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FOUNDATIONS

The characterizations of upper approximation operators based on coverings

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Abstract
In this paper, We propose a condition of symmetry for the covering $\mathscr{C}$ in a covering-based approximation space $(U, \mathscr{C})$. By using this condition, we obtain general, topological and intuitive characterizations of the covering $\mathscr{C}$ for two types of covering-based upper approximation operators being closure operators. We investigate axiomatic systems for $\overline{apr}_S$ and discuss the relationships among upper approximation operators. We also give a description of $(U, \mathscr{C})$ in terms of information exchange systems when these operators are closure ones. We also solve an open problem raised by Ge et al.

Keywords Closure operator · Covering-based upper approximation operator · Partition · Third condition of symmetry

1 Introduction and background

In order to obtain useful information and deal with uncertain data, many methods have been proposed, such as statistical methods, fuzzy set theory (Zadeh 1965, 1996), computing words, etc. (Zadeh 1996). But these methods have their limitations. In view of this, Pawlak proposed the theory of rough sets (Pawlak 1982) and did many works (Pawlak 1991; Pawlak and Skowron 2007b, c). It is a useful tool for handing uncertain things. Comparing with the above methods, the rough set theory has its advantages. For example, it does not need any additional information about data in the process of dealing with uncertain data. Therefore, it has been applied successfully in evidence theory (Skowron 1989), process control (Mrozek 1996), economics (Zhang 2017), medical diagnosis (Tsumoto 1996), biochemistry (Wang 2004), environmental science (Liu 2003), biology (Zhao 2010), chemistry (Zhang 2017), psychology (Tang 2015), conflict analysis (Gao 2008), and so on. From then on, many researchers have made some significant contributions to develop the rough theory. Kondo (2005) investigated the structure of generalized rough sets. Qin and Pei (2005) obtained the topological properties of fuzzy rough sets. Yao (1998a, b) studied fuzzy sets and rough sets and constructed algebraic methods of theory of rough sets. Zhu et al. investigated the rough sets based on coverings and discussed the relations among them (Zhu and Wang 2007; Zhu 2007; Zhu and Wang 2007; Zhu 2007, 2009a, b; Zhu and Wang 2012). Fan et al. (2011, 2012) discussed the covering approximation spaces. However, a problem with Pawlak's rough set theory is that partition or equivalence relation is explicitly used in the definition of the lower and upper approximations. Such a partition or equivalence relation is too restrictive for many applications because it can only deal with complete information systems. To address that issue, generalizations of rough set theory were considered by scholars. One approach was to extend the equivalence relation to the tolerance (Skowron and Stepaniuk 1996; Slowinski and Vanderpooten 2000; Yao 1998) and others (Zhu 2007, 2009a; Zhu and Zhang 2002; Zhang and Luo 2013). The other important approach was to relax the partition to a covering of the universe. In 1983, W. Zakowski generalized the classical rough set theory using coverings of a universe instead of partitions (Zakowski

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1983). A pair of lower and upper approximation operators are defined, but the generalized approximation operators are no longer dual to each other with respect to set complement (Yao 1998; Pomykala 1987). This new model is often referred as the first type of covering-based rough set. Such a generalization leads to various covering approximation operators with both theoretical and practical importance (Chen and Wang 2007; Ge 2010, 2014). The relationships between properties of covering-based approximation and their corresponding coverings have attracted intensive researches. Based on the mutual correspondence of the concepts of extension and intension, Bryniarski (1989) and Bonikowski et al. (1998) gave the second type of covering-based rough sets. The third and the fourth type of covering-based rough sets were introduced in Thomas (2003). Subsequently, Zhu (2007) utilized the topological method to characterize covering rough sets. Zhu and Wang discussed the relationship between properties of four types of covering-based upper approximation operators and their corresponding coverings (Zhu 2009a, b; Zhu and Wang 2012). Yang et al. (2010) investigated attribute reduction of covering information systems. Deer et al. (2016) discussed neighborhood operators for covering-based rough sets and defined the partial relation on the universe. Yang and Hu (2016) obtained some interesting results based on covering-based rough set and fuzzy lattice. Chen et al. (2015) studied the relations of reduction between covering generalized rough sets and concept lattices. Liu studied the two types of rough sets induced by coverings and obtained some interesting results. Cattaneo et al. obtained the algebraic structures of generalized rough set theory (Cattaneo 1998; Cattaneo and Ciucci 2004; Liu and Sai 2009; Liu and Zhu 2008). Liu also used the axiomatic method to characterize covering-based rough sets (Liu 2013, 2006, 2008). Bian et al. (2015) gave characterizations of covering-based approximation operators being closure operators. Ge et al. proposed not only general, but also topological characterizations of coverings for these operators being closure operators (Ge and Li 2011; Ge 2010, 2014; Ge et al. 2012). Furthermore, they gave intuitive characterizations of covering-based upper operators. At the end of Ge et al. (2012), Ge et al. raised the following question:

Question 1 (Question 9.4 of Ge et al. 2012) What are general, topological or intuitive characterizations of covering $\mathscr{C}$ for them to be closure operators? What kind of information exchanges systems does covering-based approximation space $(U, \mathscr{C})$ represent when any of them is a closure operator?

Since this question was put forward, many scholars did a lot of work about it. Their work focuses on $N(x)$ is a neighborhood of x. Yao defined the map n which may not be reflexive, therefore $n(x)$ may not be a neighborhood of x. How to deal with the above question when $n(x)$ may not be a neighborhood of x? We will investigate the approximation operators $\overline{apr}_n, \overline{apr}_{N_3}$ proposed by Yao and Yao (2012), and will focus on these two approximation operators to answer the above question.

This paper is arranged as follows: In Sect. 2, we present some basic concepts in covering-based rough sets and obtain characterization of $\overline{C}_4(X)$. Sections 3, 4 and 5 are the core content of this paper. In Sect. 3, we investigate the characterizations of coverings $\mathscr{C}$ for $\overline{apr}_n$ being a closure operator and get the general, topological, intuitive and information exchange characterization for $\overline{apr}_n$ being a closure operator. In Sect. 4, we obtain the characterizations of coverings $\mathscr{C}$ for $\overline{apr}_{N_3}$ being a closure operator. We also obtain topology characterization of $\mathscr{C}$ for $\overline{apr}_{N_3}$ to be a closure operator. In Sect. 5, we discuss the relationships between $\overline{apr}_n$ and unary covering. In Sect. 6, we obtain some propositions about $\overline{apr}^v_n$. In Sects. 7 and 9, we get the characterizations of $\overline{apr}^v_n$ and $\overline{apr}_S$ being a closure operator. In Sect. 8, we discuss the relationship between $\overline{apr}^v_n$ and $\overline{apr}_n$. In Sect. 10, we draw a conclusion.

2 Preliminaries

In this section, we introduce the fundamental concepts used in this paper. In the following discussion, unless it is mentioned specially, the universe of discourse U is considered finite. $P(U)$ denotes the family of all subsets of U. $\mathscr{C}$ is a family of subsets of U. If none of subsets in $\mathscr{C}$ is empty, and $\cup\mathscr{C} = U$, then $\mathscr{C}$ is called a covering of U.

Definition 1 (*Covering approximation space* Zhu and Wang 2012) Let U be an universe, $\mathscr{C}$ a covering of U , then we call U together with covering $\mathscr{C}$ a covering approximation space, denoted by $(U, \mathscr{C})$.

Definition 2 Let $(U, \mathscr{C})$ be a covering approximation space. For any $x \in U$ and $X \subseteq U$, the operators are defined as follows:

(1) $\underline{C}(X) = \cup\{C \in \mathscr{C} : C \subseteq X\}$;
(2) $md(x) = \{C \in \mathscr{C} : (x \in C) \wedge (\forall K \in \mathscr{C})(x \in K \wedge K \subseteq C \Rightarrow C = K)\}$;
(3) $N(x) = \cap\{C \in \mathscr{C} : x \in C\}$;
(4) $Friends(x) = \cup\{C \in \mathscr{C} : x \in C\}$;
(5) $MD(x) = \{C \in \mathscr{C} : (x \in C) \wedge (\forall K \in \mathscr{C}) \wedge (C \subseteq K) \Rightarrow C = K\}$;
(6) $n_0(MD(x)) = \cap\{C : C \in MD(x)\}$;
(7) $N_3(x) = n_0(MD(x))$.

