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FOUNDATIONS

A class of belief structures based on possibility measures

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Abstract
After discussing the basics of belief structures we introduce a new class of belief structures in which we select from among the focal elements using a possibility measure. We refer to this as a maxitive belief structure, MBS. The concepts of belief and plausibility are defined for an MBS, and it is noted how an MBS can be used to model imprecise possibility distributions. We describe various operations with these structures including arithmetic and fusion. We look at the use of the Choquet integral type aggregation for these MBS. Measures other than belief and plausibility were defined for these structures.

Keywords Uncertainty modeling · Imprecision · Theory of evidence · Imprecise possibility

1 Introduction

The classic Dempster–Shafer belief structure can be viewed as a bi-level model for representing the uncertainty associated with a variable that takes its value in the space X (Dempster 1966, 1967; Shafer 1976; Yager 1987; Smets 1988; Yager et al. 1994; Dempster 2008; Liu and Yager 2008; Yager and Liu 2008). At the first level a subset of X is randomly selected from a collection of subsets of X called the focal elements. This makes use of a probability distribution over the focal elements. Once having randomly selected this subset, in the second step an object is chosen from this subset in some unknown, indeterminate, manner. This chosen object is the value of variable of interest. This indeterminism in this second step leads to a model of uncertainty that mixes both randomness and imprecision. One use of this structure has been for modeling imprecise probabilistic information (Caselton and Luo 1992). Here we introduce a variation of the classic belief structure in which in the first step instead of selecting from among the focal elements using a probability distribution we use a possibility, maxitive, distribution. We refer to this new structure as a maxitive belief structure, MBS. We look at various properties and features of the new structure. We note that one use of the MBS is for modeling imprecise possibilistic information (Tsiporkova and Baets 1998; Dubois et al. 2013). An important real-world application of the framework developed here is in multi-criteria decision-making under uncertainty (Dammak et al. 2016). In many situations rather than precisely knowing the degree of satisfaction of a criterion by an alternative we are only able to specify this satisfaction with some uncertainty that is best modeled by an MBS.

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2 Basic belief structure

Assume $X = \{x_i \text{ for } i = 1 \text{ to } n\}$ are a set of elements. Consider a structure m defined on X consisting a collection of non-empty subsets of X, F_j for $j = 1$ to q, called the focal elements and a group of associated weights $m(F_j) = \alpha_j \in [0, 1]$ such that $\sum_{j=1}^{q} \alpha_j = 1$. This is used as the basis of model of uncertainty called a Dempster–Shafer belief structure (Dempster 1966, 1967; Shafer 1976; Dempster 2008; Liu and Yager 2008; Yager and Liu 2008). Let V be a variable whose value is determined from the set of focal elements $\mathbf{F} = \{F_1, \ldots, F_q\}$ by a probabilistic experiment where α_j is the probability that F_j will be the outcome. In addition let U be a related variable taking its value in the space X. If $V = F_{j*}$, then U is selected from F_{j*} in some indeterminate manner, that is all we know is that the value of U is some element in F_{j*}. Here we see that Poss($U = x_k$ for $x_k \in F_{j*}$) = 1.

One interpretation of the D–S belief structure is as an imprecise probability distribution on X associated with the variable U. Here instead of precisely assigning probabilities to the elements in X we indicate that an amount of probability, α_j, is allocated in some unspecified manner to the elements in F_j.

If m is a belief structure so that two of the focal elements are same $F_{k_1} = F_{k_2}$ with weights α_{k_1} and α_{k_2}, we can replace these by one focal element $F_k = F_{k_1} = F_{k_2}$ with weight $\alpha_k = \alpha_{k_1} + \alpha_{k_2}$. We shall refer to this as the compression principle.

Two concepts closely associated with a D–S structure are the measures of plausibility, denoted Pl, and belief, denoted Bel, which are set mappings from the space X into the unit interval (Shafer 1976). We recall Pl: $2^X \rightarrow [0, 1]$ and Bel: $2^X \rightarrow [0, 1]$ are such that for any subset A of X $\text{Pl}_m(A) = \sum_{j, F_j \cap A \neq \varnothing} \alpha_j$ and $\text{Bel}_m(A) = \sum_{j,\, F_j \subseteq A} \alpha_j$. We observe that for any subset $A \subseteq X$, $\text{Bel}_m(A) \leq \text{Pl}_m(A)$.

It can easily be shown that both Pl and Bel are monotonic set measures on the space X (Klir 2006):

(1) $\text{Pl}(\varnothing) = \text{Bel}(\varnothing) = 0$
(2) $\text{Pl}(X) = \text{Bel}(X) = 1$
(3) If $A \subseteq B \subseteq X$, then $\text{Pl}(A) \leq \text{Pl}(B)$ and $\text{Bel}(A) \leq \text{Bel}(A)$

We also note that Bel and Pl are duals, for any subset $A \subseteq X$, $\text{Bel}(A) = 1 - \text{Pl}(\bar{A})$.

Under the interpretation of a belief structure as an imprecise probability distribution associated with U it is well known that $\text{Pl}(A)$ is the upper bound on the probability of A and $\text{Bel}(A)$ is the lower bound on the probability of A. In this spirit $\text{Prob}(A) \in [\text{Bel}(A), \text{Pl}(A)]$. An alternate view is that $\text{Prob}(A)$ can be seen as an imprecise probability with $\text{Prob}_m(A) = [\text{Bel}_m(A), \text{Pl}_m(A)]$.

Another measure that can be associated with a belief structure is called a Pignistic measure (Smets 1992; Smets and Kennes 1994) which is defined such that $\text{Pig}_m(A) = \sum_{j=1}^{q} \frac{\text{Card}(F_j \cap A) m(F_j)}{\text{Card}(F_j)}$. We note that for any A

$$\text{Pl}_m(A) \geq \text{Pig}_m(A) \geq \text{Bel}_m(A)$$

The pignistic measure is actually a probability distribution

$$p_k = \text{Pig}_m(\{x_k\}) = \sum_{j=1}^{q} \frac{\text{Card}(F_j \cap \{x_k\}) m(F_j)}{\text{Card}(F_j)}$$

and for any subset A of X $\text{Pig}_m(A) = \sum_{x_k \in A} Pig_m(\{x_k\})$.

The concepts of possibility and certainty introduced by Zadeh (1978, 1979) are useful concepts in discussing the measures of plausibility or belief. Assume A and B are two crisp subsets X, then the possibility of A given B is defined as

$$\text{Poss}(A/B) = 1 \text{ if } A \cap B \neq \varnothing$$
$$\text{Poss}(A/B) = 0 \text{ if } A \cap B = \varnothing$$

and the certainty of A given B is defined as

$$\text{Cert}(A/B) = 1 \text{ if } B \subseteq A$$
$$\text{Cert}(A/B) = 0 \text{ if } B \not\subset A$$

Using these concepts we see that $\text{Pl}_m(A) = \sum_{j=1}^{q} \text{Poss}(A/F_j] m(F_j)$ and $\text{Bel}_m(A) = \sum_{j=1}^{q} \text{Cert}(A/F_j) m(F_j)$. Thus, the plausibility of A is the expected possibility of A over the focal elements and the belief of A is the expected certainty of A over the focal elements. We note that $\text{Certainty}(A/B) = 1 - \text{Poss}(\bar{A}/B)$.

