi}},\\ \frac{[1+r^+(h)]^{\varpi}-[1-r^+(h)]^{\varpi}}{[1+r^+(h)]^{\varpi}+[1-r^+(h)]^{\varpi}},\\ \frac{[1+s^+(h)]^{\varpi}-[1-s^+(h)]^{\varpi}}{[1+s^+(h)]^{\varpi}+[1-s^+(h)]^{\varpi}}\end{bmatrix};$$

$$[(\mu_A)]\begin{bmatrix}\frac{2[p(h)]^{\varpi}}{[(2-p(h)]^{\mu}+[p(h)]^{\varpi}},\\ \frac{2[q(h)]^{\varpi}}{[(2-q(h)]^{\varpi}+[q(h)]^{\varpi}},\\ \frac{2[r(h)]^{\varpi}}{[(2-r(h)]^{\varpi}+[r(h)]^{\varpi}},\\ \frac{2[s(h)]^{\varpi}}{[(2-s(h)]^{\varpi}+[s(h)]^{\varpi}}\end{bmatrix}=$$

$$\left\{\begin{matrix}\langle[p^-(h),q^-(h),r^-(h),s^-(h)],(I_A^-)\\ [p^+(h),q^+(h),r^+(h),s^+(h)],(I_{A^+})\\ [p(h),q(h),r(h),s(h)],(\mu_A)\rangle|h\in H\end{matrix}\right\}=A$$

TrCFEWA $(A_1, A_2, ..., A_n) = A$.
The proof is completed.

(2) (Boundary):
Let $f(x) = \frac{(1-x)}{(1+x)}$ $x \in [0,1]$; then $\frac{-2}{(1-x)^2} < 0$; that is, $f(x)$ is a decreasing function. Since $p_{\min}^- \le p_j^- \le p_{\max}^-$, then for all j, we have $f(p_{\min}^-) \le f(p_j^-) \le f(p_{\max}^-)$; that is $\frac{1-p_{\max}^-}{1+p_{\max}^-} \le \frac{1-p_j^-}{1+p_j^-} \le \frac{1-p_{\min}^-}{1+p_{\min}^-}$.
Let $w = (w_1, w_2, ..., w_n)$ be the weight vector of $(A_1, A_2, ..., A_n)$, such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$. Then, for all $w_j \in [0,1]$, we have $\left(\frac{1-p_{\max}^-}{1+p_{\max}^-}\right)^{w_j} \le \left(\frac{1-p_j^-}{1+p_j^-}\right)^{w_j} \le \left(\frac{1-p_{\min}^-}{1+p_{\min}^-}\right)^{w_j}$.
Thus

$$\prod_{j=1}^{n}\left(\frac{1-p_{\max}^-}{1+p_{\max}^-}\right)^{w_j} \le \prod_{j=1}^{n}\left(\frac{1-p_j^-}{1+p_j^-}\right)^{w_j} \le \prod_{j=1}^{n}\left(\frac{1-p_{\min}^-}{1+p_{\min}^-}\right)^{w_j}$$

$$\Leftrightarrow \frac{1-p_{\max}^-}{1+p_{\max}^-} \le \prod_{j=1}^{n}\left(\frac{1-p_j^-}{1+p_j^-}\right)^{w_j} \le \frac{1-p_{\min}^-}{1+p_{\min}^-}$$

$$\Leftrightarrow \frac{2}{1+p_{\max}^-} \le 1+ \le \prod_{j=1}^{n}\left(\frac{1-p_j^-}{1+p_j^-}\right)^{w_j} \le \frac{2}{1+p_{\min}^-}$$

$$\Leftrightarrow \frac{1+p_{\min}^-}{2} \le \frac{1}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^-}{1+p_j^-}\right)\right)^{w_j}} \le \frac{1+p_{\max}^-}{2}$$

$$\Leftrightarrow 1+p_{\min}^- \le \frac{2}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^-}{1+p_j^-}\right)\right)^{w_j}} \le 1+p_{\max}^-$$

$$\Leftrightarrow p_{\min}^- \le \frac{2}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^-}{1+p_j^-}\right)\right)^{w_j}} - 1 \le p_{\max}^-$$

that is

$$p_{\min}^- \le \frac{\prod_{j=1}^{n}\left(1+p_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+p_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^-\right)^{w_j}} \le p_{\max}^-$$

