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# Comments on *crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets*

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**Abstract**
In the paper *Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets*, D.-F. Li pointed out that some kind of definitions of operations over Atanassov's intuitionistic fuzzy sets are incorrect. We can see that, near 30 years after the first Atanassov's papers, there exist some misunderstandings related not only on the name, but also on the basic operations on IFSs. Those misunderstandings concern, this time, on the operations of the sum and product. Li also casts doubt the equivalence of the intuitionistic fuzzy sets and the interval-valued fuzzy sets. In this paper, the Li's reasoning is presented and commented.

**Keywords** Intuitionistic fuzzy sets (IFSs) · Operations on IFSs · Extension principle for IFSs · Interval-valued fuzzy sets

## 1 Introduction

Intuitionistic fuzzy sets (IFSs) were first introduced by K.T. Atanassov in 1983. The latest development of the theory is collected in the monographs (Atanassov 2012, 2017). But, we can see that, many years after the first Atanassov's papers, there exist some misunderstandings related not only on the name of the IFSs (see, e.g., Dubois et al. 2005; Atanassov 2005). Despite fairly well-defined terms, there may be still some misunderstandings regarding the operations on the intuitionistic fuzzy sets. Such misunderstandings can be found in the paper *Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets* (Li 2012a). The similar reasoning was repeated later in the monograph *Decision and Game Theory in Management with Intuitionistic Fuzzy Sets* (Li 2014).

Because a lot of citations of the original Li's papers, they will be denoted by italic font. A part of the following remarks are included in the paper (Dworniczak 2018). Present paper contains a clearly expanded comments.

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## 2 Main comments

In the paper Li (2012a)—*Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets*—the Author, Deng-Feng Li, presents the doubts related to some operations on the intuitionistic fuzzy sets. Based on the Atanassov's paper (1986), Li gives the definition of the sum $A + B$ and the products $A \cdot B$ and $\beta A$ in the form

$$A + B = \{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x) >: x \in X\}, \quad (1)$$

$$AB = \{< x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x) >: x \in X\}, \quad (2)$$

and

$$\beta A = \{< x, 1 - (1 - \mu_A(x))^\beta, (\nu_A(x))^\beta >: x \in X\}, \quad (3)$$

where $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \nu_B(x) >: x \in X\}$, and $\beta \geq 0$.

Let us note, Atanassov (1986) did not define the operation (3). The operation (3) was presented exactly first by De et al. (2000). However, the operation (3) can be viewed as a simple extension of the sum (1). Moreover, the operation (3) relates only to the situation where $\beta$ is a natural number.

In his paper, Li presented first the doubts related to operations $A + B$ and $AB$ on particular sets. The Li's examples ***(a)–(c)*** are given below (Li 2012a):

***(a)*** *If* $B = \{< x, 1, 0 >: x \in X\}$, i.*e., B is a fuzzy set, which means that every element x completely belongs to B, then according to* Eq. (1)*, we have*

$$A + B = \{< x, 1, 0 >: x \quad \in X\} = B,$$

*which also means that every element x completely belongs to A + B despite A is any Atanassov's IFS. Similarly, according to* Eq. (2)*, we have*

$$AB = A,$$

*which means that whether every element x belonging to AB completely depends on A despite B ensures that all elements x completely belong to B.*

Li calls the above set $B$ the fuzzy set. Formally, it is correct. However, the set $B$ can be called a classical set, and, moreover $B = X$.

***(b)*** *If* $B = \{< x, 0, 1 >: x \in X\}$ *, i.e., B is a fuzzy set, which means that every element x completely does not belong to B, then according to* Eq. (1)*, we have*

$$A + B = A,$$

*which means that whether every element x belonging to A + B completely depends on A despite B ensures that all elements x completely do not belong to B. Similarly, according to* Eq. (2)*, we have*

$$AB = B,$$

*which means that every element x completely does not belong to AB despite A is any Atanassov's IFS.*

Li calls the above set $B$ the fuzzy set. Formally, it is correct. However, the set $B$ is in fact the classical empty set.