3 Maxitive/possibilistic belief structures

Here we shall introduce a variation of the classic D–S belief structure also defined on the space $X = \{x_i \text{ for } i = 1 \text{ to } n\}$. Here again we have a collection of non-empty subsets of X, F_j for $j = 1$ to q, called focal elements and a collection of associated weights $\pi_j \in [0, 1]$; however, we require $\text{Max}_j[\pi_j] = 1$. Here again the variable V is a value from the space $\mathbf{F} = \{F_1, \ldots, F_q\}$; however, here it is determined via a possibility measure λ (Klir 2006; Klir and Wierman 1999; Wang and Klir 2009) on $\mathbf{F}$ such that $\lambda(\{F_j\}) = \pi_j$ and for any subset "B" of $\mathbf{F}$ we have $\lambda(\text{“}B\text{”}) = \text{Max}_{j, F_j \in \text{“}B\text{”}}[\pi_j]$.

Here again we let U be a related variable taking its value in the space X. As in the preceding if $V = F_{j*}$ then U is selected from F_{j*} in some unspecified manner, that is all we know is that U is some element in F_{j*}. We shall denote this structure as g and refer to it as a maxitive Dempster–Shafer belief structure. As we shall subsequently see one interpretation of g is as an imprecise possibility distribution.

Here we note that one conceivable source of the possibility distribution λ on V is a fuzzy subset D on the set $\mathbf{F}$ describing our knowledge of the value of the variable V. In this perspective, as noted by Zadeh (1978), the value π_j associated with the possibility measure λ would be equal to the membership grade of the focal element F_j in D, $D(F_j)$.

We note that a corresponding compression principle exists for these maxitive belief structures. If $\mathbf{F}$ has two focal elements that are same, $F_{k_1} = F_{k_2}$ with weights π_{k_1} and π_{k_2} we can replace these by one focal element $F_k = F_{k_1} = F_{k_2}$ with weight $\pi_k = \text{Max}[\pi_{k_1}, \pi_{k_2}]$.

In this framework we can define the concepts of plausibility and belief associated with a maxitive belief structure g. Assume E is any subset of X, then $\text{Pl}_g(E) = \text{Max}_{j, F_j \cap E \neq \varnothing}[\pi_j]$ and $\text{Bel}_g(E) = \text{Max}_{j, F_j \subseteq E}[\pi_j]$. We note that if "$A$" is the subset of $\mathbf{F}$ consisting of those focal elements for which $E \cap F_j \neq \varnothing$, then $\text{Pl}_g(E) = \lambda(\text{“}A\text{”})$. Similarly if

"B" is the subset of $\mathbf{F}$ consisting of those focal elements for which $F_j \subseteq$ E, then $\mathrm{Bel}_g(E) = \lambda(\text{"}B\text{"})$.

Under the interpretation of a maxitive belief structure g as an imprecise possibility distribution on X we see that $\mathrm{Pl}_g(E)$ is the upper bound on the possibility of E and $\mathrm{Bel}_g(E)$ is the lower bound on the possibility of E. Here then $\mathrm{Poss}_g(E) \in [\mathrm{Bel}_g(E),\ \mathrm{Pl}_g(E)]$. In an alternative view $\mathrm{Poss}_g(E)$ can be seen as an imprecise possibility distribution. $\mathrm{Poss}_g[E] = [\mathrm{Bel}_g[E), \mathrm{Pl}_g[E)]$

Example Assume $X = \{x_1, x_2, x_3, x_4, x_5\}$. Let g be a maxitive D–S belief structure with focal elements: $F_1 = \{x_1, x_3, x_5\}$, $F_2 = \{x_2, x_4\}$, $F_3 = \{x_1, x_4\}$, F_4=$\{x_2, x_3, x_4\}$. Here $\mathbf{F}$=$\{F_1, F_2, F_3, F_4\}$.

Assume λ is a maxitive/possibility measure on $\mathbf{F}$ such that:

$$\pi_1 = \lambda(\{F_1\}) = 0.6, \pi_2 = \lambda(\{F_2\}) = 1,$$
$$\pi_3 = \lambda(\{F_3\}) = 0.5, \pi_4 = \lambda(\{F_4\}) = 0.8$$

Since λ is a maxitive measure, then $\lambda(\text{"}A\text{"}) = \mathrm{Max}_{F_j \in \text{"}A\text{"}} [\pi_j]$ and hence

$$\lambda(\{F_1, F_2\}) = 1, \lambda(\{F_1, F_3\}) = 0.6, \lambda(\{F_1, F_4\}) = 0.8,$$
$$\lambda(\{F_2, F_3\}) = 1, \lambda(\{F_2, F_4\}) = 1, \lambda(\{F_3, F_4\}) = 0.8,$$
$$\lambda(\{F_1, F_2, F_3\}) = 1, \lambda(\{F_1, F_2, F_4\}) = 1, \lambda(\{F_1, F_3, F_4\}) = 0.8$$
$$\lambda(\{F_2, F_3, F_4\}) = 1, \lambda(\{F_1, F_2, F_3, F_4\}) = 1.$$

Assume $E_1 = \{x_2\}$, then

$\mathrm{Poss}(E_1/F_1) = 1$	$\mathrm{Cert}(E_1/F_1) = 0$
$\mathrm{Poss}(E_1/F_2) = 0$	$\mathrm{Cert}(E_1/F_2) = 0$
$\mathrm{Poss}(E_1/F_3) = 1$	$\mathrm{Cert}(E_1/F_3) = 0$
$\mathrm{Poss}(E_1/F_4) = 0$	$\mathrm{Cert}(E_1/F_4) = 0$

From this we get $\mathrm{Pl}_g(E_1) = \lambda(\{F_1, F_3)) = 0.6$ and $\mathrm{Bel}_g(E_1) = \lambda(\varnothing) = 0$

Assume $E_2 = \{x_1, x_4\}$, then

$\mathrm{Poss}(E_2/F_1) = 1$	$\mathrm{Cert}(E_2/F_1) = 0$
$\mathrm{Poss}(E_3/F_2) = 1$	$\mathrm{Cert}(E_2/F_2) = 0$
$\mathrm{Poss}(E_2/F_3) = 1$	$\mathrm{Cert}(E_2/F_3) = 1$
$\mathrm{Poss}(E_2/F_4) = 1$	$\mathrm{Cert}(E_2/F_4) = 0$

From this we get $\mathrm{Pl}_g(E_2) = \lambda(\mathbf{F}) = 1$ and $\mathrm{Bel}_g(E_2) = \lambda(\{F_3\}) = 0.5$

4 Arithmetic and other operations on maxitive belief structures

Assume V_1 and V_2 are variables whose values lie in the set R of real numbers. Let us assume our knowledge of the values of each of these variables is expressed in terms of maxitive D–S structures. Assume $V = V_1 + V_2$ where $+$ is the arithmetic addition operation. Here we provide an approach for obtaining the maxitive D–S structure for V.

