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METHODOLOGIES AND APPLICATION

Multiperiod mean absolute deviation uncertain portfolio selection with real constraints

Peng Zhang[1]

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Abstract
Absolute deviation is a commonly used risk measure, which has attracted more attentions in portfolio optimization. Most of existing mean–absolute deviation models are devoted to stochastic single-period portfolio optimization. However, practical investment decision problems often involve the uncertain dynamic information. Considering transaction costs, borrowing constraints, threshold constraints, cardinality constraints and risk control, we present a novel multiperiod mean absolute deviation uncertain portfolio selection model, which an optimal investment policy can be generated to help investors not only achieve an optimal return, but also have a good risk control. In proposed model, the return rate of asset and the risk are quantified by uncertain expected value and uncertain absolute deviation, respectively. Cardinality constraints limit the number of risky assets in the optimal portfolio. Threshold constraints limit the amount of capital to be invested in each asset and prevent very small investments in any asset. Based on uncertainty theories, the model is transformed into a crisp dynamic optimization problem. Because of the transaction costs and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence, which is "NP hard" problem that is very difficult to solve. The proposed model is approximated to a mix integer dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy and is proved linearly convergent. Finally, an example is given to illustrate the behavior of the proposed model and the designed algorithm using real data from the Shanghai Stock Exchange.

Keywords Uncertain modeling · Multiperiod portfolio optimization · Mean absolute deviation model · Uncertainty theory · The discrete iteration method

1 Introduction

Portfolio selection aims at choosing the proportions of various securities to make the portfolio better than any other ones according to some criteria. These criteria will combine the considerations of expected value and risk of the portfolio. Markowitz (1952) proposed the well-known mean–variance model by maximizing the expected return for a given risk or minimizing the risk for a given expected return. After Markowitz, variance is widely accepted as a risk measure, and most of the research is devoted to the extensions of mean–variance model. A typical extension is Konno and Yamazaki (1991), which employed absolute deviation to measure the risk of the portfolio return and formulated a mean–absolute deviation model. This model can cope with large-scale portfolio selection because it can remove most of the difficulties associated with the classical Markowitz's model while maintaining its advantages. When all the returns are normally distributed random variables, the authors showed that the mean–absolute deviation model gave essentially the same results as the mean–variance model. One of the commonly used downside risk measures is semi-variance originally introduced in Markowitz (1959). Another one is semiabsolute deviation proposed by Speranza (1993), which can be easily evaluated since it could be determined using linear programming models.

In Markowitz's theoretical framework, an implicit assumption is that future returns of securities can be correctly reflected by past performance. In other words, security returns should be represented by random variables whose characteristics such as expected value and risk may accurately be calculated based on the sample of available his-

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✉ Peng Zhang
zhangpeng300478@aliyun.com

1 School of Economics and Management, South China Normal University, Guangzhou 510006, People's Republic of China

torical data. It keeps valid when large amounts of data are available such as in the developed financial market. However, investors often encounter the situation that there is a lack of data about security returns just like in an emerging market. In many cases, security returns are beset with ambiguity and vagueness. In particularly, when little information is available, fuzzy approaches are, in general, more appropriate. Thus, it is worthwhile to use fuzzy set theory to investigate the uncertainty of financial markets. Following the widely used fuzzy set theory in Zadeh (1965), researchers used the fuzzy set theory to investigate portfolio selection problems under uncertain environment. Numerous models have been proposed by using different approaches. Qin (2017) proposed a random fuzzy mean absolute deviation portfolio selection model. Some researchers extend the single-period fuzzy portfolio selection into multiperiod setting. By using experts' judgments, Sadjadi et al. (2011) formulated a fuzzy multiperiod portfolio selection model with different rates for borrowing and lending by using fuzzy set theory. Zhang et al. (2012, 2014), and Liu et al. (2012, 2013) proposed several kinds of multiperiod fuzzy portfolio selection models, respectively. Zhang and Zhang (2014) proposed a multiperiod mean absolute deviation fuzzy portfolio selection model with cardinality constraints.

Though possibility measure has been widely used in portfolio selection, it has limitation. One great limitation is that possibility measure is not self-dual. Using possibility measure which has no self-duality property, we can find that two fuzzy events with different occurring chances may have the same possibility value. In addition, whenever the possibility value of a portfolio returns greater than a target value is lower than 1, the possibility value of the opposite event (i.e., the portfolio return less than or equal to the target value) is the maximum value of 1; or whenever the possibility value of a portfolio return less than or equal to a target value is lower than 1, the possibility value of the opposite event (i.e., the portfolio returns greater than the target value) is the maximum value of 1. These results are quite awkward and will confuse the decision maker. Thus, Huang (2008) presented credibilitic mean–semi-variance model, Li et al. (2010) proposed credibilitic cross-entropy minimization model, Zhang and Liu (2014) proposed a credibilitic multiperiod mean–variance portfolio selection model. Mehlawat (2016) proposed a multiperiod mean entropy credibilitic portfolio selection model.

Another alternative way to describe the subjective imprecise quantity is uncertainty theory proposed by Liu (2007) by estimating indeterministic quantities subject to experts' estimations. Based on this framework, much work is undertaken to develop the theory and related practical applications. Uncertainty theory is also applied to model the portfolio selection. Huang (2012) established a risk index model for uncertain portfolio selection and further employed the criterion to consider a portfolio adjusting problem. Different from a risk index model, Liu et al. (2012) presented a semi-absolute deviation of an uncertain variable to measure risk and formulated a mean–semiabsolute deviation criterion. Qin and Kar (2013) formulated the uncertain counterpart of mean–variance model. As extensions, Huang and Qiao (2012) presented a risk index model for multiperiod case. Different from these, Zhu (2010) applied uncertain optimal control to model continuous-time problem, but Yao and Ji (2014) considered the problem by using uncertain decision making.

Although the case of a long-term investment horizon is of greater importance in practice, much less has been done in that area. The first formulation of the multiperiod portfolio selection problem has already been given in the book of Markowitz (1959). Although it is heavily discussed in recent literature, to the best of our knowledge, a closed-form solution is not available in the general case up to now. Closed-form solutions are presented only under the assumption of frictionless market, i.e., Li and Ng (2000) used dynamic programming method to deal with the multiperiod mean variance portfolio selection problem by using the idea of embedding the problem in a tractable auxiliary problem. Then, they obtained breakthrough result, that is, the optimal mean–variance portfolio policy and the efficient frontier; Zhu et al. (2004) incorporated a control of the probability of bankruptcy in the generalized mean variance formulation for multiperiod portfolio optimization; Yu et al. (2010), Yu and Wang (2012) discussed a dynamic portfolio optimization problem with risk control for the absolute deviation model; Wu and Li (2012) investigate a non-self-financing portfolio optimization problem under the framework of multiperiod mean–variance with Markov regime switching and a stochastic cash flow; Li and Li (2012) represented a multiperiod portfolio optimization problem for asset–liability management of an investor who intends to control the probability of bankruptcy before reaching the end of an investment horizon. Cui et al. (2012, 2014, 2017) studied the time-consistent optimal strategies of the multiperiod mean–variance model. Gao et al. (2015) considers the time cardinality constrained mean–variance dynamic portfolio selection problem in markets with correlated returns and in which the number of time periods to invest in risky assets is limited. Zhou et al. (2016) propose a multiperiod portfolio optimization model with stochastic cash flows. Chen et al. (2014, 2016) discussed the time-consistent strategies for different portfolio optimization models. Wu and Zeng (2015) studied a generalized multiperiod mean variance portfolio selection problem within the game theoretic framework for a defined-contribution pension scheme member. For more general models, the solution is frequently determined by a numerical procedure, i.e., van Binsbergen and Brandt (2007) compared the numerical performance of value function iterations with portfolio weight iterations in the context of the simulation-based dynamic programming approach; Mansini et al. (2007) presented multiperiod mean CVaR portfolio selection model; Gülpınar and Rustem (2007) extend the multiperiod mean–variance optimization framework to worst-case design with multiple rival return and risk scenarios; Yan et al. (2007), Yan and Li (2009) proposed a hybrid genetic algorithm with particle swarm optimizer to solve a class of multiperiod semi-variance portfolio selection with a four-factor futures price model and a multiperiod semi-variance portfolio selection; Köksalan and Şakar (2016) consider expected return, conditional value at risk, and liquidity criteria in a multiperiod portfolio optimization setting modeled by stochastic programming.

In many existing models are proposed on the framework of Markowitz's mean variance with cardinality constraints, threshold constraints and so on. These real constraints come from real-world practice where the administration of a portfolio made up of many assets is clearly not desirable because of transaction costs, complexity of management, or policy of the asset management companies. Because of its practical relevance, this cardinality constrained model, and some variations have been intensively studied in the last decade. Especially from the computational viewpoint, some researchers proposed exact solution methods, i.e., Bertsimas and Shioda (2009); Li et al. (2006); Shaw et al. (2008); Murray and Shek (2012); Cesarone et al. (2013); Cui et al. (2013); Sun et al. (2013); Le Thi et al. (2009), Le Thi and Moeini (2014). Since exact solution methods are able to solve only a fraction of practically useful cardinality constrained models, many heuristic algorithms have also been proposed, i.e., Fernández and Gómez (2007); Ruiz-Torrubiano and Suarez (2010); Anagnostopoulos and Mamanis (2011); Woodside-Oriakhi et al. (2011); Deng et al. (2012). In these studies, it appears that the computational complexity for the solution of the cardinality constrained model is much greater than the one required by the classical Markowitz model or by several other of its refinements. Indeed, the standard Markowitz model is a convex quadratic programming problem, while the cardinality constrained model is a mixed integer quadratic programming problem which is a "NP hard" problem.

