

# Chance-constrained random fuzzy CCR model in presence of skew-normal distribution

Behrokh Mehrasa<sup>1</sup> · Mohammad Hassan Behzadi<sup>1</sup>

Published online: 20 October 2017  
© Springer-Verlag GmbH Germany 2017

**Abstract** Data envelopment analysis (DEA) is a mathematical method to evaluate the performance of decision-making units. In the classic DEA theory, assume deterministic and precise values for the input and output observations; however, in the real world, the observed values of the inputs and outputs data are mainly fuzzy and random. In the present paper, the fuzzy data were assumed random with a skew-normal distribution, whereas previous works have been based on the assumption of data normality, which might not be true in practice. Therefore, the use of a normal distribution would result in an incorrect conclusion. In the present work, the random fuzzy DEA models were investigated in two states of possibility–probability and necessity–probability in the presence of a skew-normal distribution with a fuzzy mean and a fuzzy threshold level. Finally, a set of numerical example is presented to demonstrate the efficacy of procedures and algorithms.

**Keywords** Data envelopment analysis · Random fuzzy variable · Skew-normal distribution · Possibility–probability · Necessity–probability

## 1 Introduction

Efficiency and productivity measurement of organizations have attracted extensive attention in recent years. DEA is

a nonparametric method with multiple inputs and outputs for evaluating the performance of commercial units; furthermore, it is a mathematical method for measuring relative efficiency of DMUs. CCR model is one of the fundamental models in DEA, which was initially proposed by [Charnes et al. \(1978\)](#). Since then, due to the extensive use of DEA in various problems, several models have been presented in this regard. Many researchers assume that the input and output data are constant and without change, while in practice, the input and output data are mainly random fuzzy (Ra-Fu). So far, various methods have been presented for solving DEA problems with fuzzy data ([Zhu 2003](#); [Wu et al. 2006](#)). [Zadeh \(1978\)](#) defined the possibility theory for fuzzy sets as a mathematical framework in modeling.

A normal distribution is a continuous distribution which is extremely important in statistics because of its behavior. It has numerous characteristics which increase the applicability. In their previous studies on DEA models with random fuzzy data, researchers have assumed that input and output variables follow a normal distribution ([Tavana et al. 2012, 2013a, b](#); [Khanjani et al. 2014a, b, 2017](#)), while this may not be true in the real world and data may be slightly skewed and asymmetric. Therefore, if an asymmetrical distribution has similar characteristics to a normal one, it can play a major role in data analysis.

One of these distributions is the skew-normal (SN) distribution, first introduced by [Azzalini \(1985\)](#) and widely considered by other researchers. This distribution has a skewness regulation parameter which, if set to 0, can result in a normal distribution. Thus, a SN distribution also includes a normal distribution. In random fuzzy environments, inputs and outputs are random fuzzy, and such data cannot be directly applied in DEA models because they lead to incorrect conclusions. In the present paper, it was assumed that the input and output variables were random fuzzy and fol-

---

Communicated by V. Loia.

---

✉ Mohammad Hassan Behzadi  
Behzadi@srbiau.ac.ir

<sup>1</sup> Department of Statistics, Science and Research branch, Islamic Azad University, Tehran, Iran

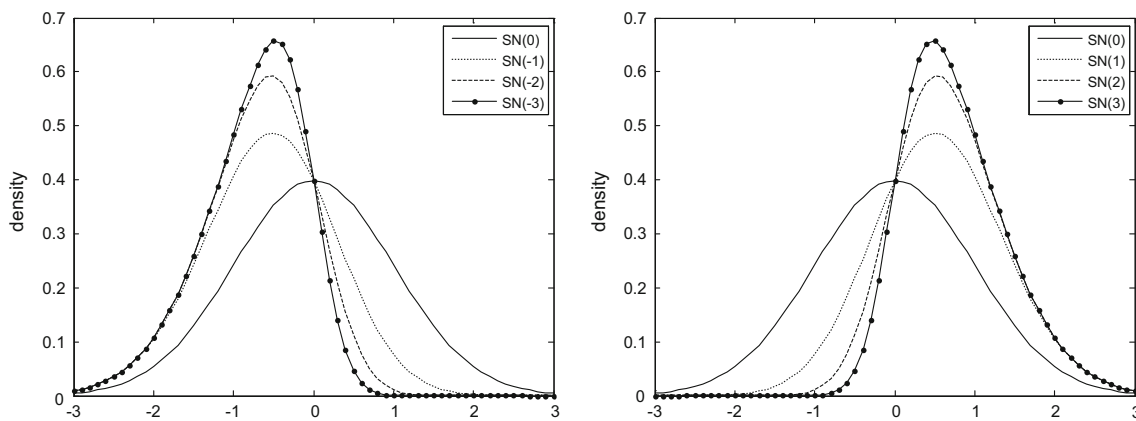


Fig. 1 Density functions of SN distribution for  $\delta$ s difference

lowed a skew-normal distribution with a fuzzy mean and a fuzzy cutoff level. Accordingly, the models were presented in possibility–probability and necessity–probability states and, finally, the proposed models were investigated in two numerical examples.

The rest of this paper is organized as follows. In the next section, the normal-skewed distribution and primary fuzzy definitions will be introduced. Section (2) presents some models in possibility–probability and necessity–probability states in the presence of the skew-normal distribution. In Sect. (4), the models presented in the previous section will be expressed in one numerical example, and finally, Sect. (5) includes the discussion and conclusion.

### 2 Preliminaries

In this section, we first review some basic definitions of fuzzy sets (Dubois and Prade 1978; Klir and Yuan 1995; Zimmermann 2001) followed by several definitions associated with fuzzy random variables (FRVs) (Kwakernaak 1978; Liu and Liu 2002, 2003) and the SN distribution (Azzalini 1985; Azzalini and Capitano 1999).

**Definition 1** SN Distribution: A random variable  $Z$  is said to have a standard SN distribution with real parameters  $\delta$ , denoted by  $Z \sim \text{SN}(\delta)$ , if its probability density function (pdf) is given by

$$f_Z(z) = 2\phi(z)\Phi(\delta z), \tag{1}$$

where  $\phi$  and  $\Phi$  are the pdf and cumulative distribution function (cdf) of standard normal distribution, respectively. The shape parameter  $\delta$  is called skewness control parameter. The pdf (1) for  $\delta > 0$  is skewness to right,  $\delta < 0$  skewness to left and, also, if  $\delta = 0$  then it is symmetric and is transformed into standard normal distribution. The pdf (1) for  $\delta$ s difference is drawn in Fig. 1.

With transformation  $W = \mu + \sigma Z, \mu \in \Re, \sigma > 0$  pdf of random variable  $W$  with location parameter  $\mu$ , scale parameter  $\sigma$  and skew parameter  $\delta$  (henceforth called the  $W \sim \text{SN}(\mu, \sigma^2, \delta)$  distribution) is:

$$f_W(w) = \frac{2}{\sigma} \phi\left(\frac{w-\mu}{\sigma}\right) \Phi\left(\delta\left(\frac{w-\mu}{\sigma}\right)\right). \tag{2}$$

If  $W \sim \text{SN}(\mu, \sigma^2, \delta)$ , then mean and variance are as follows:

$$\mathbb{E}(W) = \mu + \sigma\mu_z, \mathbb{V}(W) = \sigma^2(1 - \mu_z^2), \tag{3}$$

where  $\mu_z = \sqrt{\frac{2}{\pi}}\gamma$  and  $\gamma = \frac{\delta}{\sqrt{1+\delta^2}}$ , see Azzalini (1985).

**Definition 2** Let  $U$  be a universe set. A fuzzy set  $\tilde{A}$  of  $U$  is defined by a membership function  $\mu_{\tilde{A}}(x) \rightarrow [0, 1]$ , where  $\mu_{\tilde{A}}(x), \forall x \in U$ , indicates the degree of membership of  $A$  to  $U$ .

**Definition 3** The  $\alpha$ -cut of fuzzy set  $\tilde{A}, \tilde{A}_\alpha$ , is the crisp set  $\tilde{A}_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha\}$ . The support of  $\tilde{A}$  is the crisp set  $\text{Sup } \tilde{A} = \{x | \mu_{\tilde{A}}(x) \geq 0\}$ .