As background we note that as discussed in Moore (1966) if A and B are two discrete sets defined on the real line R, then $C = A + B$ is a subset of R where $C = \{x + y|$ for all $x \in A$ and $y \in B\}$.

Example If $A = \{10, 15, 20\}$ and $B = \{2, 4\}$, then $A + B = \{12, 14, 17, 19, 22, 24\}$

Using this we can extend the idea of addition to two maxitive belief structures. Assume the knowledge V_1 and V_2 are expressed via maxitive belief structures g_1 and g_2. Let g_1 be a maxitive belief structure with focal elements A_i for $i = 1$ to q and possibility measure λ_1. Let g_2 also be a maxitive belief structure with focal elements B_j for $j = 1$ to r and possibility measure λ_2. Here $V = V_1 + V_2$ is a maxitive belief structure g with focal elements, $F_{ij} = A_i + B_j$, for $i = 1$ to q and for $j = 1$ to r and associated possibility measure λ so that $\lambda(F_{ij}) = \mathrm{Min}[\lambda_1(A_i), \lambda_2(B_j)]$.

At times we shall find it more convenient to represent this addition operator directly in terms of the corresponding addition of maxitive belief structures, $g = g_1 + g_1$. Also at times to simplify the notation we shall use the notation $g(F_i)$ for $\lambda(\{F_i\})$ and $g(F_1, F_2, F_3)$ for $\lambda(\{F_1, F_2, F_3\})$ and $g(\text{"}B\text{"}) = \mathrm{Max}_{F_i \in \text{"}B\text{"}}[g(F_i))$

We note that the operation $\lambda(\{F_{ij}\}) = \mathrm{Min}[\lambda_1(\{A_i\}), \lambda_2(\{B_j\})]$ always results in the appropriate type of weights for a possibility measure. In particular, all $\lambda(\{F_{ij}\}) \in [0, 1]$ and since there always exists a pair A_{i^*} and B_{j^*} so that $\lambda_1(\{A_{i^*}\}) = 1$ and $\lambda_2(\{B_{j^*}\}) = 1$, then there always exists one focal element $F_{i^*j^*}$ so $\lambda(\{F_{i^*j^*}\}) = 1$. Since $F_{i^*j^*} = A_{i^*} + B_{j^*}$, then $F_{i^*j^*}$ is always non-null when A_{i^*} and B_{j^*} are non-null.

The following two properties can easily be shown to hold for the addition of maxitive belief structures:

(1) Symmetry: $g_1 + g_2 = g_2 + g_1$
(2) Associatively: $g_1 + g_2 + g_3 = (g_1 + g_2) + g_3 = g_1 + (g_2 + g_3)$

In cases when addition of maxitive belief structures leads to duplicate focal elements with different weights the compression principle can be used to eliminate the duplicates.

Example Let g_1 and g_2 be two maxitive belief structures on R. Assume that g_1 has focal elements

$$A_1 = \{1, 2, 3\}$$
$$A_2 = \{3, 4, 5\}$$
$$A_3 = \{5, 6\}$$

and possibility measure λ_1 is such $g_1(A_1) = 0.7$, $g_1(A_2) = 0.8$, and $g_1(A_3) = 1$.

In addition g_2 has focal elements $B_1 = \{10, 20\}$ and $B_2 = \{25, 35\}$ and possibility measure λ_2 with $\lambda_2(B_1) = 1$ and $\lambda_2(B_2) = 0.5$.

Here then $g = g_1 + g_2$ with focal elements

$$F_{11} = \{11, 12, 13, 21, 22, 23\}$$
$$F_{12} = \{26, 27, 28, 36, 37, 38\}$$
$$F_{21} = \{13, 14, 15, 23, 24, 25\}$$
$$F_{22} = \{28, 29, 30, 38, 39, 40\}$$
$$F_{31} = \{15, 16, 25, 26\}$$
$$F_{32} = \{30, 31, 40, 41\}$$

Further g is a maxitive measure with $g(F_{ij}) = g(A_i) \wedge g(B_j)$; hence, we have

$$g(F_{11}) = 0.7 \wedge 1 = 0.7$$
$$g(F_{12}) = 0.7 \wedge 0.5 = 0.5$$
$$g(F_{21}) = 0.8 \wedge 1 = 0.8$$
$$g(F_{22}) = 0.8 \wedge 0.5 = 0.5$$
$$g(F_{31}) = 1 \wedge 1 = 1$$
$$g(F_{32}) = 1 \wedge 0.5 = 0.5$$

The preceding approach can be extended to other binary arithmetic operations. Assume g_1 and g_2 are two maxitive belief structure with focal elements A_i, $i = 1$ to p and B_j, $j = 1$ to q, respectively. Let $\perp$ be any arithmetic operator: addition, subtraction, multiplication, division, exponential. If $g = g_1 \perp g_2$, then g is a maxitive belief structure with focal elements

$$F_{ij} = \{x \perp y | \text{ for all } x \in A_i \text{ with } y \in B_j\} \text{ and}$$
$$g(F_{ij}) = \text{Min}[g_1(A_i), g_2(B_j)) \quad \text{for all } A_i \text{ and } B_j.$$

We note here all F_{ij} are non-null and here g(F_{ij}) a valid possibility distribution.

If g_1 is a maxitive belief structure (MBS) with focal elements A_j for $j = 1$ to q all subsets of R. If b is a number, then $g = b\,g_1$ is an MBS with focal element $B_j = b\ A_j = \{bx | \text{ for all } x \in A_j\}$ and $g(B_j) = g_1(A_j)$ for $j = 1$ to q. Using this and our definition of addition of MBS we can easily obtain the weighted aggregation of MBS. Thus, if g_k for $k = 1$ to r are MBS with focal elements in R and if w_k are a set of weights such $w_k \in [0, 1]$ and $\sum_{r=1}^{r} w_k = 1$, then the weighted average of these r belief structures is $g = \sum_{k=1}^{r} w_k g_k$.

