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The information value and the uncertainties in two-stage uncertain programming with recourse

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Abstract Based on uncertainty theory, this paper mainly studies the uncertainties and the information value in the two-stage uncertain programming with recourse. We first define three fundamental concepts and investigate their theoretical properties, based on which we present two optimal indices, i.e., EVPI and VUS. Then, we introduce a method to calculate the expected value of the second-stage objective function involving discrete uncertain variables. Due to the complexity of calculation, the upper bound and lower bound for the two indices are studied, respectively. Finally, two examples are given to illustrate these concepts clearly. The results obtained in this paper can provide theoretical basis for studying uncertainties and information value in decision-making process under uncertain systems.

Keywords Uncertainty theory · Two-stage uncertain programming · Expected value · Expected value of perfect information

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1 Introduction

1.1 Motivation

Nowadays, the two-stage stochastic programming (SP) with recourse is more practical, and it has become a fundamental planning tool in the fields of engineering, economics, etc. If we neglect the randomness in the two-stage stochastic programming, such as replacing the random vectors (or variables) by their expected value, some unacceptable results may happen. Obviously, the perfect (or accurate) stochastic information in future can help to make more profits (or less cost) if it is available before taking the first-step decision; thus, how to evaluate the stochastic information value is important in the decision-making process. Under the random environment, many researchers have studied the importance of randomness and the stochastic information value in two-stage stochastic programming and obtain some results based on probability theory.

As is known to all, a fundamental premise of employing probability theory is that the estimated probability is close enough to the real frequency. Due to the lack of observed data, we have to invite some experts to provide their belief degree that each event will occur. Researches showed that human beings tend to overweight unlikely events, and the belief degree may have a much larger range than the real frequency. If we insist on considering the belief degree as probability, some counterintuitive results will happen. In this uncertain environment, how to investigate the importance of such uncertain factors and how to evaluate their perfect information value are practically important and theoretical challenging issues for the real-world applications. In addition, we know that complexity of computation is a distinctive feature of two-stage stochastic programming; therefore, under the uncertain environment, the computation of two-stage uncertain programming would be difficult. Focusing on these problems proposed above, this research will deal with them explicitly.

1.2 Literature review

An initial example of the two-stage SP problem was given by Dantzig (1955) who applied the linear programming to aircraft flight with the random factors. Then, Birge and Louveaux (2011) investigated the theoretical properties of two-stage SP with recourse and solution method. He also defined the "here-and-now" solution, "wait-and-see" solution, "expected results by using the expected value solution," which represented three types of decision-making schemes. Then, based on the three concepts, the expected value of perfect information (EVPI) and the value of stochastic solution (VSS) were introduced. Shapiro and Dentcheva (2014) investigated how to solve the two-stage SP by dual theory and presented multistage stochastic programming model. Romeijnders et al. (2014) presented an approximation method to two-stage integer programming. Leovey and Romisch (2015) presented quasi-Monte Carlo methods for linear two-stage stochastic programming. Since two-stage SP is closer to the real-world problem, recently, it has been developed rapidly in applications, such as Parisio and Jones (2015) applied the two-stage SP approach to employee scheduling; Fan et al. (2015) investigated water resource allocation planning by fuzzy two-stage SP method. Eckermann and Willan (2007) applied the EVPI to the health treatment assignment. More applications can be referred to Wolf et al. (2014), Hoomans et al. (2009), Eckermann et al. (2010), etc.

As introduced in the motivation, the frequently used probability distribution is not appropriate to the some problems due to the shortage of sample data (or even no sample). To deal with this inaccurate phenomenon, the uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010a) in 2010 based on normality, duality, subadditivity and product axioms. Since then, uncertainty theory has been developed continuously such as Liu (2013), Sheng and Yao (2014), which provides a theoretical foundation for uncertain programming just like the role of probability theory for stochastic programming. The pioneering research on uncertain programming and its application was started by Liu (2009a) in 2009, Liu and Chen (2015) investigated multiobjective programming and uncertain goal programming in 2015, Wang et al. (2015) studied the solution method to the uncertain multiobjective programming, and Zheng et al. (2017a) further proposed several types of efficient solutions to the uncertain multiobjective programming and studied their relations. The application of efficient solutions to the uncertain multiobjective programming can be referred to Zheng et al. (2016); Liu and Yao (2015) studied an uncertain multilevel programming for modeling decentralized decision systems; Zheng et al. (2017b) first proposed the model of two-stage uncertain programming and investigated its solution method; Liu (2010a) applied the uncertain programming to the machine scheduling problem with uncertain process times, vehicle routing problem and project scheduling problem; Liu (2010b) initially studied uncertain risk analysis and uncertain reliability analysis; and Zhang and Peng (2013) applied the uncertain programming to the optimal assignment problem.

1.3 Proposed approaches

To the best of our knowledge, the majority of existing literatures devoted the study of the two-stage programming problem under the environment, including the theoretical properties and algorithm. However, under the uncertain environment, there are few researches work except Zheng et al. (2017b), who presented the model of two-stage uncertain programming with recourse (UPR) and investigated its properties. Therefore, following the idea of the two-stage UP problem by Zheng et al. (2017b), this paper mainly deals with the uncertainties and the information value in the two-stage UP problem. We first define three fundamental concepts, i.e., "here-and-now" (HN) solution, "wait-and-see" (WS) solution, "expected results by the expected value" (EEV) solution, which represent three types of decision-making schemes to the two-stage UP problem, and we discuss their theoretical properties. Based on the three concepts above, we define the difference between WS and HN (i.e., HN-WS) as the expected value of perfect information (EVPI), and the difference between WS and HN (i.e., EEV-HN) as the value of uncertainty (VUS) (considering the minimum of objective function). The quantity, EVPI, measures the maximum amount that a decision maker would pay in return for accurate information, and the quantity, VUS, reflects the importance of the uncertainties in the actual problem. In addition, this paper presents a method for calculating the expected value of the second-stage objective function involving discrete uncertain variables. Duo to the complexity of calculation, we, respectively, investigate the upper bound and lower for the two indices. Finally, two examples are given to illustrate the three concepts and the two optimal indices clearly. The results obtained in this paper can provide theoretical basis for studying uncertainties and information value in decision-making process under uncertain systems.