Similarly, we have

$$[q_{\min}^- \le \frac{\prod_{j=1}^{n}\left(1+q_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+q_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^-\right)^{w_j}} \le q_{\max}^-,$$

$$r_{\min}^- \le \frac{\prod_{j=1}^{n}\left(1+r_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+r_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^-\right)^{w_j}} \le r_{\max}^-,$$

$$s_{\min}^- \le \frac{\prod_{j=1}^{n}\left(1+s_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+s_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^-\right)^{w_j}} \le s_{\max}^-],$$

$$\max[I_j^-]A_j^-$$

$$=\max[I_j^-]\left\{\begin{matrix}\frac{\prod_{j=1}^{n}\left(1+p_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+p_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^-\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+q_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+q_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^-\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+r_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+r_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^-\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+s_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^-\right)^{w_j}}{\prod_{j=1}^{n}\left(1+s_j^-\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^-\right)^{w_j}}\end{matrix}\right\};$$

Let $g(y) = \frac{(1-y)}{(1+y)}$ $y \in [0,1]$; then $\frac{-2}{(1-y)^2} < 0$;that is, $g(y)$ is a decreasing function. Since $p_{\min}^+ \le p_j^+ \le p_{\max}^+$, then for all j, we have $g(p_{\min}^+) \le g(p_j^+) \le g(p_{\max}^+)$; that is $\frac{1-p_{\max}^+}{1+p_{\max}^+} \le \frac{1-p_j^+}{1+p_j^+} \le \frac{1-p_{\min}^+}{1+p_{\min}^+}$.
Let $w = (w_1, w_2, ..., w_n)$ be the weight vector of $(A_1, A_2, ..., A_n)$, such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$.
Then, for all $w_j \in [0,1]$, we have $\left(\frac{1-p_{\max}^+}{1+p_{\max}^+}\right)^{w_j} \le \left(\frac{1-p_j^+}{1+p_j^+}\right)^{w_j} \le \left(\frac{1-p_{\min}^+}{1+p_{\min}^+}\right)^{w_j}$.

Thus

$$\prod_{j=1}^{n}\left(\frac{1-p_{\max}^{+}}{1+p_{\max}^{+}}\right)^{w_j} \leq \prod_{j=1}^{n}\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)^{w_j}$$
$$\leq \prod_{j=1}^{n}\left(\frac{1-p_{\min}^{+}}{1+p_{\min}^{+}}\right)^{w_j}$$
$$\Leftrightarrow \frac{1-p_{\max}^{+}}{1+p_{\max}^{+}} \leq \prod_{j=1}^{n}\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)^{w_j} \leq \frac{1-p_{\min}^{+}}{1+p_{\min}^{+}}$$
$$\Leftrightarrow \frac{2}{1+p_{\max}^{+}} \leq 1+ \leq \prod_{j=1}^{n}\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)^{w_j} \leq \frac{2}{1+p_{\min}^{+}}$$
$$\Leftrightarrow \frac{1+p_{\min}^{+}}{2} \leq \frac{1}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)\right)^{w_j}} \leq \frac{1+p_{\max}^{+}}{2}$$
$$\Leftrightarrow 1+p_{\min}^{+} \leq \frac{2}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)\right)^{w_j}} \leq 1+p_{\max}^{+}$$
$$\Leftrightarrow p_{\min}^{+} \leq \frac{2}{1+\prod_{j=1}^{n}\left(\left(\frac{1-p_j^{+}}{1+p_j^{+}}\right)\right)^{w_j}} - 1 \leq p_{\max}^{+}$$

that is

$$p_{\min}^{+} \leq \frac{\prod_{j=1}^{n}\left(1+p_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-p_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+p_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-p_j^{+}\right)^{w_j}} \leq p_{\max}^{+}$$