**Definition 4**  $\tilde{A}$  is a fuzzy number if and only if  $\tilde{A}$  is normal and convex fuzzy subset of the set of real numbers.

**Definition 5** A fuzzy number of  $L$ – $R$  type is denoted by  $\tilde{A} = (m, \alpha, \beta)_{LR}$  and its membership function can be expressed as

$$\mu_{\tilde{A}}(t) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & x \leq m, \\ R\left(\frac{x-m}{\beta}\right) & x \geq m. \end{cases} \tag{4}$$

where  $L$  and  $R$  are the left and right functions, respectively, and  $\alpha$  and  $\beta$  are the (nonnegative) left and right spreads, respectively.

**Definition 6** The  $\alpha$ -cut,  $\alpha \in [0, 1]$ , of a  $L$ - $R$  type fuzzy number  $\tilde{A}$  is a closed interval as follows

$$A_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\} = [A_\alpha^L, A_\alpha^R] = [m - L^{-1}(\alpha), m + R^{-1}(\alpha)] \tag{5}$$

where  $A_\alpha^L$  and  $A_\alpha^R$  are the left and right extreme points, respectively.

**Definition 7** Let  $(\Omega, A, Pr)$  be probability space where  $\Omega$  is a sample space,  $A$  is the  $\sigma$ -algebra of subset of  $\Omega$  and  $Pr$  is a probability measure on  $\Omega$ , and  $F(R)$  be the set of all fuzzy numbers in the set of real numbers  $R$ . Generally,  $F$  involves the normal convex fuzzy subsets. Thus, a fuzzy random variable (FRV) is a mapping function  $\xi : \Omega \rightarrow F$  such that for any Borel set  $B$  of  $\mathcal{R}$ , and

**Definition 8** Let  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$  be a fuzzy random vector, and  $f : \mathfrak{N}^n \rightarrow \mathfrak{N}$  be a continuous function, then  $f(\xi)$  is FRV.

**Definition 9** Given a universe set  $U$ , let  $P(U)$  be a power set of  $U$ .  $(U, P(U), Pos)$ . The triple  $(U, P(U), Pos)$  is called a possibility space where  $Pos$  is a possibility measure defined on  $P(U)$ . For any set  $A$  and  $B$ , the properties of the possibility measure are presented as

- a.  $Pos(\emptyset) = 0$  and  $Pos(U) = 1$ ;
- b. Monotonicity;  $A \subset B$  implies  $Pos(A) \leq Pos(B)$  for any  $A, B \in P(U)$ ;
- c. Subadditivity;  $Pos(A \cup B) + Pos(A \cap B) \leq Pos(A) + Pos(B)$  for any  $A, B \in P(U)$ .

The necessity measure of  $A$ , denoted by  $Nec(A)$ , is also defined on  $P(U)$  as  $Nec(A) = 1 - Pos(A^c)$  where  $A^c$  is a complement set of  $A$ . For any set  $A$  and  $B$ , the properties of the necessity measure are presented as,

- a.  $Nec(\emptyset) = 0$  and  $Nec(U) = 1$ ;
- b.  $Pos(A) \geq Nec(A)$ ;
- c. Monotonicity;  $A \subset B$  implies  $Nec(A) \leq Nec(B)$ ;
- d. Subadditivity;  $Nec(A \cup B) + Nec(A \cap B) \geq Nec(A) + Nec(B)$ .

### 3 DEA with random fuzzy data

In this section, the CCR model of DEA with random fuzzy data will be generalized. This model will be presented with possibility–probability and necessity–probability constraints with the possibility level and fuzzy necessity.

In the general state, the probability level might be inaccurate and indefinite. In this section, a DEA model with a fuzzy

threshold level will be generalized; in fact, it is assumed that  $\tilde{\delta}$  is a fuzzy number.

#### 3.1 Possibility–probability CCR model in the presence of SN distribution with Fuzzy threshold level

In this section, we evaluate the performance of decision-making units (DMUs) with random fuzzy inputs and outputs. Assume that we have  $n$  DMUs such that all DMUs are independent from one another for the  $i$ th random fuzzy input,  $i = 1, 2, \dots, m$  and the  $r$ th random fuzzy output,  $r = 1, 2, \dots, s$ . Also, assume  $\tilde{x}_{ij} \sim \text{CSN}_{1,1}(x_{ij}, \sigma_{ij}^2, \delta_{ij}, 0, \sigma_{ij}^2)$  in which there are  $\tilde{x}_{ij} = (x_{ij}^\alpha, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^\beta)_{LR}$  per  $j = 1, 2, \dots, n$ , and  $\tilde{y}_{rj} \sim \text{CSN}_{1,1}(\bar{y}_{rj}, \tau_{rj}^2, \varepsilon_{rj}, 0, \tau_{rj}^2)$ , in which there are  $\tilde{x}_{ij} = (x_{ij}^\alpha, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^\beta)_{LR}$ . Let us assume that  $\tilde{\delta}$  is illustrated as in  $\tilde{\delta} = (\delta^\alpha, \delta^{m_1}, \delta^{m_2}, \delta^\beta)$ ; then, the CCR model with possibility–probability constraints will be as model (6):

$$\begin{aligned} & \max \varphi \\ & \text{s.t.} \\ & \text{Pos} \left[ \Pr \left( \varphi \leq \sum_{r=1}^s u_r \tilde{y}_{rp} \right) \geq \tilde{\delta} \right] \geq \gamma, \\ & \text{Pos} \left[ \Pr \left( \sum_{i=1}^m v_i \tilde{x}_{ip} \leq 1 \right) \geq \tilde{\delta} \right] \geq \gamma, \\ & \text{Pos} \left[ \Pr \left( \sum_{i=1}^m v_i \tilde{x}_{ip} \geq 1 \right) \geq \tilde{\delta} \right] \geq \gamma, \\ & \text{Pos} \left[ \Pr \left( \sum_{r=1}^s u_r \tilde{y}_{rj} - \sum_{i=1}^m v_i \tilde{x}_{ij} \leq 0 \right) \geq \tilde{\delta} \right] \geq \gamma, \quad j = 1, \dots, n \\ & u_r, v_i \geq 0, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \tag{6}$$

where  $\tilde{x}_{ij}$  and  $\tilde{y}_{rj}$  have a closed skew-normal distribution with fuzzy mean and  $\tilde{\delta}$  is a fuzzy threshold level. To convert this model into a definite model, the following theorem is used.

**Theorem 1** Assume that  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_2$  are two independent random fuzzy numbers with a closed skew-normal distribution as  $\tilde{\lambda}_1 \sim \text{CSN}(\bar{\mu}_1, \sigma_1^2, \delta_1, 0, \sigma_1^2)$  and  $\tilde{\lambda}_2 \sim \text{CSN}(\bar{\mu}_2, \sigma_2^2, \delta_2, 0, \sigma_2^2)$ . Then, the means will be as  $\bar{\mu}_1 = (\lambda_1^\alpha, \lambda_1^{m_1}, \lambda_1^{m_2}, \lambda_1^\beta)_{LR}$  and  $\bar{\mu}_2 = (\lambda_2^\alpha, \lambda_2^{m_1}, \lambda_2^{m_2}, \lambda_2^\beta)_{LR}$ ; thus, we have:

(a) If  $\text{Pos} \left[ \Pr \left( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \right) \geq \tilde{\delta} \right] \geq \alpha$  is true, then:

$$\left( \lambda_1^{m_1} - \lambda_2^{m_2} + L^{-1}(\alpha) \left( \lambda_1^\alpha + \lambda_2^\beta \right) \right) + \left( \sqrt{\sigma_1^2 + \sigma_2^2} \right) \Psi^{-1} \left( \delta^{m_1} - L^{-1}(\alpha) \delta^\alpha \right) \leq 0, \quad (7)$$

where

$$\Psi^{-1}(\delta^{m_1} - L^{-1}(\alpha) \delta^\alpha) \int_{-\infty}^{\delta} f_\Delta(\delta) d\delta = \delta^{m_1} - L^{-1}(\alpha) \delta^\alpha.$$

(b) If  $\text{Nec} \left[ \Pr \left( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \right) \geq \tilde{\delta} \right] \geq \alpha$  is true, then:

$$\left( \lambda_1^{m_2} - \lambda_2^{m_1} + R^{-1}(1 - \alpha) \left( \lambda_2^\alpha + \lambda_1^\beta \right) \right) + \left( \sqrt{\sigma_1^2 + \sigma_2^2} \right) \Psi^{-1} \left( \delta^{m_2} + R^{-1}(\alpha) \delta^\beta \right) \leq 0. \quad (8)$$

where

$$\Psi^{-1}(\delta^{m_2} + R^{-1}(\alpha) \delta^\beta) \int_{-\infty}^{\delta} f_\Delta(\delta) d\delta = \delta^{m_2} + R^{-1}(\alpha) \delta^\beta$$

Note: Normal case of this theorem introduced by Tavana et al. (2013a, b).