This paper is organized as follows. In Sect. 2, we review some basic results in uncertainty theory. Two optimal indices and the related concepts are proposed in Sect. 3. Section 4 studies the properties of the concepts proposed in this paper and obtains the upper bound and lower bound on the two optimal indices. In Sect. 5, two numerical examples are given to demonstrated these concepts explicitly. Finally, a brief summary is given in Sect. 6.

2 Preliminaries

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to [0, 1] is called an uncertain measure if it satisfies the following axioms (Liu 2007):

Axiom 1 *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 *(Duality Axiom)* $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 *(Subadditivity Axiom)* For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, Liu (2009b) defined a product uncertain measure by the fourth axiom:

Axiom 4 *(Product Axiom)* Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.1 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 2.2 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x.

Definition 2.3 (Liu 2010a) An uncertain distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1.)$

Definition 2.4 (Liu 2010a) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then, the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.

Definition 2.5 (Liu 2007) Let ξ be an uncertain variable. Then, the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_0^{\infty} \mathcal{M}\{\xi \ge x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \le x\}dx \tag{2}$$

provided that at least one of the two integrals is finite.

Definition 2.6 (Liu 2007) A k-dimensional uncertain vector is a function $\boldsymbol{\xi}$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that $\{\boldsymbol{\xi} \in B\}$ is an event for any Borel set B of k-dimensional real vectors.

Theorem 2.1 (Liu 2010a) *Let ξ and η be independent uncertain variables with finite expected values. Then, for any real numbers a and b, we have*

$$\mathrm{E}[a\xi + b\eta] = a\mathrm{E}[\xi] + b\mathrm{E}[\eta]. \tag{3}$$

Theorem 2.2 (Liu 2010a) *Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value exists, then*

$$\mathrm{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

Theorem 2.3 (Liu 2010a) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_{m+2}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Theorem 2.4 (Liu 2007) *Let ξ be an uncertain variable and f a convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then*

$$f(E[\xi]) \le E[f(\xi)].$$

3 Three concepts and two optimal indices in two-stage UPR problem

In order to present the three expected value solution concepts, we first discuss the basic model of two-stage UPR problem. Under the uncertain environment, a two-stage uncertain programming with recourse problem can be formulated as follows by Zheng et al. (2017b)

$$\begin{cases} \min_x c^T x + \min_y q^T(\gamma) y \\ \text{s.t.} \quad Ax = b \\ \qquad T(\gamma)x + Wy = h(\gamma) \\ \qquad x \geq 0, y \geq 0. \end{cases} \tag{4}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$ are decision variables, c is a known vector in R^{n_1}, b is a known vector in R^{m_1}, A and W are known matrices of size $m_1 \times n_1$ and $m_2 \times n_2$, respectively, W is called *recourse matrix* which is assumed to be fixed for the convenience computation.

For each realization $\gamma \in \Gamma$, $T(\gamma)$ is $m_2 \times n_1$, $q(\gamma) \in R^{n_2}$, $h(\gamma) \in R^{m_2}$. Piecing together the uncertain components of the problem (4), we obtain an uncertain vector $\boldsymbol{\xi}^T(\gamma) = (q(\gamma)^T, h(\gamma)^T, T_1(\gamma), \ldots, T_{m_2}(\gamma))$ with $N = n_2 + m_2 + (m_2 + n_1)$ components, where $T_i(\gamma)$ is the ith row of the *technology matrix* $T(\gamma)$. Let $\Xi \subset R^n$ be the support of ξ which is the smallest closed subset in R^n such that $\mathcal{M}\{\Xi\} = 1$.

The decision-observation schemecan be described as follows

decision on x
observation of uncertain event γ
decision on y.

According to this scheme, the problem (4) obtains two unsolved optimization problems. Assuming that x and γ are given, the *second-stage problem*, or *recourse problem* can be formulated as follows

$$\begin{cases} \min_y q^T(\gamma) y \\ \text{s.t.} \quad T(\gamma)x + Wy = h(\gamma) \\ \qquad y \geq 0, \end{cases} \tag{5}$$

where x belongs to the feasible set $S_1 = \{x \mid Ax = b, x \geq 0\}$. By the above analysis, we know that the second-stage problem is a more difficult one. For each $\gamma \in \Xi$, the value $y(\gamma)$ is the solution of a linear programming. To stress this fact, sometimes we use the notion of a deterministic equivalent programming. For a given realization γ of uncertain variable ξ, let

$$Q(x, \boldsymbol{\xi}(\gamma)) = \min\{q^T(\boldsymbol{\xi}(\gamma))y | Wy = h(\boldsymbol{\xi}(\gamma)) \\ -T(\boldsymbol{\xi}(\gamma))x, y \geq 0\}$$

be the second-stage value function, where $\boldsymbol{\xi}$ is an uncertain vector.

Next, we will define three basic concepts of expected value solution for the model (4). For convenience, $Q(x, \boldsymbol{\xi}(\gamma))$ is indiscriminately denoted by $Q(x, \gamma)$ throughout this paper.

