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Uncertain programming models for fixed charge multi-item solid transportation problem

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Abstract This paper investigates the fixed charge multi-item solid transportation problem, in which the fixed charges, direct costs, transportation capacities, supply and demand are uncertain variables. Based on the uncertainty theory, expected value programming model and chance-constrained programming model for fixed charge multi-item solid transportation problem are constructed, respectively. We can obtain the optimal solution of two models via solving the relevant deterministic models. Finally, a numerical experiment is implemented to illustrate the application of the models.

Keywords Transportation problem · Uncertainty programming · Uncertain variable

1 Introduction

The transportation problem (TP) is a well-known optimization problem in operational research, which goal is to minimize the total transportation cost and improve the transportation quality. In the traditional transportation problem, source constraint and destination constraint are taken into consideration. But, in real life, there are other constraints be considered. Firstly, solid transportation problem which was defined as transportation of goods by a sequence of two different conveyances was introduced by Haley (1962). After that, Bhatia et al. (1976) gave a method and obtained the minimum time for the solid transportation problem (STP). Li et al. (1997) proposed neural network approach for multi-criteria solid transportation problems. Ojha et al. (2010) provided a solid transportation problem with discounted costs, fixed charges and vehicle costs, which was formulated as a linear programming problem. Secondly, multi-item solid transportation problem (MISTP) is discussed by Kennington and Unger (1976) and Sun et al. (1998). Fixed charge is another research aspect for TP. Since Hirsch and Dantzig (1968) proposed fixed charge transportation problem (FCTP), there are many researchers who investigated this problem: Lotif and Moghaddam (2013) proposed a genetic algorithm using priority-based encoding for linear and nonlinear fixed charge transportation problem.

In classical models of transportation problem, the parameters of the models are supposed to be crisp numbers. However, some nondeterminacy factors might occur in many situations, such as market supply and demand, weather conditions, road conditions. Some researchers believed that these nondeterministic phenomena conform to randomness, and they introduced probability theory into the transportation network problem. Williams (1963) proposed a stochastic transportation model, in which the demands were supposed to be random variables. Following Williams, Yang and Feng (2007) studied a bicriteria solid transportation problem with stochastic parameters. Romeijna and Sargutb (2011) presented a branch-and-price algorithm for solving a class of stochastic transportation problems with single-source constraints.

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However, it is not suitable to regard every nondeterminacy phenomenon as random phenomenon, because of the lack of available samples. For instance, when the extreme events occur, it is impossible to get probability distribution. But experts can estimate, based on their experience, the belief degree can be got. In order to deal with this kind of human uncertainty, (2007) founded uncertainty theory, which was refined by Liu (2010b). Since then, uncertainty theory and its application have experienced explosive growth, such as uncertain process (Liu 2008), uncertain differential equation (Chen and Liu 2010; Yao 2013; Gao 2016; Yao and Chen 2013; Yang and Ralescu 2015; Wang et al. 2015, Yang and Shen 2015), uncertain set theory (Liu 2010a; Yao 2015), uncertain programming (Liu 2009a; Li and Qin 2014; Sheng and Gao 2016), uncertain inference (Gao et al. 2010), uncertain economics (Yang and Gao 2017), uncertain management (Gao et al. 2017; Gao and Yao 2015), uncertain finance (Chen and Gao 2013; Guo and Gao 2017), uncertain statistics (Liu 2010b; Gao et al. 2016) and uncertain differential game (Yang and Gao 2013; Yang and Gao 2016). Up to now, the uncertainty theory has become a branch of mathematics.

With respect to the uncertain factors of transportation problem, some authors used the uncertainty theory to handel this problem. Sheng and Yao (2012) gave the uncertain model for fixed charge TP. Cui and Sheng (2012) considered the solid transportation problem in uncertain environment and proposed an expected constrained model. Dalman (2017) consider multi-item STP. Zhang et al. (2016) investigated the fixed charge solid transportation problem under uncertainty. Gao et al. (2016) studied uncertain models on railway transportation planning problem. Chen et al. (2017) investigated the solid transportation problem based an entropy in uncertain environment.