***(c)*** *If* $B = \{< x, 0, 0 >: x \in X\}$ *, which means that every element x cannot be completely determined whether it belongs to B or not, then according to* Eq. (1)*, we have*

$$A + B = \{< x, \mu_A(x), 0 >: x \in X\},$$

*which means that the membership degree of the element x to A+ B is the same as that of x to A whereas the non-membership degree of the element x to A + B is 0 despite A ensures that the non-membership degree of the element x to A is* $\nu_A(x)$.

*Similarly, according to* Eq. (2)*, we have*

$$AB = \{< x, 0, \nu_A(x) >: x \in X\},$$

*which means that the non-membership degree of the element x to A+B is the same as that of x to A whereas the membership degree of the element x to A + B is 0 despite A ensures that the membership degree of the element x to A is* $\mu_A(x)$.

After the next numerical example (Li 2012a), Li argues that Eqs. (1), (2), and (3) define the operations incompatible/inconsistent with the Zadeh's extension principle. The extension principle allows, in brief, to compute the membership function of the fuzzy sets which is the result of the mapping of the fuzzy set. For the intuitionistic fuzzy sets, it is mentioned exactly first by Stoeva (1999). Li presents the extension principle for IFSs in Li (2014), pp. 28–31 and also in Li (2012b).

The numerical example, given by Li, related to the extension principle is as follows.

*Let A and B be intuitionistic fuzzy sets on the universum* $X = \{3, 4, 5, 6, 7\}$ *, which mean "approximately 5," where*

$$A = \{< 3, 0.7, 0.2 >, < 4, 0.8, 0.1 >, < 5, 1, 0 >, \\ < 6, 0.8, 0.1 >\} \tag{4}$$

*and*

$$B = \{< 4, 0.7, 0.2 >, < 5, 1, 0 >, < 6, 0.9, 0.05 >, \\ < 7, 0.85, 0.1 >\}, \tag{5}$$

*respectively. Using* Eqs. (1)–(3)*, we have*

$$A + B = \{< 3, 0.7, 0 >, < 4, 0.94, 0.02 >, < 5, 1, 0 >, \\ < 6, 0.98, 0.005 >, < 7, 0.85, 0 >\}, \tag{6}$$

$$AB = \{< 3, 0, 0.2 >, < 4, 0.56, 0.28 >, < 5, 1, 0 >, \\ < 6, 0.72, 0.145 >, < 7, 0, 0.1 >\}, \tag{7}$$

*and*

$$\beta A = \{< 3, 1 - 0.3^\beta, 0.2^\beta >, < 4, 1 - 0.2^\beta, 0.1^\beta >, \\ < 5, 1, 0 >, < 6, 1 - 0.2^\beta, 0.1^\beta >\}. \tag{8}$$

*In a similar way to the extension principle of the fuzzy sets, we have*

$$A + B = \{< 3, 0.7, 0.2 >, < 7, 0.7, 0.2 >, < 8, 0.7, 0.2 >, \\ < 9, 0.8, 0.1 >, < 10, 1, 0 >, < 11, 0.9, 0.05 >, \\ < 12, 0.85, 0.1 >, < 13, 0.8, 0.1 >\}, \tag{9}$$

$$AB = \{< 12, 0.7, 0.2 >, < 15, 0.7, 0.2 >, < 16, 0.7, 0.2 >, \\ < 18, 0.7, 0.2 >, < 20, 0.8, 0.1 >, < 21, 0.7, 0.2 >, \\ < 24, 0.8, 0.1 >, < 25, 1, 0 >, < 28, 0.8, 0.1 >, \\ < 30, 0.9, 0.05 >, < 35, 0.85, 0.1 >, \\ < 36, 0.8, 0.1 >, < 42, 0.8, 0.1 >\}, \tag{10}$$

*and*

$$\beta A = \{< 3\beta, 0.7, 0.2 >, < 4\beta, 0.8, 0.1 >, < 5\beta, 1, 0 >, \\ < 6\beta, 0.8, 0.1 >\}, \quad (11)$$

*which are remarkably different from* Eqs. (6)–(8) *since* Eqs. (9)–(11) *compute elements in A and B rather than membership and non-membership degrees.*

It is really easy to see that the results (6)–(8) and (9)–(11) are different (by the way: the element $< 3, 0.7, 0.2 >$ in (9) comes probably from a copy–paste method). Li assumes correctness of the extension principle, and, because it leads, in this case, to another results, concludes that the operations (1)–(3) cannot be correct.

