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FOUNDATIONS

N-soft sets and their decision making algorithms

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Abstract In this paper, we motivate and introduce the concept of N-soft set as an extended soft set model. Some useful algebraic definitions and properties are given. We cite real examples that prove that N-soft sets are a cogent model for binary and non-binary evaluations in numerous kinds of decision making problems. Finally, we propose decision making procedures for N-soft sets.

Keywords N-soft set · Non-binary evaluation · Decision making · Choice value · Intersection and union

1 Introduction

Many problems in the fields of economic, social, medical, environmental and other sciences involve uncertainty, imprecision, or subjectivity. This paper expands the range of applications of one of the theories that can be used to deal with these characteristics, namely soft set theory. It was introduced by Molodtsov (1999), who also showed its applicability to various fields. Soft set does not require parameters specification. Instead, it accommodates all types of parameters as its benchmark. The parameters of a soft set can be numbers, words, sentences, functions, and so on. Thus, the soft set definition associates the pertinent attributes with information or knowledge about the elements in the universe.

Molodtsov's soft set is different from the idea that was proposed by Pawlak (1994) under the same name. It is also different from soft set as defined in Basu et al. (1992), where a soft set is conceived of as the ordinal approach of a vague set. Nevertheless, these notions embody a common philosophy i.e., a concept of set that is not bound to the classical concept in Cantor's set theory.

Various kinds of evaluations in soft sets generate many definitions in this field. Besides soft sets (Molodtsov 1999), we can cite probabilistic soft sets (Fatimah et al. 2017a, b; Zhou and Xu 2017; Zhu and Wen 2010), soft binary relations (Li et al. 2017), soft rough sets (Zhan et al. 2017b, c; Feng et al. 2011), fuzzy soft sets (Abbas and Ibedou 2016; Çetkin and Aygün 2016; Maji et al. 2001a), interval soft sets (Zhang 2014), soft fuzzy rough sets (Zhan et al. 2017a), soft rough fuzzy sets (Zhan and Zhu 2017), intuitionistic fuzzy soft sets (Maji et al. 2001b), interval-valued fuzzy soft sets (Yang et al. 2009), hesitant fuzzy soft sets (Babitha and John 2013), interval-valued intuitionistic hesitant fuzzy soft sets (Peng and Yang 2015a; Wang and Chen 2014), interval-valued hesitant fuzzy soft sets (Peng and Yang 2015b), vague soft sets (Wang and Aj 2015; Xu et al. 2010), neutrosophic soft set (Peng and Liu 2017; Deli and Broumi 2015), *ivnpiv*-neutrosophic soft sets (Deli et al. 2016), linguistic value soft sets (Sun et al. 2017), etc. (see (Ma et al. 2017; Zhan and Zhu

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2015) for recent surveys of hybrid soft set models). Mostly, the researchers in soft-set-inspired models used binary evaluation (either 0 or 1) or else, real numbers between 0 and 1.

But in fact, in daily life, we often find non-binary evaluations in many areas (Alcantud and Laruelle 2014; Brunelli et al. 2014; Aleskerov et al. 2010; Dokow and Holzman 2010). In a social choice environment, Alcantud and Laruelle (2014) refer to ternary voting situations. Non-binary evaluations are also frequent in ranking or rating systems. Real examples show that ratings could adopt the form of number of stars (like 'five stars,' 'four stars,' 'three stars,' 'two stars,' 'one star' in hotel descriptions), by numbers as labels (like 0 as a label for 'bad,' 1 for 'average,' and 2 for 'good'), or even by icons (like a fresh red tomato for a good film review, a splat rotten green tomato for a bad film review). Herawan and Deris (2009) mentioned about n binary-valued information system in soft sets where each of parameters has its own rankings, but it was undefined for the rank order cases like in Chen et al. (2013). Relatedly, Hakim et al. (2014a, b) used asterisks as the ranking system to evaluate the objects parameters in soft sets, although a definition about the ranking evaluation is also missing in their paper. Instead of ranking as evaluations, Ali et al. (2015) concerned about the ranking among the elements of soft sets parameters.

Clearly, the modelization by soft sets is inadequate to account for these circumstances. Therefore, the purpose of this paper is to put forward a new model that can deal with uncertainties in the form required by the cases above. To be specific, a sample of real examples motivate the need for an extension of the soft set ethos that allows for a finer granularity in the parameterizations.

To that purpose, we propose the novel extended soft set model that we call N-soft sets. In view of the importance of grades in real-world situations, N-soft sets introduce parameterized descriptions of the universe of options that depend on a finite number of grades. Such description does not belong to any of the modelizations described above, and hence it requires a separate analysis. Of course, our model can be used to make decisions in general real situations like those described above. Consequently, in addition to the formal expression that defines the new framework, we start investigating the main characteristics of N-soft sets and their operations. Then, we pose some algorithms for decision making. They are inspired by well-established mechanisms from the soft set literature, and we put forward their relationships. Our decision making procedures are flexible in the sense that they enable the practitioner to reflect his specific priorities with regard to the attributes.

The organization of this paper is as follows. Section 2 provides the reader with the relevant theoretical background about soft sets. In Sect. 3, we motivate and define our novel concept of N-soft set. To begin with, in Sect. 3.1, we discuss the paramount importance of ranking systems through real examples. Then, in Sect. 3.2 we propose our definition of extended soft sets, namely N-soft sets, as a general notion which can handle this kind of situations. Afterward, we investigate the properties and the operations of N-soft sets, as well as related notions. We give associated definitions, for example incomplete N-soft set, efficient N-soft set, normalized N-soft set, as well as equivalence under normalization, equality, weak complements and particular examples, and soft sets derived from a threshold. We also construct the restricted, resp., extended, intersection and union of N-soft sets in two different ways. The first one relies on ideas from Ali et al. (2009), the other one is altogether new and is based on N-soft sets and thresholds. Section 4 presents decision making procedures for N-soft sets. In Sect. 5, we use real data in order to compare the results of our algorithms, which proves their implementability and adaptability to the users. Finally, Sect. 6 concludes our presentation.

2 Theoretical background about soft sets

In this section, we recall some basic notions that are useful for discussion in the next sections.

Definition 1 (Molodtsov 1999) Let U be a universal set and E be a set of parameters, $A \subseteq E$. A pair (F, A) is called a soft set over U if F is a mapping from the set A to the set of all subsets of U, $F : A \to 2^U$.

Therefore, a soft set is a parameterized family of subsets of U. For each $e \in A$, we can interpret $F(e)$ as a subset of U, which is usually called the set of e-approximate elements of (F, A). But we can also regard $F(e)$ as a mapping $F(e) : U \longrightarrow \{0, 1\}$, and then $F(e)(u) = 1$ is equivalent to $u \in F(e)$. It is well known that any soft set can be represented in tabular form when both U and A are finite sets.