Similarly, we have

$$[q_{\min}^{+} \leq \frac{\prod_{j=1}^{n}\left(1+q_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-q_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+q_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-q_j^{+}\right)^{w_j}} \leq q_{\max}^{+},$$
$$r_{\min}^{+} \leq \frac{\prod_{j=1}^{n}\left(1+r_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-r_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+r_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-r_j^{+}\right)^{w_j}} \leq r_{\max}^{+},$$
$$s_{\min}^{+} \leq \frac{\prod_{j=1}^{n}\left(1+s_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-s_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+s_j^{+}\right)^{w_j} - \prod_{j=1}^{n}\left(1-s_j^{+}\right)^{w_j}} \leq s_{\max}^{+}],$$

$$\max[I_j^{+}]A_j^{+}$$
$$= \max[I_j^{+}]\left\{\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+p_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+p_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-p_j^{+}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+q_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+q_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-q_j^{+}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+r_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+r_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-r_j^{+}\right)^{w_j}},\\ \frac{\prod_{j=1}^{n}\left(1+s_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^{+}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+s_j^{+}\right)^{w_j}-\prod_{j=1}^{n}\left(1-s_j^{+}\right)^{w_j}}\end{array}\right\}$$

and
Let $h(z) = \frac{(2-z)}{z}$, $z \in [0, 1]$; then $\frac{-2}{z^2} < 0$; that is, $h(z)$ is a decreasing function. Since $p_{\min} \leq p_j \leq p_{\max}$, then for all j, we have $h(p_{\min}) \leq h(p_j) \leq h(p_{\max})$; that is $\frac{2-p_{\max}}{h} \leq \frac{2-p_j}{h} \leq \frac{2-p_{\min}}{h}$.
Let $w = (w_1, w_2, ..., w_n)$ be the weight vector of $(A_1, A_2, ..., A_n)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then, for all $w_j \in [0, 1]$,
we have

$$\left(\frac{2-p_{\max}}{p_{\max}}\right)^{w_j} \leq \left(\frac{2-p_j}{p_j}\right)^{w_j} \leq \left(\frac{2-p_{\min}}{p_{\min}}\right)^{w_j}.$$

Thus

$$\prod_{j=1}^{n}\left(\frac{2-p_{\max}}{p_{\max}^{-}}\right)^{w_j} \leq \prod_{j=1}^{n}\left(\frac{2-p_j}{p_j}\right)^{w_j}$$
$$\leq \prod_{j=1}^{n}\left(\frac{2-p_{\min}}{p_{\min}}\right)^{w_j}$$
$$\Leftrightarrow \frac{2-p_{\max}}{p_{\max}} \leq \prod_{j=1}^{n}\left(\frac{2-p_j}{p_j}\right)^{w_j} \leq \frac{2-p_{\min}}{p_{\min}}$$
$$\Leftrightarrow \frac{2}{p_{\max}} \leq \prod_{j=1}^{n}\left(\frac{2-p_j}{p_j}\right)^{w_j} + 1 \leq \frac{2}{p_{\min}}$$
$$\Leftrightarrow \frac{p_{\min}}{2} \leq \frac{1}{\prod_{j=1}^{n}\left(\left(\frac{2-p_j}{p_j}\right)\right)^{w_j}+1} \leq \frac{p_{\max}}{2}$$
$$\Leftrightarrow p_{\min} \leq \frac{2}{1+\prod_{j=1}^{n}\left(\left(\frac{2-p_j}{p_j}\right)\right)^{w_j}} \leq p_{\max}$$

that is

$$p_{\min} \leq \frac{2\prod_{j=1}^{n}(p_j)^{w_j}}{\prod_{j=1}^{n}\left(2+p_j\right)^{w_j} - \prod_{j=1}^{n}\left(p_j\right)^{w_j}} \leq p_{\max}$$