Remark 1 $MD(x)$ and $md(x)$ denote subsets of $P(U)$, the others represent subsets of U.

Definition 3 Let $(U, \mathscr{C})$ be a covering approximation space. $(U, \mathscr{C})$ is called a strongly discrete space, if for any $x \in U$, we have $\{x\} \in \mathscr{C}$.

Definition 4 Let $(U, \mathscr{C})$ be a covering approximation space and $X \subseteq U$. It is easy to check $\mathscr{C}' = \{C \cap X : C \in \mathscr{C}\}$ is a covering on X. $\mathscr{C}'$ is called the induced covering, and covering approximation space $(X, \mathscr{C}')$ is called a subspace of $(U, \mathscr{C})$.

We use $\overline{\mathscr{C}_n}$ $(1 \le n \le 8)$ to present some different types of covering-based upper approximation operators listed in Samanta and Chakraborty (2009). Note that these operators were denoted by different symbols in Samanta and Chakraborty (2009) and other rough-set literature.

Definition 5 Let $(U, \mathscr{C})$ be a covering approximation space. For any $X \subseteq U$, upper approximation operators are defined as follows:

(1) $\overline{C}_1(X) = \underline{C}(X) \cup (\cup\{\cup md(x) : x \in X \backslash \underline{C}(X)\})$;
(2) $\overline{C}_2(X) = \cup\{C \in \mathscr{C} : C \cap X \neq \emptyset\}$;
(3) $\overline{C}_3(X) = \cup\{\cup md(x) : x \in X\}$;
(4) $\overline{C}_4(X) = \underline{C}(X) \cup (\cup\{C \in \mathscr{C} : C \cap (X \backslash \underline{C}(X) \neq \emptyset\})$;
(5) $\overline{C}_5(X) = \{y : \forall C(y \in C \Rightarrow C \cap X \neq \emptyset)\}$;
(6) $\overline{C}_6(X) = \{x \in U : \forall u(u \in N(x) \rightarrow N(u) \cap X \neq \emptyset)\}$;
(7) $\overline{C}_7(X) = \cup\{N(x) : N(x) \cap X \neq \emptyset\}$;
(8) $\overline{C}_8(X) = \{z : \forall y(z \in Friends(y) \Rightarrow Friends(y) \cap X \neq \emptyset)\}$;
(9) $\overline{apr}_n(X) = \{x : n(x) \cap X \neq \emptyset\}$;
(10) $\overline{apr}_{N_3}(X) = \{x : N_3(x) \cap X \neq \emptyset\}$.

Definition 6 (Yao and Yao 2012; Lin 1997) A mapping $n : U \rightarrow P(U)$ is called a neighborhood operator.

n is said to be serial, if $n(x) \neq \emptyset$ for all $x \in U$.

n is said to be reflexive, if $x \in n(x)$ for all $x \in U$.

n is said to be transitive, if $x \in n(x)$ and $y \in n(z) \Rightarrow x \in n(z)$ for all $x, y, z \in U$.

n is said to be Euclidean, if $y \in n(x) \Rightarrow n(x) \subseteq n(y)$ for all $x, y \in U$.

Definition 7 (Yao and Yao 2012) Suppose that $n : U \rightarrow P(U)$ is a neighborhood operator. It defines an upper operator of X as follows:

$$\overline{apr}_n(X) = \{x \in U : n(x) \cap X \neq \emptyset\}.$$

Definition 8 Suppose that $n : U \rightarrow P(U)$ is a neighborhood operator. The upper operator has non-reflexive property if n is non-reflexive.

Definition 9 (Zhang et al. 2015) Suppose that $n : U \rightarrow P(U)$ is a neighborhood operator. It defines a pair of approximation operator of X as follows:

$$\overline{apr}^v_n(X) = \bigcup_{x \in X} n(x);$$
$$\underline{apr}^v_n(X) = -\overline{apr}^v_n(-X).$$

Definition 10 (Zhang et al. 2015) A family of subsets of universe U is called a closure system over U if it contains U and is closed under set intersection. Given a closure system $\overline{S}$, one can define its dual system $\underline{S}$ as follows:

$$\underline{S} = \{-X : X \in \overline{S}\}.$$

Definition 11 (*Subsystem-based definition* Zhang et al. 2015) Suppose $S = (\overline{S}, \underline{S})$ is a pair of subsystems of $\mathscr{P}(U)$, $\overline{S}$ is a closure system and $\underline{S}$ is the dual system of $\overline{S}$. A pair of lower and upper approximation operators $(\overline{apr}_S, \underline{apr}_S)$ with respect to S is defined as:

$$\overline{apr}_S(X) = \cap\{K \in \overline{S} : X \subseteq K\};$$
$$\underline{apr}_S(X) = \cup\{K \in \underline{S} : K \subseteq X\} \text{ for any } X \subseteq U.$$

The following topological concepts and facts are elementary and can be found in Engelking (1989). We list them below for the purpose of being self-contained in this paper.

1. A topological space is a pair (U, τ) consisting of a set U and a family τ of subsets of U satisfying the following conditions:
   a. $\emptyset \in \tau$ and $U \in \tau$;
   b. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$;
   c. If $\mathscr{A} \subseteq \tau$, then $\cup\mathscr{A} \in \tau$. τ is called a topology on U and the members of τ are called open sets of (U, τ).

   A set F in (U, τ) is called closed set if the complement set $-F$ of F is an open set.
2. A set F is called a clopen set, if F in (U, τ) is both an open set and a closed set.
3. A family $\mathscr{B} \subseteq \tau$ is called a base for (U, τ) if for every non-empty open subset O of U and each $x \in O$, there exists a set $B \in \mathscr{B}$ such that $x \in B \subseteq O$. Equivalently, a family $\mathscr{B} \subseteq \tau$ is a base if every non-empty open subset O of U can be represented as union of a subfamily of $\mathscr{B}$.
4. For any $x \in U$, a family $\mathscr{B} \subseteq \tau$ is called a local base at x for (U, τ) if $x \in B$ for each $B \in \mathscr{B}$, and for every open subset O of U with $x \in O$, there exists a set $B \in \mathscr{B}$ such that $B \subseteq O$.
5. If $\mathscr{P}$ is a partition of U, the topology $\tau = \{O \subseteq U$: O is the union of some members of $\mathscr{P}\} \cup \{\emptyset\}$ is called a pseudo-discrete topology in Pawlak and Skowron (2007a) (also called a closed-open topology in Pawlak 1991).
6. Let (U, τ) be a topological space. If for each pair of points $x, y \in U$ with $x \neq y$, there exist open sets O, O' such that $x \in O$, $y \in O'$ and $O \cap O' = \emptyset$, then (U, τ) is called a T_2-space and τ a T_2-topology.

Definition 12 (*Induced topology and subspace*) Let (U, τ) be a topological space and $X \subseteq U$. It is easy to check that

$\tau' = \{O \cap X : O \in \tau\}$ is a topology on X. τ' is called a topology induced by X, and the topology space (X, τ') is called a subspace of (U, τ).

Definition 13 (*Closure operator*) An operator $H: P(U) \to P(U)$ is called a closure operator on U if it satisfies the following conditions: for any $X, Y \subseteq U$,

$(H_1) H(X \cup Y) = H(X) \cup H(Y)$;

$(H_2) X \subseteq H(X)$;

$(H_3) H(\emptyset) = \emptyset$;

$(H_4) H(H(X)) = H(X)$.

Definition 14 (*Interior operator*) An operator $I: P(U) \to P(U)$ is called an interior operator on U if it satisfies the following conditions: for any $X, Y \subseteq U$,

$(I_1) I(X \cap Y) = I(X) \cap I(Y)$;

$(I_2) I(X) \subseteq X$;

$(I_3) I(U) = U$;

$(I_4) I(I(X)) = I(X)$.

Definition 15 (*Dual operator*) Assume that $H, I: P(U) \to P(U)$ are two operators on U. If for any $X \subseteq U$, $H(X) = \sim I(\sim X)$. Then we say that H, I are dual operator or H is the dual operator of I.

From the definition, it is obvious that each interior operator on U is the dual operator of a closure operator on U.

We give a characterization of $\overline{C}_4$ below.

