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METHODOLOGIES AND APPLICATION

Rough fuzzy bipolar soft sets and application in decision-making problems

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Abstract The rough set theory is a successful tool to study the vagueness in data, while the fuzzy bipolar soft sets have ability to handle the uncertainty, as well as bipolarity of the information in many situations. We connect the Pawlak's rough sets with the fuzzy bipolar soft sets and introduce the concept of rough fuzzy bipolar soft sets. We also examine some structural properties of rough fuzzy bipolar soft sets and study the effects of the equivalence relation in Pawlak approximation space on the roughness of the fuzzy bipolar soft sets. We also discuss some similarity relations among the fuzzy bipolar soft sets, based on their roughness. At the end, an application of the rough fuzzy bipolar soft sets in a decision-making problem is discussed and an algorithm for this application is proposed, supported by an example.

Keywords Rough sets · Approximation space · Fuzzy sets · Bipolar information · Fuzzy bipolar soft sets

1 Introduction

In modern society, many concepts in engineering, economics, environmental science, social science, medical science and many other fields have vagueness and uncertainty in the data collected and studied for several purposes. This vagueness as well as volume and complexity in such data is increasing rapidly. In classical mathematics, all the mathematical notions must be precise and exact. So, it is not always a successful tool for dealing with the problems having uncertainties. Over the years, many researchers and scientists were trying to find out some suitable tools to deal with these uncertainties. They created many tools for this purpose. The most successful of all those are the fuzzy set theory (Zadeh 1965), rough set theory (Pawlak 1982, 1991) and the soft set theory (Molodtsov 1999). These theories reduced the gap between the traditional mathematical designs and the vague real-world data.

The fuzzy set theory (Zadeh 1965) has become a vigorous area of research in many sciences including computer sciences (Arva and Csukas 1988; Frank and Seliger 1997), automata theory (Doostfatemeh and Kremer 2005; Li and Wang 2014), decision-making theory (Roy and Maji 2007; Cagman et al. 2010; Feng et al. 2010b), medical sciences (Kovalerchukab et al. 1997; Phuong and Kreinovich 2011), management sciences (Guiffrida and Nagi 1998), engineering (Dubois and Prade 1993), graph theory (Swaminathan 2012; Akram et al. 2017). The rough set theory (Pawlak 1982, 1991; Pawlak and Skowron 2007) successfully provided a systematic scheme to handle the imprecision and uncertainty in the data. Pawlak used the upper and lower approximations of a collection of objects to investigate how close the objects are to the information attached to them. This theory has many valuable applications. Hence, this theory attracted many scientists and researchers and initiated research in many directions. Dubois and Prade (1990) introduced the notions of rough fuzzy sets and fuzzy rough sets. Yao (2010) discussed the three-way decisions with probabilistic rough sets.

Molodtsov (1999) initiated the novel concept of soft sets, a new mathematical tool to deal with imprecisions and uncertainties. The parameters of information play a vital role while

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scrutinizing and analyzing a data or taking a decision. The theory of soft sets has been evidenced to be an adequate parameterization tool. Hence, this theory magically overcame many difficulties raised while using the old theories. Due to its diverse applications, this theory received much attention of researchers and scientists. A rapid growth in the research on soft sets can be seen in the last few years. Maji et al. (2002) defined some basic operations on soft sets. Maji et al. (2003) applied the soft set theory to a decision-making problem. Aktas and Cagman (2007) related the soft sets to the rough sets and the fuzzy sets. They also introduced the soft groups and investigated their properties. Ali et al. (2009) defined some new operations on soft sets. Feng et al. (2010a) provided a framework to combine soft sets, rough sets and fuzzy sets all together. Ali et al. (2011) discussed algebraic structures of soft sets associated with new operations. Ali (2012) presented another view on reduction in parameters in soft sets.

In many types of data analysis, bipolarity of the information is a core feature to be considered while constructing mathematical models for many problems. Bipolarity speaks about the positive and negative features of the information. The positive information represents what is guaranteed to be possible, while the negative information represents what is impossible, forbidden or surely false. The idea which lies behind the existence of *bipolar information* is that a wide variety of human decision making is based on bipolar judgmental thinking. For instance, sweetness and sourness of food, cooperation and competition, friendship and hostility, effects and side effects of medicines are the two sides of information in decision-making and coordination. The coexistence, equilibrium and harmony of these two sides are considered a key for the stability of a social system. The soft sets and the fuzzy sets, together with their compliments, are not appropriate tools to handle this bipolarity; for example, a dress which is not beautiful, may not be necessarily ugly. Zhang (1994) introduced the concept of the bipolar fuzzy sets, as an extension of fuzzy sets. Lee (2004) compared the bipolar fuzzy sets with intuitionistic fuzzy sets and interval-valued fuzzy sets. Three main types of the bipolarity were discussed by Dubois and Prade (2008). Naz and Shabir (2014) contributed toward the algebraic structure of fuzzy bipolar soft sets. The fuzzy bipolar soft sets have potential to handle the bipolarity, as well as fuzziness of the information about some objects with the help of two mapping (from the universe U of object to the collection of all fuzzy sets in U). One mapping handles the positivity of the information, while the other mapping measures the negativity. This is the chief motivation for us to introduce and study the roughness in fuzzy bipolar soft sets.

The purpose of this paper is to establish the concept of roughness in the fuzzy bipolar soft sets. The remaining part of the paper is organized as follows: Sect. 2 recalls some basic concepts and definitions. Sect. 3 is dedicated to the study of rough fuzzy bipolar soft sets by defining the lower and upper approximations of fuzzy bipolar soft sets in a Pawlak approximation space. Some similarity relations among the fuzzy bipolar soft sets are defined in Sect. 4. Section 5 presents an application of the rough fuzzy bipolar soft sets in a decision-making problem, supported by an example. The last section comprises of the conclusions.

2 Preliminaries

2.1 Rough sets

The rough set theory (Pawlak 1982, 1991; Pawlak and Skowron 2007) is based on the conjecture that we can always associate some information (data) to every object in the universe of discourse. Pawlak used the upper and lower approximations of a collection of objects to investigate how close the objects are to the information attached to them. The pair (U, R) is referred to as Pawlak approximation space, where U is a nonempty universe of objects and R is an equivalence relation defined on U. Objects characterized by the same information are indiscernible. The relation R is taken as the indiscernibility relation and serves as the foundation of the rough set theory. The equivalence classes defined by R are referred to as $R-$elementary granules and serve as the basic building blocks of the information. The equivalence class in U/R containing the element $x \in U$ will be denoted by $[x]_R$ (or sometimes by $[x]$, for convenience). With the help of this indiscernibility relation R, the following two operators on a subset X of U are defined:

$$\underline{X} = \{x \in U : [x]_R \subseteq X\}$$
$$\overline{X} = \{x \in U : [x]_R \cap X \neq \phi\}$$

The two subsets $\underline{X}$ and $\overline{X}$ of U, which are assigned to $X \subseteq U$, are called the lower and upper rough approximations of X with respect to R, respectively. Moreover, $Pos_R X = \underline{X}$, $Neg_R X = U - \overline{X}$ and $Bnd_R X = \overline{X} - \underline{X}$ are called the positive, negative and boundary regions of X in U, respectively. The semantics of these regions are as follows.

- $x \in Pos_R X$ means that X certainly contains the object x of U.
- $x \in Neg_R X$ means that X definitely does not contain x of U.
- $x \in Bnd_R X$ means that X may or may not contain x of U.

Thus, the rough set theory studies the objects whose membership to a set is uncertain.

Definition 1 (Pawlak and Skowron 2007) Let (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is $R-$definable if $\underline{X} = \overline{X}$; otherwise, X is known as a rough set.

2.2 Fuzzy sets and rough fuzzy sets

The theory of fuzzy sets (Zadeh 1965) measures the degree of uncertainty of information about the objects with the help of a mapping, termed as the membership function.

Definition 2 (Zadeh 1965) A fuzzy set μ in a nonempty universe U is defined by a membership function $\mu : U \longrightarrow [0, 1]$.

Thus, a fuzzy set μ assigns to each $x \in U$, a membership value $\mu(x)$ specifying the degree to which x is a member of μ. The set of all fuzzy sets in U is denoted by $FP(U)$. By $\mu \subseteq \nu$, we mean that $\mu(x) \leq \nu(x)$ for all $x \in U$. Clearly, $\mu = \nu$ if $\mu \subseteq \nu$ and $\nu \subseteq \mu$. The mappings $\emptyset, I : U \longrightarrow [0, 1]$ defined by $\emptyset(x) = 0$ and $I(x) = 1$ for all $x \in U$, are called the null fuzzy set and the whole fuzzy set in U, respectively. The operations of union, intersection and compliment of fuzzy sets are defined componentwise as follows:

- $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$
- $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$
- $\mu'(x) = 1 - \mu(x)$

where $\mu, \nu \in FP(U)$ and $x \in U$.

Dubois and Prade (1990) defined the lower and upper rough approximations of fuzzy sets in a Pawlak approximation space and introduced the notion of rough fuzzy sets.