In order to define of restricted and extended intersections and unions, we draw from Ali et al. (2009):

Definition 2 (Ali et al. 2009) Let (F, A) and (G, B) be two soft sets over the same universe U, such that $A \cap B \neq \emptyset$. The restricted intersection of (F, A) and (G, B) is denoted by $(F, A) \cap_{\mathscr{R}} (G, B)$, and it is defined as $(F, A) \cap_{\mathscr{R}} (G, B) = (H, A \cap B)$, where $H(e) = F(e) \cap G(e)$ for all $e \in A \cap B$.

Definition 3 (Ali et al. 2009) The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A \backslash B, \\ G(e), & \text{if } e \in B \backslash A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

It is denoted by $(F, A) \cap_{\mathscr{E}} (G, B) = (H, C)$.

Definition 4 (Ali et al. 2009) Let (F, A) and (G, B) be two soft sets over the same universe U, such that $A \cap B \neq \emptyset$. The restricted union of (F, A) and (G, B) is denoted by $(F, A) \cup_{\mathscr{R}} (G, B)$, and it is defined as $(F, A) \cup_{\mathscr{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $e \in C, H(e) = F(e) \cup G(e)$.

Definition 5 (Ali et al. 2009) The extended union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A \backslash B, \\ G(e), & \text{if } e \in B \backslash A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

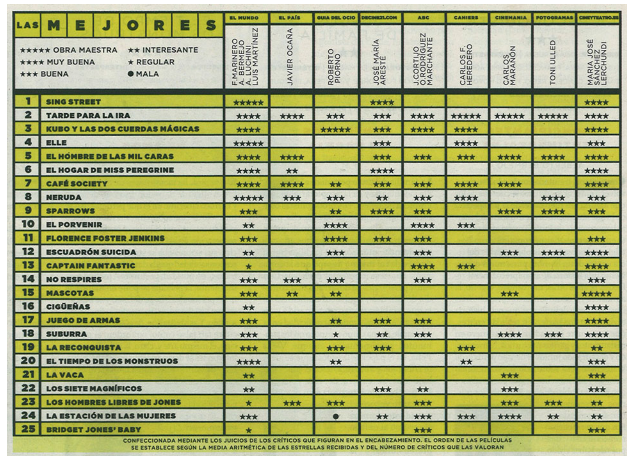
It was denoted by $(F, A) \cup_{\mathscr{E}} (G, B) = (H, C)$.

Intuitive interpretations of Definitions 2–5 are available in Akram and Nawaz (2015), Ali et al. (2015), Alkhazaleh et al. (2011), Sezgın and Atagün (2011), Jiang et al. (2010b) among others.

Standard soft sets produce a binary parameterized description of the universe of objects (Maji et al. 2002). However, many scholars argued that it is too naive a description in many decision making situations. For that reason, extended notions like fuzzy soft sets and its extensions (Akram and Shahzadi 2016; Alcantud 2016a, b, c; Muthukumar and Krishnan 2016; Alcantud 2015; Deli and Çağman 2015; Li et al. 2015; Zhang et al. 2014; Babitha and John 2013; Handaga and Deris 2012; Feng et al. 2010; Jiang et al. 2010a) have been introduced. In the next section, we give but a few real examples about the importance of grades in real-world situations. Grades produce a parameterized description of the universe of alternatives, but such parameterization does not pertain to any of the existing models. Then, we propose the novel extended soft set model that we call N-soft sets, which can be used to make decisions in these general real situations. And we also investigate their properties.

3 Extended soft set model

In this section, we first proceed to motivate the need for a new extended model of soft sets with some examples from real practice. Afterward, we propose a novel model which applies to those cases and prove some basic properties of this concept.



LAS MEJORES

★★★★★ OBRA MAESTRA ★★★★ MUY BUENA ★★★ BUENA ★★ INTERESANTE ★ REGULAR ● MALA

		EL MUNDO	EL PAÍS	GUIA DEL OCIO	DECINE21.COM	ABC	CAHIERS	CINEMANIA	FOTOGRAMAS	CINEYTEATRO.ES
		F.MARINERO A. BERMEJO A. LUCHINI LUIS MARTÍNEZ	JAVIER OCAÑA	ROBERTO PIORNO	JOSÉ MARÍA ARESTÉ	J.CORTIJO O.RODRÍGUEZ MARCHANTE	CARLOS F. HEREDERO	CARLOS MARAÑÓN	TONI ULLED	MARIA JOSÉ SÁNCHEZ LERCHUNDI
1	SING STREET	★★★★★			★★★★					★★★★
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11	FLORENCE FOSTER JENKINS	★★★		★★★★	★★★	★★★				★★★
12	ESCUADRÓN SUICIDA	★★		★★★		★★★		★★★	★★★★	★★★★
13	CAPTAIN FANTASTIC	★				★★★★	★★★			★★★★
14	NO RESPIRES	★★★	★★★	★★★		★★★				★★★
15	MASCOTAS	★★★	★★	★★				★★★		★★★★★
16	CIGÜEÑAS	★★								★★★★
17	JUEGO DE ARMAS	★★★		★★	★★★	★★★				★★★★
18	SUBURRA	★★★		★	★★	★★★		★★★★	★★★	★★★★
19	LA RECONQUISTA	★★★		★★★	★★★		★★★			★★
20	EL TIEMPO DE LOS MONSTRUOS	★★★★		★★			★★			★★★
21	LA VACA	★★						★★★		★★★
22	LOS SIETE MAGNÍFICOS	★★			★★★	★★		★★★		★★★
23	LOS HOMBRES LIBRES DE JONES	★	★★★	★★★		★★★		★★★	★★★	★★
24	LA ESTACIÓN DE LAS MUJERES	★★★		●	★★	★★★	★★★	★★★★	★★	★★
25	BRIDGET JONES' BABY	★				★★★				★★★

CONFECCIONADA MEDIANTE LOS JUICIOS DE LOS CRÍTICOS QUE FIGURAN EN EL ENCABEZAMIENTO. EL ORDEN DE LAS PELÍCULAS SE ESTABLECE SEGÚN LA MEDIA ARITMÉTICA DE LAS ESTRELLAS RECIBIDAS Y DEL NÚMERO DE CRÍTICOS QUE LAS VALORAN

Fig. 1 Ratings of movies by stars in *Metropoli* (El Mundo supplement, Spain)

3.1 Ranking systems in real-life applications

We can find many examples of real data in daily life that use rankings or ratings beyond binary evaluations. Let us see a short sample.

Example 1 Specialized magazines often rate movies by collecting movie reviews from a variety of media. In Fig. 1, we display data taken from El Mundo, one of the leading newspapers in Spain, on September 30th, 2016.

In Fig. 1, ratings are made by movie critics as listed in the table header. The footnote explains that movies are sorted based on the average of stars received. This is equivalent to an extended choice value maximizing principle, which we define in Sect. 4.