The contribution of this work is as follows. We originally represent uncertain absolute deviation to measure portfolio risk, and propose a new multiperiod mean absolute deviation uncertain portfolio selection model with borrowing constraints, transaction costs, threshold constraints and cardinality constraints. Based on uncertain theories, the model is converted to a crisp dynamic optimization problem. Because of the transaction costs and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence, which is "NP hard" problem that is very difficult to solve. The proposed model is approximated to a mix integer dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy and is proved linearly convergent. Finally, we give an example to illustrate the idea of the model and demonstrate the effectiveness of the designed algorithm.

This paper is organized as follows. In Sect. 2, several concepts, properties of uncertain measure, the definitions of the uncertain mean and the uncertain absolute deviation are introduced, respectively. In Sect. 3, the borrowing constraints, transaction costs, threshold constraints and cardinality constraints are formulated into the multiperiod portfolio, and a new multiperiod uncertain portfolio selection model with real constraints is proposed. A novel discrete iteration method is proposed to solve it in Sect. 4. In Sect. 5, a numerical example is also presented to illustrate the modeling idea and the effectiveness of the designed algorithm. Finally, some conclusions are given in Sect. 6.

2 Preliminaries

Let Γ be a nonempty set, and let A be a σ-algebra over Γ. Each element of A is called an event. A set function is called an uncertain measure (Liu 2007) if and only if it satisfies

Axiom 1. (Normality) $\mathrm{M}\{\Gamma\} = 1$;
Axiom 2. (Self-duality) $\mathrm{M}\{A\} + M\{A^c\} = 1$ for any event A;
Axiom 3. (Subadditivity) $M(\cup_i A_i) \leq \sum_{i=1}^{\infty} M(A_i)$ for any countable sequence of events $\{A_i\}$.

Definition 1 (Liu 2007) Let Γ be a nonempty set, and let A be a σ-algebra over it. If M is an uncertain measure, then the triplet (Γ, A, M) is called an uncertainty space.

Definition 2 (Liu 2007) Uncertain variable ξ is defined as a measurable function from an uncertainty space (Γ, A, M) to the set of real numbers $\Re$. That is, for any Borel set B, we have

$$\{\gamma \in \Gamma, \xi(\gamma) \in B\} \in A \tag{1}$$

Definition 3 (Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} M\{\xi \leq x\}\mathrm{d}x \tag{2}$$

provided that at least one of the two integrals is finite.

Based on Definition 3, Liu (2009) deduced the following two theorems.

Theorem 1 (Liu 2009) *Let ξ be an uncertain variable with finite expected value. Then, for any real numbers a and b, it holds that*

$$E[a\xi + b] = a\,E[\xi] + b \tag{3}$$

Theorem 2 (Linearity of Expected Value Operator, Liu 2009) *Let ξ and η be independent uncertain variables with finite expected values. Then, for any real numbers a and b, it holds that*

$$E(a\xi + b\eta) = aE(\xi) + bE(\eta) \tag{4}$$

Definition 4 (Liu 2007) An uncertain variable ξ can be characterized by an uncertainty distribution which is a function $\Phi: \Re \to [0, 1]$ is defined as

$$\Phi(t) = M\{\xi \le t\} \tag{5}$$

Definition 5 Let ξ be an uncertain variable with finite expected value e. Then the absolute deviation of ξ is defined by

$$AD(\xi) = E[|\xi - e|] \tag{6}$$

If ξ is an uncertain variable with expected value e, then its absolute deviation is used to measure the spread of its distribution about e.

Theorem 3 *Let ξ be an uncertain variable with finite expected value e. Then its uncertain absolute deviation is defined as*

$$AD(\xi) = \int_e^{+\infty} (1 - \Phi(r))\mathrm{d}r + \int_{-\infty}^{e} \Phi(r)\mathrm{d}r \tag{7}$$

Proof From Definitions 5 and 3, it follows that

$$\begin{aligned}
\mathrm{AD}(\xi) &= E[|\xi - \mathrm{e}|] \\
&= \int_0^{+\infty} M\{|\xi - \mathrm{e}| \ge x\}\mathrm{d}x - \int_{-\infty}^{0} M\{|\xi - \mathrm{e}| \le x\}\mathrm{d}x \\
&= \int_0^{+\infty} M\{|\xi - \mathrm{e}| \ge x\}\mathrm{d}x \\
&= \int_0^{+\infty} M\{\{\xi - \mathrm{e} \ge x\} \bigcup \{\xi - \mathrm{e} \le -x\}\}\mathrm{d}x \\
&= \int_0^{+\infty} M\{\{\xi \ge x + e\} \bigcup \{\xi \le e - x\}\}\mathrm{d}x \\
&= \int_e^{+\infty} (1 - M\{\xi \le r\})\mathrm{d}r + \int_{-\infty}^{e} M\{\xi \le r\}\mathrm{d}r \\
&= \int_e^{+\infty} (1 - \Phi(r))\mathrm{d}r + \int_{-\infty}^{e} \Phi(r)\mathrm{d}r
\end{aligned}$$

Thus, the proof of the theorem is ended. □

Theorem 4 *Let ξ be an uncertain variable with finite expected value e. Then for any nonnegative real numbers λ, it holds*

$$AD(\lambda\xi) = \lambda AD(\xi) \tag{8}$$

Proof From Definition 5, it follows that

$$\begin{aligned}
\mathrm{AD}(\lambda\xi) &= E[|\lambda\xi - \lambda e|] = \lambda E[|\xi - e|] \\
&= \lambda \mathrm{AD}(\xi)
\end{aligned}$$

Thus, the proof of the theorem is ended. □

Theorem 5 *Let ξ be an uncertain variable with finite expected value e. Then for any nonnegative real numbers λ and for any real numbers η, it holds*

$$AD(\lambda\xi + \eta) = \lambda AD(\xi) \tag{9}$$

Proof From Definition 5, it follows that

$$\begin{aligned}
\mathrm{AD}(\lambda\xi + \eta) &= E[|\lambda\xi + \eta - (\lambda e + \eta)|] \\
&= E[\lambda\,|\xi - e|] = \lambda E[|\xi - e|] \\
&= \lambda \mathrm{AD}(\xi)
\end{aligned}$$

Thus, the proof of the theorem is ended. □

If $r = (a, \alpha, \beta)$ be a triangular uncertain variable, then uncertainty distribution Φ(r) can be described as:

$$\Phi(r) = \begin{cases} 0, & \text{if } r \le a - \alpha, \\ \frac{r-(a-\alpha)}{2\alpha}, & \text{if } a - \alpha \le r \le a, \\ \frac{r+\beta-a}{2\beta}, & \text{if } a \le r \le a + \beta, \\ 1, & \text{if } r \ge a + \beta. \end{cases} \tag{10}$$

The triangle uncertain variable is denoted by $r(a, \alpha, \beta)$, where $\alpha \ge 0$, $\beta \ge 0$.

Theorem 6 *If $r = (a, \alpha, \beta)$ be a triangle uncertain variable, the expected value of r can be given by:*

$$E(r) = a + \frac{\beta - \alpha}{4} \tag{11}$$

Proof From Definition 3 and Theorem 6, it follows that

$$\begin{aligned}
E(r) &= \int_0^{+\infty} M\{r \ge x\}\mathrm{d}x - \int_{-\infty}^{0} M\{r \le x\}\mathrm{d}x \\
&= \int_0^{+\infty} (1 - M\{r \le x\})\mathrm{d}x - \int_{-\infty}^{0} M\{r \le x\}\mathrm{d}x \\
&= \int_0^{+\infty} (1 - \Phi(r))\mathrm{d}r - \int_{-\infty}^{0} \Phi(r)\mathrm{d}r
\end{aligned} \tag{12}$$

According to Eq. (10), the right-hand side of Eq. (12) is

$$\int_0^{+\infty} (1 - \Phi(r))\mathrm{d}r - \int_{-\infty}^{0} \Phi(r)\mathrm{d}r = \int_0^{a-\alpha} (1 - 0)\mathrm{d}r$$

$$+\int_{a-\alpha}^{a}\left(1-\frac{r-(a-\alpha)}{2\alpha}\right)\mathrm{d}r$$
$$+\int_{a}^{a+\beta}\left(1-\frac{r+a+\beta}{2\beta}\right)\mathrm{d}r+\int_{a+\beta}^{+\infty}(1-1)\mathrm{d}r$$
$$=a-\alpha+\frac{3\alpha}{4}+\frac{\beta}{4}=a+\frac{\beta-\alpha}{4} \tag{13}$$

According to Eqs. (12) and (13), we can get

$$E(r)=a+\frac{\beta-\alpha}{4}$$

Thus, the proof of the theorem is ended. □

Theorem 7 *Let $r=(a,\alpha,\beta)$ be a triangle uncertain variable, which $E(r)=a+\frac{\beta-\alpha}{4}$. Then, the uncertain absolute deviation of ξ can be given by:*

$$AD(r)=\begin{cases}\frac{4a^2-12(a-\alpha)a+4a(a+\beta)-6(a-\alpha)(a+\beta)+9(a-\alpha)^2+(a+\beta)^2}{32\alpha}, \\ \quad \text{if } \beta\le\alpha \\ \frac{(3\beta+\alpha)^2}{32\beta}, \quad \text{if } \beta\ge\alpha\end{cases} \tag{14}$$