Lemma 1 (Zhu and Wang 2012) *$\overline{C}_4$ is a closure operator if and only if $\mathscr{C}$ satisfies the following condition: for any $K_1, K_2 \in \mathscr{C}$, if $K_1 \neq K_2$ and $K_1 \cap K_2 \neq \emptyset$, then for any $x \in K_1 \cap K_2$, we have $\{x\} \in \mathscr{C}$.*

Lemma 2 *Let $(U, \mathscr{C})$ be a covering approximation space. $\overline{C}_4$ a closure operator and for any $K_1, K_2 \in \mathscr{C}$, if $K_1 \neq K_2$ and $X = K_1 \cap K_2 \neq \emptyset$, then $(X, \mathscr{C}')$ is a strongly discrete space, where $\mathscr{C}' = \{C \cap X : C \in \mathscr{C}\}$.*

Proof If X has only one point, the result is obvious. Otherwise, we assume that X has no less than two points. From Definition 2.5, $(X, \mathscr{C}')$ is a subspace of $(U, \mathscr{C})$, for any $x, y \in X$ and $x \neq y$, since $\overline{C}_4$ is a closure operator and from Lemma 1, we have $\{x\}, \{y\} \in \mathscr{C}'$. So $(X, \mathscr{C}')$ is a strongly discrete space. □

Lemma 3 *Let $(U, \mathscr{C})$ be a covering approximation space. For any $K_1, K_2 \in \mathscr{C}$, $K_1 \neq K_2$ and $X = K_1 \cap K_2 \neq \emptyset$. If $(X, \mathscr{C}')$ is a strongly discrete space, then $\overline{C}_4$ is a closure operator, where $\mathscr{C}' = \{C \cap X : C \in \mathscr{C}\}$.*

Proof For any $x \in X$, since $(X, \mathscr{C}')$ is a strongly discrete space, from Definition 2.4, we have $\{x\} \in \mathscr{C}$. From Lemma 1, $\overline{C}_4$ is a closure operator. □

Each member of U is a person or a machine (such as a computer and so on) that can exchange information with other members and each $C \in \mathscr{C}$ is an information exchange group. That is, for each pair $x, y \in U$, if there is a $C \in \mathscr{C}$ such that $x, y \in C$, we say that x and y can exchange or share certain information. If there is a $C \in \mathscr{C}$ such that $x \in C$ and $y \notin C$, we say that x has some information which cannot be shared with y.

From Lemmas 2, 3 and the above description, we obtain the following results:

Theorem 1 (Characterization of coverings $\mathscr{C}$ for $\overline{C}_4$ being a closure operator) *Let $(U, \mathscr{C})$ be a covering approximation space. $\overline{C}_4$ is a closure operator if and only if for any $K_1, K_2 \in \mathscr{C}$, if $K_1 \neq K_2$ and $X = K_1 \cap K_2 \neq \emptyset$, then $(X, \mathscr{C}')$ is a strongly discrete space, where $\mathscr{C}' = \{C \cap X : C \in \mathscr{C}\}$.*

From Lemma 2 and Theorem 1, we get the following result:

Theorem 2 (Information exchange system description of an approximation space $(U, \mathscr{C})$ for $\overline{C}_4$ being a closure operator) *For a covering-based approximation space $(U, \mathscr{C})$, $\overline{C}_4$ is a closure operator if and only if the approximation space $(U, \mathscr{C})$ can be described as an information exchange system in which for any $K_1, K_2 \in \mathscr{C}$, $K_1 \neq K_2$ and $X = K_1 \cap K_2 \neq \emptyset$, such that for any $x, y \in X$, they can not share any information with each other.*

In order to answer Question 1, substantial progress has been made on searching for general, topological and intuitive characterizations of $\mathscr{C}$, and also on giving information systems representation of $(U, \mathscr{C})$, when these operators are closure operators. We summarize these results in the following table:

3 Characterization of covering $\mathscr{C}$ for $\overline{apr}_n$ being a closure operator

Theorem 3 (General characterization of coverings $\mathscr{C}$ for $\overline{apr}_n$ being a closure operator) *Let n be a reflexive neighborhood operator, then $\overline{apr}_n$ is a closure operator if and only if for any $x, y \in U$, $n(x) \subseteq n(y)$ whenever $x \in n(y)$.*

Proof ($\Rightarrow$) For any $x, y \in U$ with $x \in n(y)$, we claim that $n(x) \subseteq n(y)$. If not, there exists a $z \in n(x)$ such that $z \notin n(y)$. Since n is a reflexive neighborhood operator, we have $x \in \overline{apr}_n(\{z\})$. So $\overline{apr}_n$ is a closure operator, thus $x \in \overline{apr}_n(\overline{apr}_n(\{z\}))$. By $x \in n(y)$, we have $n(y) \cap \overline{apr}_n(\{z\}) \neq \emptyset$. Thus $y \in \overline{apr}_n(\overline{apr}_n(\{z\}))$. Since

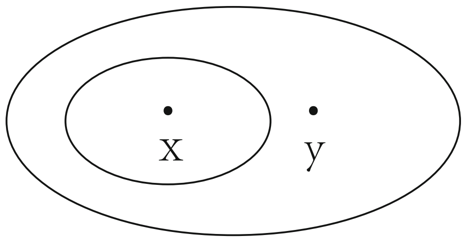

Fig. 1 The relationship

$\overline{apr}_n$ is a closure operator, so $y \in \overline{apr}_n(\{z\})$. Therefore, $n(y) \cap \{z\} \neq \emptyset$, hence $z \in n(y)$. This contradicts the assumption that $z \notin n(y)$.

($\Leftarrow$) For any $x, y \in U$, if $x \in n(y)$, then $n(x) \subseteq n(y)$. To prove $\overline{apr}_n$ is a closure operator, we only need to prove $\overline{apr}_n$ is a closure operator which satisfies Definition 6. It is easy to obtain $\overline{apr}_n(\emptyset) = \emptyset$, since n is a reflexive neighborhood operator. For any $A, B \subseteq U$, we have $A \subseteq \overline{apr}_n(A)$. By the definition of $\overline{apr}_n$, $\overline{apr}_n(A \cup B) = \overline{apr}_n(A) \cup \overline{apr}_n(B)$ is obvious for any $A, B \subseteq U$. Thus, it remains to prove that $\overline{apr}_n(\overline{apr}_n(A)) = \overline{apr}_n(A)$ for any $A \subseteq U$. From the above, we have $\overline{apr}_n(A) \subseteq \overline{apr}_n(\overline{apr}_n(A))$. We need to prove that $\overline{apr}_n(\overline{apr}_n(A)) \subseteq \overline{apr}_n(A)$. Let $x \in \overline{apr}_n(\overline{apr}_n(A))$, by the definition of $\overline{apr}_n$, $n(x) \cap \overline{apr}_n(A) \neq \emptyset$. Pick $p \in n(x) \cap \overline{apr}_n(A)$. Then $p \in n(x)$ and $n(p) \cap A \neq \emptyset$. Thus $n(p) \subseteq n(x)$, and hence $n(x) \cap A \neq \emptyset$. Therefore, we know that $x \in \overline{apr}_n(A)$. By the arbitrariness of x, we obtain that $\overline{apr}_n$ is a closure operator. □

Figure 1 gives a simple example of Theorem 3.

Lemma 4 (Engelking 1989) *Suppose we are given a set X and a family $\mathscr{B}$ of subsets of X which has properties:*

(1) *For any $U_1, U_2 \in \mathscr{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U \in \mathscr{B}$ such that $x \in U \subseteq U_1 \cap U_2$;*
(2) *For every $x \in X$ there exists a $U \in \mathscr{B}$ such that $x \in U$.*

Then $\mathscr{B}$ is a base for X.

Theorem 4 (Topological characterization of coverings $\mathscr{C}$ for $\overline{apr}_n$ being a closure operator) *Let n be a reflexive neighborhood operator. Then $\overline{apr}_n$ is a closure operator if and only if there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ).*

Proof ($\Rightarrow$) Assume that $\overline{apr}_n$ is a closure operator. Since n is a reflexive neighborhood operator, so $\{n(x) : x \in U\}$ is a covering of U. For any $n_1(x), n_2(x) \in \{n(x) : x \in U\}$ and any $y \in n_1(x) \cap n_2(x)$, we have $y \in n_1(x)$ and $y \in n_2(x)$. By Theorem 3, $n(y) \subseteq n_1(x)$ and $n(y) \subseteq n_2(x)$. Hence $n(y) \subseteq n_1(x) \cap n_2(x)$. Therefore $y \in n(y) \subseteq n_1(x) \cap n_2(x)$.