3.1 HN solution

Denote the expected second-stage value function (or *recourse function*) as

$$\mathcal{Q}_E(x) = E_{\boldsymbol{\xi}}[Q(x, \gamma)] \tag{6}$$

where $E_{\boldsymbol{\xi}}$ is the expected value operator with respect to uncertain vector $\boldsymbol{\xi}$.

Therefore, the two-stage UPR problem can be rewritten as follows

$$\begin{cases} \min_x z(x) = c^T x + \mathcal{Q}_E(x) \\ \text{s.t.} \ Ax = b \\ \qquad x \geq 0 \end{cases} \quad \text{(UPR)} \tag{7}$$

where $\mathcal{Q}_E(x) = E_{\boldsymbol{\xi}}[Q(x, \gamma)]$, and

$$\begin{cases} Q(x, \gamma) = \min_y q^T(\gamma) y \\ \text{s.t.} T(\gamma)x + Wy = h(\gamma) \\ y \geq 0. \end{cases} \tag{8}$$

Obviously, by the above discussion, the problem (7) can be rewritten as follows

$$\begin{cases} \min_x z(x) = c^T x + E_{\boldsymbol{\xi}}[\min_y q^T(\boldsymbol{\xi}) y] \\ \text{s.t.} \quad Ax = b \\ \qquad T(\boldsymbol{\xi})x + Wy = h(\boldsymbol{\xi}) \\ \qquad x \geq 0, y \geq 0 \end{cases} \quad \text{(UPR)} \tag{9}$$

where $\boldsymbol{\xi}$ is an uncertain vector defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

Definition 3.1 We define the "here-and-now" solution (HN) (or recourse problem) in the problem (9) as

$$HN = \min_x E_{\boldsymbol{\xi}} z(x, \boldsymbol{\xi}) = \min_x \{c^T x + \mathcal{Q}_E(x)\} = z(x^*)$$

with an optimal solution, x^*.

3.2 WS solution

Suppose that the uncertainties in the problem (9) can be modeled through a number of scenarios. Let $\boldsymbol{\xi}$ be the uncertain vector whose realizations correspond to the various scenarios. Define

$$\begin{cases} \min_x z(x, \boldsymbol{\xi}(\gamma)) = c^T x + \min\{q^T(\gamma) y | Wy \\ \qquad\qquad = h(\gamma) - T(\gamma)x, y \geq 0\} \\ \text{s.t.} \quad Ax = b, x \geq 0 \end{cases} \tag{10}$$

as the optimization problem associated with one particular scenario γ.

We may also reasonably assume that there exists at least one $x \in R^{n_1}$ such that $z(x, \boldsymbol{\xi}(\gamma)) < +\infty$. If not, there would exist no feasible solution for at least one scenario; in such a situation, no reasonable stochastic programming model could be established. This assumption implies that, for all $\boldsymbol{\xi}(\gamma) \in \Xi$, there exists at least one feasible solution which in turn means the existence of at least one optimal solution. Let $\bar{x}(\boldsymbol{\xi}(\gamma))$ denote some optimal solution to problem (10). As in a scenario method, we might be interested in finding all solutions $\bar{x}(\boldsymbol{\xi}(\gamma))$ of problem (10) for all scenarios and the related optimal objective values $z(\bar{x}(\boldsymbol{\xi}(\gamma)), \boldsymbol{\xi}(\gamma))$.

Here, we assume that we somehow have the ability to find these decisions $\bar{x}(\boldsymbol{\xi}(\gamma)$ and their objective value $z(\bar{x}(\boldsymbol{\xi}(\gamma)), \boldsymbol{\xi}(\gamma))$ so that we can in a position to calculate the expected value of the correspondingly optimal objective value, which is called "wait-and-see" (WS) solution.

Definition 3.2 We define the "wait-and-see" (WS) solution as

$$WS = E_{\boldsymbol{\xi}}\{\min_x z(x, \boldsymbol{\xi}(\gamma))\} = E_{\boldsymbol{\xi}} z(\bar{x}(\boldsymbol{\xi}(\gamma)), \boldsymbol{\xi}(\gamma)).$$

3.3 EEV solution

For practical purpose, we would believe that finding the WS solution is still too much work because it delivers a set of solutions instead of one solution that would be implementable. Therefore, a natural temptation is to solve a much simpler problem: the one obtained by replacing all uncertain vectors or uncertain variables in problem (9) with their expected values. This is called *expected value* (EV) problem, which is simply

$$EV = \min_x z(x, \bar{\boldsymbol{\xi}}) = z(\bar{x}(\bar{\boldsymbol{\xi}})) \tag{11}$$

with optimal solution $\bar{x}(\bar{\boldsymbol{\xi}})$, where $\bar{\boldsymbol{\xi}}$ is the expected value of $\boldsymbol{\xi}$, and $\bar{x}(\bar{\boldsymbol{\xi}})$ (called *expected value solution*(EV)) is an optimal solution to the EV problem.

In practical decision-making process, we would feel at least a little insecure about advising to take decision $\bar{x}(\bar{\boldsymbol{\xi}})$. Indeed, unless $\bar{x}(\boldsymbol{\xi})$ is somehow independent of $\boldsymbol{\xi}$, there is no reason to believe that $\bar{x}(\bar{\boldsymbol{\xi}})$ is in any way near the solution of the problem (9). Therefore, in order to precisely measure how good or, more frequently, how bad a decision $\bar{x}(\bar{\boldsymbol{\xi}})$ is in terms of the problem (9), we should find a method to evaluate the value of the uncertain solution which begins with defining the following concept of *expected results of using the EV solution*.