We now consider further operations with maxitive belief structures. Assume X_1, X_2 and X_3 are three not necessarily different sets. Let S be a set operator defined as

$$S : I^{x_1} \times I^{x_2} \to I^{x_3}$$

That is if A and B are subsets of X_1 and X_2, respectively, then $S(A, B) = C$ where C is a subset of X_3. Let g_1 and g_2 be two MBS on X_1 and X_2, respectively, with focal elements A_i for $i = 1$ to q and B_j for $j = 1$ to p. We are now interested in obtaining the MBS $g = S(g_1, g_2)$. Here we let $F_{ij} = S(A_i, B_j)$ which is a subset of X_3. We associate with this a set mapping $\hat{g}$ on X_3 so that $\hat{g}(F_{ij}) = g_1(A_i) \wedge g_2(B_j)$. We note that $\hat{g}$ is such that all $\hat{g}(F_{ij}) \in [0, 1]$ and there exists at least one $F_{i^* j^*}$ such that $\hat{g}(F_{i^* j^*}) = 1$. This occurs for any pair for where $g_1(A_{i^*}) = g_2(B_{j^*}) = 1$.

At this point we must distinguish between two classes of the operator S. We call S non-null forming if for any A and B that are not null $S(A, B) \neq \varnothing$. On the other hand we shall say that S is a null forming if for some $A \neq \varnothing$ and $B \neq \varnothing$ we can have $S(A, B) = \varnothing$. We note the union is an example of non-null forming S. Another example of non-null forming S is the Cartesian product. An example of a null forming operator is the intersection. We must use different procedures for obtaining g from $\hat{g}$ in these cases. If S is non-nulling forming, then g is $\hat{g}$; the focal elements of g are also the $F_{ij} = S(A_i, B_j)$ and $g(F_{ij}) = \hat{g}(F_{ij})$. We see in this case that g is a possibility measure, all $g(F_{ij}) \in [0, 1]$, and there is at least one F_{ij}, $g(F_{ij}) = \hat{g}(F_{ij}) = 1$. If S is null forming, we must use the following procedure for obtaining $g = S(g_1, g_2)$

(1) Focal elements of g are all the non-null F_{ij}, that is $\mathbf{F} = \{F_{ij} \text{ s.t. } F_{ij} \neq \varnothing\}$
(2) If there exists one $F_{ij} \neq \varnothing$ such that $\hat{g}(F_{ij}) = 1$, then g is an MBS with focal elements all $F_{ij} \neq \varnothing$ and for these focal elements $g(F_{ij}) = \hat{g}(F_{ij})$
(3) If there is no $F_{ij} \neq \varnothing$ such that $\hat{g}(F_{ij}) = 1$. We let $\alpha = \text{Max}[\hat{g}(F_{ij})]$ over those $F_{ij} \neq \varnothing$. Using this we define g having focal element all $F_{ij} \neq \varnothing$ and $g(F_{ij}) = \frac{\hat{g}(F_{ij})}{\alpha}$.

We note here that $g(F_{ij}) \in [0, 1]$ and there exists at least one F_{ij} with $g(F_{ij}) = 1$.

5 Fusion of multiple MBS

A fundamental issue in the classic Dempster–Shafer theory is the fusion of multiple structures. The predominant methodology to accomplish this is the Dempster's rule (Dempster 1966, 1967; Shafer 1976; Dempster 1968; Zadeh 1979; Fu and Yang 2011). Here we look at the issue of fusion multiple maxitive belief structure.

We note for the case of $S(g_1, g_2)$ where S is the intersection, then $S(g_1, g_2)$ as defined earlier becomes an extension of the Dempster's rule. More generally assume g_k for $k = 1$ to t are a collection of MBS based on possibility measures λ_k, respectively, with focal elements F_{kj} for $j = 1$ to n_k. In the following we shall denote $\lambda_k(F_{kj}) = \pi_{kj}$. We now can consider the fusion of these to obtain $g = S(g_1, \ldots, g_t)$. Here g is an MBS with focal elements E_r where each E_r is composed by forming an non-null intersection made of one focal element from each of the g_k; thus, each E_r is the intersection $\mathrm{E}_r = \bigcap_{k=1}^{t} F_{kj_k} \neq \varnothing$. Also for each E_r we obtain $\hat{g}(E_r) = \mathrm{Min}_{k=1 \text{ to } t}[\lambda_k(F_{kj_k})]$. If there exists at least one $E_r \neq \varnothing$ with $\hat{g}(E_r) = 1$, then $g(E_r) = \hat{g}(E_r)$. If there is no $E_r \neq \varnothing$ with $\hat{g}(E_r) = 1$, then we let $\alpha = \mathrm{Max}[\hat{g}(E_r)]$ for $E_r \neq \varnothing$. Using this we define $g(E_r) = \frac{\hat{g}(E_r)}{\alpha}$.

6 Choquet integral associated with an MBS

Let $F = \{F_1, \ldots, F_q\}$ be a collection of focal elements, non-empty subsets of X. Let r_j be some numeric value associated with each focal element. In many cases we are interested in some kind of average of these values. In the case when we have an MBS structure g based on the possibility measure λ with $\lambda(\{F_j\}) = \pi_j$ we often use the Choquet integral (Choquet 1953; Beliakov et al. 2007) to obtain the average; we shall denote this $\mathrm{Choq}_\lambda(r_j$ for $j = 1$ to $q)$. Assume ρ is an index function so that ρ(i) in the index of ith largest of the r_j; thus, $r_{\rho_{(i)}}$ is the ith largest of the r_j. Using this we get

$$\mathrm{Choq}_\lambda(r_j \text{ for } j = 1 \text{ to } q) = \sum\nolimits_{i=1}^{q} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”}))r_{\rho_{(i)}}$$

where “H_j” $= \{F\rho_{(1)}, \ldots, F\rho_{(i)})\}$ is the subset of focal elements with the i largest values of r_j. Denoting $w_i = \lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”})$ we see $\mathrm{Choq}_\lambda(r_j$ for $j = 1$ to $q) = \sum_{i=1}^{q} w_i r_{\rho(i)}$. It is easy to show that each $w_i \in [0, 1]$ and $\sum_{i=1}^{q} w_i = 1$. Thus, we see that the Choquet integral is essentially providing a weighted average of $r_{\rho_{(i)}}$. Here the weights are determined by the measure λ.

We further note that since λ is a possibility measure with $\lambda(\{F_j\}) = \pi_j$ and for a subset “B” of $\mathbf{F}$ we have $\lambda(\text{“}B\text{”}) = \mathrm{Max}_{F_j \in \text{“}B\text{”}}[\pi_j]$. In the special case of “H_i” we have $\lambda(\text{“}H_i\text{”}) = \mathrm{Max}_{k=1 \text{ to } i}[\pi_{\rho_{(k)}}]$. Thus, here $w_i = (\mathrm{Max}[\pi_{\rho_{(i)}}, \lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”})] = \mathrm{Max}[0, \pi\rho_{(i)} - \mathrm{Max}_{k=1 \text{ to } i-1}[\pi_{\rho_{(k)}}]]$. We observe that if $\pi_{\rho_{(i*)}} = 1$, then w_i for all i $> i^*$. Here we shall use the notational convention that in the case of an MBS g based on the possibility measure λ that $g(F_j) = \pi_j = \lambda(\{F_j\})$.