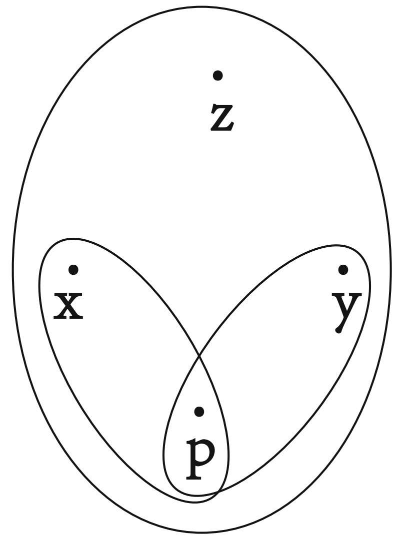

Fig. 2 The third condition of symmetry

By Lemma 4, we know that there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ).

($\Leftarrow$) Let n be a reflexive neighborhood operator, and assume there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ). By the definition of base and Lemma 4, for any $y \in n(x)$, we have $n(y) \subseteq n(x)$. From Theorem 3, $\overline{apr}_n$ is a closure operator. □

With this description, we have the following results.

Theorem 5 (Information exchange system description of approximation space $(U, \mathscr{C})$ for $\overline{apr}_n$ being a closure operator) *Let n be a reflexive neighborhood operator, $\overline{apr}_n$ is a closure operator if and only if the approximation space $(U, \mathscr{C})$ can be described as such an information exchange system that for any $x, y \in U$, if $x \in n(y)$, then y knows all the information which x has.*

4 Characterization of covering $\mathscr{C}$ for $\overline{apr}_{N_3}$ being a closure operator

Definition 16 (*The third condition of symmetry*) Let $\mathscr{C}$ be a covering of U. We say that $\mathscr{C}$ satisfies the third condition of symmetry if the following condition is true:

For any $x, y \in U$, if there exists a $p \in U$ such that $p \in C$ for any $C \in MD(p)$ with $x \in C$ or $y \in C$, then there exists a $z \in U$ such that $x \in C'$ and $y \in C'$ for any $C' \in MD(z)$.

Figure 2 gives an intuitive illustration of the third condition of symmetry.

Lemma 5 *Let $\mathscr{C}$ be a covering of U. Then $\mathscr{C}$ satisfies the third condition of symmetry if and only if for every $x, y \in U$, if $N_3(x) \cap N_3(y) \neq \emptyset$, there is a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$.*

Proof ($\Rightarrow$) Assume that $\mathscr{C}$ satisfies the third condition of symmetry. If $x, y \in U$ and $N_3(x) \cap N_3(y) \neq \emptyset$, take $p \in N_3(x) \cap N_3(y)$. For any $C \in MD(p)$, $x \in C$ or $y \in C$. By the third condition of symmetry of $\mathscr{C}$, there exists a $z \in U$ such that $x \in C'$ and $y \in C'$ for any $C' \in MD(z)$. If $C \subseteq C'$, then $N_3(x) \subseteq N_3(z)$. If $C \cap C' \neq \emptyset$, it is easy to obtain $N_3(x) \subseteq N_3(z)$. If $N_3(z) \subseteq N_3(x)$, then $y \in N_3(z)$, so there exists $C_0 \in MD(p)$ such that $x, y \in C_0$. It is a contradiction because $x \in C_0$ or $y \in C_0$. We can prove $N_3(y) \subseteq N_3(z)$ by the same method. Thus $N_3(x) \cup N_3(y) \subseteq N_3(z)$.

($\Leftarrow$) Assume that for any $x, y \in U$, and $N_3(x) \cap N_3(y) \neq \emptyset$, there exists a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$. Let $x, y \in U$. If there is a $p \in U$ such that for any $C \in MD(p)$ with $x \in C$ or $y \in C$, then $p \in N_3(x) \cap N_3(y)$, and hence $N_3(x) \cap N_3(y) \neq \emptyset$. By the assumption, there is a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$, which means that $x \in C'$ and $y \in C'$ for any $C' \in MD(z)$. □

Lemma 6 *Let $(U, \mathscr{C})$ be a covering-based approximation space. $\overline{apr}_{N_3}$ satisfies the condition $\overline{apr}_{N_3}(\overline{apr}_{N_3}(X)) = \overline{apr}_{N_3}(X)$ for every $X \subseteq U$ if and only if for every $x, y \in U$, if $N_3(x) \cap N_3(y) \neq \emptyset$, then there is a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$.*

Proof ($\Rightarrow$) Suppose that $\overline{apr}_{N_3}$ satisfies the condition $\overline{apr}_{N_3}(\overline{apr}_{N_3}(X)) = \overline{apr}_{N_3}(X)$ for every $X \subseteq U$. Take any $x, y \in U$ with $N_3(x) \cap N_3(y) \neq \emptyset$. If $N_3(x) \subseteq N_3(y)$ or $N_3(y) \subseteq N_3(x)$, we only need to take $z = x$ or $z = y$. Hence, we assume $N_3(x) \nsubseteq N_3(y)$ and $N_3(y) \nsubseteq N_3(x)$. Without loss of generality, we can pick $X = \{x\}$.

We prove that $y \in \overline{apr}_{N_3}(X)$.Otherwise, by $N_3(x) \subseteq \overline{apr}_{N_3}(X)$, $N_3(x) \cap N_3(y) \neq \emptyset$, we get $N_3(y) \cap \overline{apr}_{N_3}(X) \neq \emptyset$, and hence $N_3(y) \subseteq \overline{apr}_{N_3}(\overline{apr}_{N_3}(X))$. It follows that $y \in \overline{apr}_{N_3}(\overline{apr}_{N_3}(X))$, which contradicts the fact that $\overline{apr}_{N_3}(\overline{apr}_{N_3}(X)) = \overline{apr}_{N_3}(X)$. This contradiction shows that $y \in \overline{apr}_{N_3}(X) = \cup\{\overline{apr}_{N_3}(z) : z \in X\}$, and hence there is a $z \in U$ such that $x \in N_3(z)$ and $y \in N_3(z)$. It follows that $N_3(x) \subseteq N_3(z)$ and $N_3(y) \subseteq N_3(z)$, which means that $N_3(x) \cup N_3(y) \subseteq N_3(z)$.

($\Leftarrow$) Assume that for any $x, y \in U$ with $N_3(x) \cap N_3(y) \neq \emptyset$, there exists a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$. By the definition of $\overline{apr}_{N_3}$, it is easy to check that $\overline{apr}_{N_3}(X) = \cup\{\overline{apr}_{N_3}(x) : x \in X\}$. Thus, to prove that $\overline{apr}_{N_3}(\overline{apr}_{N_3}(X)) = \overline{apr}_{N_3}(X)$ for every $X \subseteq U$, it is enough to prove that for every $x \in U$, $\overline{apr}_{N_3}(\overline{apr}_{N_3}(\{x\})) = \overline{apr}_{N_3}(\{x\})$. It is obvious that $\overline{apr}_{N_3}(\{x\}) \subseteq \overline{apr}_{N_3}(\overline{apr}_{N_3}(\{x\}))$. So we only need to prove the converse.

Otherwise, we take $y \in \overline{apr}_{N_3}(\overline{apr}_{N_3}(\{x\})) \backslash \overline{apr}_{N_3}(\{x\})$. By $\overline{apr}_{N_3}(\{x\}) = \cup\{N_3(z) : x \in N_3(z)\}$ and $y \notin \overline{apr}_{N_3}(\{x\})$, we know that for every $z \in U$, $x \in N_3(z)$ implies $y \notin N_3(z)$. On the other hand, there is a $z_0 \in U$ such that $x \in N_3(z_0)$, $N_3(y) \cap N_3(z_0) \neq \emptyset$, $y \in \overline{apr}_{N_3}(\overline{apr}_{N_3}(\{x\}))$. By our assumption, there exists a $z' \in U$ such that $N_3(z_0) \cup N_3(y) \subseteq N_3(z')$. It follows that $x \in N_3(z_0) \subseteq N_3(z')$ and $y \in N_3(y) \subseteq N_3(z')$, which contradicts the fact that for any $z \in U$, $x \in N_3(z)$ implies $y \notin N_3(z)$. □

Theorem 6 (General characterization of coverings $\mathscr{C}$ for $\overline{apr}_{N_3}$ being a closure operator) *Let $(U, \mathscr{C})$ be a covering-based approximation space. Then $\overline{apr}_{N_3}$ is a closure operator if and only if $\mathscr{C}$ satisfies the third condition of symmetry.*

Proof ($\Rightarrow$) Assume that $\overline{apr}_{N_3}$ is a closure operator. Then by (H_4), $\overline{apr}_{N_3}(\overline{apr}_{N_3}(X)) = \overline{apr}_{N_3}(X)$ for every $X \subseteq U$. Hence by Lemma 6, for any $x, y \in U$, if $N_3(x) \cap N_3(y) \neq \emptyset$, there is a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$. By Lemma 5, $\mathscr{C}$ satisfies the third condition of symmetry.