Definition 3.3 We define the expected results of using the EV solution (EEV) to be

$$EEV = E_{\boldsymbol{\xi}} z(\bar{x}(\bar{\boldsymbol{\xi}}), \boldsymbol{\xi}).$$

3.4 The EVPI and VUS

If we can obtain the precise uncertain information in the problem (9) by purchasing or other means, there is no doubt that it is helpful to make decisions. There is a natural question: How much is the information? Based on uncertainty theory, we present a concept called the expected value of perfect information (EVPI) to solve this problem. Unfortunately, in some practical problems, we cannot obtain any information by all means. Under these circumstances, how can we evaluate the uncertainties in the problems? Therefore, the concept of the value of uncertain solution (VUS) is also proposed in this subsection.

Definition 3.4 The expected value of perfect information is the difference between the wait-and-see solution and the here-and-now solution, namely

$$EVPI = HN - WS.$$

Remark 1 The quantity, EVPI, measures the maximum amount that a decision maker would be ready to pay in return for complete or accurate information about the future.

Definition 3.5 The value of fuzzy solution is the difference between the here-and-now solution and the EEV, namely

$$VFS = EEV - HN.$$

Remark 2 The quantity, VUS, reflects the importance of the uncertainties in the problem.

4 Basic properties

Theorem 4.1 *For the two-stage UPR problem (9), we have*

$$WS \leq HN \leq EEV.$$

Proof For any realization, $\boldsymbol{\xi}(\gamma)$, we have

$$z(\bar{x}(\boldsymbol{\xi}(\gamma)), \boldsymbol{\xi}(\gamma)) \leq z(\bar{x}^*, \boldsymbol{\xi}(\gamma)),$$

where, as described before, x^* denotes the optimal solution to the here-and-now problem.

By the definition of uncertain vector, it is easy to obtain

$$E_{\boldsymbol{\xi}}[z(\bar{x}(\boldsymbol{\xi}(\gamma)), \boldsymbol{\xi}(\gamma))] \leq E_{\boldsymbol{\xi}}[z(\bar{x}^*, \boldsymbol{\xi}(\gamma))],$$

which, by the definitions, is

$$WS \leq HN.$$

On the other hand, since $\bar{x}(\bar{\boldsymbol{\xi}})$ is just one feasible solution to the two-stage UPR problem (9), evidently,

$$E_{\boldsymbol{\xi}}[z(\bar{x}^*, \boldsymbol{\xi}(\gamma))] \leq E_{\boldsymbol{\xi}}[z(\bar{x}(\bar{\boldsymbol{\xi}}), \boldsymbol{\xi}(\gamma))],$$

that is,

$$HN \leq EEV.$$

The proof is complete. □

Theorem 4.2 *For the two-stage UPR problem (9) with fixed recourse matrix W and fixed objective coefficients q, we can obtain*

$$EV \leq WS.$$

Proof We first prove that the $f(\boldsymbol{\xi}) = \min_x z(x, \boldsymbol{\xi}(\gamma))$ is a convex function with respect to $\boldsymbol{\xi}$. Since

$$\begin{aligned}\min_x z(x, \boldsymbol{\xi}(\gamma)) &= c^T x + \min\{q^T y | Wy \\ &= h(\gamma) - T(\gamma)x, y \geq 0\},\end{aligned}$$

it is evidently sufficient to prove that the second-stage optimal objective function $Q(x, \boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$.

Denote

$$g(T(\gamma), h(\gamma)) = \min\{q^T y | Wy = h(\gamma) - T(\gamma)x, y \geq 0\}.$$

Suppose that $\gamma_1, \gamma_2 \in \Gamma$, and $T(\gamma_1), h(\gamma_1), T(\gamma_2), h(\gamma_2) \in \Xi$. Denote

$$T_\lambda = \lambda T(\gamma_1) + (1-\lambda)T(\gamma_2), h_\lambda = \lambda h(\gamma_1) + (1-\lambda)h(\gamma_2).$$

If y_1^* and y_2^* are the optimal solutions to the problems $g(T(\gamma_1), h(\gamma_1))$ and $g(T(\gamma_2), h(\gamma_2))$, respectively. Then, $\forall\lambda \in (0, 1), \lambda y_1^* + (1-\lambda)y_2^*$ is a feasible solution to the problem $g(T_\lambda, h_\lambda)$. Let y^* be the optimal solution to the $g(T_\lambda, h_\lambda)$, we have

$$\begin{aligned}g(T_\lambda, h_\lambda) &= q^T y^* \leq q^T \lambda(y_1^* + (1-\lambda)y_2^*) \\ &= \lambda q^T y_1^* + (1-\lambda)q^T y_2^* \\ &= \lambda g(T(\gamma_1), h(\gamma_1)) + (1-\lambda)g(T(\gamma_2), h(\gamma_2)),\end{aligned}$$

which shows that the $g(T(\gamma), h(\gamma))$ is a convex function; thus, the function $f(\boldsymbol{\xi}) = \min_x z(x, \boldsymbol{\xi}(\gamma))$ is a convex function with respect to $\boldsymbol{\xi}$.

By Theorem 2.4, we obtain

$$f(E[\xi]) \leq E[f(\boldsymbol{\xi})],$$

namely,

$$\min_x z(x, \bar{\boldsymbol{\xi}}) \leq E[\min_x z(x, \boldsymbol{\xi}(\gamma))],$$

which, by the definitions of EV and WS, is

$$EV \leq WS.$$

The proof is complete. □

Theorem 4.2 does not hold for general two-stage UPR problem. For example, if only the q in problem (9) is uncertain, similarly, we can easily prove that $z(x, \boldsymbol{\xi})$ is a concave function and the Jensen's inequality cannot be applied.