Based on their motivations, we extend this work to fixed charge multi-item solid transportation problem (FCMISTP). Uncertain transportation models, namely expected value programming model and chance-constrained programming model, are proposed, respectively. Based on the uncertainty theory, it can be shown that the optimal solution to transportation models can be obtained via solving a relevant deterministic model. Finally, a numerical experiment is implemented to illustrate the application of the models.

The remainder of this paper is organized as follows. In Sect. 2, some basic concepts and properties in uncertainty theory used in this paper are introduced. Sect. 3 is problem description, where the fixed charge multi-item solid transportation problem is briefly explicate, the expected value programming model for FCMISTP is constructed. And a relevant deterministic model can be obtained by uncertainty theory. In Sect. 4, the chance-constrained model for FCMISTP is constructed. And a relevant deterministic model can be obtained by uncertainty theory. In Sect. 5, numerical examples are given to illustrate the models. In Sect. 6, we give a brief summary to this paper.

2 Preliminary

Uncertainty theory provides an axiomatic system to cope with the imprecise information in experts' experiment data. In this section, we state some basic concepts and properties in uncertainty theory (2007), which will be used to throughout this paper.

Definition 1 (Liu 2007) Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called uncertainty spaces. Furthermore, Liu (2009b) defined a product uncertain measure by the fourth axiom.

Axiom 4. *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where *Λ_k are arbitrarily chosen events form $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.*

Definition 2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 4 (Liu 2009b) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \min_{1\leq i\leq n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition 5 (Liu 2010b) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

We usually assume that all uncertainty distributions in practical applications are regular. Otherwise, a small perturbation can be imposed to obtain a regular one. The following Theorem 1 shows that the inverse uncertainty distribution has good operational properties, which makes it easy to obtain the solution for the uncertain programming problem.

Theorem 1 (Liu 2010b) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Definition 6 (Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2 (Liu 2007) *Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$\mathrm{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Theorem 3 (Liu 2010b) *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$\mathrm{E}[a\xi + b\eta] = a\mathrm{E}[\xi] + b\mathrm{E}[\eta].$$

3 Expected value programming model for FCMISTP

3.1 Notations

In this section, we will state fixed charge multi-item STP in detail. The fixed charge multi-item STP is to make a transport plan so that the total transportation cost is minimized. We will consider two types of costs, which are the direct cost and the fixed charge. When the conveyance moves on a link, the direct costs will arise along with the cost of per unit. The direct cost, naturally, is a function related to the conveyance type and the characteristic of links.

Different conveyances and different links have different fixed charge. Furthermore, the capacity of each link during the planning period is limited, which is mainly presented in two aspects. First, during one period, the frequency of using each link is limited. Second, on each link, the capacity of every conveyance is limited. To construct the mathematical models for FCMISTP, we need to determine the parameters as follows (Table 1).

3.2 Objective function and constraints

The total fixed charge of opening link is

$$f_1 = \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \eta_{ijp} y_{ijp},$$

If $y_{ijp} = 0$, link (i, j) will not be open in this transportation network, so there is no fixed charge in this link. If $y_{ijp} \neq 0$, the transportation activity is assigned on link (i, j), then the corresponding fixed charge will occur. The total direct transportation cost is

$$f_2 = \sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \xi_{ijp}^{k} x_{ijp}^{k}.$$

So the total relevant cost can be formulated as

$$f = \sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \xi_{ijp}^{k} x_{ijp}^{k} + \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \eta_{ijp} y_{ijp}. \quad (1)$$