Similarly, we have

$$q_{\min} \leq \frac{2\prod_{j=1}^{n}(q_j)^{w_j}}{\prod_{j=1}^{n}\left(2+q_j\right)^{w_j} - \prod_{j=1}^{n}\left(q_j\right)^{w_j}} \leq q_{\max},$$
$$r_{\min} \leq \frac{2\prod_{j=1}^{n}(r_j)^{w_j}}{\prod_{j=1}^{n}\left(2+r_j\right)^{w_j} - \prod_{j=1}^{n}\left(r_j\right)^{w_j}} \leq r_{\max},$$
$$s_{\min} \leq \frac{2\prod_{j=1}^{n}(s_j)^{w_j}}{\prod_{j=1}^{n}\left(2+s_j\right)^{w_j} - \prod_{j=1}^{n}\left(s_j\right)^{w_j}} \leq s_{\max}$$

That is $\min[\mu_j]A_j =$

$$\min[\mu_j]\left\{\begin{array}{c}\frac{2\prod_{j=1}^{n}(p_j)^{w_j}}{\prod_{j=1}^{n}(2+p_j)^{w_j}-\prod_{j=1}^{n}(p_j)^{w_j}},\\ \frac{2\prod_{j=1}^{n}(q_j)^{w_j}}{\prod_{j=1}^{n}(2+q_j)^{w_j}-\prod_{j=1}^{n}(q_j)^{w_j}},\\ \frac{2\prod_{j=1}^{n}(r_j)^{w_j}}{\prod_{j=1}^{n}(2+r_j)^{w_j}-\prod_{j=1}^{n}(r_j)^{w_j}},\\ \frac{2\prod_{j=1}^{n}(s_j)^{w_j}}{\prod_{j=1}^{n}(2+s_j)^{w_j}-\prod_{j=1}^{n}(s_j)^{w_j}}\end{array}\right\}.$$

The proof is completed

(3) (Monotonicity)

Since

$$\begin{aligned}&\{p_A^-(h) \le p_B^-(h),\\ &q_A^-(h) \le q_B^-(h), r_A^-(h) \le r_B^-(h),\\ &s_A^-(h) \le s_B^-(h)], (I_A^-) \le (I_B^-)\};\\ &\{p_A^+(h) \le p_B^+(h), q_A^+(h) \le q_B^+(h),\\ &r_A^+(h) \le r_B^+(h), s_A^+(h) \le s_B^+(h)],\\ &(I_A^+) \le (I_B^+)\}\end{aligned}$$

and

$$\begin{aligned}&\{p_A(h) \le p_B(h), q_A(h) \le q_B(h),\\ &r_A(h) \le r_B(h), s_A(h) \le s_B(h)],\\ &(\mu_A) \le (\mu_B)\}\end{aligned}$$

Since

$$\begin{aligned}&\frac{1+p_A^-(h)}{1+p_A^-(h)} \le \frac{1+p_B^-(h)}{1+p_B^-(h)},\\ &\frac{1+q_A^-(h)}{1+q_A^-(h)} \le \frac{1+q_B^-(h)}{1+q_B^-(h)},\\ &\frac{1+r_A^-(h)}{1+r_A^-(h)} \le \frac{1+r_B^-(h)}{1+r_B^-(h)},\\ &\frac{1+s_A^-(h)}{1+s_A^-(h)} \le \frac{1+s_B^-(h)}{1+s_B^-(h)},\\ &\max\{(I_A^-) \le (I_B^-)\}\\ &\frac{1+p_A^+(h)}{1+p_A^+(h)} \le \frac{1+p_B^+(h)}{1+p_B^+(h)},\\ &\frac{1+q_A^+(h)}{1+q_A^+(h)} \le \frac{1+q_B^+(h)}{1+q_B^+(h)},\\ &\frac{1+r_A^+(h)}{1+r_A^+(h)} \le \frac{1+r_B^+(h)}{1+r_B^+(h)},\\ &\frac{1+s_A^+(h)}{1+s_A^+(h)} \le \frac{1+s_B^+(h)}{1+s_B^+(h)},\\ &\max\{(I_A^+) \le (I_B^+)\}\end{aligned}$$