Theorem 4.3 *For the fixed recourse matrix W and fixed objective coefficients q, let x^* be an optimal solution to the problem (9) and $\bar{x}(\bar{\boldsymbol{\xi}})$ an optimal solution to the problem EV problem (11). Then,*

$$EEV + (x^* - \bar{x}(\bar{\boldsymbol{\xi}}))^T \eta \leq HN,$$

where $\eta \in \partial E_{\boldsymbol{\xi}}(z(\bar{x}(\bar{\boldsymbol{\xi}}), \boldsymbol{\xi}))$, the subdifferential set of $E_{\boldsymbol{\xi}}(z(x, \boldsymbol{\xi}))$ at $\bar{x}(\bar{\boldsymbol{\xi}})$.

Proof By the proof of Theorem 4.2 and properties of convex function, it is easy to obtain that $E_{\boldsymbol{\xi}}(z(x, \boldsymbol{\xi}))$ is convex. By the subgradient inequality, the relation

$$E_{\boldsymbol{\xi}} z(x_1, \boldsymbol{\xi}) + (x_2 - x_1)^T \eta \leq E_{\boldsymbol{\xi}}(z(x_2, \boldsymbol{\xi}))$$

holds at point x_1 for any x_2. Substitute $\bar{x}(\bar{\boldsymbol{\xi}})$ and x^* for the points x_1 and x_2, respectively, and we have

$$E_{\boldsymbol{\xi}} z(\bar{x}(\bar{\boldsymbol{\xi}}), \boldsymbol{\xi}) + (x_2 - x_1)^T \eta \leq E_{\boldsymbol{\xi}} z(x^*, \boldsymbol{\xi}),$$

which is

$$EEV + (x^* - \bar{x}(\bar{\boldsymbol{\xi}}))^T \eta \leq HN.$$

In order to obtain the another bound on the HN solution, we consider a slightly different version of the recourse problem defined as follows

$$\begin{cases}\min_x z_p(x, \boldsymbol{\xi}) = c^T x + \min\{q^T y | Wy \geq h(\boldsymbol{\xi}) - Tx, y \geq 0\} \\ \text{s.t. } Ax = b \\ \quad x \geq 0.\end{cases} \tag{12}$$

Compared with the problem (10), only the right-hand side of the problem (12) is uncertain and the second-stage constraints are inequalities. We can easily apply all definitions and relations to z_p. If we further assume that $h(\boldsymbol{\xi})$ is bounded above, an additional inequality results as follows. □

Theorem 4.4 *Consider the problem (12) and the related definition*

$$HN = \min_x E_{\xi} z_p(x, \xi).$$

Suppose that $h(\xi)$ is bounded above by a fixed quantity h_{max}. Let x_{max} be an optimal solution to $z_p(x, h_{max})$. Then,

$$HN \leq z_p(x_{max}, h_{max}).$$

Proof For any ξ in Ξ and $x \geq 0$, a feasible solution to $Wy \geq h_{max} - Tx$, $y \geq 0$ is also a feasible solution to $Wy \geq h(\xi) - Tx$, $y \geq 0$. Thus, $z_p(x, h_{max}) \geq z_p(x, h(\xi))$. Hence, $z_p(x, h_{max}) \geq E_{\xi} z_p(x, h(\xi))$. Evidently,

$$z_p(x_{max}, h_{max}) \geq \min_x E_{\xi} z_p(x, h(\xi)) = HN.$$

The proof is complete. □

Similarly, if we consider the following the problem

$$\begin{cases} \min_x z_q(x, \xi) = c^T x + \min\{q^T y | Wy \leq h(\xi) - Tx, y \geq 0\} \\ \text{s.t. } Ax = b \\ \quad x \geq 0, \end{cases} \tag{13}$$

then a similar conclusion can be obtained as follows.

Theorem 4.5 *Consider the problem (13) and the related definition*

$$HN = \min_x E_{\xi} z_q(x, \xi).$$

Suppose that $h(\xi)$ is bounded below by a fixed quantity h_{min}. Let x_{min} be an optimal solution to $z_q(x, h_{min})$. Then,

$$HN \leq z_q(x_{min}, h_{min}).$$

Proof Similar to Theorem 4.4, this proof can be easily obtained. The proof is complete. □

Theorem 4.6 *Suppose that the uncertain vector in the two-stage UPR problem is discrete taking on finite values γ_j with correspondingly weights p_j defined by the following Eq.(15), $j = 1, 2, \ldots, N$, we have*

$$z_* \leq WS < z^*$$

where

$$z_* = \min_j z(x(\xi(\gamma_j)), \xi(\gamma_j)), \;\; z^* = \max_j z(x(\xi(\gamma_j))\xi(\gamma_j)).$$

Proof From the following Eq.(16), we know that

$$\sum_{j=1}^{N} p_j = 1.$$

Thus, for any γ_j, we have

$$\begin{aligned} \min_j z(x(\xi(\gamma_j), \xi(\gamma_j)) &\leq \sum_{j=1}^{N} p_j \{\min z(x(\xi(\gamma_j), \xi(\gamma_j))\} \\ &\leq \max_j z(x(\xi(\gamma_j), \xi(\gamma_j)), \end{aligned}$$

namely

$$\begin{aligned} \min_j z(x(\xi(\gamma_j), \xi(\gamma_j)) &\leq \sum_{j=1}^{N} p_j z(\bar{x}(\xi(\gamma_j), \xi(\gamma_j)) \\ &\leq \max_j z(\bar{x}(\xi(\gamma_j), \xi(\gamma_j)), \end{aligned} \tag{14}$$

where $\bar{x}(\xi(\gamma_j), \xi(\gamma_j))$ is the optimal solution to the problem $\min z(x(\xi(\gamma_j), \xi(\gamma_j))$ for any γ_j. It follows from the definition of WS and the inequality (14) that

$$\min_j z(x(\xi(\gamma_j), \xi(\gamma_j)) \leq WS \leq \max_j z(\bar{x}(\xi(\gamma_j), \xi(\gamma_j));$$

thus,

$$z_* \leq WS < z^*.$$

The proof is complete. □

Theorem 4.7 *For the two-stage UPR problem (9), we have*

$$EVPI \geq 0,$$
$$VUS \geq 0.$$

Proof It is evident to obtain this conclusion by Theorem 4.1. This proof is complete. □