With a little bit of algebra we can express the Choquet integral as

$$\mathrm{Choq}_g(r_j \text{ for } j = 1 \text{ to } q) = \sum_{i=1}^{q} \lambda(\text{“}H_i\text{”})(r_{\rho_{(i)}} - r_{\rho_{(i+1)}})$$

In the case where λ is a possibility distribution, then

$$\mathrm{Choq}_q(r_j \text{ for } j = 1 \text{ to } q) = \sum_{j=1}^{q} \left(\mathop{\mathrm{Max}}_{k=1 \text{ to } i} [\pi\rho_{(k)}](r_{\rho_{(i)}} - r_{\rho_{(i+1)}}) \right)$$

We now show that the Choquet integral can provide an alternative and useful formula for the measure of plausibility and belief associated with g.

Theorem $Pl_g(E) = Choq_\lambda(Poss(E/F_j \; for \; j = 1 \; to \; q)$
$Bel_g(E) = Choq_\lambda(Cert(E/F_\lambda \; for \; j = 1 \; to \; q)$

Proof (1) Let ρ be an index function so that ρ(i) is the index of the ith largest $\mathrm{Poss}(E/F_j)$; thus, $\mathrm{Poss}(E/F\rho_{(j)})$ is the ith largest of the $\mathrm{Poss}(E/F_j)$. Here then

$$\mathrm{Choq}_\lambda(\mathrm{Poss}(E/F_j) \text{ for } j = 1 \text{ to } q) = \sum_{i=1}^{q} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”}))\mathrm{Poss}(E/F\rho_{(i)})$$

We note that if $E \cap F_j \neq \varnothing$, then $\mathrm{Poss}(E/F_j) = 1$ and if $E \cap F_j = \varnothing$, then $\mathrm{Poss}(E/F_j) = 0$. Assume j^* are the number of focal elements that intersect E. Using this we get

$$\mathrm{Choq}_\lambda(\mathrm{Poss}(E/F_j) j = 1 \text{ to } q) = \sum_{i=1}^{j^*} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”}))$$

since $\mathrm{Poss}(E/F\rho_{(j)}) = 0$ for those focal elements not intersecting E and $\mathrm{Poss}(E/F\rho_{(j)}) = 1$ for those focal elements intersecting E. We further see that

$$\sum_{i=1}^{j^*} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”})) = \lambda(\text{“}H_{j^*}\text{”}) - \lambda(\varnothing) = \lambda(H_{j^*})$$

However “H_{j^*}” $= \{F\rho_{(1)}, \ldots, F\rho_{(j*)}\}$ is the subset of focal elements that intersects E; thus, Choq_λ $(\mathrm{Poss}(E/F_j)$, $j = 1$ to $q) = \lambda(\text{“}A\text{”})$ where “A” is the subset of focal elements that intersects E, $\mathrm{Pl}_\lambda(E)$

(2) In a similar way we can show that $\mathrm{Choq}_\lambda(\mathrm{Cert}(E/F_\lambda$ for $j = 1$ to $q) = \lambda(\text{“B”}) = \mathrm{Bel}_\lambda(E)$ where “B” is the set of focal elements contained in E since $\mathrm{Cert}(E/F_j) = 1$ if $F_j \subseteq E$ and $\mathrm{Cert}(E/F_j) = 0$ otherwise.

A related integral is the Sugeno integral (Beliakov et al. 2007; Sugeno 1977; Klement et al. 2010). Here

$$\mathrm{Sug}_\lambda(r_j \text{ for } j = 1 \text{ to } q) = \mathop{\mathrm{Max}}_{i=1}^{q}[\lambda(H_i) \wedge r_{\rho_{(i)}}]$$

It is well known that the Sugeno integral is also a mean operator. In the special case where λ is a possibility measure, then

$$\text{Sug}_\lambda(r_j \text{ for } j = 1 \text{ to } q) = \underset{i=1 \text{ to } q}{\text{Max}}\left[\left(\underset{k=1 \text{ to } i}{\text{Max}}[\pi_{\rho(k)}] \wedge r_{\rho(i)}\right)\right]$$

Using this we can also show that

$$\text{Sug}_\lambda(\text{Poss}(E/F_j) \text{ for } j = 1 \text{ to } q) = \text{Pl}_\lambda(E)$$
$$\text{Sug}_\lambda(\text{Cert}(E/F_j) \text{ for } j = 1 \text{ to}) = \text{Bel}_\lambda(E)$$

□

7 Alternative measures associated with an MBS

In the preceding we showed that two measures associated with the variable U on the space X are the measures of plausibility and belief where for any subset E of X

$$\text{Pl}_g(E) = \text{Choq}_g(\text{Poss}(E/F_j) \text{ for } j = 1 \ldots q) \text{ and}$$
$$\text{Bel}_g(E) = \text{Choq}_g(\text{Cert}(E/F_j) \text{ for } j = 1 \text{ to } q).$$

Here with g an MBS based on λ having focal elements $\mathbf{F} = \{F_1, \ldots, F_q\}$ and associated with each F_j is a weight $g(F_j) = \pi_j$ we now provide a whole class of measures on X, in addition to plausibility and belief, that can be generated from g.

Let $\mathbf{W} = \langle W_1, \ldots, W_q\rangle$ be a collection of vectors called the allocation imperative. Each W_j is of dimension $n_j = |F_j|$, and the components of W_j are $w_j(k) \in [0, 1]$ with $\sum_{k=1}^{n_j} w_j(k) = 1$. If E is a subset of X, then for each focal element F_j let $z_j = \sum_{k=1}^{|F_j \cap E|} w_j(k)$. Consider the set function $\mu_{\mathbf{W},g}$ defined so that for any subset E of X

$$\mu_{W,g}(E) = \text{Choq}_g(z_j \text{ for } j = 1 \text{ to } q)$$
$$= \sum_{i=1}^{q} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”}))z_{\rho(i)}$$

where ρ(i) is the index of the ith largest z_j and “H_i” $= \{F\rho_{(1)}, \ldots, F\rho_{(i)}\}$. We now show that $\mu_{\mathbf{W},g}$ is a measure on X

(1) If $E = \varnothing$, then $F_j \cap E = \varnothing$ for j and here $z_j = 0$ for all $j = 1$ to q and hence $\mu_{W_1 g}(\varnothing) = 0$.
(2) If $E = X$, then $F_j \cap X = F_j$ and $|F_j| = n_j$ and hence $z_j = 1$ for j. In this case $\mu_{W_1 g}(X) = 1$
(3) If E_1 and E_2 are such that $E_1 \subseteq E_2$, then $(E_1 \cap F_j) \subseteq (E_2 \cap F_j)$ and $|(E_1 \cap F_j) \leq |(E_2 \cap F_j)|$ for all j. From this it follows $\sum_{k=1}^{|E_1 \cap F_j|} w_j(k) \leq \sum_{k=1}^{|E_2 \cap F_j|} w_j(k)$. From this it follows that if $E_1 \subseteq E_3$, then $\mu_{W,g}(E_1) \leq \mu_{W,g}(E_2)$. Thus, we see that $\mu_{W,g}$ is a measure on X.