Definition 3 (Dubois and Prade 1990) Let (U, R) be a Pawlak approximation space and let $\mu \in FP(U)$. The lower and upper rough approximations of μ in (U, R) are the fuzzy sets denoted by $\underline{\mu}$ and $\overline{\mu}$, respectively, and defined by

$$\underline{\mu}(x) = \bigwedge_{y \in [x]_R} \mu(y)$$

and

$$\overline{\mu}(x) = \bigvee_{y \in [x]_R} \mu(y)$$

for all $x \in U$. If $\underline{\mu} = \overline{\mu}$, the fuzzy set μ is said to be $R-$definable; otherwise, μ is a rough fuzzy set in U.

Theorem 1 (Dubois and Prade 1990) *Let (U, R) be a Pawlak approximation space and let $\mu, \nu \in FP(U)$. Then, the following hold.*

(1) $\underline{\mu} \subseteq \mu \subseteq \overline{\mu}$
(2) $\overline{(\mu')} = (\underline{\mu})'$ *and* $\underline{(\mu')} = (\overline{\mu})'$
(3) $\underline{(\underline{\mu})} = \underline{\mu} = \overline{(\underline{\mu})}$ *and* $\underline{(\overline{\mu})} = \overline{\mu} = \overline{(\overline{\mu})}$
(4) $\underline{R}(I) = I = \overline{R}(I)$ *and* $\underline{R}(\emptyset) = \emptyset = \overline{R}(\emptyset)$
(5) $\underline{\mu \cap \nu} = \underline{\mu} \cap \underline{\nu}$
(6) $\underline{\mu \cup \nu} \supseteq \underline{\mu} \cup \underline{\nu}$
(7) $\overline{\mu \cup \nu} = \overline{\mu} \cup \overline{\nu}$
(8) $\overline{\mu \cap \nu} \subseteq \overline{\mu} \cap \overline{\nu}$
(9) $\mu \subseteq \nu$ *implies that* $\underline{\mu} \subseteq \underline{\nu}$ *and* $\overline{\mu} \subseteq \overline{\nu}$.

2.3 Soft sets

Let E be a nonempty finite set of attributes (parameters, characteristics or properties) which the objects in U possess and let $P(U)$ denote the family of all subsets of U. Then, a soft set is defined with the help of a set-valued mapping, as described below.

Definition 4 (Molodtsov 1999) A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

In simple words, a soft set (F, A) over U is a parameterized family of subsets of U where each parameter $e \in A$ is associated with a subset $F(e)$ of U. The set $F(e)$ contains the objects of U having the property e and is called the set of $e-$approximate elements in (F, A).

2.4 Fuzzy bipolar soft sets

A fuzzy bipolar soft set (Naz and Shabir 2014) is obtained with the help of two set-valued mappings, by considering not only a set of parameters, but also an allied set of carefully chosen parameters with opposite meanings, termed as "not set of parameters." The material in this subsection is taken from Naz and Shabir (2014).

Definition 5 A triplet $\omega = (F, G, A)$ is called a fuzzy bipolar soft set over U, where $A \subseteq E$ and F, G are mappings given by $F : A \to FP(U)$ and $G : \neg A \to FP(U)$ such that

$$F(e)(x) + G(\neg e)(x) \leq 1$$

for all $e \in A$ and for all $x \in U$, where $\neg A$ stands for the "not set of A."

The condition $F(e)(x) + G(\neg e)(x) \leq 1$ is imposed as a consistency constraint. Here, $F(e)$ and $G(\neg e)$ represent fuzzy sets in U, $F(e)(x)$ denotes the degree of presence of a property e in an object x of U, while $G(\neg e)(x)$ denotes the degree of presence of some implicit counter property $\neg e$ in x. We describe and define the fuzzy bipolar soft set (F, G, A) with the help of these two fuzzy sets $F(e)$ and $G(\neg e)$ in U.

The sum $\sum_{e\in E} F(e)(x)$ of all the positive membership values is termed as the degree of positivity of an object x, while the sum $\sum_{\neg e\in\neg E} G(\neg e)(x)$ expresses the degree of negativity of the object x. It is worth noting that the degree of lacking of a property e in an object may not be equal to the degree of having the opposite property $\neg e$. So, we may have $F(e)(x) + G(\neg e)(x) \lneqq 1$ for some $e \in E$ and $x \in U$. This is termed as the degree of hesitation of a fuzzy bipolar soft set (F, G, A) over U and can be approximated by $h(e)(x) = 1 - (F(e)(x) + G(\neg e)(x))$ for $e \in A$ and $x \in U$. Let us denote the collection of all fuzzy bipolar soft sets over U by Ω.

Definition 6 For any two fuzzy bipolar soft sets $\omega_1 = (F_1, G_1, A_1)$ and $\omega_2 = (F_2, G_2, A_2)$ over U, we say that ω_1 is a fuzzy bipolar soft subset of ω_2, denoted by $\omega_1 \widetilde{\subseteq} \omega_2$, if

1) $A_1 \subseteq A_2$
2) $F_1(e) \subseteq F_2(e)$ and $G_1(\neg e) \supseteq G_2(\neg e)$ for all $e \in A_1$.

The fuzzy bipolar soft sets ω_1 and ω_2 are equal, if $F_1(e)(x) = F_2(e)(x)$ for all $e \in A_1$ and $G_1(\neg e)(x) = G_2(\neg e)(x)$ for all $\neg e \in \neg A_1$ and for all $x \in U$.

Definition 7 The relative whole fuzzy bipolar soft set is $(\mathcal{U}, \Phi, A)$, denoted by $\mathcal{U}_A$, where $\mathcal{U}(e) = I$ and $\Phi(\neg e) = \emptyset$ for all $e \in A$. The whole fuzzy bipolar soft set is $(\mathcal{U}, \Phi, E)$. The relative null fuzzy bipolar soft set is $(\Phi, \mathcal{U}, A)$, denoted by Φ_A, where $\Phi(e) = \emptyset$ and $\mathcal{U}(\neg e) = I$ for all $e \in A$. The null fuzzy bipolar soft set is $(\Phi, \mathcal{U}, E)$.

Definition 8 Let $\omega_1 = (F_1, G_1, A_1)$ and $\omega_2 = (F_2, G_2, A_2)$ be two fuzzy bipolar soft sets over a common universe U. Then, their unions and intersections are defined as follows.

(1) The extended union of ω_1 and ω_2, denoted by $\omega_1 \cup_\varepsilon \omega_2$, is a fuzzy bipolar soft set $(F_1 \widetilde{\cup} F_2, G_1 \widetilde{\cap} G_2, A_1 \cup A_2)$ over U, defined as:

$$(F_1\widetilde{\cup}F_2)(e) = \begin{cases} F_1(e) & \text{if } e \in A_1 - A_2 \\ F_2(e) & \text{if } e \in A_2 - A_1 \\ F_1(e) \cup F_2(e) & \text{if } e \in A_1 \cap A_2 \end{cases}$$

$$(G_1\widetilde{\cap}G_2)(\neg e) = \begin{cases} G_1(\neg e) & \text{if } \neg e \in (\neg A_1) - (\neg A_2) \\ G_2(\neg e) & \text{if } \neg e \in (\neg A_2) - (\neg A_1) \\ G_1(\neg e) \cap G_2(\neg e) & \text{if } \neg e \in \neg(A_1) \cap (\neg A_2) \end{cases}$$

(2) The restricted union of ω_1 and ω_2, denoted by $\omega_1 \cup_r \omega_2$, is a fuzzy bipolar soft set $(F_1 \widetilde{\cup} F_2, G_1 \widetilde{\cap} G_2, A_1 \cap A_2)$ over U, where $(F_1 \widetilde{\cup} F_2)(e) = F_1(e) \cup F_2(e)$ and $(G_1 \widetilde{\cap} G_2)(\neg e) = G_1(\neg e) \cap G_2(\neg e)$ for all $e \in A_1 \cap A_2$, provided $A_1 \cap A_2 \neq \phi$.

(3) The extended intersection of ω_1 and ω_2, denoted by $\omega_1 \cap_\varepsilon \omega_2$, is a fuzzy bipolar soft set $(F_1 \widetilde{\cap} F_2, G_1 \widetilde{\cup} G_2, A_1 \cup A_2)$ over U, defined as:

$$(F_1\widetilde{\cap}F_2)(e) = \begin{cases} F_1(e) & \text{if } e \in A_1 - A_2 \\ F_2(e) & \text{if } e \in A_2 - A_1 \\ F_1(e) \cap F_2(e) & \text{if } e \in A_1 \cap A_2 \end{cases}$$

$$(G_1\widetilde{\cup}G_2)(\neg e) = \begin{cases} G_1(\neg e) & \text{if } \neg e \in (\neg A_1) - (\neg A_2) \\ G_2(\neg e) & \text{if } \neg e \in (\neg A_2) - (\neg A_1) \\ G_1(\neg e) \cup G_2(\neg e) & \text{if } \neg e \in \neg(A_1) \cap (\neg A_2) \end{cases}$$

(4) The restricted intersection of ω_1 and ω_2, denoted by $\omega_1 \cap_r \omega_2$, is a fuzzy bipolar soft set $(F_1 \widetilde{\cap} F_2, G_1 \widetilde{\cup} G_2, A_1 \cap A_2)$ over U, where $(F_1 \widetilde{\cap} F_2)(e) = F_1(e) \cap F_2(e)$ and $(G_1 \widetilde{\cup} G_2)(\neg e) = G_1(\neg e) \cup G_2(\neg e)$ for all $e \in A_1 \cap A_2$, provided $A_1 \cap A_2 \neq \phi$.