Theorem 4.8 *For the two-stage UPR problem (9) with fixed recourse matrix W and fixed objective coefficients q, we can obtain*

$$EVPI \leq EEV - EV,$$
$$VUS \leq EEV - EV.$$

Proof Evidently, it can be verified by Theorem 4.2. The proof is complete. □

Remark 3 From Theorem 4.8, we can obtain that the $EVPI$ and the VUS are nonnegative (anyone would be surprised if this is not true) and are both bounded above by the same quantity. When $EEV = EV$, both the $EVPI$ and VUS are vanished. A sufficient condition to this is to have $\bar{x}(\xi)$

independent of $\boldsymbol{\xi}$. In such situations, the optimal solutions are insensitive to the value of the uncertain elements, and we can find them by any given uncertain vector such as $\bar{\xi}$. Hence, it is unnecessary to solve a recourse problem. Such extreme situations occur rarely.

5 Numerical examples

In this section, we take two examples to illustrate how to calculate the $EVPI$ and VUS. To further study the properties of the two optimal indices, these two examples are also designed to demonstrate the cases in which one of the two concepts is null and the other is positive.

Let us begin with the expected value of discrete uncertain variable in order to deal with the two-stage UPR problem involving discrete uncertain variables. Assume that the discrete uncertain variable ξ takes the expert's experimental data

$$(\xi_1, \alpha_1), (\xi_2, \alpha_2), \ldots, (\xi_n, \alpha_n),$$

which meet the following consistence conditions (perhaps after rearrangement)

$$\xi_1 \le \xi_2 \le \cdots \le \xi_n, \; 0 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n = 1.$$

Based on the experimental data, discrete uncertain variable ξ has the following experimental uncertain distribution

$$\Phi(x) = \begin{cases} \alpha_1, & \text{if } x \le \xi_1 \\ \alpha_i + \frac{(\alpha_{i+1}-\alpha_i)(x-\xi_i)}{\xi_{i+1}-\xi_i}, & \text{if } \xi_i \le x \le \xi_{i+1}, 1 \le i < n, \\ \alpha_n, & \text{if } x \ge \xi_n \end{cases}$$

where $0 = \alpha_1 < \alpha_2 \cdots < \alpha_n = 1$.

Define

$$p_i = \frac{\alpha_{i+1} - \alpha_{i-1}}{2}, \quad i = 1, 2, \ldots, n, (\alpha_0 = 0, \alpha_{n+1} = 1) \tag{15}$$

as the weights of discrete point ξ_i, $i = 1, 2, \ldots, n$, respectively.

Then, it is easy to know that the corresponding weights satisfy the following constraints

$$p_i \ge 0, \; \sum_{i=1}^{n} p_i = 1, \; i = 1, 2, \ldots, n. \tag{16}$$

By Definition 2.5, we can deduce that the expectation of discrete uncertain variable ξ is represented in the formula

$$\mathrm{E}[\xi] = \sum_{i=1}^{n} p_i \xi_i. \tag{17}$$

Assume that the second-stage value functions satisfy the condition

$$Q(x, \xi_1) \le Q(x, \xi_2) \cdots \le Q(x, \xi_n).$$

Then, the second-stage objective function $\mathcal{Q}_{\mathrm{E}}(x)$ can be calculated by the following formula

$$\mathcal{Q}_{\mathrm{E}}(x) = \sum_{i=1}^{n} p_i Q(x, \xi_i). \tag{18}$$

Next, we will take two examples to illustrate the properties of the two optimal indices.

a. $EVPI = 0$ *and* $VUS \ne 0$

Example 5.1 Consider the following two-stage UPR problem

$$\begin{cases} \min\limits_{x} z(x, \boldsymbol{\xi}) = x_1 + 4x_2 + \min\{y_1 + 10y_2^+ + 10y_2^- | y_1 \\ \qquad + y_2^- - y_2^- = \xi + x_1 - 2x_2, \, y_1 \le 2, \, y \ge 0\} \\ \text{s.t. } x_1 + x_2 = 1, \\ \qquad x \ge 0, \end{cases} \tag{19}$$

where ξ is a linear uncertain variable with the following uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & if \; x < 1 \\ (x-a)/(b-a), & if \; 1 \le x \le 3 \\ 1. & if \; x > 3 \end{cases}$$

Calculate the $EVPI$ and VUS.

For a given x and ξ, we can obtain

$$y^*(x, \xi) = \begin{cases} y_1 = \xi + x_1 - 2x_2, \; y_2 = 0, & if \; 0 \le \xi + x_1 - 2x_2 \le 2 \\ y_1 = 2, \, y_2^+ = \xi + x_1 - 2x_2 - 2, & if \; \xi + x_1 - 2x_2 > 2 \\ y_2^- = 2x_2 - x_1 - \xi. & if \; \xi + x_1 - 2x_2 < 0 \end{cases}$$

Hence,

$$z(x, \xi) = \begin{cases} 2x_1 + 2x_2 + \xi, & if \; 0 \le \xi + x_1 - 2x_2 \le 2 \\ -18 + 11x_1 - 16x_2 + 10\xi, & if \; \xi + x_1 - 2x_2 > 2 \\ -9x_1 + 24x_2 - 10\xi, & if \; \xi + x_1 - 2x_2 < 0. \end{cases} \tag{20}$$

Note that the first-stage constraint $x_1 + x_2 = 1$, evidently, we have $z(x, \xi) = 2 + \xi$ in the first of these three regions. By the first-stage constraint and the definition of the regions, we can easily check that $z(x, \xi) \ge 2 + \xi$ in the other two regions.