($\Leftarrow$) Assume that $\mathscr{C}$ satisfies the third condition of symmetry. By Lemma 5, for any $x, y \in U$ with $N_3(x) \cap N_3(y) \neq \emptyset$, there is a $z \in U$ such that $N_3(x) \cup N_3(y) \subseteq N_3(z)$. By Lemma 6, $\overline{apr}_{N_3}$ satisfies (H_4). From Definition 5, it is easy to prove (H_1),(H_2) and (H_3) holds for any covering $\mathscr{C}$, and hence $\overline{apr}_{N_3}$ is a closure operator. □

Theorem 7 (Topological characterization of coverings $\mathscr{C}$ for $\overline{apr}_{N_3}$ being a closure operator) *Let $(U, \mathscr{C})$ be a covering-based approximation space. Then $\overline{apr}_{N_3}$ is a closure operator if and only if $\mathscr{N} = \{N_3(x) : x \in U\}$ is a base for some topology τ on U such that in the topological space (U, τ), each element of $\mathscr{N}$ is contained in a clopen element of $\mathscr{N}$.*

Proof ($\Rightarrow$) Assume that $\overline{apr}_{N_3}$ is a closure operator. For any $N \in \mathscr{N}$, let $N_1 = N$. If for every $y \notin N$, $N_3(y) \cap N = \emptyset$, then $N = U \backslash \cup\{N_3(y) : y \notin N\}$. Since $\mathscr{N}$ is a base for the topology τ, both N and $\cup\{N_3(y) : y \notin N\}$ are open sets in the topological space (U, τ). Hence, N is a clopen set in (U, τ), and we can stop. Otherwise, there is a $y \in N$ with $N_3(y) \cap N \neq \emptyset$. Then by Theorem 6 and Lemma 6, there is a $z \in U$ such that $N \cup N_3(y) \subseteq N_3(z)$. Let $N_2 = N_3(z)$. If for every $w \notin N_3(z)$, $N_3(w) \cap N_3(z) = \emptyset$, then $N_3(z) = U \backslash \cup\{N_3(w) : w \notin N_3(z)\}$ is a clopen set in (U, τ) containing N, and we can stop. Otherwise, we can continue in this way. Finally, either we can find a clopen element N_i of $\mathscr{N}$ containing N for some positive integer i, or we can get infinitely many different elements N_i of $\mathscr{N}$ such that $N_1 \subseteq N_2 \subseteq N_3 \subseteq ... \subseteq N_i \subseteq N_{i+1} \subseteq ....$ However, U is a finite sets, so is $\mathscr{N}$, and hence the latter is impossible. Therefore, we must stop at getting some clopen elements of $\mathscr{N}$ containing N.

($\Leftarrow$) Assume that $\mathscr{N} = \{N_3(x) : x \in U\}$ is a base for some topology τ on U such that in the topological space (U, τ), each element of $\mathscr{N}$ is contained in a clopen element of $\mathscr{N}$. If $x, y \in U$ and $N_3(x) \cap N_3(y) \neq \emptyset$, there is a $z \in U$ such that $N(z)$ is a clopen in the topological space (U, τ) and $N_3(x) \subseteq N_3(z)$. Since $N(z)$ is clopen, $U \backslash N_3(z)$ is open, and we may assume that $U \backslash N_3(z) = \cup\{N_3(w) : w \in K\}$ for some $K \subseteq U$, because $\mathscr{N} = \{N_3(x) : x \in U\}$ is a base for τ. If $y \in U \backslash N_3(z)$, then there is a $w \in K$ such that $y \in N_3(w)$, thus $N_3(y) \subseteq N_3(w) \subseteq U \backslash N_3(z)$. It follows that $N_3(x) \cap N_3(y) = \emptyset$. By $N_3(x) \subseteq N_3(z)$, we obtain that $N_3(x) \cap N_3(y) = \emptyset$, which is a contradiction because $N_3(x) \cap N_3(y) \neq \emptyset$. This contradiction shows that $y \in N_3(z)$ and hence $N_3(y) \subseteq N_3(z)$. Thus, $N_3(x) \cup N_3(y) \subseteq N_3(z)$. By Lemma 6, $\mathscr{C}$ satisfies the third condition of symmetry. By Theorem 6, $\overline{apr}_{N_3}$ is a closure operator. □

5 The relationships among different covering approximation operators

In this section, we first give some basic concepts and discuss the relationships among different covering approximation operators. Let $(U, \mathscr{C})$ be a covering-based approximation space and $K \in \mathscr{C}$, $x \in K$, x is called a representative element of K if $\forall S \in \mathscr{C}(x \in S \Rightarrow K \subseteq S)$.

Let $\mathscr{C}$ be a covering of U. $\mathscr{C}$ is called unary if $\forall x \in U$, $|md(x)| = 1$. $CL(X) = \cup\{n(x) : n(x) \subseteq X\}$, $\underline{apr}_n(X) = \{x : n(x) \subseteq X\}$ for any $X \subseteq U$.

Lemma 7 (Zhu and Wang 2012) *$\mathscr{C}$ is a unary covering of U if and only if $N(x) \in \mathscr{C}$ for any $x \in U$.*

Theorem 8 *Let n be a reflexive neighborhood operator. Then $\{n(x) : x \in U\}$ is a unary covering of U if and only if $CL(X) = \underline{apr}_n(X)$ for any $X \subseteq U$.*

Proof ($\Rightarrow$) We prove $CL(X) \subseteq \underline{apr}_n(X)$ for any $X \subseteq U$. Otherwise, we can pick an $x \in CL(X) \backslash \underline{apr}_n(X)$. By the definition of $CL(X)$, there exists $x_0 \in X$ such that $x \in n(x_0) \subseteq X$. Since n is a reflexive neighborhood operator and $\{n(x) : x \in U\}$ is a unary covering of U, we have $x \in N(x) \subseteq n(x_0) \subseteq X$. Thus $x \in \underline{apr}_n(X)$ by the definition of $\underline{apr}_n(X)$ and Lemma 7, a contradiction. Thus $CL(X) \subseteq \underline{apr}_n(X)$ for any $X \subseteq U$. Now we prove $\underline{apr}_n(X) \subseteq CL(X)$ for any $X \subseteq U$. Otherwise, we can pick a $y \in \underline{apr}_n(X) \backslash CL(X)$. Since $y \in \underline{apr}_n(X)$, we have $n(y) \subseteq X$. By $n(y) \in \{n(x) : x \in U\}$ and the definition of CL, we have $y \in n(y) \subseteq CL(X)$, a contradiction.

($\Leftarrow$) Assume that $\{n(x) : x \in U\}$ is not a unary covering of U. Then by Lemma 7, there exists an $x_0 \in X$ such that $N(x_0) \notin \mathscr{C}$. Take $X = N(x_0)$. It is obvious that $x \in \underline{apr}_n(X)$ and $x \notin CL(X)$. It contradicts the assumption that $CL(X) = \underline{apr}_n(X)$ for any $X \subseteq U$. □

The relationship between $\overline{C_3}$ and $\underline{apr}_n$ is given by the following theorem.

Theorem 9 *Let n be a reflexive neighborhood operator. Then $\{n(x) : x \in U\}$ is a unary covering of U if and only if $\overline{C_3}(\underline{apr}_n(X)) = CL(X)$ for any $X \subseteq U$.*

Proof ($\Rightarrow$) Let n be a reflexive neighborhood operator and $\{n(x) : x \in U\}$ a unary covering of U. By Theorem 8, We have $CL(X) = \underline{apr}_n(X)$ for any $X \subseteq U$. It is easy to check that $CL(X)$ is either a union of finite union of members of $\{n(x) : x \in U\}$ or an empty set, and in both cases we have $\overline{C_3}(CL(X)) = CL(X)$ and hence $CL(X) = \overline{C_3}(CL(X)) = \overline{C_3}(\underline{apr}_n(X))$ for any $X \subseteq U$.