The above reasoning, on the similar example, has been also presented by Li in the monograph (Li 2014, pp. 33–35) and in the paper (Li 2012b).

It is understandable that "approximately 5" added to "approximately 5" must be "approximately 10", and "approximately 5" multiplied by "approximately 5" must be "approximately 25". Here the argumentation of Li is convincing, and the Atanassov's mistake is evident.

However, in fact, it is Li who makes the mistake!

Namely, he does not understand correctly the signs + and $\cdot$ (the sign $\cdot$ is by Li, typically, omitted). These signs denote not the operations of sum and product in terms of arithmetic of fuzzy numbers (or intuitionistic fuzzy numbers) but the sum and the product in the set-theoretically sense.

Here the mistake of Li is obvious and the rest of his reasoning is invalid.

This means, obviously, that the results in the earlier examples ***(a)***, ***(b)***, and ***(c)*** are not surprising. By the way–similar results we obtain based on typical sum and product of the IFSs, using the minimum and maximum operators.

It is possible that the use of the symbols + and $\cdot$ by Atanassov, without direct comments, is not fortunate, but such a notation is very often used in the case of classically fuzzy sets also.

In his paper, Li failed to notice the use of sum and product based on other t- and s-norms, over the most popular norms: minimum and maximum. Li (2012a) writes even: Eqs. (1)–(3) *are defined according to the probability sum and product*. Yes, exactly! And by this fact the recognition of these operations as the sum or product of numbers has no sense.

In general, the union and intersection of the IFSs are defined based on the IF t-norm and IF s-norm. For IFSs, these norms are considered explicitly first by Cornelis and Deschrijver (2001), Cornelis et al. (2002) and Deschrijver and Kerre (2002).

Cornelis et al. (2002), defined the intuitionistic fuzzy t-norm (IF triangular norm) on the lattice $L$ as any monotonous, commutative, associative mapping from $L^2$ to $L$ with the neutral element $< 1, 0 >$. The intuitionistic fuzzy s-norm (IF triangular co-norm) on the lattice $L$ the Authors call any monotonous, commutative, associative mapping from $L^2$ to $L$ with the neutral element $< 0, 1 >$.

The Authors formulated also the theorem as below (Cornelis and Deschrijver 2001; Deschrijver and Kerre 2002).

**Theorem 1** *Let $T_1$ and $T_2$ are t-norms, and $S_1$ and $S_2$ are s-norms.*

The mappings $\mathcal{T}, \mathcal{S} : L^2 \to L$ on the lattice $L = \{< a, b >\in [0, 1]^2 : a + b \le 1\}$, given in the form:

$$\mathcal{T}(< a, b >, < c, d >) =< T_1(a, c), S_1(b, d) >,$$

and

$$\mathcal{S}(< a, b >, < c, d >) =< S_2(a, c), T_2(b, d) >,$$

fulfilling the conditions $T_1(a, c) + S_1(b, d) \le 1$ and $S_2(a, c) + T_2(b, d) \le 1$, are the IF t-norm and IF s-norm, respectively.

The definition of the t-, s-norms and the lattice are widely known.

It is not difficult to see, in the cited formulas (1) and (2), (the (3) is some consequence of the (1)) the use of the probabilistic (product) t-norm

$$T(a, b) = a \cdot b,$$

and the s-norm

$$S(a, b) = a + b - a \cdot b,$$

to define the union and intersection of intuitionistic fuzzy sets.

Li is therefore wrong, when he wrote in the end of his paper *the addition and multiplication operations of Atanassov's IFSs are incorrect.*

In the above-mentioned monograph (Li 2014, pp. 32–33), Li considered, besides addition and multiplication, the operations of subtraction, division and calculating the power of a set. They are not compared with analogous operations given by Atanassov, because such operations simply do not exist. Li gives it only as an additional argument for that the operations over intuitionistic fuzzy sets, based on extension principle, are remarkably different from those given by Atanassov. Li writes in the end of the chapter: *They have to be cautiously chosen for applications to solving real management and decision problems*. It is true the more so, that sometimes quite different operations are marked with the same symbols.