Example 2 The 'Tomatometer rating' from the web www.rottentomatoes.com is an online aggregator of movie and TV show reviews. The ratings are denoted by a 'fresh red tomato' (the movie must be at least get 60% rating), a 'rotten green tomato splat' (59% or less rating), and a 'certified fresh' (the movie must have rating 75% or better).

As in Example 1, the granularity is finer than the binary description in soft sets.

Example 3 In online editorial systems, referees give their recommendations about manuscripts submitted for journal publication. The referees typically declare whether the manuscript is assessed as 'reject,' 'major revision,' 'minor revision,' 'accept,' or 'accept with no recommendation,' as can be seen in Figs. 2 and 3.

Sometimes, the granularity of the description of the attributes is finer in similar contexts. Bakanic et al. (1987) is a classical reference. The authors examined the final evaluations of 2337 manuscripts submitted to the American Sociological Review between 1977 and 1981. The values for

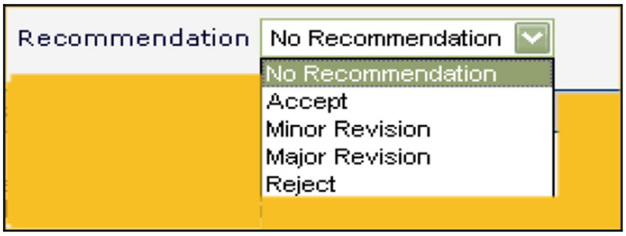

Fig. 2 Referee ranking in first online editorial system (Springer)

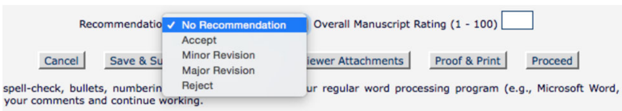

Fig. 3 Referee ranking in second online editorial system (Elsevier)

their ordinal variable were 1 = rejected without qualification; 2 = revise and resubmit, eventually rejected; 3 = conditionally accept, eventually rejected; 4 = revise and resubmit, not resubmitted and conditionally accept, author withdraws; 5 = revise and resubmit, eventually published; 6 = conditionally accept, eventually published; and 7 = unconditionally accept for publication.

Therefore, if we adopt Molodtsov's position that parameterizations of the universe of alternatives have their own advantages over, e.g., membership degrees in fuzzy set theory, it is still convenient to introduce a gradation in the parameters in order to fit real data into the model. Motivated by these requirements, we now define a novel extended notion of soft set.

3.2 N-soft sets

In this section, we generalize the concept of soft set in order to account for cases like the real data from Examples 1–3. Our target is to produce a parameterized description of the universe that is neither binary (as in soft sets) nor continuous (as in fuzzy soft sets). Instead, it is based on a finite granularity in the perception of the attributes. The formal definition is as follows:

Definition 6 Let U be a universe set of objects and E be attributes, $A \subseteq E$. Let $R = \{0, 1, \ldots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \ldots\}$. We say that (F, A, N) is an N-soft set on U if $F : A \to 2^{U \times R}$, with the property that for each $a \in A$ and there exists a unique $(u, r_a) \in U \times R$ such that $(u, r_a) \in F(a)$, $u \in U$, $r_a \in R$.

Given attribute a, every object u in U receives exactly one evaluation from the assessments space R, namely the unique r_a for which $(u, r_a) \in F(a)$. In order to make our notation as close to the soft set case as possible, we also write $F(a)(u) = r_a$ as a shorthand for $(u, r_a) \in F(a)$. Henceforth, we assume

Table 1 Tabular form of N-soft set

(F, A, N)	a_1	a_2	...	a_q
u_1	r_{11}	r_{12}	...	r_{1q}
u_2	r_{21}	r_{22}	...	r_{2q}
...	...	...	...	...
u_p	r_{p1}	r_{p2}	...	r_{pq}

that both $U = \{u_i, i = 1, 2, \ldots, p\}$ and $A = \{a_j, j = 1, 2, ..., q\}$ are finite unless otherwise stated. Clearly, in that case the N-soft set can be presented by a tabular form as well where r_{ij} means $(u_i, r_{ij}) \in F(a_j)$ or $F(a_j)(u_i) = r_{ij}$. This is symbolized by Table 1.

Remark 1 It is natural to identify a 2-soft set with a soft set. In formal terms, we identify the 2-soft set $F : A \to 2^{U \times \{0,1\}} = \mathscr{P}(U \times \{0, 1\})$ with the soft set $F' : A \to \mathscr{P}(U)$ defined through

$$F'(a) = \{u \in U : (u, 1) \in F(a)\}$$

where $\mathscr{P}$ denotes power set.

The identification stated in Remark 1 is explained with the following example stated in a voting environment:

Example 4 Let $U = \{c_1, c_2, c_3\}$ be a set of candidates, $A = \{v_1, v_2, v_3\}$ be a set of voters. Voter 1 approves candidate 1. Voter 2 approves candidate 1. Voter 3 approves candidate 2.
Let $R = \{0, 1\}$. Thus, a 2-soft set $F : A \to 2^{U \times R}$ can be used to explain this situation as follows:

$$F(v_1) = \{(c_1, 1), (c_2, 0), (c_3, 0)\},$$
$$F(v_2) = \{(c_1, 1), (c_2, 0), (c_3, 0)\},$$
$$F(v_3) = \{(c_1, 0), (c_2, 1), (c_3, 0)\}.$$

Now, this 2-soft set can be identified with the soft set (F', A) which is defined by $F'(v_1) = \{c_1\}$, $F'(v_2) = \{c_1\}$, and $F'(v_3) = \{c_2\}$.
Voting situations with finer granularity are frequent in the real world (Alcantud and Laruelle 2014). Hence, it would be simple and realistic to perform a similar analysis with 3-soft sets for example.

Score $0 \in R$ in Definition 6 does not mean there is no assessment or incomplete information. It means the lowest grade. Literally, we easy to find incomplete information in many decision making problems like has been discussed in several soft sets applications like in Alcantud and Santos-García (2016, 2017), Kong et al. (2014), Deng and Wang (2013), Qin et al. (2011). Hence, the real case study given by Example 1 suggests to extend Definition 6 to account for incomplete information in the setting of N-soft set, as follows:

Table 2 Information extracted from the real data in Fig. 1

U	a_1	a_2	a_3	a_4	a_5	a_6
u_1	****	***	***	****	*****	****
u_2	****	*****	***	****	****	****
u_3	****	**	***	***	****	****
u_4	*****	***	**	***	****	***
u_5	***	•	**	***	***	**

It can be expressed as a 6-soft set (cf., Table 3)

Table 3 The 6-soft set in Example 5

$(F, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	4	3	3	4	5	4
u_2	4	5	3	4	4	4
u_3	4	2	3	3	4	4
u_4	5	3	2	3	4	3
u_5	3	0	2	3	3	2

Definition 7 In Definition 6, if we only require the property that for each $a \in A$ and $u \in U$, there exists at most one $(u, r_a) \in U \times R$ such that $(u, r_a) \in F(a)$ then we obtain an incomplete N-soft set.