Table 1 List of notations

$i \in \{1, 2, \dots, m\}$	The index for sources
$j \in \{1, 2, \dots, n\}$	The index for destinations
$k \in \{1, 2, \dots, l\}$	The index for items to be transported
$p = \{1, 2, \dots, q\}$	The index for conveyances
z^+	The set of nonnegative integers
a_i^k	The amount of item k at source i
b_j^k	The demand of item k at destination j
c_{ijp}	The capacity of conveyance p from i to j
ξ_{ijp}^k	The cost of per unit of k from i to j by p
η_{ijp}	The fixed cost from i to j by p
x_{ijp}^k	Amount of item k from i to j by p
y_{ijp}	Frequency from i to j by p, $y_{ijp} \in z^+$
d_{ijp}	The maximum frequency from i to j by p

The total quantity carried from source i is no more than a_i^k, we can give the following constraint

$$\sum_{p=1}^{q}\sum_{j=1}^{n} x_{ijp}^k \le a_i^k. \tag{2}$$

The total quantity in destination j is less than b_j^k, so we have

$$\sum_{p=1}^{q}\sum_{i=1}^{m} x_{ijp}^k \ge b_j^k. \tag{3}$$

Also, the total amount transportation by conveyance p is no more than its transportation capacity. Then we have the following constraint

$$\sum_{k=1}^{l} x_{ijp}^k \le y_{ijp}c_{ijp}. \tag{4}$$

3.3 Expected value programming model

Let $\bar{f}$ be a predetermined maximal cost, the most transportation plan of FCMISTP is that the maximal uncertain measure of the cost less than or equal to $\bar{f}$.

Definition 7 A solution $(\mathbf{x}^*, \mathbf{y}^*)$ is called most transportation plan of FCMISTP if

$$\mathcal{M}\{f(\mathbf{x}^*, \mathbf{y}^*; \xi, \eta) \le \bar{f}\} \ge \mathcal{M}\{f(\mathbf{x}, \mathbf{y}; \xi, \eta) \le \bar{f}\}$$

holds for any feasible solution $(\mathbf{x}, \mathbf{y})$, and $\bar{f}$ is a predetermined maximal cost.

The main idea of expected value programming model is to optimize the expected value of the objective function, under the expected value constraints. We may construct the model for FCMISTP as follows:

$$\begin{cases} \max \ \ \mathrm{E}[f_1 + f_2] \\ \text{s.t. } \ f_1 = \sum\limits_{p=1}^{q}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} \eta_{ijp}y_{ijp} \\ \quad f_2 = \sum\limits_{k=1}^{l}\sum\limits_{p=1}^{q}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} \xi_{ijp}^k x_{ijp}^k \\ \quad \mathrm{E}\left[\sum\limits_{p=1}^{q}\sum\limits_{j=1}^{n} x_{ijp}^k - a_i^k\right] \le 0 \\ \quad \mathrm{E}\left[\sum\limits_{p=1}^{q}\sum\limits_{i=1}^{m} x_{ijp}^k - b_j^k\right] \ge 0 \\ \quad \mathrm{E}\left[\sum\limits_{k=1}^{l} x_{ijp}^k - y_{ijp}c_{ijp}\right] \le 0 \\ \quad 0 \le y_{ijp} \le d_{ijp}, \ \ y_{ijp}, d_{ijp} \in z^+ \\ \quad x_{ijp}^k \ge 0, \end{cases} \tag{5}$$

where $i \in [1, m]$, $j \in [1, n]$, $p \in [1, q]$, $k \in [1, l]$, $i, j, p, k \in z^+$.

3.4 Deterministic transformation

In order to solve the constructed model, we can transfer them into deterministic form.