To find the extended union of ω_1 and ω_2, as in (1) of Definition 8, we consider all the attributes of A_1 and A_2, that is, the set $A_1 \cup A_2$, and then, we partition $A_1 \cup A_2$ into three sets $A_1 - A_2$, $A_2 - A_1$ and $A_1 \cap A_2$. The fuzzy set $(F_1 \widetilde{\cup} F_2)(e)$ is evaluated as $F_1(e) \cup F_2(e)$ for the attributes e of $A_1 \cap A_2$ only, while $(F_1 \widetilde{\cup} F_2)(e)$ is same as $F_1(e)$ for the attributes $e \in A_1 - A_2$ and same as $F_2(e)$ for $e \in A_2 - A_1$. Same is the case with fuzzy set $(G_1 \widetilde{\cap} G_2)(e)$ and (3) of the same definition. We explain it in the following example.

Example 1 Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe containing five houses and $E = \{e_1 =$expensive, $e_2 =$beautiful, $e_3 =$ wooden, $e_4 =$in green surroundings, $e_5 =$in good repair$\}$ be a set of attributes for U. Let the "not set of E" be $\neg E = \{\neg e_1 =$ cheap, $\neg e_2 =$ugly, $\neg e_3 =$not wooden, $\neg e_4 =$in the commercial area, $\neg e_5 =$in bad repair$\}$. We define, here, a fuzzy bipolar soft set $\omega_1 = (F_1, G_1, A_1)$ over U, describing the opinion of Mr. X, who intends to purchase a house, preferring the attributes $A_1 = \{e_1, e_2, e_3\}$. Assume that Mr. X assigns the membership values $\{0.7, 0.6, 0.8, 0.5, 0.6\}$ and $\{0.2, 0.3, 0.1, 0.5, 0.3\}$ to the houses in U for the attribute e_1, describing the degrees of expensiveness and cheapness in the houses, respectively. Then, $F_1(e_1)$ and $G_1(\neg e_1)$ are the fuzzy sets given below.

$$F_1(e_1) = \{h_1/0.7, h_2/0.6, h_3/0.8, h_4/0.5, h_5/0.6\}$$
$$G_1(\neg e_1) = \{h_1/0.2, h_2/0.3, h_3/0.1, h_4/0.5, h_5/0.3\}$$

In the same way, we assume:

$$F_1(e_2) = \{h_1/0.8, h_2/0.7, h_3/0.8, h_4/0.6, h_5/0.6\}$$
$$G_1(\neg e_2) = \{h_1/0.1, h_2/0.1, h_3/0.2, h_4/0.2, h_5/0.3\}$$
$$F_1(e_3) = \{h_1/0.4, h_2/0.6, h_3/0.4, h_4/0.6, h_5/0.5\}$$
$$G_1(\neg e_3) = \{h_1/0.5, h_2/0.2, h_3/0.5, h_4/0.4, h_5/0.5\}$$

This fuzzy bipolar soft set can also be represented in tabular form by setting the entry against e_i and h_j as (a_{ij}, b_{ij}), where $a_{ij} = F(e_i)(h_j)$ and $b_{ij} = G(\neg e_i)(h_j)$. Hence, the tabular representation of $\omega_1 = (F_1, G_1, A_1)$ is given in Table 1.

Table 1 Fuzzy bipolar soft set $\omega_1 = (F_1, G_1, A_1)$

ω_1	h_1	h_2	h_3	h_4	h_5
e_1	(0.7, 0.2)	(0.6, 0.3)	(0.8, 0.1)	(0.5, 0.5)	(0.6, 0.3)
e_2	(0.8, 0.1)	(0.7, 0.1)	(0.8, 0.2)	(0.6, 0.2)	(0.6, 0.3)
e_3	(0.4, 0.5)	(0.6, 0.2)	(0.4, 0.5)	(0.6, 0.4)	(0.5, 0.5)

We can also take another fuzzy bipolar soft set $\omega_2 = (F_2, G_2, A_2)$ over U, with $A_2 = \{e_1, e_2\}$ as below.

$$F_2(e_1) = \{h_1/0.6, h_2/0.6, h_3/0.7, h_4/0.7, h_5/0.6\}$$
$$G_2(\neg e_1) = \{h_1/0.3, h_2/0.2, h_3/0, h_4/0.2, h_5/0.3\}$$
$$F_2(e_2) = \{h_1/0.7, h_2/0.6, h_3/0.6, h_4/0.7, h_5/0.6\}$$
$$G_2(\neg e_2) = \{h_1/0.1, h_2/0.2, h_3/0.1, h_4/0.1, h_5/0.2\}.$$

We find the restricted and extended unions of ω_1 and ω_2, described as $\omega_1 \cup_\varepsilon \omega_2 = (F_1 \tilde{\cup} F_2, G_1 \tilde{\cap} G_2, A_1 \cup A_2)$ and $\omega_1 \cup_r \omega_2 = (F_1 \tilde{\cup} F_2, G_1 \tilde{\cap} G_2, A_1 \cap A_2)$, respectively. For the restricted union, the fuzzy sets $(F_1 \tilde{\cup} F_2)(e)$ and $(G_1 \tilde{\cap} G_2)(\neg e)$ are calculated for $e \in A_1 \cap A_2$, that is, for $e = e_1, e_2$ only, by using Definition 8, as below.

$$(F_1 \tilde{\cup} F_2)(e_1) = F_1(e_1) \cup F_2(e_1)$$
$$= \{h_1/0.7, h_2/0.6, h_3/0.8, h_4/0.7, h_5/0.6\}$$
$$(F_1 \tilde{\cup} F_2)(e_2) = F_1(e_2) \cup F_2(e_2)$$
$$= \{h_1/0.8, h_2/0.7, h_3/0.8, h_4/0.7, h_5/0.6\}$$
$$(G_1 \tilde{\cap} G_2)(\neg e_1) = G_1(\neg e_1) \cap G_2(\neg e_1)$$
$$= \{h_1/0.2, h_2/0.2, h_3/0, h_4/0.2, h_5/0.3\}$$
$$(G_1 \tilde{\cap} G_2)(\neg e_2) = G_1(\neg e_2) \cap G_2(\neg e_2)$$
$$= \{h_1/0.1, h_2/0.1, h_3/0.1, h_4/0.1, h_5/0.2\}$$

For the extended union, the fuzzy sets $(F_1 \tilde{\cup} F_2)(e)$ and $(G_1 \tilde{\cap} G_2)(\neg e)$ are also calculated for the attributes of $A_1 - A_2$ and $A_2 - A_1$. As $A_1 - A_2 = \{e_3\}$ and $A_2 - A_1$ is empty, we calculate $(F_1 \tilde{\cup} F_2)(e)$ and $(G_1 \tilde{\cap} G_2)(\neg e)$ for $e = e_3$, as below.

$$(F_1 \tilde{\cup} F_2)(e_3) = F_1(e_3)$$
$$= \{h_1/0.4, h_2/0.6, h_3/0.4, h_4/0.6, h_5/0.5\}$$
$$(G_1 \tilde{\cap} G_2)(\neg e_3) = G_1(\neg e_3)$$
$$= \{h_1/0.5, h_2/0.2, h_3/0.5, h_4/0.4, h_5/0.5\}$$

Definition 9 The compliment of a fuzzy bipolar soft set $\omega = (F, G, A)$ over U is a fuzzy bipolar soft set $\omega^c = (F^c, G^c, A)$ over U, where $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in A$.

3 Rough fuzzy bipolar soft sets

Motivated by the idea of rough approximations of soft sets by Feng et al. (2010b), we define the lower and upper approximations of the fuzzy bipolar soft sets and introduce the notion of rough fuzzy bipolar soft sets as follows.