Proof From Theorem 3, it follows that

$$\mathrm{AD}(r)=\int_{e}^{+\infty}(1-\Phi(r)\mathrm{d}r+\int_{-\infty}^{e}\Phi(r)\mathrm{d}r$$
$$=\int_{a+\frac{\beta-\alpha}{4}}^{+\infty}(1-\Phi(r))\mathrm{d}r+\int_{-\infty}^{a+\frac{\beta-\alpha}{4}}\Phi(r)\mathrm{d}r \tag{15}$$

If $\beta\le\alpha$, the right-hand side of Eq. (15) is

$$\int_{a+\frac{\beta-\alpha}{4}}^{+\infty}(1-\Phi(r))\mathrm{d}r+\int_{-\infty}^{a+\frac{\beta-\alpha}{4}}\Phi(r)\mathrm{d}r=\int_{a+\frac{\beta-\alpha}{4}}^{b}\left(1-\frac{r-(a-\alpha)}{2\alpha}\right)\mathrm{d}r+\int_{a}^{a+\beta}\left(1-\frac{r+a+\beta}{2\beta}\right)\mathrm{d}r$$
$$+\int_{a+\beta}^{+\infty}(1-1)\mathrm{d}r+\int_{0}^{a}0\mathrm{d}r+\int_{a-\alpha}^{a+\frac{\beta-\alpha}{4}}\frac{r-(a-\alpha)}{2\alpha}\mathrm{d}r$$
$$=\frac{20a^2-28(a-\alpha)a-12a(a+\beta)+10(a-\alpha)(a+\beta)+9(a-\alpha)^2+(a+\beta)^2}{64\alpha}$$
$$+\frac{\beta}{4}+\frac{(3\alpha+\beta)^2}{64\alpha}$$
$$=\frac{4a^2-12(a-\alpha)a+4a(a+\beta)-6(a-\alpha)(a+\beta)+9(a-\alpha)^2+(a+\beta)^2}{32\alpha} \tag{16}$$

If $\beta\ge\alpha$, the right-hand side of Eq. (15) is

$$\int_{a+\frac{\beta-\alpha}{4}}^{+\infty}(1-\Phi(r))\mathrm{d}r+\int_{-\infty}^{a+\frac{\beta-\alpha}{4}}\Phi(r)\mathrm{d}r$$
$$=\int_{a+\frac{\beta-\alpha}{4}}^{a+\beta}\left(1-\frac{r+\beta-a}{2\beta}\right)\mathrm{d}r$$
$$+\int_{a+\beta}^{+\infty}(1-1)\mathrm{d}r+\int_{0}^{a-\alpha}0\mathrm{d}r+\int_{a-\alpha}^{a}\frac{r-(a-\alpha)}{2\alpha}\mathrm{d}r$$
$$+\int_{a}^{a+\frac{\beta-\alpha}{4}}\frac{r-(a-\alpha)}{2\alpha}\mathrm{d}r$$
$$=\frac{(3\beta+\alpha)^2}{64\beta}+\frac{(3\beta+\alpha)^2}{64\beta}$$
$$=\frac{(3\beta+\alpha)^2}{32\beta} \tag{17}$$

According to Eqs. (16) and (17), we can get

$$\mathrm{AD}(r)=\begin{cases}\frac{4a^2-12(a-\alpha)a+4a(a+\beta)-6(a-\alpha)(a+\beta)+9(a-\alpha)^2+(a+\beta)^2}{32\alpha}, \\ \quad \text{if } \beta\le\alpha \\ \frac{(3\beta+\alpha)^2}{32\beta}, \quad \text{if } \beta\ge\alpha\end{cases}$$

Thus, the proof of the theorem is ended. □

3 The multiperiod portfolio selection model

Assume that there are n risky assets and one risk-free asset in financial market for trading. An investor wants to allocate his/her initial wealth W_1 among $n+1$ assets at the beginning of period 1, and obtains the final wealth at the end of period T. He/She can reallocate his/her wealth among the $n+1$ assets at the beginning of each of the following T consecutive investment periods. Suppose that the return rates of the n risky assets at each period are denoted as triangular uncertain variables. For the sake of description, let us first introduce the following notations. Let x_{i0} be the initial investment proportion of risky asset i at period 0, x_{it} be the investment proportion of risky asset i at period t, x_{ft} be the investment proportion of risk-free asset at period, x_t be the portfolio at period t, where $x_t=(x_{ft},x_{1t},x_{2t},\ldots,x_{nt})$, x_{ft}^b be the lower bound of the investment proportion of risk-free asset at period t, where $x_{ft}^3\ge x_{ft}^b$, R_{it} be the return of risky asset i at period t, r_{pt} be the return rate of the portfolio x_t at period t, r_{bt} be the borrowing rate of the risk-free asset at period t, r_{lt} be the lending rate of the risk-free asset at period t, u_{it}

be the upper bound constraints of x_{it}, r_{Nt} be the net return rate of the portfolio x_t at period t, W_t be the crisp form of the holding wealth at the beginning of period t, c_{it} be the unit transaction cost of risky asset i at period t, K be the desired number of risky assets in the portfolio at period t.

3.1 Return, risk and real constraints

In this section, we employ the uncertain mean value of the return on the portfolio at each period to measure the return of portfolio. The risk on the return rate of portfolio at each period is quantified by the uncertain absolute deviation. The return rate of security i at period t, $R_{it} = (a_{it}, \alpha_{it}, \beta_{it})$, is triangular uncertain variable for all $i = 1, \ldots, n$ and $t = 1, \ldots T$.

The expected return rate of the portfolio $x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})'$ at period t can be expressed as

$$\begin{aligned} r_{pt} &= \sum_{i=1}^{n} E(R_{it})x_{it} + r_{ft}\left(1 - \sum_{i=1}^{n} x_{it}\right) \\ &= \sum_{i=1}^{n}\left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right)x_{it} \\ &\quad + r_{ft}\left(1 - \sum_{i=1}^{n} x_{it}\right), t = 1, \ldots, T \end{aligned} \tag{18}$$

where $r_{ft} = \begin{cases} r_{lt}, & 1 - \sum_{i=1}^{n} x_{it} \geq 0 \\ r_{bt}, & 1 - \sum_{i=1}^{n} x_{it} \leq 0 \end{cases}$, $r_{bt} \geq r_{lt}$. When $1 - \sum_{i=1}^{n} x_{it} \geq 0$, it denotes that lending is allowed on the risk-free asset; When $1 - \sum_{i=1}^{n} x_{it} \leq 0$, it represents that borrowing is allowed on the risk-free asset.

Let the preset value be x_{ft}^{b}, where $x_{ft}^{b} \leq 0$, the borrowing constraint of risk-free asset at period t is

$$x_{ft} = 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^{b} \tag{19}$$

We assume in the sequel that the transaction costs at period t is a V shape function of difference between the tth period portfolio $x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})$ and the t-1th period portfolio $x_{(t-1)} = (x_{f(t-1)}, x_{1(t-1)}, x_{2(t-1)}, \ldots, x_{n(t-1)})$. The transaction cost for asset i at period t can be expressed by

$$C_{it} = c_{it}\left|x_{it} - x_{i(t-1)}\right| \tag{20}$$

Hence, the total transaction costs of the portfolio $x_t = (x_{1t}, x_{2t}, \ldots, x_{nt})$ at period t can be represented as

$$C_t = \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|, t = 1, \ldots, T \tag{21}$$

Thus, the net expected return rate of the portfolio x_t at period t can be denoted as

$$\begin{aligned} r_{Nt} &= \sum_{i=1}^{n}\sum_{i=1}^{n}\left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right)x_{it} + r_{ft}\left(1 - \sum_{i=1}^{n} x_{it}\right) \\ &\quad - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|.t = 1, \ldots, T \end{aligned} \tag{22}$$

Then, the crisp form of the holding wealth at the beginning of the period t can be written as

$$\begin{aligned} W_{t+1} &= W_t(1 + r_{Nt}) \\ &= W_t\left(1 + \sum_{i=1}^{n}\left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right)x_{it} + r_{ft}\left(1 - \sum_{i=1}^{n} x_{it}\right)\right. \\ &\quad \left. - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|\right), t = 1, \ldots, T \end{aligned} \tag{23}$$

The absolute deviation of the portfolio x_t can be expressed as

$$\mathrm{AD}_t(x_t) = \mathrm{AD}_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \cdots + r_{nt}x_{nt}) \tag{24}$$

Threshold constraints limit the amount of capital to be invested in each stock and prevent very small investments in any stock. The threshold constraints of multiperiod portfolio selection can be expressed as

$$0 \leq x_{it} \leq u_{it} \tag{25}$$

where u_{it} is the upper bound of x_{it}.

To formulate cardinality constraints into the multiperiod portfolio model, zero-one decision variables are added as:

$$z_{it} = \begin{cases} 1 & \text{if any of asset } i \text{ of period} \\ & \quad t\ (i = 1, \ldots, n;\ t = 1, \ldots, T) \text{ is held} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where $\sum_{i=1}^{n} z_{it} \leq K$.