$$\begin{aligned}&\prod_{j=1}^{n}\left\{\frac{1+p_A^-(h)}{1+p_A^-(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+p_B^-(h)}{1+p_B^-(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}\left\{\frac{1+q_A^-(h)}{1+q_A^-(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+q_B^-(h)}{1+q_B^-(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}\left\{\frac{1+r_A^-(h)}{1+r_A^-(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+r_B^-(h)}{1+r_B^-(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}\left\{\frac{1+s_A^-(h)}{1+s_A^-(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+s_B^-(h)}{1+s_B^-(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}(I_A^-) \le \prod_{j=1}^{n}(I_B^-);\\ &\prod_{j=1}^{n}\left\{\frac{1+p_A^+(h)}{1+p_A^+(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+p_B^+(h)}{1+p_B^+(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}\left\{\frac{1+q_A^+(h)}{1+q_A^+(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+q_B^+(h)}{1+q_B^+(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}\left\{\frac{1+r_A^+(h)}{1+r_A^+(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+r_B^+(h)}{1+r_B^+(h)}\right\}^{w_j}\\ &\prod_{j=1}^{n}\left\{\frac{1+s_A^+(h)}{1+s_A^+(h)}\right\}^{w_j}\\ &\le \prod_{j=1}^{n}\left\{\frac{1+s_B^+(h)}{1+s_B^+(h)}\right\}^{w_j},\\ &\prod_{j=1}^{n}(I_A^+) \le \prod_{j=1}^{n}(I_B^+);\end{aligned}$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+p_A^-(h)}{1+p_A^-(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+p_B^-(h)}{1+p_B^-(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+q_A^-(h)}{1+q_A^-(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+q_B^-(h)}{1+q_B^-(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+r_A^-(h)}{1+r_A^-(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+r_B^-(h)}{1+r_B^-(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+s_A^-(h)}{1+s_A^-(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+s_B^-(h)}{1+s_B^-(h)}\right\}^{w_j}},$$

$$\prod_{j=1}^{n}(I_A^-) \leq \prod_{j=1}^{n}(I_B^-);$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+p_A^+(h)}{1+p_A^+(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+p_B^+(h)}{1+p_B^+(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+q_A^+(h)}{1+q_A^+(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+q_B^+(h)}{1+q_B^+(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+r_A^+(h)}{1+r_A^+(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+r_B^+(h)}{1+r_B^+(h)}\right\}^{w_j}},$$

$$\frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+s_A^+(h)}{1+s_A^+(h)}\right\}^{w_j}} \geq \frac{2}{1+\prod_{j=1}^{n}\left\{\frac{1+s_B^+(h)}{1+s_B^+(h)}\right\}^{w_j}},$$

$$\prod_{j=1}^{n}(I_A^+) \leq \prod_{j=1}^{n}(I_B^+).$$

We have

$$\frac{\prod_{j=1}^{n}[1+p_A^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-p_A^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p_A^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-p_A^-(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+p_B^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-p_B^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p_B^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-p_B^-(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+q_A^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-q_A^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q_A^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-q_A^-(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+q_B^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-q_B^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q_B^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-q_B^-(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+r_A^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-r_A^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r_A^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-r_A^-(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+r_B^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-r_B^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r_B^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-r_B^-(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+s_A^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-s_A^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s_A^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-s_A^-(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+s_B^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-s_B^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s_B^-(h)]^{\varpi}+\prod_{j=1}^{n}[1-s_B^-(h)]^{\varpi}},$$