Definition 10 Let (U, R) be a Pawlak approximation space and let $\omega = (F, G, A) \in \Omega$. The lower and upper rough approximations of ω in (U, R) are the fuzzy bipolar soft sets $\underline{\omega}_R = (\underline{F}_R, \underline{G}_R, A)$ and $\overline{\omega}^R = (\overline{F}^R, \overline{G}^R, A)$, respectively, where $\underline{F}_R(e), \overline{F}^R(e), \underline{G}_R(\neg e), \overline{G}^R(\neg e)$ are fuzzy sets in U, defined by

$$\underline{F}_R(e)(x) = \underline{F(e)}_R(x) = \bigwedge_{y \in [x]_R} F(e)(y)$$
$$\overline{F}^R(e)(x) = \overline{F(e)}^R(x) = \bigvee_{y \in [x]_R} F(e)(y)$$
$$\underline{G}_R(\neg e)(x) = \overline{G(\neg e)}^R(x) = \bigvee_{y \in [x]_R} G(\neg e)(y)$$
$$\overline{G}^R(\neg e)(x) = \underline{G(\neg e)}_R(x) = \bigwedge_{y \in [x]_R} G(\neg e)(y)$$

for all $e \in A$. If $\underline{\omega}_R = \overline{\omega}^R$, then ω is said to be $R-$definable; otherwise, ω is a rough fuzzy bipolar soft set over U.

The information about an object x of U depicted by the above defined fuzzy sets is as follows.

- $\underline{F}_R(e)(x)$ indicates the degree to which x definitely has the property e.
- $\overline{F}^R(e)(x)$ indicates the degree to which x probably has the property e.
- $\underline{G}_R(\neg e)(x)$ indicates the degree to which x probably has the property opposite to e.
- $\overline{G}^R(\neg e)(x)$ indicates the degree to which x definitely has the property opposite to e.

If the relation R is understood, we will not write R as subscript or superscript (for convenience) in the above notations. One can easily verify the following properties of the rough fuzzy bipolar soft sets.

Theorem 2 *Let (U, R) be a Pawlak approximation space and let $\omega = (F, G, A) \in \Omega$. Then, the following assertions are true.*

(1) $\underline{\omega} \tilde{\subseteq} \omega \tilde{\subseteq} \overline{\omega}$
(2) $\underline{\Phi_A} = \Phi_A = \overline{\Phi_A}$
(3) $\underline{\mathcal{U}_A} = \mathcal{U}_A = \overline{\mathcal{U}_A}$
(4) $\underline{(\underline{\omega})} = \underline{\omega} = \overline{(\underline{\omega})}$
(5) $\underline{(\overline{\omega})} = \overline{\omega} = \overline{(\overline{\omega})}$
(6) $\overline{\omega^c} = (\underline{\omega})^c$
(7) $\underline{\omega^c} = (\overline{\omega})^c$

Proof (1)–(5) These assertions can be verified by using Definitions 6, 7, 10 and Theorem 1.

(6) By using Definitions 9 and 10, the fuzzy bipolar soft sets $\overline{\omega^c}$ and $\left(\underline{\omega}\right)^c$ are described as $\overline{\omega^c} = \left(\overline{F^c}, \overline{G^c}, A\right)$ and $\left(\underline{\omega}\right)^c = ((\underline{F})^c, (\underline{G})^c, A)$. Notice that

$$\begin{aligned}\overline{F^c}(e)(x) &= \underset{y\in[x]}{\vee} F^c(e)(y) = \underset{y\in[x]}{\vee} G(\neg e)(y)\\ &= \underline{G}(\neg e)(x) = (\underline{F})^c(e)(x)\\ \overline{G^c}(\neg e)(x) &= \underset{y\in[x]}{\wedge} G^c(\neg e)(y) = \underset{y\in[x]}{\wedge} F(e)(y)\\ &= \underline{F}(e)(x) = (\underline{G})^c(\neg e)(x)\end{aligned}$$

hold for all $e \in A$ and for all $x \in U$. By Definition 6, the above assertions immediately give

$$\overline{\omega^c} = \left(\underline{\omega}\right)^c$$

(7) The proof is similar to the proof of (6). □

Remark 1 Notice that for any two fuzzy bipolar soft sets $\omega_1 = (F_1, G_1, A)$ and $\omega_2 = (F_2, G_2, B)$ over a common universe U and for any $e \in A \cup B$, we have the following assertions, by using Theorem 1.

(1) $\overline{F_1}(e) \cup \overline{F_2}(e) = (\overline{F_1 \tilde{\cup} F_2})(e)$
(2) $\overline{F_1}(e) \cap \overline{F_2}(e) \supseteq (\overline{F_1 \tilde{\cap} F_2})(e)$
(3) $\underline{F_1}(e) \cup \underline{F_2}(e) \subseteq (\underline{F_1 \tilde{\cup} F_2})(e)$
(4) $\underline{F_1}(e) \cap \underline{F_2}(e) = (\underline{F_1 \tilde{\cap} F_2})(e)$
(5) $\overline{G_1}(\neg e) \cup \overline{G_2}(\neg e) \subseteq (\overline{G_1 \tilde{\cup} G_2})(\neg e)$
(6) $\overline{G_1}(\neg e) \cap \overline{G_2}(\neg e) = (\overline{G_1 \tilde{\cap} G_2})(\neg e)$
(7) $\underline{G_1}(\neg e) \cup \underline{G_2}(\neg e) = (\underline{G_1 \tilde{\cup} G_2})(\neg e)$
(8) $\underline{G_1}(\neg e) \cap \underline{G_2}(\neg e) \supseteq (\underline{G_1 \tilde{\cap} G_2})(\neg e)$.

Theorem 3 *Let (U, R) be a Pawlak approximation space. Then, the following assertions are true for any $\omega_1 = (F_1, G_1, A)$, $\omega_2 = (F_2, G_2, B) \in \Omega$.*

(1) $\omega_1 \tilde{\subseteq} \omega_2$ *implies that* $\underline{\omega_1} \tilde{\subseteq} \underline{\omega_2}$ *and* $\overline{\omega_1} \tilde{\subseteq} \overline{\omega_2}$
(2) $\underline{\omega_1 \cap_\varepsilon \omega_2} = \underline{\omega_1} \cap_\varepsilon \underline{\omega_2}$
(3) $\underline{\omega_1 \cap_r \omega_2} = \underline{\omega_1} \cap_r \underline{\omega_2}$
(4) $\underline{\omega_1 \cup_\varepsilon \omega_2} \tilde{\supseteq} \underline{\omega_1} \cup_\varepsilon \underline{\omega_2}$
(5) $\underline{\omega_1 \cup_r \omega_2} \tilde{\supseteq} \underline{\omega_1} \cup_r \underline{\omega_2}$
(6) $\overline{\omega_1 \cap_\varepsilon \omega_2} \tilde{\subseteq} \overline{\omega_1} \cap_\varepsilon \overline{\omega_2}$
(7) $\overline{\omega_1 \cap_r \omega_2} \tilde{\subseteq} \overline{\omega_1} \cap_r \overline{\omega_2}$
(8) $\overline{\omega_1 \cup_\varepsilon \omega_2} = \overline{\omega_1} \cup_\varepsilon \overline{\omega_2}$
(9) $\overline{\omega_1 \cup_r \omega_2} = \overline{\omega_1} \cup_r \overline{\omega_2}$.

Proof (1) Given that $\omega_1 \tilde{\subseteq} \omega_2$, that is, $(F_1, G_1, A) \tilde{\subseteq} (F_2, G_2, B)$. Then, $F_1(e), F_2(e), G_1(\neg e), G_2(\neg e)$ are fuzzy sets in U, such that $F(e) \subseteq F_2(e)$ and $G(\neg e) \supseteq G_1(\neg e)$ for all $e \in A$, where $A \subseteq B$. By using Definition 10 and Theorem 1, we get

$$\underline{F}(e) = \underline{F(e)} \subseteq \underline{F_2(e)} = \underline{F_2}(e)$$

and

$$\underline{G}(\neg e) = \overline{G(\neg e)} \supseteq \overline{G_1(\neg e)} = \underline{G_1}(\neg e)$$

for all $e \in A$. Thus, $\underline{\omega_1} \tilde{\subseteq} \underline{\omega_2}$ by Definition 6. Similarly, one can verify that $\overline{\omega_1} \tilde{\subseteq} \overline{\omega_2}$.

(2) By using Definition 8, the fuzzy bipolar soft sets $\underline{\omega_1 \cap_\varepsilon \omega_2}$ and $\underline{\omega_1} \cap_\varepsilon \underline{\omega_2}$ are described as $\underline{\omega_1 \cap_\varepsilon \omega_2} = (\underline{F_1 \tilde{\cap} F_2}, \underline{G_1 \tilde{\cup} G_2}, A \cup B)$ and $\underline{\omega_1} \cap_\varepsilon \underline{\omega_2} = (\underline{F_1} \tilde{\cap} \underline{F_2}, \underline{G_1} \tilde{\cup} \underline{G_2}, A \cup B)$. Now, Remark 1 states that the equations

$$(\underline{F_1 \tilde{\cap} F_2})(e) = \underline{F_1}(e) \cap \underline{F_2}(e) = (\underline{F_1} \tilde{\cap} \underline{F_2})(e)$$

and

$$(\underline{G_1 \tilde{\cup} G_2})(\neg e) = \underline{G_1}(\neg e) \cup \underline{G_2}(\neg e) = (\underline{G_1} \tilde{\cup} \underline{G_2})(\neg e)$$

hold for all $e \in A \cup B$. By Definition 6, the above equations assert that

$$\underline{\omega_1 \cap_\varepsilon \omega_2} = \underline{\omega_1} \cap_\varepsilon \underline{\omega_2}$$

(3) This equation can be deduced from (2).