3.2 The basic multiperiod portfolio optimization models

When the investors can give a tolerable level of risk at period t, and want to maximize the terminal wealth at the given level of risk, the multiperiod uncertain mean absolute deviation model with real constraints is as follows:

$$\begin{aligned} \max \prod_{t=1}^{T} &\left[\sum_{i=1}^{n}\left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right)x_{it} + r_{ft}\left(1 - \sum_{i=1}^{n} x_{it}\right)\right. \\ &\left. - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|\right] \end{aligned}$$

$$\text{s.t}\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{n}\left(a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right)x_{it}\right.\right. \\ \quad \left.\left.+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right)\right)W_t \quad \text{(a)} \\ \mathrm{AD}_t(r_{1t}x_{1t}+r_{2t}x_{2t}+\cdots+r_{nt}x_{nt})\leq \mathrm{AD}_{0t} \quad \text{(b)} \\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b} \quad \text{(c)} \\ \sum_{i=1}^{n}z_{it}\leq K,\ z_{it}\in\{0,1\} \quad \text{(d)} \\ 0\leq x_{it}\leq u_{it}z_{it},\ i=1,\ldots,n,\ t=1,\ldots,T \quad \text{(e)} \end{cases} \tag{27}$$

where AD_{0t} denotes the maximum risk level the investors can tolerate. The model (27) consists of an objective, namely the maximization of the investors' terminal wealth. Constraint (a) denotes the wealth accumulation constraint; constraint (b) states that the absolute deviation of the portfolio x_t cannot exceed the given minimum risk AD_{0t} at each period; constraint (c) indicates that the investment proportion of risk-free asset at period t must exceed the given lower bound; constraint (d) represents the desired number of assets in the portfolio must not exceed a given value K; constraint (e) states the lower and upper of x_{it}.

According to Qin et al. (2011), if $r_{1t}, r_{2t}, \ldots, r_{nt}$ are independent triangular uncertain variables, and $x_{it} \geq 0, i = 1, \ldots, n$,

$$\mathrm{AD}_t(r_{1t}x_{1t}+r_{2t}x_{2t}+\cdots+r_{nt}x_{nt}) = \sum_{i=1}^{n}x_{it}\mathrm{AD}_t(r_{it}). \tag{28}$$

where

$$AD(r_{it}) = \begin{cases} \frac{4a_{it}^2-12(a_{it}-\alpha_{it})a_{it}+4a_{it}(a_{it}+\beta_{it})}{32\alpha} \\ -\frac{6(a_{it}-\alpha_{it})(a_{it}+\beta_{it})+9(a_{it}-\alpha_{it})^2+(a_{it}+\beta_{it})^2}{32\alpha}, & \text{if } \beta_{it}\leq\alpha_{it} \\ \frac{(3\beta_{it}+\alpha_{it})^2}{32\beta_{it}}, & \text{if } \beta_{it}\geq\alpha_{it} \end{cases}. \tag{29}$$

According to Eq. (28), the Model (27) can be turned into as follows:

$$\max \prod_{t=1}^{T}\left[\sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right]$$
$$\text{s.t}\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{n}\left(a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right)x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right)\right)W_t \\ \sum_{i=1}^{n}\mathrm{AD}_t(r_{it})x_{it}\leq \mathrm{AD}_{0t} \\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b} \\ \sum_{i=1}^{n}z_{it}\leq K,\ z_{it}\in\{0,1\} \\ 0\leq x_{it}\leq u_{it}z_{it},\ i=1,\ldots,n,\ t=1,\ldots,T \end{cases} \tag{30}$$

4 Solution algorithm

In this section, the multiperiod mean absolute deviation uncertain portfolio selection model with real constraints will be approximated into a mix integer dynamic programming problem with linear inequality constraints. A novel discrete iteration method will be proposed to solve the problem, and will be proved the linear convergence.

4.1 The proposed model approximated to dynamic programming problem

The sub-problem of Model (30) at period t is as follows:

$$\max \sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|$$
$$\text{s.t}\begin{cases} \sum_{i=1}^{n}\mathrm{AD}_t(r_{it})x_{it}\leq \mathrm{AD}_{0t} \\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b} \\ \sum_{i=1}^{n}z_{it}\leq K,\ z_{it}\in\{0,1\},\ i=1,\ldots,n \\ 0\leq x_{it}\leq u_{it}z_{it},\ i=1,\ldots,n \end{cases} \tag{31}$$

Let $x_{it-1} = \overline{x_{it-1}}$, where $\overline{x_{it-1}}$ is preset value, $\sum_{i=1}^{n}\overline{x_{it-1}} = 1 - x_{ft}^{b}$, the Model (30) can be approximated into the following model:

$$\max \prod_{t=1}^{T} \left[\sum_{i=1}^{n} \left[a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right] x_{it} + r_{ft} \left(1 - \sum_{i=1}^{n} x_{it} \right) - \sum_{i=1}^{n} c_{it} \left| x_{it} - \overline{x_{i(t-1)}} \right| \right]$$

$$\text{s.t} \begin{cases} W_{t+1} = (1 + \left(\sum_{i=1}^{n} \left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \left| x_{it} - \overline{x_{i(t-1)}} \right| \right)) W \\ \sum_{i=1}^{n} \text{AD}_t(r_{it}) x_{it} \le \text{AD}_{0t} \\ 1 - \sum_{i=1}^{n} x_{it} \ge x_{ft}^b \\ \sum_{i=1}^{n} z_{it} \le K \\ z_{it} \in \{0, 1\}, i = 1, \dots, n \\ 0 \le x_{it} \le u_{it} z_{it}, i = 1, \dots, n \end{cases} \tag{32}$$

The sub-problem of model (32) at period t is as follows:

$$\max \left[\sum_{i=1}^{n} \left[a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right] x_{it} + r_{ft} \left(1 - \sum_{i=1}^{n} x_{it} \right) - \sum_{i=1}^{n} c_{it} \left| x_{it} - \overline{x_{i(t-1)}} \right| \right]$$

$$\text{s.t} \begin{cases} \sum_{i=1}^{n} \text{AD}_t(r_{it}) x_{it} \le \text{AD}_{0t} \\ 1 - \sum_{i=1}^{n} x_{it} \ge x_{ft}^b \\ \sum_{i=1}^{n} z_{it} \le K, z_{it} \in \{0, 1\}, i = 1, \dots, n \\ 0 \le x_{it} \le u_{it} z_{it}, i = 1, \dots, n \end{cases} \tag{33}$$

Theorem 8 *Let the optimal solutions and objective function values of Model (31) and Model (33), respectively be x^{1*}, $G(x^{1*})$, and x^{2*}, $F(x^{2*})$. Then $G(x^{1*}) - G(x^{2^*}) \le 4\max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b)$.*

Proof Because the feasible solution set of the Model (31) is same as Model (33), x^{1*} and x^{2*} are the feasible solutions of Model (31) and Model (33), respectively. Then,

$G(x^{1*}) \ge G(x^{2*})$ and $F(x^{2*}) \ge F(x^{1*})$

then $G(x^{1*}) + F(x^{2*}) \ge G(x^{2*}) + F(x^{1*})$

so

$$G(x^{1*}) - G(x^{2*}) + F(x^{2*}) - F(x^{1*}) \ge 0 \tag{34}$$

The left-hand side of Eq. (34) is

$$\begin{aligned} & G(x^{1*}) - G(x^{2*}) + F(x^{2*}) - F(x^{1*}) \\ & = \left[\sum_{i=1}^{n} c_{it} \left| x_{it}^{1*} - x_{i(t-1)} \right| - \sum_{i=1}^{n} c_{it} \left| x_{it}^{1*} - \overline{x_{i(t-1)}} \right| \right] \\ & + \left[\sum_{i=1}^{n} c_{it} \left| x_{it}^{2*} - \overline{x_{i(t-1)}} \right| - \sum_{i=1}^{n} c_{it} \left| x_{it}^{2*} - x_{i(t-1)} \right| \right] \\ & \le 2 \sum_{i=1}^{n} c_{it} \left| x_{i(t-1)} - \overline{x_{i(t-1)}} \right| \end{aligned} \tag{35}$$

Because $x_{i(t-1)} \ge 0$, $x_{it} \ge 0$,

$$\begin{aligned} & 2 \sum_{i=1}^{n} c_{it} \left| x_{i(t-1)} - \overline{x_{i(t-1)}} \right| \le 2 \sum_{i=1}^{n} c_{it} x_{i(t-1)} \\ & + 2 \sum_{i=1}^{n} c_{it} \overline{x_{i(t-1)}} \\ & = 2 \max_{i=1}^{n}\{c_{it}\} \sum_{i=1}^{n} x_{it} + 2 \max_{i=1}^{n} \sum_{i=1}^{n} \overline{x_{i(t-1)}} \\ & \le 2 \max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b) + 2 \max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b) \\ & = 4 \max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b) \end{aligned}$$

So $G(x^{1*}) - G(x^{2^*}) \le 4\max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b)$.
Which ends the proof.

Because $\max_{i=1}^{n}\{c_{it}\} << r_{it}$, where asset $i \in$ efficient asset set of portfolio,
$4\max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b)$ is small, and $G(x^{1*}) - G(x^{2^*})$ is also small.
If $c_{it} = 0.003$, $x_{ft}^b = -0.5$, $G(x^{1*}) - G(x^{2^*}) \le 4\max_{i=1}^{n}\{c_{it}\}(1 - x_{ft}^b) \le 4 \times 0.003 \times 1.5 = 0.018$.