*Proof* Assume that  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_2$  are two independent random fuzzy numbers with a closed skew-normal (CSN) distribution as  $\tilde{\lambda}_1 \sim \text{CSN}(\tilde{\mu}_1, \sigma_1^2, \delta_1, 0, \sigma_1^2)$  and  $\tilde{\lambda}_2 \sim \text{CSN}(\tilde{\mu}_2, \sigma_2^2, \delta_2, 0, \sigma_2^2)$ ; thus,  $\tilde{h} = \tilde{\lambda}_1 - \tilde{\lambda}_2$  will have a closed skew-normal distribution with the following parameters:

$$\mu_{\tilde{h}} = E \left[ \tilde{\lambda}_1 - \tilde{\lambda}_2 \right] = \tilde{\mu}_1 - \tilde{\mu}_2 = \left( \lambda_1^\alpha + \lambda_2^\beta, \lambda_1^{m_1} - \lambda_2^{m_2}, \lambda_1^{m_2} - \lambda_2^{m_1}, \lambda_1^\beta + \lambda_2^\alpha \right)$$

$$\sigma_{\tilde{h}}^2 = \text{Var} \left[ \tilde{\lambda}_1 - \tilde{\lambda}_2 \right] = \sigma_1^2 + \sigma_2^2$$

$$D^* = \frac{1}{\sigma_1^2 + \sigma_2^2} \begin{bmatrix} \delta_1 \sigma_1^2 \\ -\delta_2 \sigma_2^2 \end{bmatrix}$$

$$v^* = [0, 0]'$$

$$\Delta^* = \begin{bmatrix} \sigma_1^2 + \delta_1^2 \sigma_1^2 - \frac{\delta_1^2 \sigma_1^4}{\sigma_1^2 + \sigma_2^2} & \frac{\delta_1 \delta_2 \sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \\ \frac{\delta_1 \delta_2 \sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} & \sigma_2^2 + \delta_2^2 \sigma_2^2 - \frac{\delta_2^2 \sigma_2^4}{\sigma_1^2 + \sigma_2^2} \end{bmatrix}$$

The CSN distribution  $\tilde{h}$  with mean fuzzy parameters is as follows:

$$f(\tilde{h}) = c\phi \left( \tilde{h}; \mu_{\tilde{h}}, \sigma_{\tilde{h}}^2 \right) \Phi_2 \left( D^* \left( \tilde{h} - \mu_{\tilde{h}} \right); v^*, \Delta^* \right).$$

$$c^{-1} = \Phi_2 \left( 0; v^*, \Delta^* + D^* \sigma_{\tilde{h}}^2 D^{*'} \right)$$

So that in this state and in  $c^{-1} = 2^{-2}$ ,

By changing the variable  $W = \frac{\tilde{h} - \mu_{\tilde{h}}}{\sigma_{\tilde{h}}}$ , we will have the following equation:

$$\begin{aligned} & \left[ P \left( \tilde{h} \leq 0 \right) \right]_\alpha \\ &= \left\{ \int_{-\infty}^{\frac{-\mu_{\tilde{h}}}{\sigma_{\tilde{h}}}} f(w) dw \mid \mu_{\tilde{h}} \in \left( \mu_{\tilde{h}} \right)_\alpha; \int_{-\infty}^{+\infty} f(w) dw = 1 \right\} \\ &= \left\{ \Psi \left( \frac{-\mu_{\tilde{h}}}{\sigma_{\tilde{h}}} \right) \mid \mu_{\tilde{h}} \in \left( \mu_{\tilde{h}} \right)_\alpha; \int_{-\infty}^{+\infty} f(w) dw = 1 \right\} \end{aligned}$$

And  $\Psi$  is the cumulative distribution function (CDF) of the standard closed skew-normal distribution of variable  $W$ . Also,  $\Psi$  is an additive continuous function. Using the definition of  $\alpha$ -cut, we will have the following equation:

$$\begin{aligned} & \left[ P \left( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \right) \right]_\alpha = \left[ \Psi \left( \frac{-\mu_{\tilde{h}}}{\sigma_{\tilde{h}}} \right)_\alpha^R, \Psi \left( \frac{-\mu_{\tilde{h}}}{\sigma_{\tilde{h}}} \right)_\alpha^L \right] \\ & \text{Pos} \left[ \Pr \left( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \right) \geq \tilde{\delta} \right] \geq \alpha \Rightarrow \left[ \Pr \left( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \right) \right]_\alpha^R \geq \left( \tilde{\delta} \right)_\alpha^L \\ & \Psi \left( \frac{-\mu_{\tilde{h}}}{\sigma_{\tilde{h}}} \right)_\alpha^L \geq \left( \tilde{\delta} \right)_\alpha^L \Leftrightarrow \left( \mu_{\tilde{h}} \right)_\alpha^L + \sigma_{\tilde{h}} \Psi^{-1} \left( \left( \tilde{\delta} \right)_\alpha^L \right) \leq 0 \end{aligned}$$

where

$$\begin{aligned} \left( \mu_{\tilde{h}} \right)_\alpha^L &= \lambda_1^{m_1} - \lambda_2^{m_2} + L^{-1}(\alpha) \left( \lambda_1^\alpha + \lambda_2^\beta \right) \\ \sigma_{\tilde{h}} &= \sqrt{\sigma_1^2 + \sigma_2^2} \\ \left( \tilde{\delta} \right)_\alpha^L &= \delta^{m_1} - L^{-1}(\alpha) \delta^\alpha \end{aligned}$$

Proof of the second part is similar to the first part.  $\square$

Now, let us assume that there are  $n$  DMUs with  $m$  inputs and  $s$  outputs as follows:

$$\begin{aligned} \tilde{x}_{ij} &\sim \text{CSN}_{1,1} \left( \tilde{x}_{ij}, \sigma_{ij}^2, \delta_{ij}, 0, \sigma_{ij}^2 \right), \bar{x}_{ij} \\ &= \left( x_{ij}^\alpha, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^\beta \right)_{LR} \end{aligned}$$

$$\begin{aligned} \tilde{y}_{rj} &\sim \text{CSN}_{1,1} \left( \tilde{y}_{rj}, \tau_{rj}^2, \varepsilon_{rj}, 0, \tau_{rj}^2 \right), \bar{y}_{rj} \\ &= \left( y_{rj}^\alpha, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^\beta \right)_{LR} \end{aligned}$$

Using the above theorem on the constraints of model (6), the following definite model will be obtained:

**Table 1** Parameters of inputs SN distribution

DMUs	$X_{1j} \sim SN(\bar{\mu}, \sigma^2, \delta)$	$X_{2j} \sim SN(\bar{\mu}, \sigma^2, \delta)$	$X_{3j} \sim SN(\bar{\mu}, \sigma^2, \delta)$
DMU1	SN((8.999,16.8,999), 0.0033, - 0.1)	SN(12.15,12), 5.052, - 10.455)	SN((7.056,16.20.1), 0.044, 0.683)
DMU2	SN((9.477,11.33.4), 0.06, - 0.973)	SN((9.081,16.23.56), 19.211, - 17.716)	SN((0.415.0.7.0.87), 0.019, - 0.009)
DMU3	SN((4.366,17.25.09), 0.103, - 0.971)	SN((4.192,11.2.23.15), 10.702, 17.971)	SN((8.145,10.04,14.16), 0.0551, - 0.594)
DMU4	SN((9.677,10.09,13.0034), 0.0871, 0.985)	SN((12.82213,15.36,18.08), 5.954, - 11.5)	SN((58.486,59.22,60.21), 0.067, - 0.177)
DMU5	SN (6.944 ,7.012,7.098), 0.114, 1)	SN(7.827,8.657,4.326), 1.367, - 5.832)	SN((9.505,9.74,9.998), 2.893, - 6.97)
DMU6	SN((7.004,7.665,8.85), 0.084, - 0.964)	SN((18.687,18.94,19.01), 1.444, 3.413)	SN((565.63,567.12,589.34), 33.937, - 17.255)
DMU7	SN((10.973,11.32,12.67), 0.009, 0.999)	SN((24.588,25.44,28.45), 9.758, - 16.394)	SN((548.45,557.55,598.44), 16.506, - 17.566)
DMU8	SN((15.002,16.76,16.89), 0.03, 0.707)	SN((119.557,121.45,132.22), 61.342, - 18.595)	SN((13.956,14.32,15.33), 0.031, 0.599)
DMU9	SN((7.707,7.67,8.55), 0.05, - 0.268)	SN((31.479,32.44,33.09), 13.745, - 15.666)	SN((358.35,369.66,370.31), 30.974, - 18.32)
DMU10	SN((18.925,19.65,19.98), 0.04, - 0.367)	SN((41.726,45.47,45.95), 17.995, - 10.568)	SN((10.573,10.65,11.58), 0.092, - 1)
DMU11	SN((17.893,17.953,18.05), 0.038, - 0.659)	SN((40.652,40.688,42.36), 17.838, 18.858)	SN((23.984,23.98,24.75), 0.014, 1.599)
DMU12	SN((19.657,19.89,23.53), 0.041, - 0.979)	SN((95.136,96.23,97.07), 51.122, - 16.125)	SN((120.344,120.87,121.01), 33.828, 16.314)
DMU13	SN((16.388,16.97,17.03), 0.449, - 3.463)	SN((16.239,16.543,17.17), 5.581, - 10.948)	SN((74.813,74.98,76.84), 1.548, - 6.773)
DMU14	SN((6.636,6.85,7.53), 0.072, 1.618)	SN((29.417,30.31,31.65), 11.234, - 15.63)	SN((112.443,113.58,115.43), 1.128, - 4.938)
DMU15	SN((18.031,18.95,19.09), 0.067, 1.054)	SN((48.254,48.95,50.05), 26.871, 18.853)	SN((56.218,57.43,59.09), 0.336, - 1.798)
DMU16	SN((7.703,7.89,8.03), 0.094, 0.992)	SN((47.9,49.85,50.23), 46.122, 3.882)	SN((38.15,39.06,40.74), 19.963, - 18.574)
DMU17	SN((8.989,9.73,9.977), 0.0175, - 0.755)	SN((26.67,27.85,27.98), 13.721, - 16.834)	SN((167.43,168.76,169.01), 1.587, - 6.582)
DMU18	SN((13.06,13.84,14.16), 0.054, 1.044)	SN((35.42,35.87,36.006), 16.786, - 17.244)	SN((6.637,7.98,8.32), 0.126, - 0.979)
DMU19	SN((9.55,10.57,11.03), 0.08, 0.633)	SN((21.759,23.64,24.83), 10.996, - 15.647)	SN((30.914,31.49,33.31), 0.016, - 0.4)
DMU20	SN((6.762,7.73,8.09), 0.057, 0.488)	SN((13.198,14.58,16.83), 5.102, - 10.354)	SN((10.559,11.32,13.74), 0.733, - 9.191)
DMU21	SN((7.737,7.97,8.941), 0.002, 1.432)	SN((25.305,26.65,27.79), 1.525, 1.341)	SN((60.16,61.13,62.69), 3.918, 1.65)
DMU22	SN((7.001,7.96,8.36), 0.009, - 0.461)	SN((60.543,60.75,62.41), 272.308, 5.761)	SN((160.87,160.47,162.33), 4.127, 3.045)
DMU23	SN((11.94,12.56,14.53), 0.023, 2.11)	SN((20.97,21.42,22.64), 316.476, - 10.361)	SN((8.231,8.43,9.56), 1.0838, - 1.231)
DMU24	SN((7.3.8.2.9.8), 0.009, 0.671)	SN((18.402,19.54,20.84), 630.893, - 6.219)	SN((160.874,163.85,164.28), 4.66, 2.075)
DMU25	SN((14.511,15.55,17.48), 0.018, 1.275)	SN((25.86,26.63,27.31), 137.986, 10.671)	SN((175.16,176.65,174.12), 44.856, - 5.821)

**Table 2** Parameters of outputs SN distribution

DMUs	$Y_{1j} \sim SN(\bar{\eta}_1, \tau^2, \varepsilon)$	$Y_{2j} \sim SN(\bar{\eta}_2, \tau^2, \varepsilon)$	$Y_{3j} \sim SN(\bar{\eta}_3, \tau^2, \varepsilon)$	$Y_{4j} \sim SN(\bar{\eta}_4, \tau^2, \varepsilon)$	$Y_{5j} \sim SN(\bar{\eta}_5, \tau^2, \varepsilon)$
DMU1	SN((134.3,135.3,137.9), 22.71, -19.8)	SN((47.491,48.85,49.08), 3.122, -8.1)	SN((3.706,3.98,4.76), 0.0523, 0.2)	SN((3.812,3.54,5.61), 0.0519, -1.1)	SN((28.47,29.53,30.01), 1.152, -5.47)
DMU2	SN((53.49,54.64,55.97), 3.028, 8.61)	SN((72.59,74.42,76.03), 0.203, -1.27)	SN((2.06,2.09,3.01), 0.011, 0.657)	SN((1.72,2.86,3.01), 0.096, -1.577)	SN((4.648,4.95,5.63), 0.059, -0.97)
DMU3	SN((262.286,293), 50.746, 23.18)	SN((101.9,102.3,103.01), 9.54, -14.38)	SN((5.923,6.73,7.18), 0.009, 0.12)	SN((11.355,12.46,13.09), 0.105, 1.36)	SN((3.08,4.76,5.71), 0.062, -0.013)
DMU4	SN((135.3,137.5,138.9), 6.2, 12.261)	SN((43.975,44.67,45.97), 0.192, 1.46)	SN((3.673,3.76,4.09), 0.09, 0.245)	SN((3.941,4.65,5.82), 0.072, -0.18)	SN((61.26,64.79,68.9), 1.696, -5.05)
DMU5	SN((92.767,96.42,97.06), 2.204, -6.1)	SN((30.8,31.23,32.58), 0.258, -0.662)	SN((1.69,2.3,3.13), 0.049, 0.419)	SN((8.4419,34,10.53), 0.161, -1.63)	SN((61.56,62.32,63.90), 1.42, -4.32)
DMU6	SN((125.9,126.3,128), 3.09, 4.933)	SN((86.673,87.4,88.3), 11.988, 18)	SN((9.801,9.32,10), 1.151, -5.35)	SN((39.703,40.54,41.2), 0.561, -2.8)	SN((286.4,287.43,289.08), 4.257, -10.4)
DMU7	SN((188.211,189.87,190.87), 33.8, -18)	SN((74.5,76.3,78.9), 7.98, -12.201)	SN((5.457,6.74,7.009), 0.315, 2.57)	SN((14.15,16), 0.182, -1.805)	SN((6.01,7.45,8.211), 0.434, -2.319)
DMU8	SN((720.24,721.34,725), 77.8, -18.5)	SN((213.2,216.218), 12.96, -16.403)	SN((31.98,32.46,33.69), 1.799, 6.14)	SN((12.56,13.53,14.07), 0.591, -3.3)	SN((358.9,359.09,360.98), 2.936, -9.23)
DMU9	SN((546.549,560), 62.37, -10.82)	SN((82.521,84.54,85.96), 2.456, -7.8)	SN((19.639,20.75,24.42), 0.34, 3.038)	SN((82.95484,43.86,86), 1.4, -5.569)	SN((561.96,562.32,564.14), 37.27, -18)
DMU10	SN((123.126,129), 19.6, -18.86)	SN((20.4,21.3,22), 36.5, 27.192)	SN((23.49,23.46,24.76), 1.067, 5.3)	SN((82.8,87.5,86.2), 3.038, -8.399)	SN((598.601,602), 23.752, -15.09)
DMU11	SN((128.137,140), 411.89, 0.9)	SN((158.139,159.683,164), 43.63, -19)	SN((167.176,180), 6.988, -11.21)	SN((6.179,7.562,8.09), 2.723, -7.64)	SN((114.6,117.5,120), 7.142, -12.5)
DMU12	SN((188.211,189.8,230), 101, -17.513)	SN((244,247,250), 10.306, -12.74)	SN((46.5,48.3,50.9), 0.271, 0.973)	SN((28.52,29.6,32), 1.95, -6.006)	SN((93.9,96.4,97.1), 18.4, -13.336)
DMU13	SN((459.5,458.1,460), 59.864, -1.3)	SN((99.534,103.1,110), 11.317, -15)	SN((7.987,8.341,9.54), 0.724, 4.31)	SN((27.084,28.43,30.71), 2.705, -7.5)	SN((326.7,329.5,330.4), 36.44, -21.2)
DMU14	SN((255.99,260.4,263.61), 7.3, -12.52)	SN((82.8,85.3,89.1), 8.609, -12.75)	SN((7.825,8.43,10.1), 0.082, 0.973)	SN((5.769,6.481,7.761), 0.085, -1.24)	SN((35.072,37.48,38.30), 0.262, -1.6)
DMU15	SN((98.103,110), 4431, -18.67)	SN((162.884,168.4,170.4), 5.8, -9.77)	SN((65.34,67.5,70.1), 1.753, 6.31)	SN((129.131,134), 6.603, -9.228)	SN((916.9,18,920), 47.75, -19.812)
DMU16	SN((703.544,709.8,710.2), 15.6, -13)	SN((154.7,156.4,159.6), 13.2, -15.66)	SN((35.18,37.4,39.6), 0.296, 1.6)	SN((8.53,9.67,10.3), 1.531, -6.61)	SN((76.07,78.5,81.3), 5.4, -10.458)
DMU17	SN((305.309,312), 16.04, -13.34)	SN((103.112,116), 17.26, -15.295)	SN((10.7,12.5,14.3), 0.068, 0.856)	SN((9.83,10.3,12.4), 1.84, -7.743)	SN((326.330,336), 53.213, -25.51)