Let us look at some special cases of **W**. First consider the case where **W** is such for all W_j, $w_j(1) = 1$. Here we see that if F_j is such that if $E \cap F_j \neq \varnothing$, then $|E \cap F_j| \geq 1$ and $z_j = 1$. While if $E \cap F_j = \varnothing$, then $|E \cap F_j| = 0$ and $z_j = 0$. Here we see for this **W** that z_j is essentially the $\text{Poss}(E/F_j)$. Thus, in this case

$$\mu_{\mathbf{W},g}(E) = \text{Choq}_g(\text{Poss}(E/F_j) \text{ for } j = 1 \text{ to } q) = \text{Pl}_g(E)$$

Thus, in this special case of $\mathbf{W}_j$ where $w_j(1) = 1$ for all j gives us the plausibility measure.

Consider now the special case where for all W_j we have $w_j(n_j) = 1$. Here we observe that if $F_j \subseteq E$, then $E \cap F_j = F_j$ and thus $|E \cap F_j| = n_j$ and $z_j = 1$. While if $F_j \not\subseteq$ E, then $|E \cap F_j| < n_j$ and $z_j = 0$. Thus, we see z_j is essentially the $\text{Cert}(E/F_j)$. Thus, in this case

$$\mu_{W,g}(E) = \text{Choq}_g(\text{Cert}(E/F_j) \text{ for } j = 1 \text{ to } q) = \text{Bel}_g(E)$$

We note that if E is a singleton, $E = \{x^*\}$, then $\text{Pl}_g(\{x^*\}) = \text{Max}_j[\pi_j]$ over all j so that $x^* \in F_j$. On the other hand in this case $\text{Bel}_g(\{x^*\}) = \text{Max}_j[\pi_j]$ over all j so that so that $F_j = \{x^*\}$

Another special case of interest is one in which each vector W_j is such that $w_j(k) = \frac{1}{n_j}$ for all $k = 1$ to n_j. Here we see that $z_j = \frac{|F_j \cap E|}{n_j}$, z_j is the proportion of elements in F_j that are in E. Let ρ be an index function on the focal elements so that $\rho(i)$ is the index of the focal element with the ith largest z_j. Thus, $z_{\rho(i)} = \frac{|F_{\rho(i)} \cap E|}{n_{\rho(i)}}$ and with “H_i”$= \{F\rho_{(1)}, \ldots, F\rho_{(i)}\}$ we have

$$\mu_{W,g}(E) = \sum_{i=1}^{q} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”}))z_{\rho(i)}$$
$$= \sum_{i=1}^{q}\left(\text{Max}\left[0, \left(\pi_{\rho(i)} - \underset{k=1 \text{ to } i-1}{\text{Max}}[\pi_{\rho(k)}]\right)\right]\right) \times \frac{|F_{\rho(i)} \cap E|}{n_{\rho(i)}}.$$

A standardization of the determination of the vectors W_j can be had using a methodology introduced by O'Hagan (1990). Here we provide a parameter $\beta \in [0, 1]$ called the attitudinal character or degree of optimism. The larger the β, the more optimistic the resulting measure. Here for each W_j we obtain its n_j components, the w_j(k), by solving the following optimization problem.

$$\text{Maximize: } -\sum_{k=1}^{n_j} w_j(k)\ln(w_j(k))$$
$$\text{subject to: } \sum_{k=1}^{n_j}\frac{n-k}{n-1}=\beta$$
$$\sum_{k=1}^{n_j} w_j(k)=1$$
$$w_j(k)\geq 0 \quad \text{for } k=1 \text{ to } n_j$$

We shall here refer to the measure obtained using this β across all W_j as $\mu_{\beta,g}$. We note that the following properties can be shown

(1) if $\beta=1$, then all $w_j(1)=1$ and $\mu_{\beta,g}(E)=\text{Pl}(E)$
(2) If $\beta=0$, then all $w_j(n_j)=1$ and $\mu_{\beta,g}(E)=\text{Bel}(E)$
(3) If $\beta=0.5$, then all $w_j(k)=\frac{1}{n_j}$
(4) If $\beta_1>\beta_2$, then $\mu_{\beta_1,g}(E)\geq(E)\mu_{\beta_2,g}(E)$ for all E

Another method for standardization of the W_j can be had using a weight generating function $f:[0,1]\rightarrow[0,1]$ such that (1) $f(0)=0$, (2) $f(1)=1$ and (3) $f(a)\geq f(b)$ if $a>b$ (Yager 2017). Using this function we can generate the OWA weights for each W_j, $w_j(k)$ for $k=1$ to n_j as (Yager 1996)

$$w_j(i)=f\left(\frac{1}{n_j}\right)-f\left(\frac{(i-1)}{n_j}\right).$$

We note that the function f^* such that $f^*(x)=1$ for all $x>0$ has $w_j(1)=1$ and generates the plausibility measure. The function f_* such that $f_*(x)=0$ for all $x<1$ has $w_j(n_j)=1$ and generates the belief measure. In addition the function $f(x)=x$ generates the weights $w_j(k)=\frac{1}{n_j}$ for all k. It can be shown that if f_1 and f_2 are two function such that $f_1(x)\geq f_2(x)$ for all x, then the associated generated measures, $\mu_{f_1,g}$ and $\mu_{f_2,g}$, are such that $\mu_{f_1,g}(E)\geq\mu_{f_2,g}(E)$ for all E. In addition we can associate with every weight generating function f a measure of optimism $\beta=\int_0^1 f(x)\mathrm{d}x$. We see here that if $f_1(x)\geq f_2(x)$ for all x, then $\beta_1=\int_0^1 f_1(x)\mathrm{d}x\geq\int_0^1 f_2(x)\mathrm{d}x\geq\beta_2$. Here also note that $\beta=\int_0^1 f^*(x)\mathrm{d}x=1$, $\beta=\int_0^1 f_*(x)\mathrm{d}x=0$ and $iff(x)=x$, then $\beta=0.5$

Consider a weight generating function f_1 and the function $f_2(x)=1-f_1(1-x)$. We see

(1) $f_2(1)=1-f_1(0)=1$
(2) $f_2(0)=1-f_1(1-0)=1-f_1(1)=0$
(3) If $x\geq y$, then $f_2(x)\geq f_2(y)$. We see this as follows. Since $f_2(x)=1-f_1(1-x)$ and $f_2(y)=1-f_1(1-y)$ when $x\geq y$, then $1-x\leq 1-y$ and hence $f_1(1-x)\leq f_1(1-y)$

but $f_2(x)=1-f_1(1-x)\geq 1-f_1(1-y)=f_2(y)$.