4.2 The smallest and biggest value of state variable at every period

In Model (32), investors can choose W_t between $W_t^{\min}$ and $W_t^{\max}$. $W_t^{\min}$ and $W_t^{\max}$ can be, respectively, obtained as follows:

The investor considers to maximize the expected return of the portfolio at period t.

$$\max \sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right) - \sum_{i=1}^{n}c_{it}\left|x_{it}-\overline{x_{i(t-1)}}\right|$$
$$\text{s.t}\begin{cases}1-\sum\limits_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ \sum\limits_{i=1}^{n}z_{it}\leq K, z_{it}\in\{0,1\}, i=1,\ldots,n, t=1,\ldots,T\\ 0\leq x_{it}\leq u_{it}z_{it}, i=1,\ldots,n, t=1,\ldots,T\end{cases} \tag{36}$$

Let $y_{it} = \left|x_{it}-\overline{x_{i(t-1)}}\right|$. Then the Model (36) can be turned into as follows.

$$\max \sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}y_{it}$$
$$\text{s.t}\begin{cases}1-\sum\limits_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ y_{it}\geq x_{it}-\overline{x_{i(t-1)}}\\ y_{it}\geq -(x_{it}-\overline{x_{i(t-1)}})\\ \sum\limits_{i=1}^{n}z_{it}\leq K, z_{it}\in\{0,1\}, i=1,\ldots,n\\ 0\leq x_{it}\leq u_{it}z_{it}, i=1,\ldots,n\end{cases} \tag{37}$$

$x_t^{\max *}$ (the optimal solution $x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})'$) can be obtained solving Model (37) by the CPLEX. $r_{Nt}^{\max}$ (the biggest of $\sum_{i=1}^{n}[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}]x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}y_{it}$) can also be obtained. Then, $W_{t+1}^{\max}$ can be obtained as follows:

$$W_{t+1}^{\max} = W_t^{\max}\left(1+\sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}^{\max} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}^{\max}\right)-\sum_{i=1}^{n}c_{it}y_{it}^{\max}\right), t=1,\ldots,T \tag{38}$$

where W_1 is initial wealth, which is preset value.

The biggest value of the absolute deviation of the portfolio selection at period $t(\sum_{i=1}^{n}x_{it}^{\max}\text{AD}_t(r_{it}))$ can be also obtained.

The investor only considers to minimize the absolute deviation of the portfolio at period t, that is, the smallest value of the r_{Nt} can be obtained as follows:

$$\min \sum_{i=1}^{n}\text{AD}_t(r_{it})x_{it}$$
$$\text{s.t}\begin{cases}1-\sum\limits_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ \sum\limits_{i=1}^{n}z_{it}\leq K, z_{it}\in\{0,1\}, i=1,\ldots,n\\ 0\leq x_{it}\leq u_{it}z_{it}, i=1,\ldots,n\end{cases} \tag{39}$$

$x_t^{\min *}$(the optimal solution $x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})'$) can be obtained solving the Model (39) by the CPLEX. Simultaneously, $r_{Nt}^{\min}$ (the smallest of $\sum_{i=1}^{n}[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}]x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}\left|x_{it}-\overline{x_{i(t-1)}}\right|$) is also obtained. Then, $W_{t+1}^{\min}$ can be obtained as follows:

$$W_{t+1}^{\min} = W_t^{\min}\left(1+\sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}^{\min} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}^{\min}\right)-\sum_{i=1}^{n}c_{it}\left|x_{it}^{\min}-\overline{x_{i(t-1)}^{\min}}\right|\right), \quad t=1,\ldots,T \tag{40}$$

where W_1 is initial wealth, which is preset value.

The smallest value of the absolute deviation of the portfolio selection at period $t(\sum_{i=1}^{n}x_{it}^{\min}\text{AD}_t(r_{it}))$ can be also obtained.

4.3 The discrete iteration method

In this section, the max-plus algebra, which is proposed by Heidergott et al. (2006), will be used. The method solving the longest path of the multiperiod weighted digraph, which is from the starting point to the ending point, will be proposed, and some definitions will be introduced first. Then, the method of finding the longest path of the multiperiod weighted digraph can be obtained. Finally, $k+1$th iteration method will be presented.

Definition 6 A semi-field (or semi-ring) is a triplet $A = \{Q, \oplus, \otimes\}$ consisting of nonempty set Q and two binary operations $\oplus$ and $\otimes$, called modi-addition and modi-multiplication, respectively, defined on Q, such that for all a_1, a_2, a_3 in Q

(i) each of the operations $\oplus$ and $\otimes$ is commutative

$$a_1\oplus a_2 = a_2\oplus a_1, a_1\otimes a_2 = a_2\otimes a_1$$

(ii) each of the operations $\oplus$ and $\otimes$ is associative

$$(a_1 \oplus a_2) \oplus a_3 = a_1 \oplus (a_2 \oplus a_3), (a_1 \otimes a_2) \otimes a_3 = a_1 \otimes (a_2 \otimes a_3)$$

(iii) the operation $\otimes$ is distributive with respect to the operation $\oplus$

$$a_1 \otimes (a_2 \oplus a_3) = a_1 \otimes a_2 \oplus a_1 \otimes a_3$$

(iv) there exists an element in Q which is the zero element, denoted by ε, such that for all a in Q, we have $\varepsilon \oplus a = a, \varepsilon \otimes a = \varepsilon$
If there exists an element denoted by e in a semi-field $A = \{Q, \oplus, \otimes\}$, such that for all $a$in Q, we have $e \otimes a{=}a$
then e is called an identity element of the semi-field.

Definition 7 $a_1, a_2 \in R$, in semi-field $A = \{R, \max, +\}$, then $\varepsilon = -\infty$ and $e = 0$.

Definition 8 Let us consider matrices $A_{n\times n} = (a_{ij})_{n\times n}$, $B_{n\times n} = (b_{ij})_{n\times n}, a_{ij}, b_{ij} \in R$, in semi-field $A = \{R, \max, +\}$. Then $C = A \oplus B, C_{n\times n} = (c_{ij})_{n\times n}$, where $c_{ij} = \max\{a_{il}, b_{lj}\}$.

Definition 9 Let us consider matrices $A_{n\times m} = (a_{ij})_{n\times m}$, $B_{m\times k} = (b_{ij})_{m\times k}, a_{ij}, b_{ij} \in R$, in semi-field $A = \{R, \max, +\}$. Then $C = A \otimes B, C_{n\times k} = (c_{ij})_{n\times k}$, where $c_{ij} = \max_{l=1}^{m}\{a_{il} + b_{lj}\}$.

The Model (32) is a mix integer dynamic programming problem with linear inequality constraints, the optimal solution can't be obtained by the dynamic programming recursive relationship. In this section, a novel discrete iteration method is proposed. The method is as follows: Firstly, according to the network approach, discretizes the state variables and transforms the model into multiperiod weighted digraph. Secondly, the max-plus algebra is used to obtain the largest path that is the admissible solution. Thirdly, based on the admissible solution, continues iterating until the two admissible solutions are real near. Finally, the method is proved linearly convergent.

The state variable W_t of the period t is discretized into four intervals of same widths from the smallest value to the biggest one. It means that there are five discrete values for the state variable in every period. In this way, Model (32) is transformed into a multiperiod weighted digraph as shown in Fig. 1. The investment period, the value of the objective function of the period t and a discrete value of the state variable are represented by the stage, the weight of the period t and the point of the multiperiod weighted digraph, respectively.

In this section, a discrete iteration method will be proposed to solve the Model (32).

Step 1: The discrete state variables at period t ($t = 2, \ldots, T+1$) can be obtained by discretizing the interval value of $W_t^{\max} - W_t^{\min}$ into four equalities. That is

$$W_{it} = W_t^{\min} + (W_t^{\max} - W_t^{\min})(i-1)/4, i = 1, \ldots, 5$$

Step 2: The weight of the arcs in Fig. 1 can be obtained following three steps:

Step 2.1: The net expected return of the portfolio $r_{N1}(1, j)$, $j = 1, \ldots, 5$, can be obtained as follows:

$$r_{N1}(1, j) = W_{j2}/W_1 - 1$$

Step 2.2: The net expected return of portfolio $r_{Nt}(j, k)$ at period t, which j is the number of the point at period t, $j = 1, \ldots, 5$ and k is the number of the point at period $t+1$, $k = 1, \ldots, 5$, can be obtained as follows:

$$r_{Nt}(j, k) = W_{kt+1}/W_{jt} - 1$$

Step 2.3: The weights of the side at period t, which the values of the objective function $F_1(1, j)$ and $F_t(j, k)$ in Fig. 1, can be obtained as follows:

When $r_{Nt}(k, l)$ is known, the sub-problem at period t of the Model (32) can be turned into

$$\max \sum_{i=1}^{n}\left[a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right]x_{it} - \sum_{i=1}^{n} c_{it}\left|x_{it} - \overline{x_{i(t-1)}}\right|$$
$$\text{s.t}\begin{cases} \max \sum_{i=1}^{n}\left[a_{it} + \frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it} \\ -\sum_{i=1}^{n} c_{it}\left|x_{it} - \overline{x_{it-1}}\right| \geq r_{Nt}(k, l) \\ 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^{b} \\ \sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\} \\ 0 \leq x_{it} \leq u_{it}z_{it}, i = 1, \ldots, n \end{cases} \tag{41}$$