Theorem 4 *We assume that ξ_{ijp}^k, η_{ijp}, a_i^k, b_j^k, c_{ijp} are independent uncertain variables with regular uncertainty distributions Φ_{ijp}^k, Ψ_{ijp}, Υ_i^k, Λ_j^k, Θ_{ijp}, respectively, then the model (5) is equivalent to the following deterministic transportation* *model*

$$\begin{cases} \min & \left(\sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ijp}^k \int_0^1 (\Phi_{ijp}^k)^{-1}(\alpha)\mathrm{d}\alpha \right. \\ & \left. + \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} y_{ijp} \int_0^1 (\Psi_{ijp})^{-1}(\alpha)\mathrm{d}\alpha \right) \\ \text{s.t.} & \sum_{p=1}^{q}\sum_{j=1}^{n} x_{ijp}^k - \int_0^1 (\Upsilon_i^k)^{-1}(\alpha)\mathrm{d}\alpha \le 0 \\ & \sum_{p=1}^{q}\sum_{i=1}^{m} x_{ijp}^k - \int_0^1 (\Lambda_j^k)^{-1}(\alpha)\mathrm{d}\alpha \ge 0 \\ & \sum_{k=1}^{l} x_{ijp}^k - y_{ijp} \int_0^1 (\Theta_{ijp})^{-1}(\alpha)\mathrm{d}\alpha \le 0 \\ & 0 \le y_{ijp} \le d_{ijp}, \ \ y_{ijp}, d_{ijp} \in z^+ \\ & x_{ijp}^k \ge 0, \end{cases} \tag{6}$$

where $i \in [1, m]$, $j \in [1, n]$, $p \in [1, q]$, $k \in [1, l]$, $i, j, p, k \in z^+$, *and* $(\Phi_{ijp}^k)^{-1}$, $(\Psi_{ijp})^{-1}$, $(\Upsilon_i^k)^{-1}$, $(\Lambda_j^k)^{-1}$, $(\Theta_{ijp})^{-1}$ *are the inverse distribution of the distribution* Φ_{ijp}^k, Ψ_{ijp}, Υ_i^k, Λ_j^k, Θ_{ijp}, *respectively.*

Proof By Theorem 2 and 3, we can get

$$\mathrm{E}\left[\sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \xi_{ijp}^k x_{ijp}^k + \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \eta_{ijp} y_{ijp}\right] = \sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ijp}^k \int_0^1 (\Phi_{ijp}^k)^{-1}(\alpha)\mathrm{d}\alpha + \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} y_{ijp} \int_0^1 (\Psi_{ijp})^{-1}(\alpha)\mathrm{d}\alpha.$$

Similarly, we can prove that $E\left[\sum_{p=1}^{q}\sum_{j=1}^{n} x_{ijp}^k - a_i^k\right] \le 0$ is equivalent to

$$\sum_{p=1}^{q}\sum_{j=1}^{n} x_{ijp}^k - \int_0^1 (\Upsilon_i^k)^{-1}(\alpha)\mathrm{d}\alpha \le 0$$

$\mathrm{E}\left[\sum_{p=1}^{q}\sum_{i=1}^{m} x_{ijp}^k - b_j^k\right] \ge 0$ is equivalent to

$$\sum_{p=1}^{q}\sum_{i=1}^{m} x_{ijp}^k - \int_0^1 (\Lambda_j^k)^{-1}(\alpha)\mathrm{d}\alpha \ge 0,$$

and $\mathrm{E}\left[\sum_{k=1}^{l} x_{ijp}^k - y_{ijp} c_{ijp}\right] \le 0$ is equivalent to

$$\sum_{k=1}^{l} x_{ijp}^k - y_{ijp} \int_0^1 (\Theta_{ijp})^{-1}(\alpha)\mathrm{d}\alpha \le 0.$$

We complete the proof. □

4 Chance-constrained programming model for FCMISTP

4.1 Chance-constrained programming model

Let α be a predetermined confidence level, with $\alpha \in (0, 1)$. The decision maker hopes to get a smallest value $\bar{f}$ such that uncertain variable $f(\mathbf{x}^*, \mathbf{y}^*; \xi, \eta)$ is less than or equal to $\bar{f}$ with predetermined confidence level α.