In the point 3 of the paper, Li (2012a) casts doubt the equivalence of the intuitionistic fuzzy sets and the interval-valued fuzzy sets (IVFSs) pointed out by Deschrijver and Kerre (2003).

Based on the above paper, Li denotes that the IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is equivalent to the IVFS $D = \{< x, [\mu_A(x), 1 - \nu_A(x)] >: x \in X >$, where $[\mu_A(x), 1 - \nu_A(x)]$ is an subinterval of the interval [0, 1]. In fact Deschrijver and Kerre (2003) use the word *equivalent* just in a single sentence writing: …*so IVFS theory is equivalent to IFS theory, which on its turn is equivalent to vague set theory.* The Authors used otherwise the word *isomorphism*. But it can be accepted that the isomorphism is some kind of the equivalence.

In opposition to the paper by Deschrijver and Kerre (2003), Li (2012a) claims that the IFSs and IVFSs are not mathematically equivalent. In the three points (denoted ***(a)***, ***(b)*** and ***(c)***, same as in part 2 of the Li's paper), he considers the differences between the IFSs and the IVFSs for the singleton $A = \{< x_0, \mu_A(x_0), \nu_A(x_0) >\}$ denoted, for short, as $A =< \mu, \nu >$.

The reasoning of Li is as follows:

***(a)*** *The Atanassov's IFS* $A = < \mu, \nu >$ *is a two-tuple, which may be mathematically regarded as a point of the two-dimension space. The IVFS* $D = [\mu, 1 - \nu]$ *is a subinterval of the unit interval* [0, 1]*, which is a range of one dimension.*

This reasoning is wrong because the equivalence does not means the equality. Even in the first line of ***(a)*** Li have mentioned some equivalence (…*may be regarded*…) but not the equality—the pair of numbers is not equal to a geometrical object such as a point.

Subsequently, in ***(b)*** Li writes:

***(b)*** *The Atanassov's IFS* $A = < \mu, \nu >$ *means that the membership degree of the element* $x_0$ *belonging to the given set is the exact value (i.e., real number)* $\mu$*, the non-membership degree of the element* $x_0$ *belonging to the given set is the exact value* $\nu$ (…)*. However, the IVFS* $D = [\mu, 1 - \nu]$ *means that the membership degree of the element* $x_0$ belonging to the given set is any value in the subinterval $[\mu, 1 - \nu] \subseteq [0, 1]$ *besides* $\mu$*, the non-membership degree of the element* $x_0$ *belonging to the given set is any value in the subinterval* $[\nu, 1 - \mu]$ *(besides* $\nu$*)* (…).

The first sentence cited in ***(b)*** is a basic interpretation in the IFS theory. However, the second is at least problematic. Namely, even in the classical handbooks (Dubois and Prade 2000; Klir and Yuan 1995) we have a definition of the IVFS in the other sense as presented above.

In Dubois and Prade (2000), the Authors write: "An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and forms an interval in the membership scale."

In the same sense the IVFS is defined in Deschrijver and Kerre (2003). This paper is cited by Li (2012a). In Klir and Yuan (1995), the Authors write: "A membership function based on the latter approach does not assign to each element of the universal set one real number, but a closed interval of real numbers between the identified lower and upper bounds. Fuzzy sets denned by membership functions of this type are called *interval-valued fuzzy sets.*" All the above-mentioned Authors clearly define the membership function of the IVFS not as a real-valued function, but as an interval-valued function. But, even if an interpretation of a single value from an interval can be considered, then one cannot accept the independence between the membership value and the non-membership value. For example, the IF pair $< \mu, \nu >=< 0.3, 0.3 >$ corresponds to the intervals $[\mu, 1 - \nu] = [0.3, 0.7]$ and $[\nu, 1 - \mu] = [0.3, 0.7]$. However, Li writes: *the membership degree (…) is any value in the subinterval* $[\mu, 1 - \nu](\ldots)$ *the non-membership degree (…) is any value in the subinterval* $[\nu, 1 - \mu]$ *(…).* This means that, without assuming some relationship between $\mu$ and $\nu$, one can take any values from these intervals, for example $\mu = 0.7$ and $\nu = 0.7$. But, at this point, the sum of $\mu$ and $\nu$ exceeds the value 1 and is not considered in the FS theory (IVFS theory). The reasoning presented in ***(b)*** is so in any case wrong.