($\Leftarrow$) Assume that $\{n(x) : x \in U\}$ is not a unary covering of U. Then by Lemma 7, there is an $x \in X$ such that $N(x) \notin \{n(x) : x \in U\}$. Take $X = N(x)$. It is easy to prove that $\underline{apr}_n(X) = X$ and hence $x \in X \subseteq \overline{C_3}(\underline{apr}_n(X))$. On the other hand, since $\forall K \in \{n(x) : x \in U\}$ with $x \in K$, we have $X \subseteq K$ and $X \neq K$, and it follows that $x \notin CL(X)$, which contradicts the assumption that $CL(X) = \overline{C_3}(\underline{apr}_n(X))$ for any $X \subseteq U$. □

Theorem 10 *Let n be a reflexive neighborhood operator. Then $\{n(x) : x \in U\}$ is a unary covering of U if and only if $\{md(x) : x \in U\}$ is a base for some topology τ on U and $|md(x)| = 1$ for each $x \in U$.*

Proof ($\Rightarrow$) Since n is a reflexive neighborhood operator and $\{n(x) : x \in U\}$ is a unary covering of U, we have $\{md(x) : x \in U\}$ is a cover of U and $|md(x)| = 1$ for each $x \in U$. For any $md(y), md(z) \in \{md(x) : x \in U\}$ and any $p \in md(y) \cap md(z)$, we have $md(p) \subseteq md(y) \cap md(z)$. Otherwise, it contradicts $\{n(x) : x \in U\}$ be a unary covering of U.

($\Leftarrow$) Since $\{md(x) : x \in U\}$ is a base for some topology τ on U, we have $\cup\{md(x) : x \in U\} = U$. $\{md(x) : x \in U\} \subseteq \{n(x) : x \in U\}$, thus $\{n(x) : x \in U\}$ is a covering of U. From $|md(x)| = 1$ for each $x \in U$, we have $\{n(x) : x \in U\}$ is a unary covering. □

Theorem 11 *Let n be a reflexive neighborhood operator and $\{n(x) : x \in U\}$ is a unary covering of U. Then for any $x, y, z \in U$ and $z \in n(y) \cap n(x)$, there exists $x_0 \in U$ such that $z \in n(x_0) \subseteq n(y) \cap n(x)$.*

Proof Since n is a reflexive neighborhood operator and $\{n(x) : x \in U\}$ is a unary covering of U, we have $|md(z)| = 1$ and $N(z) = md(z)$. By Lemma 7, we have $N(z) \in \{n(x) : x \in U\}$. For any $x, y, z \in U$ and $z \in n(y) \cap n(x)$, we have $N(z) \subseteq n(y)$ and $N(z) \subseteq n(x)$. Thus $N(z) \subseteq n(y) \cap n(x)$. Pick $n(x_0) = N(z)$, therefore $z \in n(x_0) \subseteq n(y) \cap n(x)$.

□

6 Some propositions of $\overline{apr}_n^v$

Zhang et al. defined $\overline{apr}_n^v$ and discussed the relationships between generalized rough sets based on covering and reflexive neighborhood system (Zhang et al. 2015). We will investigate the properties of $\overline{apr}_n^v$.

Proposition 6.1 *Suppose that $n : U \to P(U)$ is a neighborhood operator. For any $X, Y \in U$, we have the following statements:*

(1) $\overline{apr}_n^v(\emptyset) = \emptyset$;
(2) $\underline{apr}_n^v(U) = U$;
(3) $\overline{apr}_n^v(X) = \bigcup_{x\in X} \overline{apr}_n^v(\{x\})$;
(4) *If* $X \subseteq Y$, *then* $\overline{apr}_n^v(X) \subseteq \overline{apr}_n^v(Y)$;
(5) $\overline{apr}_n^v(X \cup Y) = \overline{apr}_n^v(X) \cup \overline{apr}_n^v(Y)$.

Proof For (1) and (2), they are easy to prove by Definition 9.

(3) For any $y \in \overline{apr}_n^v(X)$, by Definition 9, there exists $x_0 \in X$ such that $y \in n(x_0) = \overline{apr}_n^v(\{x_0\})$. So $\overline{apr}_n^v(X) \subseteq \overline{apr}_n^v(X)$.

We only need to prove the converse. For any $y \in \bigcup_{x\in X} \overline{apr}_n^v(\{x\})$, there exists $x_0 \in X$ such that $y \in \overline{apr}_n^v(\{x_0\}) = n(x_0) \subseteq \bigcup_{x\in X} n(x) = \overline{apr}_n^v(X)$. Therefore $\overline{apr}_n^v(X) = \bigcup_{x\in X} \overline{apr}_n^v(\{x\})$.

(4) The proof is obvious by Definition 9.

(5) $\overline{apr}_n^v(X) \cup \overline{apr}_n^v(Y) \subseteq \overline{apr}_n^v(X \cup Y)$ is obvious by (3).

We only need to prove the converse. For any $y \in \overline{apr}_n^v(X \cup Y) = \bigcup_{x\in X\cup Y} \overline{apr}_n^v(X \cup Y)$, there exists $x_0 \in X \cup Y$ such that $y \in \overline{apr}_n^v(\{x_0\}) = n(x_0)$. If $x_0 \in X$, then $y \in \overline{apr}_n^v(X)$; if $x_0 \in Y$, then $y \in \overline{apr}_n^v(Y)$, therefore $\overline{apr}_n^v(X \cup Y) = \overline{apr}_n^v(X) \cup \overline{apr}_n^v(Y)$. □

However, the following properties may not hold:

(1) $\overline{apr}_n^v(U) = U$;
(2) $X \subseteq \overline{apr}_n^v(X)$;
(3) $\overline{apr}_n^v(\overline{apr}_n^v)(X) = \overline{apr}_n^v(X)$;
(4) $\underline{apr}_n^v(X) \subseteq X$.

Example 6.1 Let $U = \{a, b, c\}$ and $n : U \to P(U)$ is a neighborhood operator.

$n : a \mapsto \{a\}$;
$b \mapsto \{a\}$;
$c \mapsto \{a\}$.

It is easy to see $\overline{apr}_n^v(U) = \{a\} \neq U$.

Example 6.2 Let $U = \{a, b, c\}$ and $n : U \to P(U)$ is a neighborhood operator. Pick $X = \{a, b\}$ and

$n : a \mapsto \{c\}$;
$b \mapsto \{c\}$;
$c \mapsto \{c\}$.

It is easy to see $\overline{apr}_n^v(X) = \{c\}$. Therefore $X \nsubseteq \overline{apr}_n^v(X)$.

Example 6.3 Let $U = \{a, b, c\}$ and $n : U \to P(U)$ is a neighborhood operator. Pick $X = \{a, b\}$ and

$n : a \mapsto \{c\}$;
$b \mapsto \{c\}$;
$c \mapsto U$.

It is easy to see $\overline{apr}_n^v(\overline{apr}_n^v(X)) = U$ and $\overline{apr}_n^v(X) = \{c\}$. $\overline{apr}_n^v(\overline{apr}_n^v(X)) \neq \overline{apr}_n^v(X)$.

Example 6.4 Let $U = \{a, b, c\}$ and $n : U \to P(U)$ is a neighborhood operator. Pick $X = \{a, b\}$ and

$n : a \mapsto U$;
$b \mapsto U$;
$c \mapsto \{a\}$,then $-X = \{c\}$.

By Definition 9, it is easy to see $\underline{apr}_n^v(X) = \{b, c\}$. Therefore $\underline{apr}_n^v(X) \nsubseteq X$.

Proposition 6.2 *Let $n : U \to P(U)$ is a neighborhood operator of U. Then the following are equivalent:*

(1) *n is reflexive;*
(2) $X \subseteq \overline{apr}_n^v(X)$;
(3) $\underline{apr}_n^v(X) \subseteq X$.

Proof (1) ⇒ (2) Since n is reflexive, we have $x \in n(x)$ for any $x \in X$. Thus $x \in \bigcup_{x\in X} n(x) = \overline{apr}_n^v(X)$.

(2) ⇒ (1) For any $x \in U$, $x \in \overline{apr}_n^v(\{x\}) = \bigcup_{x\in\{x\}} n(x) = n(x)$, we have $x \in n(x)$. Therefore n is reflexive.