(4) The fuzzy bipolar soft sets $\underline{\omega_1 \cup_\varepsilon \omega_2}$ and $\underline{\omega_1} \cup_\varepsilon \underline{\omega_2}$ in the equation to be proved are described as $\underline{\omega_1 \cup_\varepsilon \omega_2} = (\underline{F_1 \tilde{\cup} F_2}, \underline{G_1 \tilde{\cap} G_2}, A \cup B)$ and $\underline{\omega_1} \cup_\varepsilon \underline{\omega_2} = (\underline{F_1} \tilde{\cup} \underline{F_2}, \underline{G_1} \tilde{\cap} \underline{G_2}, A \cup B)$. Remark 1 states that the expressions

$$(\underline{F_1 \tilde{\cup} F_2})(e) \supseteq \underline{F_1}(e) \cup \underline{F_2}(e) = (\underline{F_1} \tilde{\cup} \underline{F_2})(e)$$

and

$$(\underline{G_1 \tilde{\cap} G_2})(\neg e) \subseteq \underline{G_1}(\neg e) \cap \underline{G_2}(\neg e) = (\underline{G_1} \tilde{\cap} \underline{G_2})(\neg e)$$

hold for all $e \in A \cup B$. By Definition 6, the above expressions prove that

$$\underline{\omega_1 \cap_\varepsilon \omega_2} \tilde{\supseteq} \underline{\omega_1} \cap_\varepsilon \underline{\omega_2}$$

(5) This expression can be deduced from (4).

(6–9) These assertions can be verified in the same way as the assertions (2–5) above. □

For the illustration of the above theorem, we consider the following example.

Example 2 Consider the universe U of five houses, E, $\neg E$ and the fuzzy bipolar soft set ω_1, as defined in Example 1. Let the house h_1 be in some locality A, the houses h_2 and h_3 be in a locality B and the houses h_4 and h_5 be in a locality C. We define a binary relation R on U, such that two houses are related in R if they are in same locality. Then, R is an equivalence relation on U, given by $\{(h_1, h_1), (h_2, h_2), (h_3, h_3), (h_4, h_4), (h_5, h_5), (h_2, h_3), (h_3, h_2), (h_4, h_5), (h_5, h_4)\}$. The equivalence classes defined by R are $[h_1], [h_2, h_3], [h_4, h_5]$. In this example, we show how to determine the lower and upper rough approximations of a fuzzy bipolar soft set by verifying the assertion (1) of Theorem 2. In order to determine the lower rough approximation $(\underline{F_1}, \underline{G_1}, A_1)$ and the upper rough approximation $(\overline{F_1}, \overline{G_1}, A_1)$ of $\omega_1 = (F_1, G_1, A_1)$, in the approximation space (U, R), the fuzzy sets $\underline{F_1}(e)$, $\underline{G_1}(\neg e)$, $\overline{F_1}(e)$ and $\overline{G_1}(\neg e)$ are to be calculated for each $e \in A_1$ by using Definition 10 and the relation R. First, we calculate the fuzzy set $\underline{F_1}(e)$ for $e = e_1$, in detail.

$$\begin{aligned}\underline{F_1}(e_1) &= \Big\{h_1/\bigwedge_{y\in[h_1]} F_1(e_1)(y), h_2/\bigwedge_{y\in[h_2]} F_1(e_1)(y), \ldots,\\ &\quad h_5/\bigwedge_{y\in[h_5]} F_1(e_1)(y)\Big\}\\ &= \Big\{h_1/\bigwedge_{y=h_1} F_1(e_1)(y), h_2/\bigwedge_{y=h_2,h_3} F_1(e_1)(y), \ldots,\\ &\quad h_5/\bigwedge_{y=h_4,h_5} F_1(e_1)(y)\Big\}\\ &= \{h_1/0.7, h_2/(0.6\wedge 0.8), h_3/(0.6\wedge 0.8),\\ &\quad h_4/(0.5\wedge 0.6), h_5/(0.5\wedge 0.6)\}\\ &= \{h_1/0.7, h_2/0.6, h_3/0.6, h_4/0.5, h_5/0.5\}\end{aligned}$$

In the same way, the following fuzzy sets are calculated for all $e \in A_1$.

$$\begin{aligned}\underline{F_1}(e_2) &= \{h_1/0.8, h_2/0.7, h_3/0.7, h_4/0.6, h_5/0.6\}\\ \underline{F_1}(e_3) &= \{h_1/0.4, h_2/0.4, h_3/0.4, h_4/0.5, h_5/0.5\}\\ \underline{G_1}(\neg e_1) &= \{h_1/0.2, h_2/0.3, h_3/0.3, h_4/0.5, h_5/0.5\}\\ \underline{G_1}(\neg e_2) &= \{h_1/0.1, h_2/0.2, h_3/0.2, h_4/0.3, h_5/0.3\}\\ \underline{G_1}(\neg e_3) &= \{h_1/0.5, h_2/0.5, h_3/0.5, h_4/0.5, h_5/0.5\}\\ \overline{F_1}(e_1) &= \{h_1/0.7, h_2/0.8, h_3/0.8, h_4/0.6, h_5/0.6\}\\ \overline{F_1}(e_2) &= \{h_1/0.8, h_2/0.8, h_3/0.8, h_4/0.6, h_5/0.6\}\\ \overline{F_1}(e_3) &= \{h_1/0.4, h_2/0.6, h_3/0.6, h_4/0.6, h_5/0.6\}\\ \overline{G_1}(\neg e_1) &= \{h_1/0.2, h_2/0.1, h_3/0.1, h_4/0.3, h_5/0.3\}\\ \overline{G_1}(\neg e_2) &= \{h_1/0.1, h_2/0.1, h_3/0.1, h_4/0.2, h_5/0.2\}\\ \overline{G_1}(\neg e_3) &= \{h_1/0.5, h_2/0.2, h_3/0.2, h_4/0.4, h_5/0.4\}\end{aligned}$$

By comparing membership values of the above fuzzy sets, one can see that $\underline{F_1}(e) \subseteq F_1(e) \subseteq \overline{F_1}(e)$ and $\underline{G_1}(\neg e) \supseteq G_1(\neg e) \supseteq \overline{G_1}(\neg e)$ for all $e \in A_1$. This verifies $\underline{\omega_1} \tilde{\subseteq} \omega_1 \tilde{\subseteq} \overline{\omega_1}$, by using Definition 6.

Now we verify (5) of Theorem 3 for the fuzzy bipolar soft sets ω_1 and ω_2, given in Example 1. We have already determined $\underline{\omega_1}$. The restricted union $\omega_1 \cup_r \omega_2$ is determined in Example 1. So, we need to determine only $\underline{\omega_2}$, $\underline{\omega_1} \cup_r \underline{\omega_2}$ and $\underline{\omega_1 \cup_r \omega_2}$. First, the fuzzy sets $\underline{F_2}(e)$ and $\underline{G_2}(e)$ of $\underline{\omega_2} = (\underline{F_2}, \underline{G_2}, A_2)$ are calculated for $e \in A_2$, as below.

$$\begin{aligned}\underline{F_2}(e_1) &= \{h_1/0.6, h_2/0.6, h_3/0.6, h_4/0.6, h_5/0.6\}\\ \underline{F_2}(e_2) &= \{h_1/0.7, h_2/0.6, h_3/0.6, h_4/0.6, h_5/0.6\}\\ \underline{G_2}(\neg e_1) &= \{h_1/0.3, h_2/0.2, h_3/0.2, h_4/0.3, h_5/0.3\}\\ \underline{G_2}(\neg e_2) &= \{h_1/0.1, h_2/0.2, h_3/0.2, h_4/0.2, h_5/0.2\}\end{aligned}$$

The fuzzy sets $(\underline{F_1}\tilde{\cup}\underline{F_2})(e)$ and $(\underline{G_1}\tilde{\cap}\underline{G_2})(e)$ of the restricted union $\underline{\omega_1} \cup_r \underline{\omega_2} = (\underline{F_1}\tilde{\cup}\underline{F_2}, \underline{G_1}\tilde{\cap}\underline{G_2}, A_1 \cap A_2)$ are calculated for $e \in A_1 \cap A_2$, as below.

$$\begin{aligned}(\underline{F_1}\tilde{\cup}\underline{F_2})(e_1) &= \{h_1/0.7, h_2/0.6, h_3/0.6, h_4/0.6, h_5/0.6\}\\ (\underline{F_1}\tilde{\cup}\underline{F_2})(e_2) &= \{h_1/0.8, h_2/0.7, h_3/0.7, h_4/0.6, h_5/0.6\}\\ (\underline{G_1}\tilde{\cap}\underline{G_2})(\neg e_1) &= \{h_1/0.2, h_2/0.2, h_3/0.2, h_4/0.3, h_5/0.3\}\\ (\underline{G_1}\tilde{\cap}\underline{G_2})(\neg e_2) &= \{h_1/0.1, h_2/0.2, h_3/0.2, h_4/0.2, h_5/0.2\}\end{aligned}$$