Table 2 continued

DMUs	$Y_{1j} \sim \text{SN}(\bar{\eta}, \tau^2, \varepsilon)$	$Y_{2j} \sim \text{SN}(\bar{\eta}, \tau^2, \varepsilon)$	$Y_{3j} \sim \text{SN}(\bar{\eta}, \tau^2, \varepsilon)$	$Y_{4j} \sim \text{SN}(\bar{\eta}, \tau^2, \varepsilon)$	$Y_{5j} \sim \text{SN}(\bar{\eta}, \tau^2, \varepsilon)$
DMU18	SN((257.8,260.4,263.8), 51.17, -19.3)	SN((77.96,79.3,80.5), 3.592, -9.22)	SN((7.706,8.43,10.1), 1.172, 6.048)	SN((7.22,8.55,10.4), 0.152, -1.82)	SN((102.46,105.6,107), 9.21, -13.5)
DMU19	SN((308.4,312.3,316.7), 74.29, -16.3)	SN((68.51,69.6,72.1), 3.968, -10)	SN((4.58,3.42,8.54), 0.067, -0.97)	SN((46.87,49.5,53.1), 3.242, -8.04)	SN((574.6,580.4,583.4), 29.66, -16.3)
DMU20	SN((395.63,403.1,410.1), 4.996, -9.9)	SN((37.93,39.5,42.4), 1.855, -6.98)	SN((11.95,13.5,15.3), 0.108, 1.581)	SN((3.68,4.32,6.75), 0.432, -3.26)	SN((78.96,84.6,90.3), 7.859, -11)
DMU21	SN((50.92,53.8,55.1), 21533, 2.1)	SN((60.69,62.6,65.9), 5.27, 3.129)	SN((3.30,5.62,7.14), 0.002, 0.127)	SN((16.23,17.3,18.5), 0.0016, 1.5)	SN((11.73,12.47,14.62), 1.549, -2.34)
DMU22	SN((161.169,175), 10.66, 1.718)	SN((73.04,75.05,78.1), 69.677, -2.2)	SN((140,146,152), 16.2, 4.832)	SN((12.4,14.3,16.5), 0.037, -1.581)	SN((2.65,3.42,5.33), 0.012, 0.957)
DMU23	SN((336.345,350), 781, 11.751)	SN((80.5,82.1,84), 36.296, 5.123)	SN((175.1,18.4,185.3), 47, 17.1, -2.9)	SN((1.91,3.4,6.2), 1.606, 0.328)	SN((175.8,180.2,183.1), 7.644, 1.981)
DMU24	SN((456.462,473), 7120, 8.673)	SN((22.79,25.37,29.10), 0.159, 3.19)	SN((38.36,42.3,44.58), 0.367, 1.73)	SN((1.99,2.98,5.36), 0.388, 2.11)	SN((50.48,54.38,60.1), 207, 3.238)
DMU25	SN((403.427,457), 16.32, -3.34)	SN((25.6,30.5,35.2), 1.247, -4.289)	SN((4.95,5.96,6.39), 0.002, -0.842)	SN((25.73,26.47,29.51), 0.03, -3.129)	SN((28.03,31.36,34.61), 207.42, -2)

$$\begin{aligned}
 & \max \varphi \\
 & \text{s.t.} \\
 & \varphi - \sum_{r=1}^s u_r \left( y_{rp}^{m_2} - L^{-1}(\gamma) y_{rp}^\beta \right) + \sigma_p(u) \Psi^{-1}(\delta)_\gamma^L \leq 0, \\
 & \sum_{i=1}^m v_i \left( x_{ip}^{m_1} - L^{-1}(\gamma) x_{ip}^\alpha \right) + \sigma_p(v) \Psi^{-1}(\delta)_\gamma^L \leq 1, \\
 & \sum_{i=1}^m v_i \left( x_{ip}^{m_1} + L^{-1}(\gamma) x_{ip}^\alpha \right) - \sigma_p(v) \Psi^{-1}(\delta)_\gamma^L \geq 1, \\
 & \sum_{r=1}^s u_r y_{rj}^{m_1} - \sum_{i=1}^m v_i x_{ij}^{m_2} - L^{-1}(\gamma) \left( \sum_{r=1}^s u_r y_{rj}^\alpha + \sum_{i=1}^m v_i x_{ij}^\beta \right) \\
 & \quad + \sigma_j(u, v) \Psi^{-1}(\delta)_\gamma^L \leq 0, j = 1, \dots, n, \\
 & \quad u_r, v_i, \sigma_p(u), \sigma_p(v), \sigma_j(u, v) \\
 & \quad \geq 0, r = 1, \dots, s; i = 1, \dots, m; j = 1, \dots, n \tag{9}
 \end{aligned}$$

And  $(\delta)_\gamma^L$  is the lower bound of  $\gamma$ -cutoff of the following range:

$$\begin{aligned}
 \delta &= \left[ (\delta)_\gamma^L, (\delta)_\gamma^R \right] = \left[ \delta^{m_1} - L^{-1}(\gamma) \delta^\alpha, \delta^{m_2} + R^{-1}(\gamma) \delta^\beta \right], \\
 \sigma_p(u) &= \sqrt{\sum_{r=1}^s u_r^2 \tau_{rp}^2}, \\
 \sigma_p(v) &= \sqrt{\sum_{i=1}^m v_i^2 \sigma_{ip}^2}, \\
 \sigma_j(u, v) &= \sqrt{\sum_{r=1}^s u_r^2 \tau_{rj}^2 + \sum_{i=1}^m v_i^2 \sigma_{ij}^2}, j = 1, \dots, n.
 \end{aligned}$$