Definition 8 A solution $(\mathbf{x}^*, \mathbf{y}^*)$ is called $\alpha-$ transportation plan of FCMISTP if

$$\min\{\bar{f} | \mathcal{M}\{f(\mathbf{x}^*, \mathbf{y}^*; \xi, \eta) \le \bar{f}\} \ge \alpha\} \le \min\{\bar{f} | \mathcal{M}\{f(\mathbf{x}, \mathbf{y}; \xi, \eta) \le \bar{f}\} \ge \alpha\}$$

holds for any feasible solution $(\mathbf{x}, \mathbf{y})$ and $\alpha \in (0, 1)$ is a predetermined confidence level.

In order to deal with the optimal problem with uncertain variables, we choose the chance-constrained programming model. When the decision maker gets a transportation plan under the chance-constrained, the model for FCMISTP can be constructed as follows:

$$\begin{cases} \min & \bar{f} \\ \text{s.t.} & \mathcal{M}\{\tilde{f} \le \bar{f}\} \ge \alpha \\ & \tilde{f} = \sum_{k=1}^{l}\sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \xi_{ijp}^k x_{ijp}^k \\ & \quad + \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{i=1}^{m} \eta_{ijp} y_{ijp} \\ & \mathcal{M}\left\{\sum_{p=1}^{q}\sum_{j=1}^{n} x_{ijp}^k \le a_i^k\right\} \ge \alpha_i^k \\ & \mathcal{M}\left\{\sum_{p=1}^{q}\sum_{i=1}^{m} x_{ijp}^k \ge b_j^k\right\} \ge \beta_j^k \\ & \mathcal{M}\left\{\sum_{k=1}^{l} x_{ijp}^k \le y_{ijp} c_{ijp}\right\} \ge \gamma_{ijp} \\ & 0 \le y_{ijp} \le d_{ijp}, \ \ y_{ijp}, d_{ijp} \in z^+ \\ & x_{ijp}^k \ge 0, \end{cases} \tag{7}$$

where $i \in [1, m]$, $j \in [1, n]$, $p \in [1, q]$, $k \in [1, l]$, $i, j, p, k \in z^+$, and $\alpha, \alpha_i^k, \beta_j^k, \gamma_{ijp}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $p = 1, 2, \ldots, q$, $k = 1, 2, \ldots, l$ are predetermined confidence levels.

4.2 Deterministic transformation

Theorem 5 *We assume that* $\xi_{ijp}^k, \eta_{ijp}, a_i^k, b_j^k, c_{ijp}$ *are independent uncertain variables with regular uncertainty distributions* Φ_{ijp}^k, Ψ_{ijp}, Υ_i^k, Λ_j^k, Θ_{ijp}, *respectively, then the model*

(7) is equivalent to the following deterministic transportation model

$$\begin{cases} \min \left(\sum_{k=1}^{l} \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ijp}^k (\Phi_{ijp}^k)^{-1}(\alpha) \right. \\ \qquad \left. + \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} y_{ijp} (\Psi_{ijp})^{-1}(\alpha) \right) \\ \text{s.t.} \sum_{p=1}^{q} \sum_{j=1}^{n} x_{ijp}^k - (\Upsilon_i^k)^{-1}(1-\alpha_i^k) \le 0 \\ \quad \Lambda_j^k)^{-1}(\beta_j^k) - \sum_{p=1}^{q} \sum_{i=1}^{n} x_{ijp}^k \le 0 \\ \quad \sum_{k=1}^{l} x_{ijp}^k - y_{ijp}(\Theta_{ijp})^{-1}(1-\gamma_{ijp}) \le 0 \\ \quad 0 \le y_{ijp} \le d_{ijp}, y_{ijp}, d_{ijp} \in z^+ \\ \quad x_{ijp}^k \ge 0, \end{cases} \tag{8}$$

where $i \in [1, m]$, $j \in [1, n]$, $p \in [1, q]$, $k \in [1, l]$, $i, j, p, k \in z^+$, and $(\Phi_{ijp}^k)^{-1}$, $(\Psi_{ijp})^{-1}$, $(\Upsilon_i^k)^{-1}$, $(\Lambda_j^k)^{-1}$, $(\Theta_{ijp})^{-1}$ are the inverse distribution of the distribution Φ_{ijp}^k, Ψ_{ijp}, Υ_i^k, Λ_j^k, Θ_{ijp}, respectively.