Thus, $\forall x^* \in \{(x_1, x_2) | x_1 + x_2 = 1, x \ge 0\}$ is an optimal solution to the problem (19) for $-x_1 + 2x_2 \le \xi \le 2 - x_1 + 2x_2$, or equivalently, for $2 - 3x_1 \le \xi \le 4 - 3x_1$.

By the analysis on the solutions, we can obtain that the solution $\bar{x} = (1/3, 2/3)$ is the optimal solution to the problem (19) for all ξ. We can also obtain different optimal solutions for various uncertain variable ξ. For example, $\bar{x}_1^* = (0, 1)$ is optimal for $\xi \in [2, 3]$, and $\bar{x}_2^* = (1, 0)$ is optimal for only $\xi = 1$.

Note that the solution $\bar{x} = (1/3, 2/3)$ is optimal for all ξ, by the definition of HN solution and WS solution we can conclude that HN=WS. Next we will calculate the objective value of the optimal solution $\bar{x} = (1/3, 2/3)$ to the initial problem (19), i.e., the HN solution. Evidently, in this situation, we know that $z(x^*, \xi) = 2 + \xi$ for all $\xi \in [1, 3]$. By Definition 2.4, we can obtain the universe distribution of linear uncertain variable ξ as follows

$$\Phi_{\xi}^{-1}(\alpha) = 2\alpha + 1.$$

By Theorem 2.2, the expected value of ξ can be obtained as follows

$$\mathrm{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha = \int_0^1 (2\alpha + 1)d\alpha = 2.$$

It follows from Theorem 2.1 that

$$\min_x E[z(x, \xi)] = E[\min_x z(x, \xi)] = E[2 + \xi] = 4,$$

which implies that HN=WS=4.

Hence,

$$EVPI = HN - WS = 0.$$

Next, we will discuss the EEV solution.

In the problem (19), we replace the uncertain variable ξ with its expected value, i.e., $\bar{\xi} = 2$, and then obtain the following deterministic programming problem

$$z(x, \bar{\xi} = 2) = \begin{cases} 4, & if\ 0 \le x_1 \le \frac{2}{3} \\ 27x_1 - 14, & if\ \frac{2}{3} \le x_1 \le 1. \end{cases} \tag{21}$$

Easily, we can obtain that the optimal solutions to the problem (19) are $\bar{x}(2) = \{x | x_1 + x2 = 1, 0 \le x \le 2/3\}$ and the corresponding optimal objective value is 4, i.e., $EV = \min_x E[z(x, \bar{\xi})] = 4$.

Without loss of generality, take the optimal solution $\bar{x}(\bar{\xi}) = (2/3, 1/3)$ in the problem (19), and we have

$$z(x(\bar{\xi}), \xi) = \begin{cases} 2 + \xi, & if\ 1 \le \xi < 2 \\ -16 + 10\xi, & if\ 2 \le \xi \le 3. \end{cases}$$

Hence, by the definition of EEV, we have

$$\begin{aligned} EEV &= E_\xi[z(\bar{x}(\bar{\xi}), \xi)] \\ &= E_{1\le\xi\le2}[z(\bar{x}(\bar{\xi}), \xi)] + E_{2\le\xi\le3}[z(\bar{x}(\bar{\xi}), \xi)] \\ &= \int_0^{\infty} \mathcal{M}\{z(\bar{x}(\bar{\xi}), \xi) \ge r\}dr = \frac{17}{4}. \end{aligned}$$

It follows from the definition of VUS that

$$VUS = EEV - HN = \frac{17}{4} - 4 = \frac{1}{4}.$$

b. $EVPI \neq 0$ *and* $VUS = 0$

Example 5.2 Replace the uncertain variable in the Example 5.1 with discrete uncertain variable $\xi \sim (0, 3/2, 2, 3)$ which has belief degrees 0, 1/2, 2/3, 1, respectively, and then calculate the EVPI and VUS.

By the analysis above, we can obtain:

$$\begin{aligned} &\text{For } \xi = 0, \bar{x}(0) = \left\{x | x_1 + x_2 = 1,\ \tfrac{2}{3} \le x_1 \le 1\right\}; \\ &\text{For } \xi = \tfrac{3}{2}, \bar{x}(\tfrac{3}{2}) = \left\{x | x_1 + x_2 = 1,\ \tfrac{1}{6} \le x_1 \le \tfrac{5}{6}\right\}; \\ &\text{For } \xi = 2, \bar{x}(2) = \left\{x | x_1 + x_2 = 1,\ 0 \le x_1 \le \tfrac{2}{3}\right\}; \\ &\text{For } \xi = 3, \bar{x}(3) = \left\{x | x_1 + x_2 = 1,\ 0 \le x_1 \le \tfrac{1}{3}\right\}. \end{aligned}$$

It follows from Eq.(17) that the expected value of ξ is obtained as follows

$$\begin{aligned} \bar{\xi} &= E[\xi] = \sum_{i=1}^{4} p_i \xi_i = \frac{1}{4} \times 0 + \frac{1}{3} \times \frac{3}{2} + \frac{1}{4} \times 2 + \frac{1}{6} \times 3 \\ &= \frac{3}{2}. \end{aligned}$$