In particular, Example 1 describes a real case study which can be stated as an incomplete 6-soft set. As in the case of incomplete soft set, we can use a tabular representation of incomplete N-soft set when both A and U are finite too.

We now give a real example of Definition 6 that goes beyond the standard soft set model:

Example 5 We draw a reduced 'cinema table' from Fig. 1 and then transfer its information to the language of 6-soft set. Let U be the universe of movies, $U = \{u_1 =$Tarde para la ira, $u_2 =$Kubo y las dos cuerdas magicas, $u_3 =$Café society, $u_4 =$Neruda, $u_5 =$La estación de las mujeres$\}$. Let E be the set of attributes 'evaluations of movies by media,' and $A \subseteq E$ be such that $A = \{ a_1 =$El mundo, $a_2 =$Guia del ocio, $a_3 =$ Decine21.com, $a_4 =$ ABC, $a_5 =$ Cahiers, $a_6 =$ Cineyteatro.es $\}$. In relation to these elements, a 6-soft set can be defined from Table 2, where five stars means 'obra maestra' (masterpiece), four stars means 'muy buena' (very good), three stars means 'buena' (good), two stars means 'interesante' (interesting), one star means 'regular' (average), and big dot means 'mala' (bad).

The graded evaluation by stars in Example 5 can be easily identified with numbers $R = \{0, 1, 2, 3, 4, 5\}$, where 0 holds for '$\bullet$,' 1 holds for '$*$', 2 holds for '$**$', 3 holds for '$***$', 4 holds for '$****$', and 5 holds for '$*****$'. Therefore, a 6-soft set $(F, A, 6)$ may be considered as follows,

$$F(a_1) = \{(u_1, 4), (u_2, 4), (u_3, 4), (u_4, 5), (u_5, 3)\},$$
$$F(a_2) = \{(u_1, 3), (u_2, 5), (u_3, 2), (u_4, 3), (u_5, 0)\},$$
$$F(a_3) = \{(u_1, 3), (u_2, 3), (u_3, 3), (u_4, 2), (u_5, 2)\},$$
$$F(a_4) = \{(u_1, 4), (u_2, 4), (u_3, 3), (u_4, 3), (u_5, 3)\},$$
$$F(a_5) = \{(u_1, 5), (u_2, 4), (u_3, 4), (u_4, 4), (u_5, 3)\},$$
$$F(a_6) = \{(u_1, 4), (u_2, 4), (u_3, 4), (u_4, 3), (u_5, 2)\}.$$

Under this natural conventions, Table 2 can be adapted to our 6-soft set model easily (cf., Table 3).

Remark 2 Any N-soft set can be naturally considered as an $(N+1)$-soft set, or in general, as an N'-soft set with $N' > N$ arbitrary. Thus, for example, an inspection of Table 3 shows that the 6-soft set in Example 5 can be considered as a 7-soft set on the same universe, and with the same attributes. The only formal difference is that in the latter case, there is a 6 grade available for us, which is never needed.

Motivated by Remark 2, we define:

Definition 8 An N-soft set is efficient if $F(a_j)(u_i) = N-1$ for the some $a_j \in A, u_i \in U$. If (F, A, N) is an N-soft set on U, its minimized soft set is the efficient N-soft set (F_m, A, M) on U defined by $M = \max_{i,j} F(a_j)(u_i) + 1$, $F_m(a_j)(u_i) = F(a_j)(u_i)$, for all $a_j \in A, u_i \in U$.

The motivation for the latter notion is that in some cases, the top grades are available but never used. The 6-soft set in Example 5 is efficient. Any efficient N-soft set coincides with its minimized soft set. We return to this issue in Example 7.

Definition 8 introduces some ideas about grades that are available but are actually superfluous in the parameterization of the universe of options. That definition is concerned with top grades. We now present the formalization of a similar idea with respect to the bottom grades of the parameterization:

Definition 9 Define (F^0, Q, N) the normalized N-soft set from (F, A, N), by the expression: for all $a_i \in A, u_i \in U$, $F^0(a_j)(u_i) = F(a_j)(u_i) - m$, where $m = \min r_{ij}$ in the tabular representation of the original (F, A, N) and $Q = \{\mathbf{1,2,\ldots,q}\}$.

Definition 10 Two N-soft sets (F, A, N) and (F', A', N') are equal over the same universal U if and only if $F = F', A = A'$, and $N = N'$. This is denoted by $(F, A, N) = (F', A', N')$.

Definition 11 We say that two N-soft sets (F, A, N) and (G, A', N) on U are equivalent under normalization if $(F^0, Q, N) = (G^0, Q', N)$.

Definition 11 implies that (F, A, N) and (F^0, Q, N) are equivalent under normalization, for each N-soft set (F, A, N). The following example illustrates Definitions 9 and 11.

Table 4 $(F, A, 6)$ in Example 6

$(F, A, 6)$	a_1	a_2
u_1	1	2
u_2	3	2

Table 5 $(G, A', 6)$ in Example 6

$(G, A', 6)$	b_1	b_2
u_1	3	4
u_2	5	4

Table 6 The normalized 6-soft set in Example 6

$(F^0, Q, 6) = (G^0, Q', 6)$	1	2
u_1	0	1
u_2	2	1

Example 6 Let $A = \{a_1, a_2\}$ and $A' = \{b_1, b_2\}$. Assume that two 6-soft sets $(F, A, 6)$ and $(G, A', 6)$ on $U = \{u_1, u_2\}$ are defined as in Tables 4 and 5, respectively. The normalized 6-soft set (Definition 9) derived from Tables 4 ($m = 1$) and 5 ($m = 3$) is the same, and it is defined by Table 6. Therefore, $(F, A, 6)$ and $(G, A', 6)$ on U are equivalent under normalization (Definition 11).

Later on, in Sect. 4, we observe the importance of equivalence under normalization in the context of decision making.

Definition 12 A weak complement of the N-soft set (F, A, N) is any N-soft set, (F^{c}, A, N), where $F^{\mathsf{c}}(a) \cap F(a) = \emptyset$, for each $a \in A$.

In tabular form, the notion of a weak complement is simple to explain. It derives from any table with the same universe and set of attributes, where the number in each cell is always different from the number in the corresponding cell of the original tabular representation. Let us illustrate this notion in Example 7 below:

Example 7 Two weak complements of the 6-soft set in Example 5, namely $(F_1^c, A, 6)$ and $(F_2^c, A, 6)$, are defined in Tables 7 and 8, respectively. Obviously, the collection of weak complements of the 6-soft set in Example 5 is much larger. Observe that $(F_2^c, A, 6)$ is not efficient, whereas $(F_1^{\mathsf{c}}, A, 6)$ is efficient. The minimized soft set associated with $(F_2^c, A, 6)$ is given by Table 9.

Despite the plurality of weak complements of a given N-soft set, we can pinpoint some noteworthy examples.