(2) ⇔ (3) It can be obtained by the duality. □

Corollary 6.1 *Let $n : U \to P(U)$ is a neighborhood operator of U. If n is reflexive, then $\overline{apr}_n^v(X) \subseteq \overline{apr}_n^v(\overline{apr}_n^v(X))$.*

Proposition 6.3 *Let $n : U \to P(U)$ is a neighborhood operator of U. Then the following are equivalent:*

(1) *n is reflexive and transitive;*
(2) $\overline{apr}_n^v(\overline{apr}_n^v(X)) = \overline{apr}_n^v(X)$ *for any* $X \subseteq U$.

Proof (1) ⇒ (2) For any $X \subseteq U$ and $y \in \overline{apr}_n^v(\overline{apr}_n^v(X))$, by Definition 9, there exists $y_0 \in \overline{apr}_n^v(X)$ such that $y \in n(y_0)$. It must be $y_0 \in \overline{apr}_n^v(X)$, there exists $x_0 \in X$ such that $y_0 \in n(x_0)$. Since n is transitive, we have $x \in n(x_0)$. By Definition 9, we can obtain $\overline{apr}_n^v(\overline{apr}_n^v(X)) \subseteq \overline{apr}_n^v(X)$. From Proposition 6.2, we have $\overline{apr}_n^v(X) \subseteq \overline{apr}_n^v(\overline{apr}_n^v(X))$. Therefore $\overline{apr}_n^v(\overline{apr}_n^v(X)) = \overline{apr}_n^v(X)$ for any $X \subseteq U$.

(2) ⇒ (1) For any $X \subseteq U$, since $\overline{apr}_n^v(\overline{apr}_n^v(X)) = \overline{apr}_n^v(X)$, by Proposition 6.2, we obtain that n is reflexive. We only need to prove n is transitive.

For any $x, y, z \in U$, $x \in n(y)$ and $y \in n(z)$. Pick $X = \{z\}$. According to Definition 9 and (2), we have $y \in \overline{apr}_n^v(X) =$

$n(z) = \overline{apr}_n^v(\overline{apr}_n^v(X)) = \bigcup_{p \in n(z)} n(p)$. Thus $n(y) \subseteq n(z)$, therefore $x \in n(z)$ and n is transitive. □

Proposition 6.4 *Let $n : U \to P(U)$ is a neighborhood operator of U. Then the following are equivalent:*

(1) *n is symmetric;*
(2) *$\overline{apr}_n^v(\underline{apr}_n^v(X)) \subseteq X$;*
(3) *$X \subseteq \underline{apr}_n^v(\overline{apr}_n^v(X))$ for any $X \subseteq U$.*

Proof (1) ⇒ (2) For any $X \subseteq U$ and $x \in \overline{apr}_n^v(\underline{apr}_n^v(X))$. By Definition 9, there exists $y_0 \in \underline{apr}_n^v(X)$ such that $x \in n(y_0)$. Since $y_0 \in \underline{apr}_n^v(X) = -\overline{apr}_n^v(-X)$, so $y_0 \notin \overline{apr}_n^v(-X)$. Thus for any $a \in U \setminus X$, $y_0 \notin n(a)$. From n is symmetric, we have $y_0 \in n(x)$, therefore $x \in X$, it follows that $\overline{apr}_n^v(\underline{apr}_n^v(X)) \subseteq X$.

(2) ⇒ (1) For any $x, y \in U$ and $x \in n(y)$, we claim that $y \in n(x)$. If not, let $A = U \setminus \{x\}$. Since $y \notin n(x)$, we have $y \notin \overline{apr}_n^v(U \setminus A) = \overline{apr}_n^v(\{x\})$. From $x \in n(y)$, it can obtain $x \in \overline{apr}_n^v(\underline{apr}_n^v(A)) = \overline{apr}_n^v(\underline{apr}_n^v(U \setminus \{x\})) \subseteq U \setminus \{x\}$. This contradiction shows that $y \in n(x)$.

(2) ⇔ (3) It can be obtained by the duality. □

7 Characterizations of $\overline{apr}_n^v$ being a closure operator

Theorem 12 (General characterization of $\overline{apr}_n^v$ being a closure operator) *Let n be a neighborhood operator of U. Then $\overline{apr}_n^v$ is a closure operator if and only if n is reflexive and transitive.*

Proof It is easy to prove by Propositions 6.2 and 6.3. □

Lemma 8 *Engelking (1989) Suppose we are given a set X and a family $\mathscr{B}$ of subsets of X which has properties:*

(1) *For any $U_1, U_2 \in \mathscr{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U \in \mathscr{B}$ such that $x \in U \subseteq U_1 \cap U_2$;*
(2) *For every $x \in X$ there exists a $U \in \mathscr{B}$ such that $x \in U$.*

Then $\mathscr{B}$ is a base for X.

Theorem 13 (Topological characterization of $\overline{apr}_n^v$ being a closure operator) *Let n be a reflexive neighborhood operator. Then $\overline{apr}_n^v$ is a closure operator if and only if there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ).*

Proof (⇒) Assume that $\overline{apr}_n^v$ is a closure operator. Since n is a reflexive neighborhood operator, so $\{n(x) : x \in U\}$ is a covering of U. For any $n_1(x), n_2(x) \in \{n(x) : x \in U\}$ and any $y \in n_1(x) \cap n_2(x)$, we have $y \in n_1(x)$ and $y \in n_2(x)$. By Theorem 12, there exists $n(y)$ such that $n(y) \subseteq n_1(x)$ and $n(y) \subseteq n_2(x)$. Hence $n(y) \subseteq n_1(x) \cap n_2(x)$. Therefore $y \in n(y) \subseteq n_1(x) \cap n_2(x)$. By Lemma 8, we know that there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ).

(⇐) Let n be a reflexive neighborhood operator, and assume there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ). By the definition of base and Lemma 8, for any $y \in n(x)$, we have $n(y) \subseteq n(x)$. Therefore n is transitive. From Theorem 12, $\overline{apr}_n^v$ is a closure operator. □

8 The relation between $\overline{apr}_n^v$ and $\overline{apr}_n$

$\overline{apr}_n$ was proposed and investigated by Yao and Yao (2012). $\overline{apr}_n^v$ was proposed and investigated by Zhang et al. (2015). In this section, we will discuss the relation between $\overline{apr}_n^v$ and $\overline{apr}_n$. The following example shows that they are different.

Example 8.1 Let $U = \{a, b, c\}$ and $n : U \to P(U)$ is a neighborhood operator.

$n : a \mapsto \emptyset$;
$b \mapsto \{a\}$;
$c \mapsto \{c\}$.

Let $X = \{b\}$, then $\overline{apr}_n^v(X) = \{a\}$, but $\overline{apr}_n(X) = \{y : n(y) \cap X \neq \emptyset\} = \emptyset$. Therefore they are different.

It is natural to ask: What condition is necessary to make them equal? The following proposition gives positive answer.

Proposition 8.1 *Let $n : U \to P(U)$ be a neighborhood operator of U. If n is reflexive and Euclidean, then $\overline{apr}_n^v(X) = \overline{apr}_n(X)$ for any $X \subseteq U$.*

Proof For any $X \subseteq U$ and $y \in \overline{apr}_n^v(X) = \bigcup_{x \in X} n(x)$, there exists an $x_0 \in X$ such that $y \in n(x_0)$. Since n is a reflexive and Euclidean neighborhood operator, so $x_0 \in n(x_0) \subseteq n(y)$ and $n(x_0) \cap X \neq \emptyset$. Thus $n(y) \cap X \neq \emptyset$. Therefore $\overline{apr}_n^v(X) \subseteq \overline{apr}_n(X)$ for any $X \subseteq U$.