By changing the variables of  $(\theta_p^o)^2 = (\sigma_p(u))^2$ ,  $(\theta_p^l)^2 = (\sigma_p(v))^2$  and  $(\lambda_j)^2 = (\sigma_j(u, v))^2$ , the following second-order programming will be obtained.

$$\begin{aligned}
 & \max \varphi \\
 & \text{s.t.} \\
 & \varphi - \sum_{r=1}^s u_r \left( y_{rp}^{m_2} - L^{-1}(\gamma) y_{rp}^\beta \right) + \theta_p^o \Psi^{-1}(\delta^{m_1} - L^{-1}(\gamma) \delta^\alpha) \leq 0, \\
 & \sum_{i=1}^m v_i \left( x_{ip}^{m_1} - L^{-1}(\gamma) x_{ip}^\alpha \right) + \theta_p^l \Psi^{-1}(\delta^{m_1} - L^{-1}(\gamma) \delta^\alpha) \leq 1, \\
 & \sum_{i=1}^m v_i \left( x_{ip}^{m_1} + L^{-1}(\gamma) x_{ip}^\alpha \right) - \theta_p^l \Psi^{-1}(\delta^{m_1} - L^{-1}(\gamma) \delta^\alpha) \geq 1, \\
 & \sum_{r=1}^s u_r y_{rj}^{m_1} - \sum_{i=1}^m v_i x_{ij}^{m_2} - L^{-1}(\gamma) \left( \sum_{r=1}^s u_r y_{rj}^\alpha + \sum_{i=1}^m v_i x_{ij}^\beta \right) \\
 & \quad + \lambda_j \Psi^{-1}(\delta^{m_1} - L^{-1}(\gamma) \delta^\alpha) \leq 0, j = 1, \dots, n, \\
 & (\theta_p^o)^2 = \sum_{r=1}^s u_r^2 \tau_{rp}^2,
 \end{aligned}$$

**Table 3** Possibility–probability scores of the DMUs with their fuzzy thresholds

$\tilde{\delta}$	(0.1, 0.1, 0.1, 0.1)			(0.1, 0.1, 0.1, 0.1)		(0.1, 0.3, 0.3, 0.1)	(0.1, 0.7, 0.7, 0.1)
$\gamma$	0.1	0.3	0.9	0.5	0.5	0.5	0.5
DMU1	0.9007	0.8466	0.7095	0.8757	0.8186	0.7740	
DMU2	0.8229	0.7902	0.7920	0.8609	0.7862	0.7308	
DMU3	0.9212	0.8693	0.7099	1.1625	0.8334	0.7784	
DMU4	1.0653	1.0181	0.8877	1.0797	1.0020	0.9417	
DMU5	1.2470	1.1892	1.0324	3.2838	1.4912	1.0473	
DMU6	0.9828	0.9399	0.8239	1.3384	0.9324	0.8657	
DMU7	0.8463	0.8003	0.6727	0.8534	0.7832	0.7307	
DMU8	0.7968	0.7638	0.6699	0.8379	0.7605	0.7134	
DMU9	1.0145	0.9730	0.8553	1.0501	0.9645	0.7989	
DMU10	1.1737	1.0877	0.8704	1.1229	1.0401	0.9760	
DMU11	1.2942	1.2161	1.1430	1.0237	1.2944	1.1430	
DMU12	1.0657	1.0092	0.8564	1.0745	0.9879	0.9215	
DMU13	1.1467	1.0904	0.9392	1.1684	1.0702	1.0009	
DMU14	0.6862	0.6528	0.5634	0.6900	0.6398	0.6007	
DMU15	1.3053	1.2274	1.0225	1.2841	1.1898	1.1181	
DMU16	1.0986	1.0490	0.9203	1.1149	1.0341	0.9718	
DMU17	0.7567	0.7221	0.6209	0.7743	0.7121	0.6654	
DMU18	1.0449	1.0038	0.8838	1.1077	1.0024	0.9255	
DMU19	0.7683	0.7338	0.6409	0.8007	0.7282	0.6747	
DMU20	1.3443	1.2557	1.0260	1.3197	1.2131	1.1317	
DMU21	0.8085	0.7785	0.6901	0.8595	0.7789	0.7197	
DMU22	0.7162	0.6644	0.5503	0.7034	0.6468	0.6042	
DMU23	0.8743	0.8385	0.7370	0.9037	0.8309	0.7197	
DMU24	1.0564	0.9717	0.7606	1.0012	0.9238	0.8641	
DMU25	0.9476	0.8935	0.7550	0.9288	0.8661	0.8171	

$$\begin{aligned}
 (\theta_p^t)^2 &= \sum_{i=1}^m v_i^2 \sigma_{ip}^2, \\
 (\lambda_j)^2 &= \sum_{r=1}^m u_r^2 \tau_{rj}^2 + \sum_{i=1}^m v_i^2 \sigma_{ij}^2, \\
 u_r, v_i, \theta_p^o, \theta_p^t, \lambda_j &\geq 0, r = 1, \dots, s; i = 1, \dots, m; j = 1, \dots, n.
 \end{aligned}
 \tag{10}$$

**Definition 10** DMU is called efficient at probability level  $\gamma$  and fuzzy possibility level  $\tilde{\delta}$  if the objective function of model (10) is equal to or bigger than 1; otherwise, it is called inefficient.

**3.2 Necessity–probability CCR model in the presence of SN distribution with Fuzzy threshold level**

In this section, the CCR model with necessity–probability constraints and fuzzy threshold level will be generalized in order to evaluate the DMUs. Therefore, the CCR model with necessity–probability constraints in the presence of SN distribution with fuzzy level  $\tilde{\delta} = (\delta^\alpha, \delta^{m_1}, \delta^{m_2}, \delta^\beta)$  is as follows:

$$\begin{aligned}
 &\max \varphi \\
 &\text{s.t.} \\
 &\text{Nec} \left[ \Pr \left( \varphi \leq \sum_{r=1}^s u_r \tilde{y}_{rp} \right) \geq \tilde{\delta} \right] \geq \gamma, \\
 &\text{Nec} \left[ \Pr \left( \sum_{i=1}^m v_i \tilde{x}_{ip} \leq 1 \right) \geq \tilde{\delta} \right] \geq \gamma, \\
 &\text{Nec} \left[ \Pr \left( \sum_{i=1}^m v_i \tilde{x}_{ip} \geq 1 \right) \geq \tilde{\delta} \right] \geq \gamma, \\
 &\text{Nec} \left[ \Pr \left( \sum_{r=1}^s u_r \tilde{y}_{rj} - \sum_{i=1}^m v_i \tilde{x}_{ij} \leq 0 \right) \geq \tilde{\delta} \right] \\
 &\geq \gamma, j = 1, \dots, n, \\
 &u_r, v_i \geq 0, r = 1, \dots, s, i = 1, \dots, m.
 \end{aligned}
 \tag{11}$$

Again, let us consider the assumption of the previous section, a summary of which is as follows:



**Table 4** Necessity–probability scores of the DMUs with their fuzzy thresholds

$\bar{\delta}$	(0.1, 0.1, 0.1, 0.1)			(0.1, 0.1, 0.1, 0.1)	(0.1, 0.3, 0.3, 0.1)	(0.1, 0.7, 0.7, 0.1)
$\gamma$	0.1	0.3	0.9	0.5	0.5	0.5
DMU1	0.5455	0.5523	0.5339	0.6050	0.5731	0.5392
DMU2	0.5138	0.5256	0.5526	0.6189	0.5654	0.5154
DMU3	0.5598	0.5189	0.5515	0.8046	0.5872	0.5154
DMU4	0.6303	1.6556	0.7433	0.7881	0.7417	0.6958
DMU5	0.8310	0.8178	0.8577	1.8803	1.0881	1.8181
DMU6	0.6560	0.6541	0.6013	0.9266	0.6872	0.6218
DMU7	0.5032	0.5181	0.5617	0.6080	0.5574	0.5028
DMU8	0.4919	0.5026	0.5316	0.5956	0.5412	0.4904
DMU9	0.6100	0.6231	0.5983	0.7037	0.6596	0.6153
DMU10	0.6121	0.6257	0.6408	0.6693	0.7006	0.6506
DMU11	0.7463	0.7478	0.7487	0.8586	0.7853	0.7178
DMU12	0.6126	0.6405	0.7266	0.7419	0.6954	0.6374
DMU13	1.6774	1.6919	1.7563	1.8179	1.7475	1.6472
DMU14	0.4030	0.4080	0.4235	0.4693	0.4320	0.3958
DMU15	0.7278	0.7407	0.7890	0.8559	0.8086	0.7614
DMU16	0.7139	0.7260	0.7456	0.8069	0.7596	0.7124
DMU17	0.4630	0.4780	0.5171	0.5616	0.5263	0.4265
DMU18	0.6702	0.6279	0.7045	1.8254	0.7443	0.6439
DMU19	0.4906	1.4806	0.4159	0.5463	0.4953	1.4311
DMU20	1.7248	0.7344	0.7516	0.8528	0.7793	0.6750
DMU21	0.4919	0.5026	0.5121	0.6301	0.5694	0.4958
DMU22	0.4083	0.4144	0.4575	0.4915	1.4519	0.4143
DMU23	0.5270	0.5457	0.5983	0.6254	0.5860	0.5476
DMU24	0.5337	0.5491	0.5923	0.6201	0.5842	0.5412
DMU25	0.5799	0.5448	0.5379	0.6347	0.5997	0.5644

$$\begin{aligned} \tilde{x}_{ij} &\sim \text{CSN}_{1,1}(\bar{x}_{ij}, \sigma_{ij}^2, \delta_{ij}, 0, \sigma_{ij}^2), \bar{x}_{ij} \\ &= (x_{ij}^\alpha, x_{ij}^{m_1}, x_{ij}^{m_2}, x_{ij}^\beta)_{LR}, \\ \tilde{y}_{rj} &\sim \text{CSN}_{1,1}(\bar{y}_{rj}, \tau_{rj}^2, \varepsilon_{rj}, 0, \tau_{rj}^2), \bar{y}_{rj} \\ &= (y_{rj}^\alpha, y_{rj}^{m_1}, y_{rj}^{m_2}, y_{rj}^\beta)_{LR}. \end{aligned}$$

By applying the second part of Theorem 1 and solving the CCR model with necessity–probability constraints with fuzzy probability level, the following definite model is obtained:

$$\begin{aligned} \max \quad & \bar{\varphi} \\ \text{s.t.} \quad & \\ & \bar{\varphi} - \sum_{r=1}^s u_r (y_{rp}^{m_2} + R^{-1}(1-\gamma)y_{rp}^\beta) \\ & + \rho_p^o \Psi^{-1}(\delta^{m_2} + R(\gamma)\delta^\beta) \leq 0, \\ & \sum_{i=1}^m v_i (x_{ip}^{m_2} + R^{-1}(1-\gamma)x_{ip}^\beta) \end{aligned}$$

$$\begin{aligned} & + \rho_p^l \Psi^{-1}(\delta^{m_2} + R(\gamma)\delta^\beta) \leq 1, \\ & \sum_{i=1}^m v_i (x_{ip}^{m_1} + R^{-1}(1-\gamma)x_{ip}^\beta) \\ & - \rho_p^l \Psi^{-1}(\delta^{m_2} + R(\gamma)\delta^\beta) \geq 1, \\ & \sum_{r=1}^s u_r y_{rj}^{m_1} \\ & - \sum_{i=1}^m v_i x_{ij}^{m_2} + R^{-1}(1-\gamma) \left( \sum_{r=1}^s u_r y_{rj}^\beta + \sum_{i=1}^m v_i x_{ij}^\alpha \right) \\ & + h_j \Psi^{-1}(\delta^{m_2} + R(\gamma)\delta^\beta) \leq 0, j = 1, \dots, n, \\ & (\rho_p^o)^2 = \sum_{r=1}^m u_r^2 \tau_{rp}^2 \\ & (\rho_p^l)^2 = \sum_{i=1}^m v_i^2 \sigma_{ip}^2 \\ & (h_j)^2 = \sum_{r=1}^m u_r^2 \tau_{rj}^2 + \sum_{i=1}^m v_i^2 \sigma_{ij}^2 \end{aligned}$$

**Table 5** Possibility–probability scores of the DMUs with their fuzzy thresholds and normal distribution

$\tilde{\delta}$	(0.1, 0.1, 0.1, 0.1)			(0.1, 0.1, 0.1, 0.1)	(0.1, 0.3, 0.3, 0.1)	(0.1, 0.7, 0.7, 0.1)
$\gamma$	0.1	0.3	0.9	0.5	0.5	0.5
DMU1	1.8906	1.8466	1.7095	1.8757	1.8186	1.7970
DMU2	1.9223	1.7902	1.7920	1.8609	1.7862	1.7591
DMU3	0.6243	0.6389	0.9079	1.5216	0.9348	0.8042
DMU4	1.1467	1.0904	0.9392	1.1684	1.0702	0.9728
DMU5	0.7043	0.2891	1.0324	3.2838	1.9421	1.4331
DMU6	0.7280	0.7863	0.9250	1.6672	0.5409	0.6920
DMU7	0.7981	0.7003	0.5691	0.7591	0.5698	0.6519
DMU8	0.7968	0.7638	0.5509	0.7062	0.7605	0.7326
DMU9	1.3275	0.9960	0.9642	1.1576	0.9886	0.9256
DMU10	1.1903	1.3581	0.9837	1.2644	1.1009	1.5538
DMU11	1.3572	1.6511	1.2580	1.1362	1.1294	1.0653
DMU12	1.0073	1.0001	0.7629	1.0083	0.7803	0.8791
DMU13	1.3782	1.0081	0.8920	1.3100	1.1151	1.0366
DMU14	0.4716	0.7491	0.6152	0.8791	0.7191	0.7820
DMU15	1.6763	1.2274	1.0225	1.2841	1.1898	1.1547
DMU16	1.3617	1.3610	0.9203	1.1149	1.0341	1.0039
DMU17	0.7865	0.7221	0.6209	0.7743	0.7121	0.6894
DMU18	1.7623	1.2006	0.9875	1.3042	1.0024	0.9647
DMU19	0.3871	0.8931	0.6409	0.8007	0.7282	0.7021
DMU20	1.5447	1.6557	1.0046	1.2644	1.4371	1.1736
DMU21	1.0001	0.8884	0.5994	0.8935	0.6698	0.6920
DMU22	0.8158	0.7680	0.4450	0.7051	0.7220	0.6260
DMU23	0.9589	0.9875	0.6537	0.9537	0.8664	0.7942
DMU24	1.1148	1.7393	0.7213	1.0012	0.8226	0.8948
DMU25	0.9177	0.9274	0.7426	0.6953	0.7559	0.8424

$$\begin{aligned}
 &u_r, v_i, \theta_p^o, \theta_p^l, \lambda_j \geq 0, r = 1, \dots, s; \\
 &i = 1, \dots, m; j = 1, \dots, n
 \end{aligned} \quad (12)$$

And  $\Psi$  is the cumulative distribution function (CDF) of the standard closed skew-normal distribution.

**Definition 11** DMU is called efficient at probability level  $\gamma$  and fuzzy necessity level  $\tilde{\delta}$  if the objective function of model (12) is equal to or bigger than 1; otherwise, it is called inefficient.

We should note that although we used this models in the paper, we could alternatively use the envelopment models.

#### 4 Numerical examples

In this paper, we developed the possibility–probability and necessity–probability DEA models with Ra-Fu parameters in presence of SK distribution with fuzzy thresholds. We provide a numerical example to illustrate the applications of models.

In this section, we will provide an example and compares the results of the efficiency of DMUs when inputs and outputs are normal distributed or SN distribution. Suppose that 25 DMUs with three inputs and five outputs. All these data are stochastic, and also, inputs and outputs for different DMUs are independent. These data are given in Tables 1 and 2.

By using the data of Tables 1 and 2, we apply model (10) and model (12) the results with what is obtained from solving the models with input orientation. The computational results of models (10) and (12) are presented in Tables 3 and 4. Information of Table 3 is the results of efficiency and slacks of DMUs with real distribution of data (SN distribution).