$$\max\{(I_A^-) \geq (I_B^-)\}$$

$$\frac{\prod_{j=1}^{n}[1+p_A^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-p_A^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p_A^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-p_A^+(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+p_B^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-p_B^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+p_B^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-p_B^+(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+q_A^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-q_A^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q_A^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-q_A^+(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+q_B^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-q_B^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+q_B^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-q_B^+(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+r_A^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-r_A^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r_A^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-r_A^+(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+r_B^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-r_B^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+r_B^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-r_B^+(h)]^{\varpi}},$$

$$\frac{\prod_{j=1}^{n}[1+s_A^+(h)]^{\varpi}-\prod_{j=1}^{n}[1-s_A^+(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s_A^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-s_A^+(h)]^{\varpi}} \geq \frac{\prod_{j=1}^{n}[1+s_B^-(h)]^{\varpi}-\prod_{j=1}^{n}[1-s_B^-(h)]^{\varpi}}{\prod_{j=1}^{n}[1+s_B^+(h)]^{\varpi}+\prod_{j=1}^{n}[1-s_B^+(h)]^{\varpi}},$$

$$\max\{(I_A^+) \geq (I_B^+)\}$$

Since

$$\frac{2-p_B(h)}{p_B(h)} \geq \frac{2-p_A(h)}{p_A(h)},$$
$$\frac{2-q_B(h)}{q_B(h)} \geq \frac{2-q_A(h)}{q_A(h)},$$
$$\frac{2-r_B(h)}{r_B(h)} \geq \frac{2-r_A(h)}{r_A(h)},$$
$$\frac{2-s_B(h)}{s_B(h)} \geq \frac{2-s_A(h)}{s_A(h)}, \min\{(\mu_A) \geq (\mu_B)\}$$

then $\left\{\frac{2-p_B(h)}{p_B(h)}\right\}^{w_j} \geq \left\{\frac{2-p_A(h)}{p_A(h)}\right\}^{w_j}$

$$\prod\nolimits_{j=1}^{n}\left\{\frac{2-p_B(h)}{p_B(h)}\right\}^{w_j} \geq \prod\nolimits_{j=1}^{n}\left\{\frac{2-p_A(h)}{p_A(h)}\right\}^{w_j}$$
$$\frac{1}{\prod_{j=1}^{n}\left\{\frac{2-p_B(h)}{p_B(h)}\right\}^{w_j}+1} \frac{1}{\prod_{j=1}^{n}\left\{\frac{2-p_A(h)}{p_A(h)}\right\}^{w_j}+1};$$
$$\left\{\frac{2-q_B(h)}{q_B(h)}\right\}^{w_j} \geq \left\{\frac{2-q_A(h)}{q_A(h)}\right\}^{w_j} \prod\nolimits_{j=1}^{n}\left\{\frac{2-q_B(h)}{q_B(h)}\right\}^{w_j} \geq \prod\nolimits_{j=1}^{n}\left\{\frac{2-q_A(h)}{q_A(h)}\right\}^{w_j}$$
$$\frac{1}{\prod_{j=1}^{n}\left\{\frac{2-q_B(h)}{q_B(h)}\right\}^{w_j}+1} \frac{1}{\prod_{j=1}^{n}\left\{\frac{2-q_A(h)}{q_A(h)}\right\}^{w_j}+1};$$
$$\left\{\frac{2-r_B(h)}{r_B(h)}\right\}^{w_j} \geq \left\{\frac{2-r_A(h)}{r_A(h)}\right\}^{w_j} \prod\nolimits_{j=1}^{n}\left\{\frac{2-r_B(h)}{r_B(h)}\right\}^{w_j} \geq \prod\nolimits_{j=1}^{n}\left\{\frac{2-r_A(h)}{r_A(h)}\right\}^{w_j}$$
$$\frac{1}{\prod_{j=1}^{n}\left\{\frac{2-r_B(h)}{r_B(h)}\right\}^{w_j}+1} \frac{1}{\prod_{j=1}^{n}\left\{\frac{2-r_A(h)}{r_A(h)}\right\}^{w_j}+1};$$
$$\left\{\frac{2-s_B(h)}{s_B(h)}\right\}^{w_j} \geq \left\{\frac{2-s_A(h)}{s_A(h)}\right\}^{w_j} \prod\nolimits_{j=1}^{n}\left\{\frac{2-s_B(h)}{s_B(h)}\right\}^{w_j} \geq \prod\nolimits_{j=1}^{n}\left\{\frac{2-s_A(h)}{s_A(h)}\right\}^{w_j}$$
$$\frac{1}{\prod_{j=1}^{n}\left\{\frac{2-s_B(h)}{s_B(h)}\right\}^{w_j}+1} \frac{1}{\prod_{j=1}^{n}\left\{\frac{2-s_A(h)}{s_A(h)}\right\}^{w_j}+1},$$
$$\min\{(\mu_A) \geq (\mu_B)\}$$
$$\prod\nolimits_{j=1}^{n}\{(\mu_A)\}^{w_j} \geq \prod\nolimits_{j=1}^{n}\{(\mu_B)\}^{w_j}.$$