Proof Let

$$\tilde{\xi} = \sum_{k=1}^{l} \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ijp}^k x_{ijp}^k + \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} \eta_{ijp} y_{ijp},$$

which is a continuous strictly increasing function. Since ξ_{ijp}^k, η_{ijp} are independent uncertain variables with regular uncertainty distributions, we can suppose $\tilde{\xi}$ have a regular uncertainty distributions Γ, which has inverse uncertainty distributions Γ^{-1}. And by Theorem 1, we can prove that

$$\mathcal{M}\left\{\tilde{\xi} \le \bar{f}\right\} \ge \alpha$$

is equivalent to

$$\Gamma^{-1}(\alpha) \le \bar{f}.$$

So we proved that

$$\mathcal{M}\left\{ \sum_{k=1}^{l} \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ijp}^k x_{ijp}^k + \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} \eta_{ijp} y_{ijp} \le \bar{f} \right\} \ge \alpha$$

is equivalent to

$$\sum_{k=1}^{l} \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ijp}^k (\Phi_{ijp}^k)^{-1}(\alpha) + \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} y_{ijp} (\Psi_{ijp})^{-1}(\alpha) \le \bar{f}.$$

Since a_i^k is a strictly increasing continuous function, and a_i^k is uncertain variables with inverse uncertainty distributions $(\Upsilon_i^k)^{-1}$, by Theorem 1, we have

$$\mathcal{M}\{a_i^k \ge \sum_{p=1}^{q} \sum_{j=1}^{n} x_{ijp}^k\} \ge \gamma_i^k$$

is equivalent to

$$(\Upsilon_i^k)^{-1}(1-\alpha_i^k) \ge \sum_{p=1}^{q} \sum_{j=1}^{n} x_{ijp}^k.$$

By the same way, we can prove that

$$\mathcal{M}\left\{ b_j^k \le \sum_{p=1}^{q} \sum_{i=1}^{m} x_{ijp}^k \right\} \ge \beta_j^k$$

is equivalent to

$$(\Lambda_j^k)^{-1}(\beta_j^k) \le \sum_{p=1}^{q} \sum_{i=1}^{n} x_{ijp}^k$$

and

$$\mathcal{M}\left\{ y_{ijp} c_{ijp} \ge \sum_{k=1}^{l} x_{ijp}^k \right\} \ge \gamma_{ijp}$$

is equivalent to

$$y_{ijp}(\Theta_{ijp})^{-1}(1-\gamma_{ijp}) \ge \sum_{k=1}^{l} x_{ijp}^k.$$

So the model (7) is equivalent to the deterministic transportation model

$$\begin{cases} \min \ \bar{f} \\ \text{s.t.} \left(\sum_{k=1}^{l} \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ijp}^k (\Phi_{ijp}^k)^{-1}(\alpha) \right. \\ \qquad \left. + \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{m} y_{ijp} (\Psi_{ijp})^{-1}(\alpha) \right) \le \bar{f} \\ \quad \sum_{p=1}^{q} \sum_{j=1}^{n} x_{ijp}^k - (\Upsilon_i^k)^{-1}(1-\alpha_i^k) \le 0 \\ \quad (\Lambda_j^k)^{-1}(\beta_j^k) - \sum_{p=1}^{q} \sum_{i=1}^{n} x_{ijp}^k \le 0 \\ \quad \sum_{k=1}^{l} x_{ijp}^k - y_{ijp}(\Theta_{ijp})^{-1}(1-\gamma_{ijp}) \le 0 \\ \quad 0 \le y_{ijp} \le d_{ijp}, \ y_{ijp}, d_{ijp} \in z^+ \\ \quad x_{ijp}^k \ge 0, \end{cases} \tag{9}$$

Table 2 The supply of item k in source i (a_i^k)