In the last point ***(c)*** Li (2012a) writes:

*Let* $A_1 =< \mu_1, \nu_1 >$ *and* $A_2 =< \mu_2, \nu_2 >$ *be two Atanassov's IFSs. Using* Eqs. (1) *and* (3), *we have*

$$A_1 + A_2 =< \mu_1 + \mu_2 - \mu_1\mu_2, \nu_1\nu_2 > \tag{12}$$

*and*

$$\beta A_1 =< 1 - (1 - \mu_1)^\beta, (\nu_1)^\beta > \tag{13}$$

*On the other hand, the Atanassov's IFSs* $A_1 =< \mu_1, \nu_1 >$ *and* $A_2 =< \mu_2, \nu_2 >$ *may be mathematically written as the IVFSs* $D_1 = [\mu_1, 1 - \nu_1]$ and $D_2 = [\mu_2, 1 - \nu_2]$, *respectively. According to the operations over the intervals, we have*

$$D_1 + D_2 = [\mu_1 + \mu_2, 2 - \nu_1 - \nu_2] \tag{14}$$

*and*

$$\beta D_1 = [\beta\mu_1, \beta - \beta\nu_1], \tag{15}$$

*which may be mathematically expressed as the following Atanassov's IFSs:*

$$\hat{A}_1 + \hat{A}_2 =< \mu_1 + \mu_2, -1 + \nu_1 + \nu_2 > \tag{16}$$

*and*

$$\beta\hat{A}_1 =< \beta\mu_1, 1 - \beta + \beta\nu_1 >, \tag{17}$$

*respectively. Here,* $\beta$ *is required to satisfy the condition:* $\beta - \beta\nu_1 \le 1$.

*Obviously,* Eqs. (12) *and* (13) *remarkably differ from* Eqs. (16) *and* (17), *respectively. In other words, the*

*Atanassov's IFSs expressed by* Eqs. (12) *and* (13) *are not equal to those expressed by* Eqs. (16) *and* (17), *respectively.*

In the point ***(c)***, Li makes a mistake analogous to the mistake in the example with the using of the extension principle. In both cases (12)–(14), and (13)–(15), Li makes different operations. In (12)–(13), we are dealing with a sum of sets on the singleton-universum $\{x_0\}$, based on probabilistic norms. In (14)–(15), we are dealing with the addition of intervals (addition of numbers). Li notices although some inconsistency in (15) and introduces the condition $\beta - \beta\nu_1 \leq 1$. The similar conditions should be, based on similar reasoning, given earlier, for (14), as: $\mu_1 + \mu_2 \leq 1$, and $2 - \nu_1 - \nu_2 \leq 1$ and $\mu_1 + \mu_2 \leq 2 - \nu_1 - \nu_2$. Also in (15) it should be $\beta\mu_1 \leq 1$, and $\beta\mu_1 \leq \beta - \beta\nu_1$. In fact, the all above conditions are factitious, and, in the context of the sum of sets, they do not have sense. The reasoning presented in the point ***(c)*** have nothing to do with the justification of mathematical equivalence of IFSs and IVFSs.

## 3 Conclusion

In this paper, we can see that, near 30 years after the first Atanassov's papers, there exist some misunderstandings related, for example, on the basic operations on IFSs. Those misunderstandings can be found not only in the papers of young adepts of science, but also in the papers written by really creative researcher in the field of intuitionistic fuzzy sets theory. It is worth noticing that, despite of several years that passed by from the Li's papers (2012a, 2012b, 2014) being published, the misunderstanding described above was not commented in the known literature. In general, the evolution of every theory sometimes leads to mistakes. This is a natural phenomenon. However, once found, they should be immediately pointed out and commented. They also may be a foundation for further discussion.

### Compliance with ethical standards

**Conflict of interest** Author P. Dworniczak declares that he has no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

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