Definition 13 The top weak complement of (F, A, N) is (F^t, A, N) where

$$F^t(a_j)(u_i) = \begin{cases} N-1, & \text{if } \ F(a_j)(u_i) < N-1, \\ 0, & \text{if } \ F(a_j)(u_i) = N-1. \end{cases}$$

Table 7 A weak complement of the 6-soft set in Example 5

$(F_1^{\mathsf{c}}, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	5	4	4	3	4	3
u_2	5	4	4	3	3	3
u_3	5	1	4	2	3	3
u_4	3	2	1	4	2	4
u_5	2	1	3	2	2	1

Table 8 Another weak complement of the 6-soft set in Example 5

$(F_2^{\mathsf{c}}, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	3	2	2	3	4	3
u_2	2	3	1	2	2	2
u_3	2	0	1	1	2	2
u_4	3	1	0	1	2	1
u_5	4	1	3	4	4	3

Table 9 The minimized soft set $(F_{2(m)}^{\mathsf{c}}, A, 5)$ of $(F_2^{\mathsf{c}}, A, 6)$ in Table 8

$(F_{2(m)}^{\mathsf{c}}, A, 5)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	3	2	2	3	4	3
u_2	2	3	1	2	2	2
u_3	2	0	1	1	2	2
u_4	3	1	0	1	2	1
u_5	4	1	3	4	4	3

Table 10 The top weak complement of the 6-soft set in Example 5

$(F^t, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	5	5	5	5	0	5
u_2	5	0	5	5	5	5
u_3	5	5	5	5	5	5
u_4	0	5	5	5	5	5
u_5	5	5	5	5	5	5

Definition 14 The bottom weak complement of (F, A, N) is (F^b, A, N) where

$$F^b(a_j)(u_i) = \begin{cases} 0, & \text{if } \ F(a_j)(u_i) > 0, \\ N-1, & \text{if } \ F(a_j)(u_i) = 0. \end{cases}$$

Example 8 The top and the bottom weak complements of the 6-soft set in Example 5 are given in Tables 10 and 11, respectively.

Not only the model by N-soft sets includes standard soft sets. Even if we are not bound by any finiteness assumption, we can also associate uniquely determined soft sets with each N-soft set as follows:

Table 11 The bottom weak complement of the 6-soft set in Example 5

$(F^b, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	0	0	0	0	0	0
u_2	0	0	0	0	0	0
u_3	0	0	0	0	0	0
u_4	0	0	0	0	0	0
u_5	0	5	0	0	0	0

Table 12 Tabular representation of $(G, B, 4)$ in Examples 9, 10 and 11

$(G, B, 4)$	a_2	a_3	b
u_1	3	2	3
u_2	3	1	1
u_3	2	3	2
u_4	1	3	1
u_5	2	1	0

Definition 15 Let (F, A, N) be an N-soft set, and $0 < T < N$ be a threshold. The soft set associated with (F, A, N) and T is denoted by (F^T, A), and it is defined by the expression $u \in F^T(a)$ if and only if $r_a \geq T$ when $(u, r_a) \in F(a)$. In other words,

$$(F^T, A) = \begin{cases} 1, & \text{if } r_a \geq T, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we say that (F^1, A) is the bottom soft set associated with (F, A, N), and (F^{N-1}, A) is the top soft set associated with (F, A, N).

The following example produces the soft sets associated with (F, A, N) and all possible thresholds.

Example 9 Consider the 4-soft set $(G, B, 4)$ described in tabular form by Table 12.

Thus, the range of the threshold is $0 < T < 4$.

All the possible soft sets associated with $(G, B, 4)$ and feasible thresholds are defined in tabular form by Table 13.

In the theoretical analysis of soft sets, union and intersection are concepts that have deserved the attention of many researchers (Zhan et al. 2017d; Ali et al. 2015; Sezgın and Atagün 2011; Ali et al. 2009; Maji et al. 2003). We proceed to show that the model that we have introduced provides a very rich environment to develop these notions.

Inspired by ideas from Ali et al. (2009) (cf., Definitions 2–5), we can define the restricted, resp., extended intersection and union of N-soft sets. Observe that in order to define these notions, we are not restricted by any finiteness assumption either.

Definition 16 Let U be a fixed universe of objects. The restricted intersection of (F, A, N_1) and (G, B, N_2) is denoted by $(F, A, N_1) \cap_{\mathscr{R}} (G, B, N_2)$. It is defined as $(H, A \cap B, \min(N_1, N_2))$ where for all $a \in A \cap B$ and $u \in U$,

$$(u, r_a) \in H(a) \Leftrightarrow r_a = \min(r_a^1, r_a^2), \text{ if } (u, r_a^1) \in F(a) \text{ and } (u, r_a^2) \in G(a)$$

Table 13 Tabular representation of the possible soft sets associated with $(G, B, 4)$ and thresholds in Example 9

(G^1, B)	a_2	a_3	b
u_1	1	1	1
u_2	1	1	1
u_3	1	1	1
u_4	1	1	1
u_5	1	1	0
(G^2, B)	a_2	a_3	b
u_1	1	1	1
u_2	1	0	0
u_3	1	1	1
u_4	0	1	0
u_5	1	0	0
(G^3, B)	a_2	a_3	b
u_1	1	0	1
u_2	1	0	0
u_3	0	1	0
u_4	0	1	0
u_5	0	0	0

Table 14 Tabular representation of $(F, A, 6) \cap_{\mathscr{R}} (G, B, 4)$, i.e., the restricted intersection of $(F, A, 6)$ described by Table 3, and $(G, B, 4)$ described by Table 12

$(H, A \cap B, 4)$	a_2	a_3
u_1	3	2
u_2	3	1
u_3	2	3
u_4	1	2
u_5	0	1

Definition 16 is easy to implement when N-soft sets are written in tabular form. Then, and only for attributes that belong to the parameter spaces of both tables, we minimize the values in the corresponding columns. Let us illustrate this definition with an example:

Example 10 Consider $(F, A, 6)$ in Example 5, and the 4-soft set $(G, B, 4)$ described in tabular form by Table 12. Then, $(F, A, 6) \cap_{\mathscr{R}} (G, B, 4) = (H, A \cap B, \min(6, 4)) = (H, \{a_1, a_2\}, 4)$ is defined in tabular form by Table 14.

Definition 17 Let U be a fixed universe of objects. The extended intersection of (F, A, N_1) and (G, B, N_2) is denoted by $(F, A, N_1) \cap_{\mathscr{E}} (G, B, N_2)$.