Thus, when f_1 is a weight generating function, then $f_2(x)=1-f_1(1-x)$ is also a weight generating function; in this case we shall refer to f_2 as the dual of f_1. If f_1 and f_2 are dual weight generating functions and $\mu_{f_1,g}$ and $\mu_{f_2,g}$ are the measures obtained from an MBS g using these functions, then it can be shown that $\mu_{f_1,g}$ and $\mu_{f_2,g}$ are duals (Yager 2017).

If f_1 and f_2 are dual weight generating functions, $f_2(x)=1-f_1(1-x)$, then $\beta_2=1-\beta_1$. If f_1 and f_2 are dual pairs of weight generating functions such that $f_1(x)\geq f_2(x)$ for all x, then (1) $\mu_{f_1,g}(E)\geq\mu_{f_2,g}(E)$ for all E and (2) $\beta_1\geq\beta_2$. Thus, in this case of $f_1(x)\geq f_2(x)$ a natural relationship exists between their respective measures. The prototypical example of this is f^* and f_* and their associated measures of plausibility and belief.

We shall find it convenient to assume the notation $\hat{f}$ for the dual of f. In addition we shall use the notation $\hat{\mu}_{f,g}$ for $\mu_{\widehat{f},g}$. If f and $\hat{f}$ are duals let $\Delta=|\beta-\hat{\beta}|$.

Consider the weight generation function $f(x)=x$. Here we see that $\hat{f}(x)=x$. Thus, $f(x)=x$ is self-dual, in this case $\mu_{f,g}(E)=\hat{\mu}_{f,g}(E)$. We also observe in this case $\hat{\beta}=\beta=0.5$. We see that Δ takes its maximum of one for f_* and f^* and its minimum value of zero for the self-dual case $f=x$.

A useful class of weight generating functions are $f(x)=x^t$ for $t\in(0,\infty)$. Here we note that for $t\rightarrow 0$ we get f^* and for $t\rightarrow\infty$ we get f_*. In addition for $t=1$ we get the function $f(x)=x$. Here we observe that $\beta=\int_0^1 x^t\mathrm{d}x=\frac{1}{t+1}$. We note here we have as dual $\hat{f}(x)=1-(1-x)^t$ and $\hat{\beta}=1-\frac{1}{t+1}=\frac{t}{t+1}$. In this case $\Delta=|\frac{t-1}{t+1}|$. Thus, if $t\leq 1$, then $\Delta=\frac{1-t}{t+1}$ and if $t\geq 1$, then $\Delta=\frac{t-1}{t+1}$.

8 Approximating belief plausibility range

Consider now an MBS g on X; as we noted, this can be used to model an imprecise possibility distribution. In this perspective we see that for any subset E of X we can say

$$\text{Bel}(E)\leq\text{Poss}(E)\leq\text{Pl}(E) \text{ or } \text{Poss}(E)\in[\text{Bel}(E),\text{Pl}(E))]$$

Consider any dual pair f and $\hat{f}$ such that $f(x)\geq\hat{f}(x)$ for all x in this case we have $\mu_{f,g}(E)\geq\hat{\mu}_{f,g}(E)$. Further since $f^*(x)\geq f(x)$ for all x, then $\text{Pl}_g(E)\geq\mu_{f,g}(E)$ or all E. In addition since $f(x)\geq f_*(x)$ for all x, then $\hat{\mu}_{f,g}(E)\geq\text{Bel}_g(E)$ for all E. Consider now the interval

$$\text{Range}_{f,g}(E)=[\hat{\mu}_{f,g}(E),\mu_{f,g}(E)]$$

we see that the $\text{Range}_{f,g}(E) \subseteq [\text{Bel}_g(E), \text{Pl}_g(E)]$. Here we see the $\text{Range}_{f,g}(E)$ provides a narrower interval for the value Poss(E). In some situations we may prefer to use a narrower interval for expressing the possibility of E. We should note that formally we cannot be certain that $\text{Poss}(E) \in \text{Rang}_{f,g}(E)$. Here $\text{Rang}_{f,g}(E)$ is only providing an approximation to the true interval value of $\text{Poss}(E)$. In this spirit we see if β is the degree of optimism associated with the function f, then $\Delta = |\beta - \hat{\beta})$ can provide some indication of confidence in using the range value $\text{Range}_{f,g}(E)$ as our approximation. Here we note that in the special case where f is self-dual, $\hat{f} = f$, then $\mu_{f,g}(E) = \mu_{\hat{f},g}(E)$ and we have no interval for Poss(E), here we have $\text{Poss}(E) = \mu_{f,g}(E)$.

We note that we can use the O'Hagan approach to obtain dual measures. Let $\mathbf{W} = \langle W_1, \ldots, W_q \rangle$ be a collection of OWA weight vectors obtained by using an optimism degree of $\beta \in [0, 1]$ in the algorithm used for calculating the $w_j(k)$. Let $\hat{W} = \langle \hat{W}_1, \ldots, \hat{W}_q \rangle$ be a collection of OWA weight vector such that W_j and $\hat{W}_j$ are duals. In this case we obtain the vector components $\hat{w}_j(k) = w_j(n - k + 1)$. From these then we calculate $z_j = \sum_{k=1}^{|F_j \cap E)} w_j(k)$ and $\hat{z}_j = \sum_{k=1}^{|F_j \cap E)} \hat{w}_j(k)$. Using these we calculate $\mu_{B,g}(E) = \sum_{i=1}^{q} (\lambda(\text{“}H_i\text{”}) - \lambda(\text{“}H_{i-1}\text{”})) z_{\rho_{(i)}}$ where ρ(i) is the index of ith largest z_j and “H_i” $= \{F\rho_{(1)}, \ldots, F\rho_{(i)}\}$. We also now calculate $\hat{\mu}_{B,g}(E) = \sum_{i=1}^{q} (\lambda_i(\text{“}\hat{H}_i\text{”}) - \lambda(\text{“}\hat{H}_{i-1}\text{”}))\hat{z}_{\hat{\rho}(i)}$ where $\hat{\rho}$(i) is the index of the ith largest $\hat{z}_j$ and “$\hat{H}_i$” $= \{F_{\hat{\rho}(1)}, \ldots, F_{\hat{\rho}(i)}\}$.

9 Random method of selecting object from chosen focal element

Here we shall consider a variation of the maxitive belief structure in which we use a different approach for selecting the element from the chosen focal element. Again we have a collection $\mathbf{F} = \{F_1, \ldots, F_q\}$ of focal elements and a maxitive measure λ for choosing the value V from $\mathbf{F}$. We shall again let $\lambda(F_j) = \pi_j$ where at least one of these has value one. Again we let U be a related variable taking its value in $X = \{x_1, \ldots, x_n\}$. Here, however, if $V = F_k$ we select the value U from F_k in the manner described below.