Now we calculate the fuzzy sets $(\underline{F_1\tilde{\cup}F_2})(e)$ and $(\underline{G_1\tilde{\cap}G_2})(e)$ of $\underline{\omega_1 \cup_r \omega_2} = (\underline{F_1\tilde{\cup}F_2}, \underline{G_1\tilde{\cap}G_2}, A_1 \cap A_2)$ for $e \in A_1 \cap A_2$, as below.

$$\begin{aligned}(\underline{F_1\tilde{\cup}F_2})(e_1) &= \{h_1/0.7, h_2/0.6, h_3/0.6, h_4/0.6, h_5/0.6\}\\ (\underline{F_1\tilde{\cup}F_2})(e_2) &= \{h_1/0.8, h_2/0.7, h_3/0.7, h_4/0.6, h_5/0.6\}\\ (\underline{G_1\tilde{\cap}G_2})(\neg e_1) &= \{h_1/0.2, h_2/0.2, h_3/0.2, h_4/0.3, h_5/0.3\}\\ (\underline{G_1\tilde{\cap}G_2})(\neg e_2) &= \{h_1/0.1, h_2/0.1, h_3/0.1, h_4/0.2, h_5/0.2\}\end{aligned}$$

Notice that

$$(\underline{G_1\tilde{\cap}G_2})(\neg e_2)(h_2) \lneqq (\underline{G_1}\tilde{\cap}\underline{G_2})(\neg e_2)(h_2)$$

By Definition 6, we immediately get

$$\underline{\omega_1 \cup_r \omega_2} \tilde{\supseteq} \underline{\omega_1} \cup_r \underline{\omega_2}$$

The other assertions can also be observed by doing the same calculations.

Theorem 4 *Let (U, R) be a Pawlak approximation space and let $\omega \in \Omega$. Then, the following assertions are equivalent.*

(1) $\overline{\omega} \tilde{\subseteq} \omega$
(2) $\omega \tilde{\subseteq} \underline{\omega}$
(3) *ω is $R-$definable.*

Proof This proof follows from Theorem 2 and Theorem 3. □

Theorem 5 *Let (U, R) be a Pawlak approximation space.*

(1) *If R is the identity relation on U, then each fuzzy bipolar soft set over U is $R-$definable.*

(2) *If R is the universal binary relation $U \times U$, then the only $R-$definable fuzzy bipolar soft sets over U are $\{\mathcal{U}_A, \Phi_A : A \subseteq E\}$.*

Proof Straightforward. □

The above theorem demonstrates that if the relation R in the Pawlak approximation space is identity relation, the no fuzzy bipolar soft set is rough. On the other hand, each fuzzy bipolar soft set over U is rough, except the relative whole and relative null fuzzy bipolar soft sets if the relation R is the universal binary relation on U.

It is worth noting that in the Pawlak approximation space, if we replace the equivalence relation R by some other equivalence relation σ on U containing R, then the upper rough approximation of a fuzzy bipolar soft set ω with respect to σ also contains the upper rough approximation of ω with respect to R. But, this order is reversed in the case of lower rough approximation. This interesting result is highlighted in the following theorem.

Theorem 6 *Let (U, R) be a Pawlak approximation space and let σ be an equivalence relation on U such that $R \subseteq \sigma$. Then, $\underline{\omega}_\sigma \tilde{\subseteq} \underline{\omega}_R$ and $\overline{\omega}^R \tilde{\subseteq} \overline{\omega}^\sigma$ for any fuzzy bipolar soft set ω over U.*

Proof Take $\omega = (F, G, A) \in \Omega$ for any $A \subseteq E$. Since $R \subseteq \sigma$, we have $[x]_R \subseteq [x]_\sigma$ for all $x \in U$. Thus, we get

$$\underline{F}_\sigma(e)(x) = \bigwedge_{y \in [x]_\sigma} F(e)(y) \leq \bigwedge_{y \in [x]_R} F(e)(y) = \underline{F}_R(e)(x)$$

for all $x \in U$ and for all $e \in A$. Hence, $\underline{F}_\sigma(e) \subseteq \underline{F}_R(e)$ for all $e \in A$. Similarly, $\underline{G}_\sigma(\neg e) \supseteq \underline{G}_R(\neg e)$ for all $\neg e \in \neg A$. Thus, $\underline{\omega}_\sigma \tilde{\subseteq} \underline{\omega}_R$. In the same way, one can verify $\overline{\omega}^R \tilde{\subseteq} \overline{\omega}^\sigma$. □

4 Similarity relations associated with rough fuzzy bipolar soft sets

Feng et al. (2010a) studied some binary relations between the soft rough sets. In this section, we define some binary relations between the fuzzy bipolar soft sets based on their rough approximations and investigate their properties.

Definition 11 Let (U, R) be a Pawlak approximation space. We define the following binary relations for $\omega_1, \omega_2 \in \Omega$,

$$\omega_1 \simeq \omega_2 \; if \; and \; only \; if \; \underline{\omega_1} = \underline{\omega_2}$$
$$\omega_1 \eqsim \omega_2 \; if \; and \; only \; if \; \overline{\omega_1} = \overline{\omega_2}$$
$$\omega_1 \approx \omega_2 \; if \; and \; only \; if \; \underline{\omega_1} = \underline{\omega_2} \; and \; \overline{\omega_1} = \overline{\omega_2}.$$

These binary relations may be called as the lower RFBS (rough fuzzy bipolar soft) similarity relation, upper RFBS similarity relation and RFBS similarity relation, respectively. Obviously, ω_1 and ω_2 are RFBS similar if and only if they are both, lower and upper RFBS similar.

Proposition 1 *The relations $\simeq$, $\eqsim$ and $\approx$ are equivalence relations on Ω.*

Proof Straightforward. □

Theorem 7 *Let (U, R) be a Pawlak approximation space and let $\{\omega_i = (F_i, G_i, A_i) : i = 1, 2, 3, 4\} \subseteq \Omega$. Then, the following assertions hold.*

(1) *$\omega_1 \eqsim \omega_2$ if and only if $\omega_1 \eqsim (\omega_1 \cup_\varepsilon \omega_2) \eqsim \omega_2$*
(2) *$\omega_1 \eqsim \omega_2$ and $\omega_3 \eqsim \omega_4$ imply that $(\omega_1 \cup_\varepsilon \omega_3) \eqsim (\omega_2 \cup_\varepsilon \omega_4)$*
(3) *$\omega_1 \tilde{\subseteq} \omega_2$ and $\omega_2 \eqsim \Phi_{A_2}$ imply that $\omega_1 \eqsim \Phi_{A_1}$*
(4) *$\omega_1 \tilde{\subseteq} \omega_2$ and $\omega_1 \eqsim \mathcal{U}_{A_1}$ imply that $\omega_2 \eqsim \mathcal{U}_{A_2}$, provided that $A_1 = A_2$*
(5) *$(\omega_1 \cup_\varepsilon \omega_2) \eqsim \Phi_{A_1 \cup A_2}$ if and only if $\omega_1 \eqsim \Phi_{A_1}$ and $\omega_2 \eqsim \Phi_{A_2}$*
(6) *$(\omega_1 \cap_\varepsilon \omega_2) \eqsim \mathcal{U}_{A_1 \cup A_2}$ implies that $\omega_1 \eqsim \mathcal{U}_{A_1}$ and $\omega_2 \eqsim \mathcal{U}_{A_2}$.*

Proof (1) Let $\omega_1 \eqsim \omega_2$. Then, $\overline{\omega_1} = \overline{\omega_2}$. By Theorem 3, we get

$$\overline{\omega_1 \cup_\varepsilon \omega_2} = \overline{\omega_1} \cup_\varepsilon \overline{\omega_2} = \overline{\omega_1} = \overline{\omega_2}$$

So $\omega_1 \eqsim (\omega_1 \cup_\varepsilon \omega_2) \eqsim \omega_2$.

Converse holds by transitivity of the relation $\eqsim$.

(2) Given that $\omega_1 \eqsim \omega_2$ and $\omega_3 \eqsim \omega_4$. Then, $\overline{\omega_1} = \overline{\omega_2}$ and $\overline{\omega_3} = \overline{\omega_4}$. By Theorem 3, we get

$$\overline{\omega_1 \cup_\varepsilon \omega_3} = \overline{\omega_1} \cup_\varepsilon \overline{\omega_3} = \overline{\omega_2} \cup_\varepsilon \overline{\omega_4} = \overline{\omega_2 \cup_\varepsilon \omega_4}$$

Thus, $(\omega_1 \cup_\varepsilon \omega_3) \eqsim (\omega_2 \cup_\varepsilon \omega_4)$.