Let $y_{it} = \left|x_{it} - \overline{x_{i(t-1)}}\right|$. Then the Model (41) can be turned into as follows.

$$\max \sum_{i=1}^{n}\left[a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right]x_{it} - \sum_{i=1}^{n} c_{it}y_{it}$$

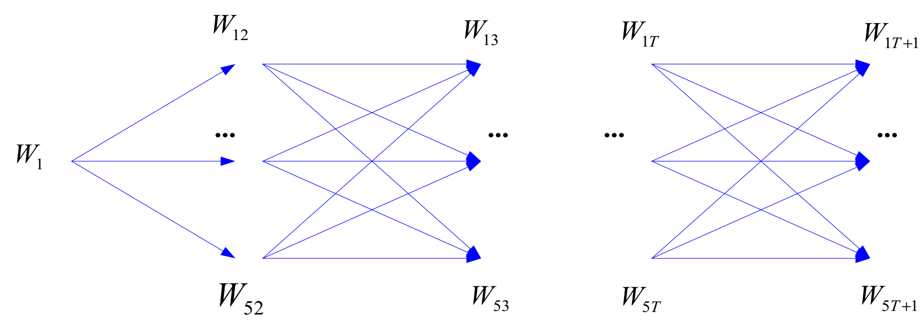

Fig. 1 The multiperiod weighted digraph

$$\text{s.t}\begin{cases}\left(\sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)\right.\\ \left.-\sum_{i=1}^{n}c_{it}y_{it}\right)\geq r_{Nt}(k,l)\\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ y_{it}\geq x_{it}-\overline{x_{i(t-1)}}\\ y_{it}\geq-(x_{it}-\overline{x_{i(t-1)}})\\ \sum_{i=1}^{n}z_{it}\leq K,z_{it}\in\{0,1\}\\ 0\leq x_{it}\leq u_{it}z_{it},i=1,\ldots,n\end{cases} \tag{42}$$

x_t^*(the optimal solution $x_t=(x_{ft},x_{1t},x_{2t},\ldots,x_{nt})'$) can be obtained solving the Model (42) by the CPLEX. Simultaneously, the objective function value $F_t(j,k)$ can be obtained as follows:

$$F_t(j,k)=\sum_{i=1}^{n}\left[a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right]x_{it}^*-\sum_{i=1}^{n}c_{it}y_{it}^*.$$

Step 3: Calculation of the longest path of the multiperiod weighted digraph

According to Definition 9, the longest path $F^{(1)}$ of the multiperiod weighted digraph of the first iteration can be obtained as follows:

$$F^{(1)}=F_1^{(1)}\otimes F_2^{(1)}\otimes\cdots\otimes F_T^{(1)} \tag{43}$$

where $F_1^{(1)}=(F^{(1)}(1,j))_{1\times5}$, $F_2^{(1)}=(F_2^{(1)}(i,j))_{5\times5},\ldots,$ $F_T^{(1)}=(F_T^{(1)}(i,j))_{5\times5}$.

Step 4: The discrete iteration method $k+1$th iteration can be described as follows:

Let the longest path of the kth iteration be $W_1\to W_{i_22}^{(k)}\to W_{i_33}^{(k)}\to\cdots\to W_{i_{T+1}T+1}^{(k)}$. The optimal solutions of the longest path of Fig. 1 are also feasible solutions of the multiperiod mean–absolute deviation portfolio selection model. Based on the $(W_1,W_{i_22}^{(k)},W_{i_33}^{(k)},\ldots,W_{i_{T+1}T+1}^{(k)})$, the state variables from period 1 to period T are discretized into four intervals as following three steps.

Step 4.1: $W_2^{\min}$ and $W_{i_22}^{(k)}$, $W_{i_22}^{(k)}$ and $W_2^{\max}$ are discretized into two same internals, respectively. The five discrete points of S_2, i.e., $W_2^{\min}$, $W_{22}^{(k+1)}$, $W_{i_12}^{(k+1)}$, $W_{32}^{(k+1)}$, $W_2^{\max}$ can be obtained.

Step 4.2: Based on $(W_{i_33}^{(k)},\ldots,W_{i_{T+1}T+1}^{(k)})$, using the same method, the state variables from period 3 to period $T+1$ are discretized into the five points, respectively. The wealth of period $t$can also be obtained.

Step 4.3: The longest path of the k+1th iteration $F^{(k+1)}$ and another feasible solution can be obtained as follows:

$$F^{(k+1)}=F_1^{(k+1)}\otimes F_2^{(k+1)}\otimes\cdots\otimes F_T^{(k+1)} \tag{44}$$

where $F_1^{(k+1)}=(F^{(k+1)}(1,j))_{1\times5}$, $F_2^{(k+1)}=(F_2^{(k+1)}(i,j))_{5\times5},\ldots,$ $F_T^{(k+1)}=(F_T^{(k+1)}(i,j))_{5\times5}$.

If $\left|F^{(k+1)}-F^{(k)}\right|\leq10^{-6}$, then the optimal solution of the longest path $F^{(k+1)}$ is also the optimal solution of the Model (32). Otherwise turn Step 2.

4.4 Convergence property of the discrete iteration method

Theorem 9 *The discrete iteration method is linearly convergent.*

Proof Let the longest path in period 1 and the longest path in period t, respectively be $F_1^{\max}(1,j_2)$, and $F_t^{\max}(i_t,j_{t+1})$, $t=2,\ldots,T$. Then the upper bound of the solution of Model (32) is

$$F_1^{\max}(1,j_2)\times F_2^{\max}(i_2,j_3)\times\cdots\times F_T^{\max}(i_T,j_{T+1})$$

The longest path of the multiperiod weighted digraph of the kth iteration $F^{(k)}$ is obtained as follows:

$$F^{(k)}=F_1^{(k)}\otimes F_2^{(k)}\otimes\cdots\otimes F_T^{(k)} \tag{45}$$

where $F_1^{(k)}=(F^{(k)}(1,j))_{1\times5}$, $F_2^{(k)}=(F_2^{(k)}(i,j))_{5\times5},\ldots,$ $F_T^{(k)}=(F_T^{(k)}(i,j))_{5\times5}$.

Let the longest path of the kth iteration be $W_1\to W_{i_22}^{(k)}\to W_{i_33}^{(k)}\to\cdots\to W_{i_{T+1}T+1}^{(k)}$. Using the **Step 4**, the multiperiod weighted digraph of the k+1th iteration can be obtained. $F^{(k+1)}$, which is the value of the longest path of the multiperiod weighted digraph of the k+1th iteration, can be

Table 1 The optimal solution when $K = 4$, $\mathrm{AD}_t = 0.01$

t	Asset i				
	The optimal investment proportions				
1	Asset3	Asset 17		x_{f1}	Otherwise asset
	0.1853448	0.2		0.6146552	0
2	Asset 15	Asset 17		x_{f2}	Otherwise asset
	0.0772222	0.2		0.7227778	0
3	Asset 15	Asset 24		x_{f3}	Otherwise asset
	0.2	0.1408867		0.6591133	0
4	Asset 20	Asset 25		x_{f4}	Otherwise asset
	0.1985612	0.2		0.6014388	0
5	Asset 20	Asset 25	Asset 30	x_{f5}	Otherwise asset
	0.2	0.2	0.094241	0.505759	0

obtained by Eq. (45). So $F^{(k+1)} \geq F^{(k)}$. The solution is becoming bigger and bigger. Because the solutions of Model (32) are increasing sets and there is an upper bound of the solution of Model (32), the discrete iteration method is convergent.

Let the optimal value of period t of Model (32) be F_t^*, $F_t^{(k)}$ be the optimal solution of the kth iteration at period t.

Because the objective function of Model (32) is linear, $\left|F_t^{(k+1)} - F_t^*\right| \leq \left|F_t^{(k)} - F_t^*\right|$. Because $F_t^* \geq F_t^{(k+1)}$, $F_t^* \geq F_t^{(k)}$, $F_t^* - F_t^{(k+1)} \leq F_t^* - F_t^{(k)}$, ie., $0 \leq \frac{\sum_{t=1}^{T}\left|F_t^{(K+1)} - F_t^*\right|}{\sum_{t=1}^{T}\left|F_t^{(K)} - F_t^*\right|} \leq 1$. So the discrete iteration method is linearly convergent. Thus, the proof of Theorem 9 is ended. □

5 Numerical example

In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from Shanghai Stock Exchange for his investment. The stocks codes are, respectively, $S_1, \ldots, S_{30}$. He/She intends to make five periods of investment with initial wealth $W_1 = 1$ and his wealth can be adjusted at the beginning of each period. He/she assumes that the returns and risk of the thirty stocks at each period are represented as triangular uncertain numbers. We collect historical data of them from April 2006 to March 2017 and set every 3 months as a period to handle the historical data. By using the estimation method in Vercher et al. (2007) to handle their historical data, the triangular uncertain distributions of the return rates of assets at each period can be obtained as shown in Appendix A. According to Eq. (14) and Appendix A, $AD_t(r_{it})$ $(i = 1, \ldots, 30; t = 1, \ldots, 5)$ can be obtained as shown in Appendix B.