We used  $\tilde{\delta} = (0.1, 0.1, 0.1)$  for  $\gamma = 0.1, 0.3$  and  $0.9$  and for fixed  $\gamma = 0.5$  and three fuzzy numbers for  $\tilde{\delta} = \{(0.1, 0.1, 0.1), (0.1, 0.3, 0.1), (0.1, 0.7, 0.1)\}$  such that  $(0.1, 0.1, 0.1) \leq (0.1, 0.3, 0.1) \leq (0.1, 0.7, 0.1)$ . The results derived from model (10) are given in Table 3. As given in Table 3, for  $\tilde{\delta} = (0.1, 0.1, 0.1)$  and  $\gamma = 0.1, 0.3, 0.5$  and  $0.9$ , DMUs 5 and 14 have the highest and lowest scores. The results of the model (12) in Table 4 are lower compared

**Table 6** Necessity–probability scores of the DMUs with their fuzzy thresholds and normal distribution

$\delta$	(0.1, 0.1, 0.1, 0.1)			(0.1, 0.1, 0.1, 0.1)	(0.1, 0.3, 0.3, 0.1)	(0.1, 0.7, 0.7, 0.1)
$\gamma$	0.1	0.3	0.9	0.5	0.5	0.5
DMU1	0.6546	0.99414	0.6235	0.6406	0.6189	0.8471
DMU2	0.61656	0.9460	0.6469	0.6631	0.6106	0.8666
DMU3	0.6717	0.9340	0.6577	0.6617	0.6341	1.1264
DMU4	0.7563	1.1800	0.8523	0.8916	0.8010	1.1034
DMU5	0.9971	1.4720	0.9648	1.0224	1.1748	2.6324
DMU6	0.7872	1.1773	0.7731	0.72155	0.7421	1.2924
DMU7	0.6038	0.9325	0.6434	0.67404	0.6019	0.8512
DMU8	0.5902	0.9046	0.6208	0.6379	0.5844	0.8338
DMU9	0.7320	1.1215	0.7695	0.7796	0.7123	0.9858
DMU10	0.7345	1.1262	1.8127	0.7896	0.7566	0.9302
DMU11	0.8955	1.3460	0.9059	0.8844	0.8481	1.2204
DMU12	0.7351	1.1529	0.8114	0.8719	0.7510	1.3866
DMU13	0.8128	1.2454	0.8632	1.9075	0.8073	1.1450
DMU14	0.4836	0.7343	0.4994	0.5082	0.4665	0.6570
DMU15	0.8733	1.3332	0.9479	1.9468	1.8732	1.1982
DMU16	0.8566	1.3068	0.8887	0.8947	0.8203	1.1296
DMU17	0.5556	0.8603	0.5972	0.6205	1.5684	0.7862
DMU18	0.8042	1.1302	0.8524	0.8454	0.8034	1.1556
DMU19	0.5887	0.8650	0.5676	0.4990	1.5349	0.7648
DMU20	0.8697	1.3219	0.8999	0.9019	0.8416	1.1939
DMU21	0.5902	0.9046	0.6342	0.6145	0.6149	0.8821
DMU22	0.4899	0.7459	0.5227	0.5469	0.4880	0.6880
DMU23	0.6324	0.9822	0.6342	0.7179	0.6328	0.8755
DMU24	0.6404	0.9883	0.6828	0.7107	0.6336	0.8681
DMU25	0.6958	0.9806	0.7031	0.6454	0.7676	0.8888

to the results of the model (10) in Table 3. The efficiency of model (12) is less than the efficiency of model (10); when  $\gamma = 0.5$  is fixed and  $\delta$  is increased, the score of model (12) is decreased.

If skewness control parameter equals to zero, we have normal distribution. Using normal data, we fit the models in normal mode (Tavana et al. 2013a, b). Results are presented in Tables 5 and 6. The results of Table 5 show that DMUs 1 and 2 are efficient, but in Table 3 these DMUs are inefficient. DMU 14 with real distribution of data in Table 3 is efficient, whereas with unsuitable choice of distribution it is inefficient. The results show that an inefficient DMU may be wrongly diagnosed as an efficient DMU and vice versa.

### 5 Discussion and conclusion

In the classic DEA theory, it is assumed that the data are definite. The nonparametric programming method in DEA

is so popular due to its simplicity and applicability. The use of DEA in recent years in various fields including economy, operational research, statistics, marketing, and public services has led to completely new results; furthermore, the use of DEA as a powerful tool for evaluating the performance of DMUs with the applied programs has resulted in a new challenge. The real input data were mostly vague, random, and/or fuzzy, and the outputs were the same. In a real problem, we may face the two random and fuzzy states simultaneously. In previous works, it has been assumed that the random fuzzy data follow a normal distribution so that if the fitness of this distribution is not suitable for the data, it will lead to incorrect conclusions. However, in the present paper, it was assumed that the input and output variables were random fuzzy and followed a SN distribution; furthermore, the possibility–probability and necessity–probability models were fitted with the conditions that the use of the skew-normal distribution in random fuzzy state instead of the previous method would be more

general and better. In other words, this method embraced the previous methods in a specific state. The advantage of this theorem was also represented in two numerical examples.

#### Compliance with ethical standards

**Conflict of interest** The authors declare that they have no competing interest.

#### References

- Azzalini A (1985) A class of distribution which includes the normal ones. *Scand J Stat* 12:171–178
- Azzalini A, Capitanò A (1999) Statistical application of the multivariate skew normal distribution. *J R Stat Soc B* 61:579–602
- Charnes A, Cooper WW, Rhodes E (1978) Measuring the efficiency of decision making units. *Eur J Oper Res* 2(6):429–444
- Dubois D, Prade H (1978) operations on fuzzy numbers. *Int J Syst Sci* 9(6):291–297
- Khanjani SR, Jalalzadeh L, Paryab K, Fukuyama H (2014a) Imprecise data envelopment analysis model with bifuzzy variables. *J Intell Fuzzy Syst* 27(1):37–48
- Khanjani SR, Charles V, Jalalzadeh L (2014b) Fuzzy rough DEA model: a possibility and expected value approaches. *Expert Syst Appl* 41(2):434–444
- Khanjani SR, Tavana M, Di Caprio D (2017) Chance-constrained data envelopment analysis modeling with random-rough data. *RAIRO-Oper Res*. doi:[10.1051/ro/2016076](https://doi.org/10.1051/ro/2016076)
- Klir GJ, Yuan B (1995) *Fuzzy sets and fuzzy logic: theory and application*. Prentice-Hall, New York
- Kwakernaak H (1978) Fuzzy random variables. Part I: definitions and theorems. *Inf Sci* 15(1):1–29
- Liu B, Liu YK (2002) Expected value of fuzzy variable and fuzzy expected value models. *IEEE Trans Fuzzy Syst* 10(4):445–450
- Liu YK, Liu B (2003) Fuzzy random variables: a scalar expected value operator. *Fuzzy Optim Decis Mak* 2(2):143–160
- Tavana M, Shiraz RK, Hatami-Marbini A, Agrell PJ, Paryab K (2012) Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC). *Expert Syst Appl* 39:12247–12259
- Tavana M, Shiraz RK, Hatami-Marbini A, Agrell PJ, Paryab K (2013a) Chance-constrained DEA models with random fuzzy inputs and outputs. *J Knowl Based Syst* 52:32–52
- Tavana M, Shiraz RK, Hatami-Marbini A (2013b) A new chance-constrained DEA model with birandom input and output data. *J Oper Res Soc*. doi:[10.1057/jors.2013.157](https://doi.org/10.1057/jors.2013.157)
- Wu D, Yang Z, Liang L (2006) Efficiency analysis of cross-region bank branches using fuzzy data envelopment analysis. *Appl Math Comput* 181(1):271–281
- Zadeh LA (1978) Fuzzy sets as a basis for theory of possibility. *Fuzzy Sets Syst* 1(1):3–28
- Zimmermann HJ (2001) *Fuzzy sets theory and its applications*, 4th edn. Kluwer Academic Publishers, Dordrecht
- Zhu J (2003) Imprecise data envelopment analysis (IDEA): a review and improvement with an application. *Eur J Oper Res* 144:513–529