(3) Given that $\omega_2 \eqsim \Phi_{A_2}$. This implies that $\overline{\omega_2} = \overline{\Phi_{A_2}} = \Phi_{A_2}$. Also $\omega_1 \tilde{\subseteq} \omega_2$ implies that $\overline{\omega_1} \tilde{\subseteq} \overline{\omega_2} = \Phi_{A_2}$. Restricting the attribute set of Φ_{A_2} to $A_1 \subseteq A_2$, we get $\overline{\omega_1} \tilde{\subseteq} \Phi_{A_1}$. But, $\Phi_{A_1} \tilde{\subseteq} \overline{\omega_1}$. So, $\overline{\omega_1} = \Phi_{A_1} = \overline{\Phi_{A_1}}$ which shows $\omega_1 \eqsim \Phi_{A_1}$.

(4) $\omega_1 \eqsim \mathcal{U}_{A_1}$ implies that $\overline{\omega_1} = \overline{\mathcal{U}_{A_1}} = \mathcal{U}_{A_1}$. By $A_1 = A_2$, we get $\mathcal{U}_{A_1} = \mathcal{U}_{A_2}$. Also given that $\omega_1 \tilde{\subseteq} \omega_2$. So, we get $\overline{\omega_2} \tilde{\subseteq} \overline{\mathcal{U}_{A_2}} = \mathcal{U}_{A_2} = \mathcal{U}_{A_1} = \overline{\omega_1} \tilde{\subseteq} \overline{\omega_2}$. This gives $\overline{\omega_2} = \overline{\mathcal{U}_{A_2}}$, and hence, $\omega_2 \eqsim \mathcal{U}_{A_2}$.

(5) Let $\omega_1 \eqsim \Phi_{A_1}$ and $\omega_2 \eqsim \Phi_{A_2}$. Then, $\overline{\omega_1} = \overline{\Phi_{A_1}} = \Phi_{A_1}$ and $\overline{\omega_2} = \overline{\Phi_{A_2}} = \Phi_{A_2}$. By Theorem 3, we get

$$\overline{\omega_1 \cup_\varepsilon \omega_2} = \overline{\omega_1} \cup_\varepsilon \overline{\omega_2} = \Phi_{A_1} \cup_\varepsilon \Phi_{A_2} = \Phi_{A_1 \cup A_2} = \overline{\Phi_{A_1 \cup A_2}}$$

Thus, $(\omega_1 \cup_\varepsilon \omega_2) \eqsim \Phi_{A_1 \cup A_2}$. Converse follows from (3).

(6) This assertion follows from (4). □

Note that in (1) and (2) of Theorem 7, $\omega_1 \approx \omega_2$ means that $\overline{\omega_1} = \overline{\omega_2}$ which indicates $A_1 = A_2$ by using Definition 6. Thus, the attribute sets of RFBS similar (lower, upper or both) fuzzy bipolar soft sets are same; hence, their restricted and extended unions as well as intersections coincide. Same is the case when $\omega_1 \simeq \omega_2$ or $\omega_1 \approx \omega_2$.

Theorem 8 *Let (U, R) be a Pawlak approximation space and let $\{\omega_i = (F_i, G_i, A_i) : i = 1, 2, 3, 4\} \subseteq \Omega$. Then, the following assertions hold.*

(1) $\omega_1 \simeq \omega_2$ *if and only if* $\omega_1 \simeq (\omega_1 \cap_\varepsilon \omega_2) \simeq \omega_2$
(2) $\omega_1 \simeq \omega_2$ *and* $\omega_3 \simeq \omega_4$ *imply that* $(\omega_1 \cap_\varepsilon \omega_3) \simeq (\omega_2 \cap_\varepsilon \omega_4)$
(3) $\omega_1 \tilde{\subseteq} \omega_2$ *and* $\omega_2 \simeq \Phi_{A_2}$ *imply that* $\omega_1 \simeq \Phi_{A_1}$
(4) $\omega_1 \tilde{\subseteq} \omega_2$ *and* $\omega_1 \simeq \mathcal{U}_{A_1}$ *imply that* $\omega_2 \simeq \mathcal{U}_{A_2}$, *provided that* $A_1 = A_2$
(5) $(\omega_1 \cup_\varepsilon \omega_2) \simeq \Phi_{A_1 \cup A_2}$ *implies that* $\omega_1 \simeq \Phi_{A_1}$ *and* $\omega_2 \simeq \Phi_{A_2}$
(6) $(\omega_1 \cap_\varepsilon \omega_2) \simeq \mathcal{U}_{A_1 \cup A_2}$ *if and only if* $\omega_1 \simeq \mathcal{U}_{A_1}$ *and* $\omega_2 \simeq \mathcal{U}_{A_2}$.

Proof The proof is similar to the proof of Theorem 7. □

Theorem 9 *Let (U, R) be a Pawlak approximation space and let $\{\omega_i = (F_i, G_i, A_i) : i = 1, 2, 3, 4\} \subseteq \Omega$. Then, the following assertions hold.*

(1) $\omega_1 \approx \omega_2$ *if and only if* $\omega_1 \approx (\omega_1 \cup_\varepsilon \omega_2) \approx \omega_2$ *and* $\omega_1 \simeq (\omega_1 \cap_\varepsilon \omega_2) \simeq \omega_2$
(2) $\omega_1 \tilde{\subseteq} \omega_2$ *and* $\omega_2 \approx \Phi_{A_2}$ *imply that* $\omega_1 \approx \Phi_{A_1}$
(3) $\omega_1 \tilde{\subseteq} \omega_2$ *and* $\omega_1 \approx \mathcal{U}_{A_1}$ *imply that* $\omega_2 \approx \mathcal{U}_{A_2}$, *provided that* $A_1 = A_2$
(4) $(\omega_1 \cup_\varepsilon \omega_2) \approx \Phi_{A_1 \cup A_2}$ *implies that* $\omega_1 \approx \Phi_{A_1}$ *and* $\omega_2 \approx \Phi_{A_2}$
(5) $(\omega_1 \cap_\varepsilon \omega_2) \approx \mathcal{U}_{A_1 \cup A_2}$ *implies that* $\omega_1 \approx \mathcal{U}_{A_1}$ *and* $\omega_2 \approx \mathcal{U}_{A_2}$.

Proof The proof is direct consequence of Theorems 7 and 8. □

5 Applications in decision-making problems

Decision making is a major area to be conferred in almost all kinds of data analysis. The researchers and experts use their knowledge to design algorithms in order to find a wise decision. As far as the information system (U, E) is concerned, one often requires to decide for the best optimum object in U. But sometimes, one may be unable to take the best decision, even when the best decision is known. In that case, it may be helpful if the worst decision also becomes visible. We propose an algorithm which provides the best, as well as, the worst decision. With the help of this algorithm, one can avoid taking the worst decision as well. Let U be the sets of objects under consideration and $E = \{e_i : 1 \leq i \leq n\}$ be the set of attributes for U. The information about the objects is represented by a fuzzy bipolar soft set $\omega = (F, G, E)$. In this section, we use the tabular representation of the fuzzy bipolar soft set as given in Table 1 of Example 1. It is already discussed that the objects having same characteristics are indiscernible. First we assign an indiscernibility value to each object and then define the indiscernibility relations associated with ω.

Definition 12 The indiscernibility parameter N has the values n_j corresponding to each object $x_j \in U$, given by

$$n_j = \sum_{i=1}^{n} (a_{ij} - b_{ij})$$

Adjoin the row of N at the bottom of the table of ω to get the indiscernibility table of ω.

This parameter represents the difference between the degree of positivity and the degree of negativity for each object x_j. In the same way, the values of the parameter $\underline{N}$ can be calculated by $\underline{n_j} = \Sigma_i(\underline{a_{ij}} - \underline{b_{ij}})$ and the parameter $\overline{N}$ by $\overline{n_j} = \Sigma_i(\overline{a_{ij}} - \overline{b_{ij}})$, where $(\underline{a_{ij}}, \underline{b_{ij}})$ and $(\overline{a_{ij}}, \overline{b_{ij}})$ are the $(i, j)th$ entries in the tables of $\underline{\omega}$ and $\overline{\omega}$, respectively.

Definition 13 The decision parameter D has the values d_j corresponding to each object $x_j \in U$, given by

$$d_j = \underline{n_j} + \overline{n_j}$$

Now we give the concept of indiscernibility relations on U, associated with ω. For $e_i \in E$, denote

$$C_1(e_i) = \{x_j \in U : a_{ij} \gneqq b_{ij}\}$$
$$C_2(e_i) = \{x_j \in U : a_{ij} = b_{ij}\}$$
$$C_3(e_i) = \{x_j \in U : a_{ij} \lneqq b_{ij}\}$$

Then, for each $e_i \in E$, the set U can be partitioned into the (atmost three) classes, $C_1(e_i)$, $C_2(e_i)$ and $C_3(e_i)$ due to the indiscernibility in U. Clearly, each $e_i \in E$ corresponds to an equivalence relation $\xi(e_i)$ on U, such that two objects are $\xi(e_i)$ equivalent if they belong to the same class $C_1(e_i)$, $C_2(e_i)$ or $C_3(e_i)$. Denote

$$IND(E) = \bigcap_{i=1}^{n} \xi(e_i)$$

Then $IND(E)$ is also an equivalence relation on U. The indiscernibility table of ω is consistent if and only if $IND(E) \subseteq IND(N)$, where $IND(N)$ is the equivalence relation on U, dividing U into the classes having same values n_j.