Suppose that the unit transaction costs of assets of the two periods investment take the same value $c_{it} = 0.003$ $(i = 1, \ldots, 30; t = 1, \ldots, 5)$, the lower bound of the investment proportion of risk-free asset $x_{ft}^b = -0.5$, the borrowing rate of the risk-free asset $r_{bt} = 0.017$, the lending rate of the risk-free asset $r_{lt} = 0.009$, $t = 1, \ldots, 5$, upper bound constraints $u_{it} = 0.2 (i = 1, \ldots, 30; t = 1, \ldots, 5)$, the desired number of risky assets in the portfolio $K = 0, \ldots, 9$ at period t, $t = 1, \ldots, 5$.

In case when the $K = 4$, $\mathrm{AD}_t = 0.01$, the multiperiod uncertain mean absolute deviation portfolio selection model maximizing the terminal utility is set as follows:

If $K = 4$, $\mathrm{AD}_t = 0.01$, the optimal solution of Model (32) will be obtained as Table 1 using the discrete iteration method.

When $K = 4$, $\mathrm{AD}_t = 0.01$, the optimal investment strategy at period 1 is $x_{31} = 0.1853448$, $x_{171} = 0.2$, $x_{f1} = 0.6146552$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 3, asset 17, risk-free asset and otherwise asset by the proportions of 18.53448, 20, 61.46552% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 1, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.321556.

If $K = 4$, $\mathrm{AD}_t = 0.03$, the optimal solution of Model (32) will be obtained as Table 2 using the discrete iteration method.

When $K = 4$, $\mathrm{AD}_t = 0.03$, the optimal investment strategy at period 1 is $x_{31} = 0.2$, $x_{171} = 0.2$, $x_{281} = 0.1935484$, $x_{f1} = 0.4064516$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 3, asset 17, asset 28, risk-free asset and otherwise asset by the proportions of 20, 20, 19.35484, 40.64516% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 2, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.804082.

Table 2 The optimal solution when $K = 4$, $\mathrm{AD}_t = 0.03$

t	Asset i					
	The optimal investment proportions					
1	Asset13	Asset 17	Asset 28		x_{f1}	Otherwise asset
	0.2	0.2	0.1935484		0.4064516	0
2	Asset13	Asset 15	Asset 17	Asset 24	x_{f2}	Otherwise asset
	0.1892683	0.2	0.2	0.2	0.2107317	0
3	Asset13	Asset 15	Asset 17	Asset 24	x_{f3}	Otherwise asset
	0.1857385	0.2	0.2	0.2	0.2142615	0
4	Asset13	Asset 15	Asset 20	Asset 25	x_{f4}	Otherwise asset
	0.2	0.2	0.2	0.2	0.2	0
5	Asset1	Asset13	Asset 17	Asset 20	x_{f5}	Otherwise asset
	0.1995261	0.2	0.2	0.2	0.2004739	0

Table 3 The optimal terminal wealth and risk of the portfolio when $AD_t = 0.07$, $K = 0, 1, 2, \ldots, 9$

K	0	1	2	3	4	5	6	7	8
W_6	1.045817	1.315544	1.505725	1.706080	1.923746	2.158884	2.408406	2.652415	2.764959
K	9	10							
W_6	2.765641	2.765641							

W_6 is denoted the terminal wealth of the portfolio

For a given investment period t, if different parameters of AD_t, $t = 1, \ldots, 5$ are specified for the risk of the portfolio, the corresponding optimal strategies and terminal wealth can also be derived. The detailed results are shown in Tables 1 and 2. For some rational investors, they might not only consider the expectations the terminal wealth of the portfolio, but also concern about the risk, and we can see when the absolute deviations of the portfolio AD_t, $t = 1, \ldots, 5$ increase, the amounts of borrowing risk-free asset decrease. From Tables 1 and 2, it is easy to find that the terminal wealth become bigger when the AD_t become bigger.

If $\mathrm{AD}_t = 0.07$, $K = 0, 1, 2, \ldots, 9$, the optimal solution of Model (32) will be obtained as the Table 3 using the discrete iteration method.

From Table 3, Fig. 2 which reflects the relationship between K and the terminal wealth of the Model (32) can be obtained as follows.

In the used data sets, the experiments in this paper correspond to the values of K in the interval [0, 9]. It can be seen that, as will be seen in Fig. 2, the terminal wealth becomes bigger, when $0 \le K \le 9$, become larger; the terminal wealth is same, when $K \ge 9$; which reflects the influence of K on portfolio selection.

If $K = 4$, $\mathrm{AD}_t = 0, 0.005, 0.01, \ldots, 0.07$, the optimal solution of Model (32) will be obtained as the Table 4 using the discrete iteration method.

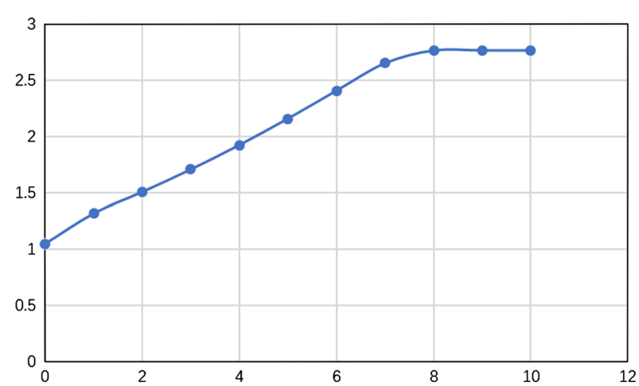

Fig. 2 The relationship between the K and the terminal wealth of the model (32), where x axis is K, y axis is W_6

From Table 4, Fig. 3 which reflects the relationship between AD_t and the terminal wealth of the Model (32) can be obtained as follows.

In the used data sets, the experiments in this paper correspond to the values of AD_t in the interval [0, 0.7]. It can be seen that, as will be seen in Fig. 3, the terminal wealth becomes bigger, when $0 \le AD_t \le 0.05$, become larger, the terminal wealth is same, when $\mathrm{AD}_t \ge 0.05$; which reflects the influence of AD_t on portfolio selection.

Table 4 the optimal terminal wealth and risk of the portfolio when $K = 4$, $\mathrm{AD}_t = 0, 0.005, 0.01, \ldots, 0.07$

AD_t	0	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04
W_6	1.045817	1.204630	1.321556	1.442145	1.566482	1.687916	1.804082	1.886100	1.908496
AD_t	0.045	0.05	0.055	0.06	0.065	0.07			
W_6	1.920493	1.923746	1.923746	1.923746	1.923746	1.923746			

W_6 is denoted the terminal wealth of the portfolio

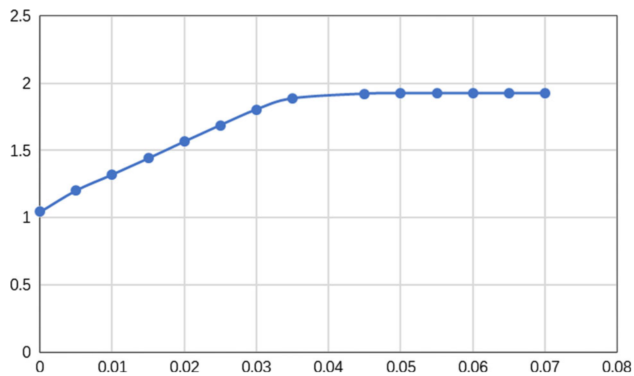

Fig. 3 The relationship between the AD_t and the terminal wealth of the model (32), where x axis is AD_t, y axis is W_6

6 Conclusions

In this paper, we consider the multiperiod portfolio selection problem in uncertain environment. We measure the return and the risk of the multiperiod portfolio using the uncertain mean value and the absolute deviation, respectively. A new multiperiod portfolio optimization model with transaction cost, borrowing constraints, threshold constraints and cardinality constraints is proposed. Based on the uncertain theories, the proposed model is transformed into a crisp dynamic optimization problem. Because of the transaction costs and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence, which is "NP hard" problem. The model is approximated to a mix integer dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy and is proved linearly convergent. Finally, we give an example to illustrate the idea of the model and demonstrate the effectiveness of the designed algorithm

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Compliance with ethical standards

Conflict of interest Peng Zhang declares that he/she has no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

Appendix A

The codes of thirty stocks are, respectively, S_1 (600,000), S_2 (600,005), S_3 (600,015), S_4 (600,016), S_5 (600,019), S_6 (600,028), S_7 (600,030), S_8 (600,036), S_9 (600,048), S_{10} (600,050), S_{11} (600,104), S_{12} (600,362), S_{13} (600,519), S_{14} (600,900), S_{15} (601,088), S_{16} (601,111), S_{17} (601,166), S_{18} (601,168), S_{19} (601,318), S_{20} (601,328), S_{21} (601,390), S_{22} (601,398), S_{23} (601,600), S_{24} (601,601), S_{25} (601,628), S_{26} (601,857), S_{27} (601,919), S_{28} (601,939), S_{29} (601,988), S_{30} (601,998). The triangle uncertain distributions, $\xi_{it} = (a_{it}, \alpha_{it}, \beta_{it})$, of the return rates of assets at each period can be obtained as shown in Tables 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14.