$$\frac{2}{\prod_{j=1}^{n}\left\{\frac{2-p_B(h)}{p_B(h)}\right\}^{w_j}+1} \leq \frac{2}{\prod_{j=1}^{n}\left\{\frac{2-p_B(h)}{p_B(h)}\right\}^{w_j}+1};$$
$$\frac{2}{\prod_{j=1}^{n}\left\{\frac{2-q_B(h)}{q_B(h)}\right\}^{w_j}+1} \leq \frac{2}{\prod_{j=1}^{n}\left\{\frac{2-q_B(h)}{q_B(h)}\right\}^{w_j}+1},$$
$$\frac{2}{\prod_{j=1}^{n}\left\{\frac{2-r_B(h)}{r_B(h)}\right\}^{w_j}+1} \leq \frac{2}{\prod_{j=1}^{n}\left\{\frac{2-r_B(h)}{r_B(h)}\right\}^{w_j}+1},$$
$$\frac{2}{\prod_{j=1}^{n}\left\{\frac{2-s_B(h)}{s_B(h)}\right\}^{w_j}+1} \leq \frac{2}{\prod_{j=1}^{n}\left\{\frac{2-s_B(h)}{s_B(h)}\right\}^{w_j}+1}.$$
$$\min[\mu_A] \leq \min[\mu_B];$$
$$\frac{2\prod_{j=1}^{n}[p_B(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-p_B(h)]^{\mu}+\prod_{j=1}^{n}[p_B(h)]^{\varpi}} \leq \frac{2\prod_{j=1}^{n}[p_A(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-p_A(h)]^{\mu}+\prod_{j=1}^{n}[p_A(h)]^{\varpi}},$$
$$\frac{2\prod_{j=1}^{n}[q_B(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-q_B(h)]^{\mu}+\prod_{j=1}^{n}[q_B(h)]^{\varpi}} \leq \frac{2\prod_{j=1}^{n}[q_A(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-q_A(h)]^{\mu}+\prod_{j=1}^{n}[q_A(h)]^{\varpi}},$$
$$\frac{2\prod_{j=1}^{n}[r_B(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-r_B(h)]^{\mu}+\prod_{j=1}^{n}[r_B(h)]^{\varpi}} \leq \frac{2\prod_{j=1}^{n}[r_A(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-r_A(h)]^{\mu}+\prod_{j=1}^{n}[r_A(h)]^{\varpi}},$$
$$\frac{2\prod_{j=1}^{n}[s_B(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-s_B(h)]^{\mu}+\prod_{j=1}^{n}[s_B(h)]^{\varpi}} \leq \frac{2\prod_{j=1}^{n}[s_A(h)]^{\varpi}}{\prod_{j=1}^{n}[(2-s_A(h)]^{\mu}+\prod_{j=1}^{n}[s_A(h)]^{\varpi}},$$

We can get TrCFEWA $(A_1, A_2, ..., A_n) \leq$ TrCFEWA $(B_1, B_2, ..., B_n)$,
which complete the proof . □

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