Table 15 Tabular representation of $(F, A, 6) \cap_{\mathscr{E}} (G, B, 4)$, i.e., the extended intersection of $(F, A, 6)$ described by Table 3, and $(G, B, 4)$ described by Table 12

$(J, A \cup B, 6)$	a_1	a_2	a_3	a_4	a_5	a_6	b
u_1	4	3	2	4	5	4	3
u_2	4	3	1	4	4	4	1
u_3	4	2	3	3	4	4	2
u_4	5	1	2	3	4	3	1
u_5	3	0	1	3	3	2	0

It is $(J, A \cup B, \max(N_1, N_2))$ which is defined by

$$J(a) = \begin{cases} F(a), \text{ if } a \in A \setminus B, \\ G(a), \text{ if } a \in B \setminus A, \\ (u, r_a) \text{ such that } r_a = \min(r_a^1, r_a^2), \text{ where} \\ \qquad (u, r_a^1) \in F(a) \text{ and } (u, r_a^2) \in G(a). \end{cases}$$

Example 11 below illustrates Definition 17:

Example 11 Consider $(F, A, 6), (G, B, 4)$ as in Example 10. Then, $(J, A \cup B, \max(6, 4)) = (J, \{a_1, \dots, a_6, b\}, 6)$, their extended intersection, is defined in tabular form by Table 15.

We observe that the columns representing common attributes coincide with the grade assignments in the restricted intersection (cf., Table 14 from Example 10), whereas the columns representing attributes that only describe the universe of options in one case coincide with the corresponding description (i.e., either Table 3 or 12).

Definition 18 Let U be a fixed universe of objects. The restricted union of (F, A, N_1) and (G, B, N_2) is denoted by $(F, A, N_1) \cup_{\mathscr{R}} (G, B, N_2)$.

It is defined as $(K, A \cap B, \max(N_1, N_2))$ where for all $a \in A \cap B$ and $u \in U$,

$$(u, r_a) \in K(a) \Leftrightarrow r_a = \max(r_a^1, r_a^2), \text{ if } (u, r_a^1) \in F(a) \text{ and } (u, r_a^2) \in G(a)$$

Definition 18 is easy to implement when N-soft sets are written in tabular form. Then, and only for attributes that belong to the parameter spaces of both tables, we maximize the values in the corresponding columns. Let us illustrate this definition with an example:

Example 12 Consider $(F, A, 6)$ in Example 5, and the 4-soft set $(G, B, 4)$ described in tabular form by Table 12. Then, $(F, A, 6) \cup_{\mathscr{R}} (G, B, 4) = (K, A \cap B, \max(6, 4)) = (K, \{a_1, a_2\}, 6)$ is defined in tabular form by Table 16.

Definition 19 Let U be a fixed universe of objects. The extended union of (F, A, N_1) and (G, B, N_2) is denoted by $(F, A, N_1) \cup_{\mathscr{E}} (G, B, N_2)$.

Table 16 Tabular representation of $(F, A, 6) \cup_{\mathscr{R}} (G, B, 4)$, i.e., the restricted union of $(F, A, 6)$ described by Table 3, and $(G, B, 4)$ described by Table 12

$(K, A \cap B, 6)$	a_2	a_3
u_1	3	3
u_2	5	3
u_3	2	3
u_4	3	3
u_5	2	2

Table 17 Tabular representation of $(F, A, 6) \cup_{\mathscr{E}} (G, B, 4)$, i.e., the extended union of $(F, A, 6)$ described by Table 3, and $(G, B, 4)$ described by Table 12

$(L, A \cup B, 6)$	a_1	a_2	a_3	a_4	a_5	a_6	b
u_1	4	3	3	4	5	4	3
u_2	4	5	3	4	4	4	1
u_3	4	2	3	3	4	4	2
u_4	5	3	3	3	4	3	1
u_5	3	2	2	3	3	2	0

It is $(L, A \cup B, \max(N_1, N_2))$, and it is defined by

$$J(a) = \begin{cases} F(a), \text{ if } a \in A \setminus B, \\ G(a), \text{ if } a \in B \setminus A, \\ (u, r_a) \text{ such that } r_a = max(r_a^1, r_a^2), \text{ where} \\ \qquad (u, r_a^1) \in F(a) \text{ and } (u, r_a^2) \in G(a). \end{cases}$$

Example 13 illustrates Definition 19:

Example 13 Consider $(F, A, 6)$ and $(G, B, 4)$ as in Example 10. Then, their extended union $(L, A \cup B, \max(6, 4))$, i.e., $(L, \{a_1, \dots, a_6, b\}, 6)$ is defined in tabular form by Table 17.

We observe that the columns representing common attributes coincide with the grade assignments in the restricted union (cf., Table 16 from Example 12), whereas the columns representing attributes that only describe the universe of options in one case coincide with the corresponding description (i.e., either Table 3 or 12).

So far, we have exploited the ideas from Ali et al. (2009) concerning intersection and union concepts in order to adapt them to the new model that we have introduced. But in this framework, we can take a completely different position in order to define the novel restricted, resp., extended T-intersection and restricted, resp., extended T-union of two N- and N'-soft sets as follows.

Definition 20 The restricted T-intersection of (F, A, N) and (G, B, N') over a common universe U, where $T \leqslant \min(N, N')$, is the restricted intersection of the soft sets (F^T, A) and (G^T, B). It is denoted by $(F, A, N) \cap_{\mathscr{R}}^T (G, B, N')$.

Similarly, we can define the extended T-intersection of (F, A, N) and (G, B, N') as the extended intersection of (F^T, A) and (G^T, B). It is denoted by $(F, A, N) \cap^T_{\mathscr{E}} (G, B, N')$.

Definition 21 The restricted T-union of (F, A, N) and (G, B, N') over a common universe U, where we request $T \leqslant \min(N, N')$, is the restricted union of the soft sets (F^T, A) and (G^T, B). It is denoted by $(F, A, N) \cup^T_{\mathscr{R}} (G, B, N')$.

Similarly, we can define the extended T-union of (F, A, N) and (G, B, N') as the extended union of (F^T, A) and (G^T, B). It is denoted by $(F, A, N) \cup^T_{\mathscr{E}} (G, B, N')$.

We emphasize that we are not bound by any finiteness assumption in Definitions 20 and 21.

4 Decision making procedures for N-soft sets

Soft sets have been applied in many real decision making problems (Sutoyo et al. 2016; Alcantud et al. 2015; Xiao et al. 2009). In this section, we propose some decision making procedures for N-soft sets. From each N-soft set (F, A, N) or its tabular representation, we can trivially define a matrix of grades, namely

$$(r_{ij})_{p\times q} = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1q} \\ r_{21} & r_{22} & \dots & r_{2q} \\ \dots & \dots & \dots & \dots \\ r_{p1} & r_{p2} & \dots & r_{pq} \end{pmatrix}$$

4.1 N-soft sets choice values

This decision making procedure is the extension of the decision making procedure for soft sets in Maji et al. (2002) without reduction of the parameters. It ranks the alternatives by their extended choice values (ECVs), or more generally, by their extended weight choice values (EWCVs), in Algorithms 1 and 2, respectively. The detailed steps of these two algorithms are presented as follows.