Table 2 Fuzzy bipolar soft set $\omega = (F, G, A)$

ω	c_1	c_2	c_3	c_4	c_5	c_6
e_1	(0.6, 0.2)	(0.5, 0.5)	(0.6, 0.3)	(0.3, 0.5)	(0.6, 0.2)	(0.4, 0.4)
e_2	(0.6, 0.4)	(0.5, 0.4)	(0.6, 0.2)	(0.7, 0.3)	(0.5, 0.5)	(0.3, 0.4)
e_4	(0.7, 0.1)	(0.4, 0.4)	(0.6, 0.2)	(0.3, 0.5)	(0.5, 0.4)	(0.4, 0.4)
e_6	(0.5, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.5, 0.4)
e_7	(0.4, 0.5)	(0.3, 0.6)	(0.6, 0.2)	(0.7, 0.2)	(0.6, 0.4)	(0.4, 0.4)
e_{10}	(0.7, 0.1)	(0.6, 0.3)	(0.5, 0.3)	(0.5, 0.4)	(0.4, 0.5)	(0.3, 0.5)

Table 3 Indiscernibility table of ω

ω	c_1	c_2	c_3	c_4	c_5	c_6
e_1	(0.6, 0.2)	(0.5, 0.5)	(0.6, 0.3)	(0.3, 0.5)	(0.6, 0.2)	(0.4, 0.4)
e_2	(0.6, 0.4)	(0.5, 0.4)	(0.6, 0.2)	(0.7, 0.3)	(0.5, 0.5)	(0.3, 0.4)
e_4	(0.7, 0.1)	(0.4, 0.4)	(0.6, 0.2)	(0.3, 0.5)	(0.5, 0.4)	(0.4, 0.4)
e_6	(0.5, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.5, 0.4)
e_7	(0.4, 0.5)	(0.3, 0.6)	(0.6, 0.2)	(0.7, 0.2)	(0.6, 0.4)	(0.4, 0.4)
e_{10}	(0.7, 0.1)	(0.6, 0.3)	(0.5, 0.3)	(0.5, 0.4)	(0.4, 0.5)	(0.3, 0.5)
N	1.7	0.4	1.6	0.9	0.5	−0.2

Definition 14 Let T be the consistent indiscernibility table of the fuzzy bipolar soft set ω and T_r be a table obtained from T by deleting the row of an attribute $r \in E$. Then, r is dispensable in T if

(1) T_r is consistent, that is, $IND(E - r) = IND(N_r)$
(2) $IND(N) = IND(N_r)$

Otherwise, r is indispensable or core attribute. The set of all core attributes of E is denoted by $CORE(E)$.

Algorithm 1 *The algorithm to decide for the best and the worst object in U is as follows.*

(1) *Input the set of choice attributes $A \subseteq E$.*
(2) *Input the fuzzy bipolar soft set $\omega = (F, G, A)$.*
(3) *Adjoin the row of the parameter N at the bottom of the table of ω to get the indiscernibility table of ω.*
(4) *Check the consistency of the table of ω. Identify the core attributes and delete the rows of dispensable attributes.*
(5) *Evaluate $\underline{(F, G, CORE(A))}$ and $\overline{(F, G, CORE(A))}$ for the fuzzy bipolar soft set $(F, G, CORE(A))$ obtained in step 4, using the equivalence relation $R = IND(CORE(A))$. Also find the values $\underline{n_j}$ and $\overline{n_j}$.*
(6) *Find the decision values $d_j = \underline{n_j} + \overline{n_j}$ for each object $x_j \in U$.*
(7) *Construct the decision table having columns of U and the decision parameter D only, by rearranging in the descending order with respect to the decision values d_j. Choose k and l, so that $d_k = \max_j d_j$ and $d_l = \min_j d_j$. Then, x_k is the best optimal object, while x_l is the worst optimal object to be decided. If k has more than one values, then any one of x_k's can be chosen.*

For the illustration, we apply this algorithm to an example.

Example 3 Let $U = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ be a collection of some construction companies considered by Mr. X for the construction of his home and consider the attribute set $E = \{e_1 =$ strong structure, $e_2 =$ innovative designs, $e_3 =$ high-quality materials, $e_4 =$ good reputation, $e_5 =$ well organized, $e_6 =$ competitive pricing, $e_7 =$ having own crew, $e_8 =$ decisiveness, $e_9 =$ flexibility, $e_{10} =$ skilled crew$\}$ and $\neg E = \{\neg e_1 =$ weak structure, $\neg e_2 =$ traditional designs, $\neg e_3 =$ low-quality materials, $\neg e_4 =$ ill reputation, $\neg e_5 =$ disorganized, $\neg e_6 =$ high pricing, $\neg e_7 =$ not having own crew, $\neg e_8 =$ indecisive, $\neg e_9 =$ rigidity, $\neg e_{10} =$ unskilled crew$\}$. Let the "Quality Analysis" of construction work be described by a bipolar soft set $\omega = (F, G, A)$.

(1) Input $A = \{e_1, e_2, e_4, e_6, e_7, e_{10}\}$.
(2) Input the bipolar soft set $\omega = (F, G, A)$ as shown in Table 2.
(3) The indiscernibility table of ω is given in Table 3:
(4) We find that

$$IND(A) = \{(c_1, c_1), (c_2, c_2), (c_3, c_3), (c_4, c_4), (c_5, c_5), (c_6, c_6)\} = IND(N)$$

which indicates that the indiscernibility table of ω is consistent. Also note that $CORE(A) = A$.

Table 4 Calculation of decision values

U	c_1	c_2	c_3	c_4	c_5	c_6
e_1	(0.6, 0.2)	(0.5, 0.5)	(0.6, 0.3)	(0.3, 0.5)	(0.6, 0.2)	(0.4, 0.4)
e_2	(0.6, 0.4)	(0.5, 0.4)	(0.6, 0.2)	(0.7, 0.3)	(0.5, 0.5)	(0.3, 0.4)
e_4	(0.7, 0.1)	(0.4, 0.4)	(0.6, 0.2)	(0.3, 0.5)	(0.5, 0.4)	(0.4, 0.4)
e_6	(0.5, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.5, 0.4)
e_7	(0.4, 0.5)	(0.3, 0.6)	(0.6, 0.2)	(0.7, 0.2)	(0.6, 0.4)	(0.4, 0.4)
e_{10}	(0.7, 0.1)	(0.6, 0.3)	(0.5, 0.3)	(0.5, 0.4)	(0.4, 0.5)	(0.3, 0.5)
N	1.7	0.4	1.6	0.9	0.5	−0.2
D	3.4	0.8	3.2	1.8	1.0	−0.4

Table 5 Decision table of ω

U	D
c_1	3.4
c_3	3.2
c_4	1.8
c_5	1.0
c_2	0.8
c_6	−0.4

(5) Note that $R = IND(CORE(A))$ is the identity relation. So ω is $R-$definable by Theorem 5, that is, $\underline{\omega} = \overline{\omega}$. This gives $\underline{n_j} = n_j = \overline{n_j}$ for each $c_j \in U$.

(6) The decision values are $d_j = \underline{n_j} + \overline{n_j} = 2n_j$ for each $c_j \in U$. These values are calculated in Table 4.

(7) Table 5 is the decision table.

We get $\max_j d_j = d_1 = 3.4$ and $\min_j d_j = d_6 = -0.4$. Hence, $k = 1$ and $l = 6$. Thus, the company c_1 is the best selection. If Mr. X could not make a deal with c_1 for some reason, then c_3 will be the second best decision. But, in any case, he must not go for c_6.

6 Conclusions

The rough set theory is emerging as a powerful theory and has diverse applications in many areas. On the other hand, the fuzzy bipolar soft sets are a suitable mathematical model to handle the uncertainty along with the bipolarity, that is, the positivity and negativity of the information or data. In this study, we have applied Pawlak's concept of rough sets on the fuzzy bipolar soft sets and introduced the rough fuzzy bipolar soft sets by defining the rough approximation of a fuzzy bipolar soft set in a Pawlak approximation space. This work may be viewed as the extension of Feng et al. (2010a). We have also examined their structural properties and investigated how the equivalence relation affects the rough approximations of a fuzzy bipolar soft set. In addition, some similarity relations between the fuzzy bipolar soft sets regarding their rough approximations are studied. At the end, an application of the rough fuzzy bipolar soft sets in a decision-making problem is presented and an algorithm for that application is proposed. This algorithm not only decides for the best object, but is also capable of identifying the worst object so that the worst decision may also be avoided. We have applied this algorithm to an example. Further study can be done to investigate the roughness in different bipolar fuzzy and soft substructures of U to establish fruitful results utilizing the notions put forth.

Compliance with ethical standards

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