$i \backslash k$	1	2
1	(36, 45, 52)	(50, 55, 52)
2	(52, 58, 66)	(33, 35, 36)

Table 3 The demand of item k in destination j (b_j^k)

$j \backslash k$	1	2
1	(16, 20, 22)	(14, 18, 20)
2	(25, 28, 31)	(20, 23, 25)
3	(33, 35, 38)	(18, 20, 24)

Table 4 The capacity of conveyance 1 from i to j (c_{ij1})

$i \backslash j$	1	2	2
1	(20, 22, 23)	(23, 25, 27)	(20, 22, 25)
2	(18, 20, 26)	(17, 23, 25)	(21, 25, 28)

Table 5 The capacity of conveyance 2 from i to j (c_{ij2})

$i \backslash j$	1	2	3
1	(22, 26, 28)	(23, 25, 28)	(20, 24, 26)
2	(23, 27, 30)	(22, 26, 34)	(18, 22, 24)

where $i \in [1, m]$, $j \in [1, n]$, $p \in [1, q]$, $k \in [1, l]$, $i, j, p, k \in z^+$. Model (9) is equivalent to (8). We complete the proof. □

5 Numerical experiments

In this section, a numerical example of uncertain transportation problem is presented to show the application of the models. We consider two items to be transported by two distinct conveyances from two sources to three destinations. The decision maker should make a transportation plan such that the transportation cost minimized. Assume that all uncertain variables are independent zigzag uncertain variables, which are listed in Tables 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11, respectively.

Table 6 The direct cost of item 1 by conveyance 1 (ξ_{ij1}^1)

$i \backslash j$	1	2	2
1	(10, 12, 15)	(8, 10, 13)	(6, 8, 12)
2	(7, 9, 10)	(6, 7, 10)	(8, 10, 14)

Table 7 The direct cost of item 2 by conveyance 1 (ξ_{ij1}^2)

$i \backslash j$	1	2	2
1	(4, 6, 10)	(5, 8, 12)	(6, 8, 12)
2	(6, 7, 10)	(8, 10, 13)	(8, 10, 13)

Table 8 The fixed charge by conveyance 1 (η_{ij1})

$i \backslash j$	1	2	2
1	(6, 8, 12)	(8, 10, 13)	(6, 8, 9)
2	(7, 9, 10)	(7, 9, 12)	(7, 9, 12)

Table 9 The direct cost of item 1 by conveyance 2 (ξ_{ij2}^1)

$i \backslash j$	1	2	2
1	(8, 10, 12)	(10, 13, 17)	(6, 10, 12)
2	(6, 8, 12)	(6, 9, 10)	(10, 12, 13)

The expected value model (6) for FCMISTP can be converted as follows

$$\begin{cases} \min \ \left(\sum_{k=1}^{2} \sum_{p=1}^{2} \sum_{j=1}^{3} \sum_{i=1}^{2} x_{ijp}^k \int_0^1 (\Phi_{ijp}^k)^{-1}(\alpha) \mathrm{d}\alpha \right. \\ \quad \left. + \sum_{p=1}^{2} \sum_{j=1}^{3} \sum_{i=1}^{2} y_{ijp} \int_0^1 (\Psi_{ijp})^{-1}(\alpha) \mathrm{d}\alpha \right) \\ \text{s.t.} \ \sum_{p=1}^{2} \sum_{j=1}^{3} x_{ijp}^k - \int_0^1 (\Upsilon_i^k)^{-1}(\alpha) \mathrm{d}\alpha \le 0 \\ \quad \sum_{p=1}^{2} \sum_{i=1}^{2} x_{ijp}^k - \int_0^1 (\Lambda_j^k)^{-1}(\alpha) \mathrm{d}\alpha \ge 0 \\ \quad \sum_{k=1}^{2} x_{ijp}^k - y_{ijp} \int_0^1 (\Theta_{ijp})^{-1}(\alpha) \mathrm{d}\alpha \le 0 \\ \quad 0 \le y_{ijp} \le d_{ijp}, y_{ijp} \in z^+, d_{ijp} \in z^+ \\ \quad x_{ijp}^k \ge 0 \\ \quad i = 1, 2, \ j = 1, 2, 3, \ p = 1, 2, \ k = 1, 2. \end{cases} \tag{10}$$