Algorithm 1 *The algorithm of ECVs.*

1. *Input* $U = \{u_1, \dots, u_p\}$ *and* $A = \{a_1, \dots, a_q\}$.
2. *Input the N-soft set* (F, A, N), *with* $R = \{0, 1, \dots, N-1\}$, $N \in \{2, 3, \dots\}$, *so that* $\forall\, u_i \in U, a_j \in A,\ \exists!\, r_{ij} \in R$.
3. *For each* u_i, *compute its ECV* $\sigma_i = \sum_{j=1}^{q} r_{ij}$.
4. *Find all indices k for which* $\sigma_k = \max_{i=1,\dots,p} \sigma_i$.
5. *The solution is any u_k from Step 4.*

In Example 1, we explain that Algorithm 1 is used in the real situation that it captures. Relatedly, we have the following case study:

Table 18 The extended choice values in Example 14

$(F, A, 6)$	a_1	a_2	a_3	a_4	a_5	a_6	σ_i
u_1	4	3	3	4	5	4	23
u_2	4	5	3	4	4	4	*24*
u_3	4	2	3	3	4	4	20
u_4	5	3	2	3	4	3	20
u_5	3	0	2	3	3	2	13

Example 14 The choice values of the 6-soft set in Example 5 can be seen in Table 18. The movie u_2 is selected, and the ranking decision is $u_2 > u_1 > u_3 = u_4 > u_5$.

Algorithm 2 *The algorithm of EWCVs.*

1. *Input* $U = \{u_1, \dots, u_p\}$ *and* $A = \{a_1, \dots, a_q\}$, *and a weight w_j for each parameter j.*
2. *Input the N-soft set* (F, A, N), *with* $R = \{0, 1, \dots, N-1\}$, $N \in \{2, 3, \dots\}$, *so that* $\forall\, u_i \in U, a_j \in A,\ \exists!\, r_{ij} \in R$.
3. *For each* u_i, *compute its EWCV* $\sigma_i^w = \sum_{j=1}^{q} w_j r_{ij}$.
4. *Find all indices k for which* $\sigma_k^w = \max_{i=1,\dots,p} \sigma_i^w$.
5. *The solution is any u_k from Step 4.*

Example 15 Assume that we give the following weights for each of the media a_j, $j = 1, 2, \dots, 6$, in Example 5: $w_1 = 0.1$, $w_2 = 0.3$, $w_3 = 0.3$, $w_4 = 0.1$, $w_5 = 0.1$, $w_6 = 0.1$. Thus, we can compute the corresponding extended weight choice values of the 6-soft set in Example 5, which is given in Table 19. We obtain the same decision as in Example 14.

Remark 3 It is immediate to check that if two N-soft sets (F, A, N) and (G, A', N) are equivalent under normalization, then the extended choice value ranking of U under both N-soft sets is the same.

Therefore in particular, the results using Algorithm 1 (resp., Algorithm 2) for an N-soft set and its normalized N-soft set are the same.

4.2 N-soft sets T-choice values

It is also possible to rank the alternatives in U from the information in (F, A, N) and a threshold T, by routine application of CVs to the soft set (F^T, A).

To that purpose, we denote by σ_i^T the CV at option i of the soft set (F^T, A) derived by Definition 15. And we call σ_i^T the T-choice value of (F, A, N) at option i.

With this notation, we can endorse the following new algorithm:

Algorithm 3 *The algorithm of T-choice values.*

1. *Input* $U = \{u_1, \dots, u_p\}$ *and* $A = \{a_1, \dots, a_q\}$.

Table 19 The extended weight choice values in Example 15

$(F, A, 6)$	a_1 $w_1 = 0.1$	a_2 $w_2 = 0.3$	a_3 $w_3 = 0.3$	a_4 $w_4 = 0.1$	a_5 $w_5 = 0.1$	a_6 $w_6 = 0.1$	σ_i^w
u_1	0.4	0.9	0.9	0.4	0.5	0.4	3.5
u_2	0.4	1.5	0.9	0.4	0.4	0.4	*4*
u_3	0.4	0.6	0.9	0.3	0.4	0.4	3
u_4	0.5	0.9	0.6	0.3	0.4	0.3	3
u_5	0.3	0	0.6	0.3	0.3	0.2	1.7

Table 20 The 4-choice values in Example 16

(F^4, A)	a_1	a_2	a_3	a_4	a_5	a_6	σ_i^4
u_1	1	0	0	1	1	1	4
u_2	1	1	0	1	1	1	*5*
u_3	1	0	0	0	1	1	3
u_4	1	0	0	0	1	0	2
u_5	0	0	0	0	0	0	0

They are computed from (F^4, A) in Example 5

2. *Input the N-soft set (F, A, N), with $R = \{0, 1, \ldots, N-1\}$, $N \in \{2, 3, \ldots\}$, so that $\forall\, u_i \in U, a_j \in A$, $\exists!\, r_{ij} \in R$.*
3. *Input the T threshold.*
4. *Compute the r_{ij}^T values as $r_{ij}^T = \begin{cases} 1, \text{if } r_{ij} \geq T, \\ 0, \text{otherwise.} \end{cases}$*
5. *For each u_i, compute its T-choice value σ_i^T, where $\sigma_i^T = \sum_{j=1}^{q} r_{ij}^T$.*
6. *Find k for which $\sigma_k^T = \max_{i=1,\ldots,p} \sigma_i^T$.*
7. *The solution is any u_k from Step 6.*

Example 16 Table 20 gives the 4-choice values of the 6-soft set in Example 5. The ranking that this decision making procedure suggests is $u_2 > u_1 > u_3 > u_4 > u_5$.

5 Comparisons

In this section, we use real data to compare the results of our algorithms. We draw information from Kimovil (`https://www.kimovil.com`). This webpage is a mobile comparator which summarizes the performance of hundreds of models of mobile phones regarding 6 characteristics: Cost-effectiveness (attribute a_1), Design & Materials (attribute a_2), Performance & Hardware (attribute a_3), Camera (attribute a_4), Connectivity (attribute a_5), Battery (attribute a_6). Therefore, we can model this information as a 101-soft set, since their set of grades has granularity 101 which we identify by $R = \{0, 1, \ldots, 100\}$. We proceed to give a formal model and decision making comparison for this real situation in our next example:

Table 21 The 101-soft set and associated ECVs in Example 17

$(F, A, 101)$	a_1	a_2	a_3	a_4	a_5	a_6	σ_i
u_1	100	97	100	100	99	85	581
u_2	100	99	97	100	98	90	*584*
u_3	100	99	97	99	98	90	583
u_4	100	99	97	99	99	70	564
u_5	100	99	97	100	94	90	580
u_6	100	99	97	99	94	89	578
u_7	100	99	97	99	94	70	559
u_8	100	100	96	97	99	71	563
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{21}	98	96	97	94	100	93	578
u_{22}	98	97	98	98	98	83	572
u_{23}	99	99	92	97	98	91	576
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{26}	98	96	95	95	99	92	575
u_{27}	98	96	97	94	100	92	577
u_{28}	98	96	99	94	100	92	579
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{32}	97	95	93	97	94	97	573
...	...	...	...	...	...	...	...
u_{71}	80	89	85	91	60	73	478

The first 8 elements are depicted in Figs. 4 and 5. Data retrieved on July 5, 2017

Fig. 4 Real data in Example 17. *Source*: `https://www.kimovil.com`, retrieved on July 5th, 2017

Example 17 Assume a customer wants to purchase a mobile phone with its price between 400 and 500 euros. The real data sets that he uses are retrieved on July 5th, 2017 from the

Fig. 5 Real data in Example 17 (cont.). *Source*: `https://www.kimovil.com`, retrieved on July 5, 2017

Kimovil webpage (Figs. 4 and 5). The buyer obtains a list of 72 mobile phone models (one of which has incomplete information). They are characterized by 6 attributes.