Table 5 The uncertain return rates on assets of five periods investment

	Asset t								
	Asset 1			Asset 2			Asset 3		
1	0.1430	0.1049	0.1156	0.0750	0.0657	0.1664	0.1083	0.0832	0.060
2	0.1449	0.0881	0.1136	0.0813	0.0708	0.1600	0.1085	0.0681	0.0603
3	0.1458	0.0800	0.1127	0.0857	0.0666	0.1556	0.1139	0.0725	0.0548
4	0.1516	0.0620	0.1070	0.0930	0.0579	0.1483	0.1152	0.0560	0.0540
5	0.1532	0.0609	0.1054	0.1053	0.0662	0.1359	0.1172	0.0570	0.0516

Table 6 The uncertain return rates on assets of five periods investment

	Asset *t*								
	Asset 4			Asset 5			Asset 6		
1	0.1172	0.0731	0.0813	0.0801	0.0791	0.0616	0.1064	0.1635	0.0616
2	0.1203	0.0743	0.0782	0.0847	0.0772	0.0571	0.1073	0.1634	0.0608
3	0.1255	0.0749	0.0730	0.0900	0.0503	0.0517	0.1083	0.1093	0.0598
4	0.1274	0.0733	0.0710	0.0906	0.0507	0.0512	0.1091	0.0763	0.0590
5	0.1289	0.0633	0.0700	0.0926	0.0495	0.0492	0.1129	0.0727	0.0551

Table 7 The uncertain return rates on assets of five periods investment

	Asset *t*								
	Asset 7			Asset 8			Asset 9		
1	0.0798	0.0562	0.1694	0.1238	0.0815	0.1023	0.0639	0.1522	0.0951
2	0.0907	0.0643	0.1750	0.1259	0.0760	0.1003	0.0790	0.0673	0.0866
3	0.0992	0.0555	0.1500	0.1277	0.0765	0.0985	0.0818	0.0606	0.0839
4	0.1029	0.0551	0.1462	0.1383	0.0538	0.0878	0.0861	0.0645	0.0800
5	0.1069	0.0534	0.1423	0.1457	0.0612	0.0805	0.0884	0.0650	0.0773

Table 8 The uncertain return rates on assets of five periods investment

	Asset *t*								
	Asset 10			Asset 11			Asset 12		
1	0.0377	0.0325	0.0414	0.0575	0.0403	0.1282	0.1243	0.1170	0.1840
2	0.0410	0.0292	0.0379	0.0592	0.0403	0.1264	0.1303	0.1007	0.1781
3	0.0469	0.0318	0.0321	0.0669	0.0374	0.1188	0.1380	0.0827	0.1704
4	0.0480	0.0314	0.0309	0.0724	0.0400	0.1133	0.1491	0.0843	0.1593
5	0.0492	0.0318	0.0298	0.0741	0.0453	0.1116	0.1540	0.0752	0.1544

Table 9 The uncertain return rates on assets of five periods investment

	Asset *t*								
	Asset 13			Asset 14			Asset 15		
1	0.2049	0.1244	0.1331	0.0254	0.1000	0.0743	0.0893	0.2079	0.1463
2	0.2102	0.1182	0.1277	0.0667	0.0604	0.0443	0.1518	0.1015	0.0859
3	0.2194	0.1236	0.1186	0.0700	0.0563	0.0411	0.1538	0.1033	0.0840
4	0.2225	0.1248	0.1154	0.0716	0.0500	0.0395	0.1565	0.0534	0.0812
5	0.2238	0.1029	0.1142	0.0731	0.0399	0.0379	0.1600	0.0553	0.0778

Table 10 The uncertain return rates on assets of five periods investment

	Asset *t*								
	Asset 16			Asset 17			Asset 18		
1	0.0615	0.0622	0.4819	0.1665	0.1188	0.0375	0.0675	0.1157	0.0586
2	0.0625	0.2795	0.2306	0.1550	0.0984	0.0772	0.1183	0.1137	0.4232
3	0.0656	0.0514	0.2276	0.1553	0.0893	0.0769	0.1349	0.1165	0.4066
4	0.0747	0.0460	0.2185	0.1575	0.0844	0.0747	0.1467	0.1225	0.3947
5	0.0835	0.0506	0.2096	0.1579	0.0535	0.0744	0.1664	0.1122	0.3750

Table 11 The uncertain return rates on assets of five periods investment

	Asset t								
	Asset 19			Asset 20			Asset 21		
1	0.2500	0.0736	0.0896	0.0825	0.1559	0.0853	0.0536	0.0817	0.0403
2	0.0916	0.0716	0.0634	0.1217	0.0633	0.0619	0.0704	0.0544	0.1220
3	0.0928	0.0708	0.0622	0.1218	0.0620	0.0618	0.0838	0.0666	0.1087
4	0.0940	0.0500	0.0610	0.1243	0.0516	0.0593	0.0880	0.0681	0.1045
5	0.0952	0.0472	0.0600	0.1269	0.0267	0.0568	0.0917	0.0703	0.1007

Table 12 The uncertain return rates on assets of five periods investment

	Asset t								
	Asset 22			Asset 23			Asset 24		
1	0.1083	0.1197	0.0684	0.0413	0.0149	0.0749	0.1430	0.0458	0.0141
2	0.1170	0.0618	0.0739	0.0454	0.1137	0.2040	0.0947	0.0637	0.0393
3	0.1200	0.0623	0.0708	0.0531	0.1209	0.1963	0.0984	0.0628	0.0355
4	0.1235	0.0656	0.0674	0.0574	0.0930	0.1920	0.1005	0.0548	0.0334
5	0.1254	0.0488	0.0654	0.0718	0.0716	0.1777	0.1021	0.0493	0.0319

Table 13 The uncertain return rates on assets of five periods investment

	Asset t								
	Asset 25			Asset 26			Asset 27		
1	0.0589	0.1432	0.1024	0.0783	0.1712	0.1096	0.0690	0.0047	0.1634
2	0.1021	0.0652	0.0591	0.1276	0.0906	0.1263	0.0438	0.1623	0.1828
3	0.1037	0.0567	0.0574	0.1329	0.0949	0.1210	0.0506	0.1526	0.1760
4	0.1044	0.0314	0.0567	0.1432	0.0812	0.1105	0.0562	0.1232	0.1694
5	0.1090	0.0243	0.0521	0.1445	0.0777	0.1093	0.0619	0.0511	0.1647

Table 14 The uncertain return rates on assets of five periods investment

	Asset t								
	Asset 28			Asset 29			Asset 30		
1	0.1551	0.1128	0.0498	0.0994	0.1233	0.0677	0.0674	0.1355	0.0854
2	0.1382	0.0789	0.0969	0.1123	0.0641	0.0498	0.1037	0.0636	0.0438
3	0.1395	0.0679	0.0956	0.1134	0.0648	0.0488	0.1048	0.0645	0.0426
4	0.1426	0.0379	0.0924	0.1157	0.0416	0.0464	0.1060	0.0574	0.0414
5	0.1470	0.0364	0.0880	0.1175	0.0312	0.0446	0.1061	0.0350	0.0413

Appendix B

According Tables 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14, and Eq. (25), $\mathrm{AD}_t(R_{it})(i = 1, \ldots, 30; t = 1, \ldots, 5)$ can be obtained as shown in Tables 15, 16, 17 and Table 18.

Table 15 The uncertain absolute deviation of assets of five periods investment

i	Asset t							
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8
1	0.0551	0.0599	0.0232	0.0386	0.0179	0.0280	0.0588	0.0461
2	0.0506	0.0593	0.0268	0.0381	0.0183	0.0282	0.0620	0.0443
3	0.0504	0.0571	0.0261	0.0331	0.0255	0.0263	0.0532	0.0439
4	0.0428	0.0533	0.0333	0.0340	0.0255	0.0245	0.0521	0.0358
5	0.0422	0.0516	0.0330	0.0395	0.0259	0.0258	0.0507	0.0356

Table 16 The credibilistic absolute deviation of assets of five periods investment

i	Asset t							
	Asset 9	Asset 10	Asset 11	Asset 12	Asset 13	Asset 14	Asset 15	Asset 16
1	0.0374	0.0185	0.0440	0.0760	0.0563	0.0228	0.0573	0.0894
2	0.0386	0.0168	0.0435	0.0708	0.0615	0.0143	0.0360	0.0765
3	0.0363	0.0160	0.0408	0.0647	0.0589	0.0150	0.0357	0.0740
4	0.0362	0.0156	0.0398	0.0620	0.0588	0.0162	0.0399	0.0704
5	0.0356	0.0123	0.0405	0.0587	0.0543	0.0195	0.0335	0.0688

Table 17 The credibilistic absolute deviation of assets of five periods investment

i	Asset t							
	Asset 17	Asset 18	Asset 19	Asset 20	Asset21	Asset 22	Asset 23	Asset 24
1	0.0285	0.0666	0.0409	0.0356	0.0119	0.0229	0.0240	0.0230
2	0.0361	0.0592	0.0338	0.0342	0.0453	0.0339	0.0319	0.0197
3	0.0393	0.0608	0.0217	0.0349	0.0443	0.0333	0.0311	0.0203
4	0.0417	0.0801	0.0278	0.0278	0.0435	0.0333	0.0346	0.0229
5	0.0322	0.0735	0.0269	0.0214	0.0430	0.0287	0.0373	0.0255

Table 18 The credibilistic absolute deviation of assets of five periods investment

i	Asset t					
	Asset 25	Asset 26	Asset 27	Asset 28	Asset 29	Asset 30
1	0.0240	0.0456	0.0568	0.0434	0.0279	0.0379
2	0.0254	0.0545	0.0638	0.0441	0.0273	0.0233
3	0.0285	0.0542	0.0599	0.0411	0.0272	0.0271
4	0.0224	0.0482	0.0581	0.0426	0.0288	0.0257
5	0.0196	0.0470	0.0564	0.0411	0.0292	0.0191

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