Table 10 The direct cost of item 2 by conveyance 2 (ξ_{ij2}^2)

$i \backslash j$	1	2	2
1	(6, 8, 9)	(8, 9, 11)	(8, 9, 12)
2	(5, 8, 10)	(6, 8, 9)	(10, 12, 13)

Table 11 The fixed charge by conveyance 2 (η_{ij2})

$i\backslash j$	1	2	2
1	(7, 9, 12)	(6, 8, 9)	(7, 10, 12)
2	(5, 8, 10)	(7, 9, 10)	(8, 9, 12)

Table 12 The conveyance and it's frequency($p(d)$) to transport item k from i to j of model (10)

$k\backslash(i, j)$	(1, 1)	(1, 3)	(2, 1)	(2, 2)
1	-	1(1)	2(1)	1(2)
2	1(1)	1(1), 2(1)	–	2(1)

Table 13 The conveyance and it's frequency($p(d)$) to transport item k from i to j of model (11)

$k\backslash(i, j)$	(1, 1)	(1, 3)	(2, 1)	(2, 2)
1	–	1(1)	1(2)	1(1), 2(1)
2	2(1)	1(1), 2(1)	–	2(1)

In the following model, suppose that the total cost at confidence levels α is not less than 0.9 , and assume that $\alpha_i^k = \beta_j^k = r_{ijp} = 0.9,\ i = 1, 2,\ j = 1, 2, 3,\ p = 1, 2,\ k = 1, 2.$ Thus, the corresponding equivalent model to model (8) is constructed as follows:

$$\begin{cases} \min \ \left(\sum_{k=1}^{2} \sum_{p=1}^{2} \sum_{j=1}^{3} \sum_{i=1}^{2} x_{ijp}^k (\Phi_{ijp}^k)^{-1}(\alpha) \right. \\ \quad \left. + \sum_{p=1}^{2} \sum_{j=1}^{3} \sum_{i=1}^{2} y_{ijp} (\Psi_{ijp})^{-1}(\alpha) \right) \\ \text{s.t.} \ \sum_{p=1}^{2} \sum_{j=1}^{3} x_{ijp}^k - (\Upsilon_i^k)^{-1}(1-\alpha_i^k) \le 0 \\ \quad (\Lambda_j^k)^{-1}(\beta_j^k) - \sum_{p=1}^{2} \sum_{i=1}^{2} x_{ijp}^k \le 0 \\ \quad \sum_{k=1}^{2} x_{ijp}^k - y_{ijp} (\Theta_{ijp})^{-1}(1-\gamma_{ijp}) \le 0 \\ \quad 0 \le y_{ijp} \le d_{ijp}, y_{ijp} \in z^+, d_{ijp} \in z^+ \\ \quad x_{ijp}^k \ge 0 \\ \quad i = 1, 2,\ j = 1, 2, 3,\ p = 1, 2,\ k = 1, 2. \end{cases} \tag{11}$$

For expected value model (10) and chance-constrained model (11), we can use LINGO to obtain the optimal plan in Tables 12 and 13, respectively.

We can also get the optimal transportation cost of model (10) and (11), as follows

$$f_1^* = 1219.69, \quad f_2^* = 1665.16.$$

6 Conclusion

This paper mainly investigated the multi-item solid transportation problem with uncertainty theory. Based on the uncertainty theory, uncertain transportation models, namely expected value programming model and chance-constrained programming model, are proposed, respectively. In order to solve the model conveniently, we transformed them into its equivalent deterministic form. Finally, as an application of the model, we presented a actual transportation problem as example, the excepted value model and the chance-constrained model were employed as the experimental models.

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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