The customer discards the mobile phone without complete information. Thus, we have 71 alternatives. The resulting 101-soft set is given by Table 21. For convenience, we trim it in order to avoid distracting information.

Suppose now that according to his personal likes and dislikes, the customer gives the following weights for each of the attributes: $w_1 = 0.3$, $w_2 = 0.1$, $w_3 = 0.2$, $w_4 = 0.2$, $w_5 = 0.1$, $w_6 = 0.1$. Thus, we can compute the corresponding extended weight choice values of this 101-soft set, which are summarized in Table 22.

Suppose a threshold $T = 90$. The 90-choice value of $(F, A, 101)$ can be seen in Table 23.

If we compare the three algorithms (Table 24), we observe that the results are quite different. This is a proof of the flexibility and adaptability of our decision making algorithms. Such feature is desirable because different users have different needs, therefore the allocation of weights to the characteristics or the choice of a threshold are simple and efficient ways to introduce this flexibility in the decision making mechanism.

6 Conclusion

In this paper, we propose a new extended and applicable version of soft sets that can handle both binary and non-binary evaluations. Several real examples have served us to show that this model is realistic. Our novel proposal owes to the spirit of soft set theory in the sense that it is phrased in terms of parameterizations. However, soft sets parameterize the universe of alternatives in binary terms (that is to say,

Table 22 The extended weight choice values in Example 17 with our choice of weights

$(F, A, 101)$	a_1 $w_1 = 0.3$	a_2 $w_2 = 0.1$	a_3 $w_3 = 0.2$	a_4 $w_4 = 0.2$	a_5 $w_5 = 0.1$	a_6 $w_6 = 0.1$	σ_i^w
u_1	30.0	9.7	20.0	20.0	9.9	8.5	*98.1*
u_2	30.0	9.9	19.4	20.0	9.8	9.0	*98.1*
u_3	30.0	9.9	19.4	19.8	9.8	9.0	97.9
u_4	30.0	9.9	19.4	19.8	9.9	7.0	96.0
u_5	30.0	9.9	19.4	20.0	9.4	9.0	97.7
u_6	30.0	9.9	19.4	19.8	9.4	8.9	97.4
u_7	30.0	9.9	19.4	19.8	9.4	7.0	95.5
u_8	30.0	10.0	19.2	19.4	9.9	7.1	95.6
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{21}	29.4	9.6	19.4	18.8	10.0	9.3	96.5
u_{22}	29.4	9.7	19.6	19.6	9.8	8.3	96.4
u_{23}	29.7	9.9	18.4	19.4	9.8	9.1	96.3
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{26}	29.4	9.6	19.0	19.0	9.9	9.2	96.1
u_{27}	29.4	9.6	19.4	18.8	10.0	9.2	96.4
u_{28}	29.4	9.6	19.8	18.8	10.0	9.2	96.8
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{32}	29.1	9.5	18.6	19.4	9.4	9.7	95.7
...	...	...	...	...	...	...	...
u_{71}	24.0	8.9	17.0	18.2	6.0	7.3	81.4

Table 23 The 90-choice values computed from (F^{90}, A) in Example 17

(F^{90}, A)	a_1	a_2	a_3	a_4	a_5	a_6	σ_i^{90}
u_1	1	1	1	1	1	0	5
u_2	1	1	1	1	1	1	*6*
u_3	1	1	1	1	1	1	*6*
u_4	1	1	1	1	1	0	5
u_5	1	1	1	1	1	1	*6*
u_6	1	1	1	1	1	0	5
u_7	1	1	1	1	1	0	5
u_8	1	1	1	1	1	0	5
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{21}	1	1	1	1	1	1	*6*
u_{22}	1	1	1	1	1	0	5
u_{23}	1	1	1	1	1	1	*6*
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{26}	1	1	1	1	1	1	*6*
u_{27}	1	1	1	1	1	1	*6*
u_{28}	1	1	1	1	1	1	*6*
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
u_{32}	1	1	1	1	1	1	*6*
...	...	...	...	...	...	...	...
u_{71}	0	0	0	1	0	0	1

Table 24 The results of the best performance of mobile phones in Example 17

No.	Algorithm	Decision
1	ECVs	u_2
2	EWCVs	u_1 or u_2
3	T-CV, $T = 90$	$u_2, u_3, u_5, u_{21}, u_{23}$, u_{26}, u_{27}, u_{28}, or u_{32}

either belongingness or non-belongingness). Although these dichotomies suffice to describe many standard cases (Alcantud et al. 2013), other relevant real applications demand a more flexible description. To that purpose, we have argued that the concept of N-soft set is a novel and adequate specification.

Subsequently, we have defined associated notions as well as operations among N-soft sets. Finally, their properties have been investigated and three decision making procedures have been presented and illustrated with examples.

Our algorithms are in the tradition of soft set decision making, and they rely on natural extensions of its main ideas. They are flexible and allow the practitioner to introduce subjectivity in order to account for his or her personal preferences.

Our definition opens up new avenues for research. Within this model, it is feasible to study entropy, correlation and similarity indices between N-soft sets as a tool to help in the decision making processes. Parameter reduction in N-soft sets can also be defined and investigated. This analysis is in continuation of the original approach in Maji et al. (2002) for soft sets. It addresses the problem in real situations like that in Fig. 1. But we can also go beyond our framework. It is possible to develop a theory on N-soft sets under incomplete information. This theory would extend existing literature in incomplete soft sets (Alcantud and Santos-García 2016, 2017; Han et al. 2014; Qin et al. 2011; Zou and Xiao 2008). And it means a useful complement to the theory about incomplete fuzzy soft sets (Liu et al. 2017; Yang et al. 2015; Deng and Wang 2013). Last but not least, hesitancy is basically reduced to lack of information in the soft set environment. Because the parameterized descriptions are binary, we can only introduce hesitancy by being fully indeterminate about the belongingness of the alternatives to the sets of e-approximate elements. Such hesitancy can be rightfully interpreted in terms of incomplete information. However, it is only natural to introduce genuine hesitancy in the context of N-soft sets, because in an environment with multiplicity of the parameterizations one can doubt about which one is correct. We expect to return to these issues in the future.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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