Associated with each x_i in X is a probability $\alpha_i \in [0, 1]$. We select the element x from F_k based on a random experiment where the probability of selecting x_i given $U = F_k$ is $p_{ik} = \frac{\alpha_i \text{Poss}[\{x_i\}/F_k)}{\sum_{i=1}^{n} \alpha_i \text{Poss}(\{x_i\}/F_k\})}$. We see p_{ik} is the normalized probability of selecting x_i from F_k.

Note: The α_i need not be a probability distribution but can be a finite nonnegative collection of weights.

In the following we find it convenient to use the notation, $F_k(x_i)$ for $\text{Poss}(\{x_i\}/F_k)$. It is essentially the membership grade of x_i in F_k. Using this we see that $p_{ik} = \frac{\alpha_i F_k(x_i)}{\sum_{i=1}^{n} \alpha_i F_k(x_i)}$.

Denoting $T_k = \sum_{i=1}^{n} \alpha_i F_k(x_i)$, the sum of the α_is of the elements in F_k, we see $p_{ik} = \frac{\alpha_i F_k(x_i)}{T_k}$. Please note that T_k is not the probability of selecting F_k, as F_k is chosen using the measure λ.

Our interest here is in determining the anticipation that $U \in E$ where E is some subset of X. We see that the probability of that $U \in E$ given F_j is $\text{Prob}(E/F_j) = \sum_{x_i \in E} p_{ij} = \frac{1}{T_j} \sum_{x_i \in E} \alpha_i F_j(x_i)$. To obtain the anticipation that $U \in E$, denoted Ant(E), we calculate the mean of the $\text{Prob}(E/F_j)$ with respect to the measure λ. We can obtain this value using the Choquet integral of the $\text{Prob}(E/F_j)$ with respect to the measure λ; using this we have

$$\begin{aligned}\text{Ant}(E) &= \text{Choq}_\lambda(\text{Prob}(E/F_j) \text{ for } j = 1 \text{ to } q)\\ &= \sum_{i=1}^{q} (\lambda(\text{“}H_i\text{”})) - \lambda(\text{“}H_{j(-1}\text{”})) \,\text{Prob}(E/F\rho_{(i)})\end{aligned}$$

where ρ is an index function on the focal elements so that ρ(i) is the index of the focal element with ith largest value for $\text{Prob}(E/F_j)$ and “H_i” $= \{F\rho_{(1)}, \ldots, F\rho_{(i)}\}$.

We note here that the value of Ant(E) is not an interval but a specific value. The reason for this is the value of U is selected from F_k in some well-defined manner.

We also note here that Ant is a measure on the space X since we can easily show that

1. $\text{Ant}(X) = 1$, 2. $\text{Ant}(\varnothing) = 0$ and 3. If $E_1 \subset E_2$, then $\text{Ant}(E_1) \leq \text{Ant}(E_2)$.

Let us look at Ant(E) for some notable cases of E. Consider the case where $E = \{x_1\}$, some arbitrary element in X, here $\text{Prob}(E/F_j) = \text{Prob}(\frac{(\{x_1\})}{F_j}) = \frac{\alpha_1}{T_j} F_j(x_1) = p_{1j}$ and

$$\begin{aligned}\text{Ant}(\{x_1\}) &= \text{Choq}_\lambda(p_{1j}, \text{ for } j = 1 \text{ to } q)\\ &= \text{Choq}_\lambda\left(\frac{\alpha_1}{T_j} F_j(x_1) \text{ for } j = 1 \text{ to } q\right).\end{aligned}$$

Let id_1 be an index function so that id_1(i) is the index of the ith largest p_{1j} and let $H_{1i} = \{F_{id_1(1)}, \ldots, F_{id_1(i)}\}$. Using this we have

$$\begin{aligned}\text{Ant}(\{x_1\}) &= \text{Choq}_\lambda(p_{1j}, \text{ for } j = 1 \text{ to } q)\\ &= \sum_{i=1}^{n} (\lambda(H_{1i})) - \lambda(H_{1(i-1)}))p_{1\text{id}_1(i)}\})\\ \text{Ant}(\{x_1\}) &= \sum_{i=1}^{n} (\lambda(H_{1i}) - \lambda(H_{1i-1}))\frac{\alpha_1 F_{id_1(i)}(x_1)}{T_{id_1(i)}}\end{aligned}$$

Assume n_1 is the number of focal elements containing x_1. We see that $F_{id_1(i)}(\text{i})(x_1) = 0$ for $i > n_1$ and $F_{id_2(i)}(x_1) = 1$

for $i \leq n_1$, from this we have $\text{Ant}(\{x_1\}) = \sum_{i=1}^{n_1} (\lambda(H_{1i}) - \lambda(H_{1i-1})) \frac{\alpha_1}{\hat{p}_{id_1(i)}}$.

For the case $E = \{x_2\}$ we get $\text{Ant}(\{x_2\}) = \sum_{i=1}^{n_2} (\lambda(H_{2i}) - \lambda(H_{2i-1})) \frac{\alpha_2 F_{id_1(i)}(x_2)}{T_{id_1(i)}}$

Let us consider the case with $E = \{x_1, x_2\}$. Here $\text{Prob}(\{x_1, x_2\}/F_j) = p_{1j} + p_{2j} = \frac{\alpha_1 F_j(x_1) + \alpha_2 F_j(x_2)}{T_j}$ and hence $\text{Ant}(\{x_1, x_2\}\} = \text{Choq}_\lambda(\frac{\alpha_1 F_j(x_1) + \alpha_2 F_j(x_2)}{T_j}$ for $j = 1$ to $q)$

A natural question is that is there any simple relationship between the values $\text{Ant}(\{x_1\})$ and $\text{Ant}(\{x_2\})$ and the value $\text{Ant}(\{x_1, x_2\})$. Unfortunately, the complexity of the Choquet integral with respect to possibility measure λ precludes the uncovering of any natural relationship. In particular what is clear is that in general $\text{Ant}(\{x_1\}) + \text{Ant}(\{x_2\}) \neq \text{Ant}(\{x_1, x_2\})$. Thus, while Ant is a measure it is not a probability measure.

10 Conclusion

We introduced a new class of belief structures in which we select from among the focal elements using a possibility measure instead of a probability measure. We referred to this as a maxitive belief structure, MBS. The concepts of belief and plausibility were defined for an MBS, and it was noted how an MBS can be used to model imprecise possibility distributions. We described various operations that can be performed with these structures including arithmetic and fusion. We looked at the use of the Choquet integral type aggregation for these MBS. Measures other than belief and plausibility were defined for these structures.

Compliance with ethical standards

Conflicts of interest The author declare that he has no conflict of interest.

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