For any $X \subseteq U$ and $y \in \overline{apr}_n(X) = \{x \in U : n(x) \cap X \neq \emptyset\}$. Then $n(y) \cap X \neq \emptyset$. There exists a $x_0 \in n(y) \cap X$ such that $x_0 \in n(y)$ and $x_0 \in X$. Since n is a reflexive and Euclidean neighborhood operator, so $y \in n(y) \subseteq n(x_0) \subseteq \bigcup_{x \in X} n(x) = \overline{apr}_n^v(X)$. Therefore $\overline{apr}_n(X) \subseteq \overline{apr}_n^v(X)$ for any $X \subseteq U$. □

9 Characterizations of $\overline{N}_S$ being a closure operator

Yu et al. defined 1-neighborhoods systems and $\overline{N}_S$ in Yu et al. (2013). They called the set $\{y \in U : n(y) = n(x)\}$ the core of $n(x)$ and denoted it with $cn(x)$, where $n : U \to P(U)$ is a neighborhood operator of U. Let $\{n(x) : x \in U\}$ be a 1-neighborhoods system of universe U. They got the following facts:

Table 1 The results of operators

A. op.	G.Ch.	Top.Ch.	In.Ch.	In.S.Re.
$\overline{C_1}$	Zhu and Wang (2012)	Ge et al. (2012), Zhu (2009b)	/	/
$\overline{C_2}$	Zhu and Wang (2012)	Ge et al. (2012)	Ge et al. (2012)	Ge et al. (2012)
$\overline{C_3}$	Zhu (2009b)	Ge (2014)	/	/
$\overline{C_4}$	Zhu (2009b), Ge et al. (2012)	Ge et al. (2012)	Ge et al. (2012)	Ge et al. (2012)
$\overline{C_5}$	Zhu and Wang (2012)	Zhu (2009b), Ge et al. (2012)	/	/
$\overline{C_6}$	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)
$\overline{C_7}$	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)
$\overline{C_8}$	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)	Bian et al. (2015)
$\overline{apr}_n$	–	–	–	–
$\overline{apr}_{N_3}$	–	–	–	–

Fact 1 $x \in cn(x)$ *for any* $x \in U$.

Fact 2 $\{cn(x) : x \in U\}$ *forms a partition of* U.

Definition 17 (Yu et al. 2013) Let $\{n(x) : x \in X\}$ be a neighborhood on U. For any set $X \subseteq U$, it defines an upper operator of X as follows:

$$\overline{N}_s(X) = \bigcup\{cn(y) : cn(y) \cap X \neq \emptyset\} \cup \bigcup\{cn(y) : n(y) \cap X \neq \emptyset\}.$$

Theorem 14 (General characterization of $\overline{N}_S$ being a closure operator) *$\{n(x) : x \in U\}$ be a 1-neighborhoods system of universe U. For any $X \subseteq U$, $\overline{N}_S$ is a closure operator if and only if for any $x, y \in U$, $x \in n(y)$ implies that $n(x) = n(y)$ or $n(x) \subset n(y)$.*

Proof ($\Leftarrow$) Assume that for any $x, y \in U$, $x \in n(y)$ implies that $n(x) = n(y)$ or $n(x) \subset n(y)$. We show that $\overline{N}_S$ satisfies Definition 17. By the definition, it is easy to check that $\overline{N}_S$ satisfies (H_1),(H_2),(H_3). We only need to prove $\overline{N}_S$ satisfies (H_4). By (H_2), it is easy to prove $\overline{N}_S(X) \subseteq \overline{N}_S(\overline{N}_S(X))$ for any $X \subseteq U$. We only need to prove the converse.

For any $y \in \overline{N}_S(\overline{N}_S(X))$, from Definition 17, we have $cn(y) \cap \overline{N}_S(X) \neq \emptyset$ or $n(y) \cap \overline{N}_S(X) \neq \emptyset$.

Case 1 $cn(y) \cap \overline{N}_S(X) \neq \emptyset$.

There exists $x_0 \in cn(y) \cap \overline{N}_S(X)$ such that $n(y) = n(x_0)$ and $cn(x_0) \cap X \neq \emptyset$ or $n(x_0) \cap X \neq \emptyset$. If $n(y) = n(x_0)$ and $cn(x_0) \cap X \neq \emptyset$, then $cn(x_0) = cn(y)$ and $cn(y) \cap X \neq \emptyset$. Therefore $y \in \overline{N}_S(X)$; If $n(y) = n(x_0)$ and $n(y) \cap \overline{N}_S(X) \neq \emptyset$, then there exists $p \in n(y) \cap \overline{N}_S(X)$ such that $p \in n(y)$ and $cn(p) \cap X \neq \emptyset$ or $n(p) \cap X \neq \emptyset$. If $p \in n(y)$ and $n(p) = n(y)$, we have $cn(y) \cap X \neq \emptyset$. Thus $y \in \overline{N}_S(X)$; If $p \in n(y)$ and $n(p) \subseteq n(y)$, then $n(y) \cap X \neq \emptyset$, Thus $y \in \overline{N}_S(X)$. So $\overline{N}_S$ is a closure operator.

Case 2 $n(y) \cap \overline{N}_S(X) \neq \emptyset$.

There exists $x_0 \in n(y) \cap \overline{N}_S(X)$ such that $x_0 \in n(y)$ and $cn(x_0) \cap X \neq \emptyset$ or $n(x_0) \cap X \neq \emptyset$. If $n(y) = n(x_0)$ and $cn(x_0) \cap X \neq \emptyset$, then $cn(x_0) = cn(y)$. Thus $cn(y) \cap X \neq \emptyset$. Therefore $y \in \overline{N}_S(X)$; If $n(y) = n(x_0)$ and $n(x_0) \cap X \neq \emptyset$, then $n(y) \cap X \neq \emptyset$, so $y \in \overline{N}_S(X)$; If $n(x_0) \subset n(y)$ and $n(x_0) \cap X \neq \emptyset$, then we have $n(y) \cap X \neq \emptyset$. Thus $y \in \overline{N}_S(X)$. So $\overline{N}_S$ is a closure operator.

($\Rightarrow$) Assume that $\overline{N}_S$ is a closure operator. We only need to prove that for any $x, y \in U$, $x \in n(y)$ implies that $n(x) = n(y)$ or $n(x) \subset n(y)$. If not, there exists $z \in U$ such that $z \in n(x)$ but $z \notin n(y)$. By Definition 17, we have $x \in \overline{N}_S(\{z\})$. Since $\overline{N}_S$ is a closure operator, so $x \in \overline{N}_S(\overline{N}_S(\{z\}))$. Thus $cn(x) \cap \overline{N}_S(\{z\}) \neq \emptyset$ or $n(x) \cap \overline{N}_S(\{z\}) \neq \emptyset$. If $n(x) \cap \overline{N}_S(\{z\}) \neq \emptyset$ and $x \in n(y)$, then $n(y) \cap \overline{N}_S(\{z\}) \neq \emptyset$. Thus $y \in \overline{N}_S(\overline{N}_S(\{z\})) = \overline{N}_S(\{z\})$. Therefore $cn(y) \cap \{z\} \neq \emptyset$ or $n(y) \cap \{z\} \neq \emptyset$. We have $n(y) = n(x)$ or $z \in n(y)$. It is contradiction to $z \notin n(y)$.

If $cn(x) \cap \overline{N}_S(\{z\}) \neq \emptyset$, by Fact 1 and the above proof, we have $x \in cn(x)$ and $x \in \overline{N}_S(\{z\})$. Since $x \in n(y)$, so $y \in \overline{N}_S(\overline{N}_S(\{z\})) = \overline{N}_S(\{z\})$. Therefore $z \in cn(y)$ or $z \in n(y)$. It is contradiction to $z \notin n(y)$. □

Theorem 15 (Topological characterization for $\overline{N}_S$ being a closure operator) *Let $\{n(x) : x \in U\}$ be a 1-neighborhoods system of universe U. Then $\overline{N}_S$ is a closure operator if and only if there exists a topology τ on U such that $\{n(x) : x \in U\}$ is a base of (U, τ).*

Proof The proof is similar to that of Theorem 13. □

10 Conclusions and future work

In this paper, we not only give general characterization of covering $\mathscr{C}$ for the fourth type of covering-based upper approximation operator $\overline{C_4}$, but also describe information exchange system when $\overline{C}_4$ is a closure operator. Besides this, we obtain general, topological characterizations of covering $\mathscr{C}$ for two types of covering-based upper approximation operators $\overline{apr}_n$ and $\overline{apr}_{N_3}$ to be closure operators. We also propose intuitive characterizations of covering $\mathscr{C}$ for $\overline{apr}_n$ or $\overline{apr}_{N_3}$ to be a closure operator. Using these characterizations, we describe covering-based approximation space $(U, \mathscr{C})$ as some special types of information exchange systems when $\overline{apr}_n$ is a closure operator. The results in this paper have solved Question 1. Beside this, we also discuss the relationship between $\overline{apr}_n$ and unary covering. We shall further investigate the relations among the covering-based upper approximation operators listed in Table 1.

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