Let us take $\bar{x}(\bar{\xi}) = (2/3, 1/3)$, and it follows from the definition of EV that

$$EV = z(\bar{x}(\bar{\xi}), 3/2) = 2 \times \frac{2}{3} + 2 \times \frac{1}{3} + \frac{3}{2} = \frac{7}{2}.$$

Evidently,

$$\begin{aligned} z(\bar{x}(\bar{\xi}), 0) &= 2 + 0 \le z\left(\bar{x}(\bar{\xi}), \frac{3}{2}\right) = 2 + \frac{3}{2} \\ &\le z(\bar{x}(\bar{\xi}), 2) = 2 + 2 \le z(\bar{x}(\bar{\xi}), 3) = -16 + 30. \end{aligned}$$

It follows from Eq.(18) that

$$\begin{aligned} EEV &= E_\xi[z(\bar{x}(\bar{\xi}), \xi)] = \sum_{i=1}^{4} p_i z\left(\left(\frac{2}{3}, \frac{1}{3}\right), \xi_i\right) \\ &= p_1 z\left(\left(\frac{2}{3}, \frac{1}{3}\right), 0\right) + p_2 z\left(\left(\frac{2}{3}, \frac{1}{3}\right), \frac{3}{2}\right) \\ &\quad + p_3 z\left(\left(\frac{2}{3}, \frac{1}{3}\right), 2\right) + p_4 z\left(\left(\frac{2}{3}, \frac{1}{3}\right), 3\right) \\ &= \frac{1}{4}(2 + 0) + \frac{1}{3}\left(2 + \frac{3}{2}\right) + \frac{1}{4}(2 + 2) \\ &\quad + \frac{1}{6}(-16 + 30) = 5. \end{aligned}$$

In order to obtain the WS solution, we should consider the different optimal solutions for all three cases. Evidently, there are many optimal solutions, such as $\bar{x}(0) = (1, 0)$, $\bar{x}(3/2) = (1/2, 1/2)$, $\bar{x}(2) = (1/4, 3/4)$, $\bar{x}(3) = (0, 1)$ for the three cases, respectively.

Evidently,

$$\begin{aligned} z(\bar{x}(\xi_1), \xi_1) &= 2 + \xi_1 \le z(\bar{x}(\xi_2), \xi_2) = 2 + \xi_2 \\ &\le z(\bar{x}(\xi_3), \xi_3) = 2 + \xi_3 \le z(\bar{x}(\xi_4), \xi_4) \\ &= 2 + \xi_4. \end{aligned}$$

Hence, by the definition of WS solution and Eq.(18), we can get

$$\begin{aligned} WS &= E_\xi z(\bar{x}(\xi), \xi) \\ &= \sum_{i=1}^{4} z(\bar{x}(\xi_i), \xi_i) \\ &= p_1 z(\bar{x}(\xi_1), \xi_1) + p_2 z(\bar{x}(\xi_2), \xi_2) + p_3 z(\bar{x}(\xi_3), \xi_3) \\ &\quad + p_4 z(\bar{x}(\xi_4), \xi_3) \\ &= \frac{1}{4}(2+0) + \frac{1}{3}\left(2 + \frac{3}{2}\right) + \frac{1}{4}(2+2) + \frac{1}{6}(2+3) = \frac{7}{2}. \end{aligned}$$

In order to obtain the HN solution, we should solve the uncertain programming $\min_x E_\xi[z(x, \xi)]$. For any given x and ξ, we have

$$\begin{aligned} HN &= \min_x E_\xi[z(x,\xi)] \\ &= \min_x \{p_1 z(x,\xi_1) + p_2 z(x,\xi_2) + p_3 z(x,\xi_3) + p_4 z(x,\xi_4)\} \\ &= \min_x \left\{ \begin{array}{l} \frac{1}{4}\begin{cases} 2x_1 + 2x_2 + \xi_1, & if\ 0 \le \xi_1 + x_1 - 2x_2 \le 2 \\ -18 + 11x_1 - 16x_2 + 10\xi_1, & if\ \xi_1 + x_1 - 2x_2 > 2 \\ -9x_1 + 24x_2 - 10\xi_1, & if\ \xi_1 + x_1 - 2x_2 < 0. \end{cases} \\ +\frac{1}{3}\begin{cases} 2x_1 + 2x_2 + \xi_2, & if\ 0 \le \xi_2 + x_1 - 2x_2 \le 2 \\ -18 + 11x_1 - 16x_2 + 10\xi_2, & if\ \xi_2 + x_1 - 2x_2 > 2 \\ -9x_1 + 24x_2 - 10\xi_2, & if\ \xi_2 + x_1 - 2x_2 < 0. \end{cases} \\ +\frac{1}{4}\begin{cases} 2x_1 + 2x_2 + \xi_3, & if\ 0 \le \xi_3 + x_1 - 2x_2 \le 2 \\ -18 + 11x_1 - 16x_2 + 10\xi_3, & if\ \xi_3 + x_1 - 2x_2 > 2 \\ -9x_1 + 24x_2 - 10\xi_3, & if\ \xi_3 + x_1 - 2x_2 < 0. \end{cases} \\ +\frac{1}{6}\begin{cases} 2x_1 + 2x_2 + \xi_4, & if\ 0 \le \xi_4 + x_1 - 2x_2 \le 2 \\ -18 + 11x_1 - 16x_2 + 10\xi_4, & if\ \xi_4 + x_1 - 2x_2 > 2 \\ -9x_1 + 24x_2 - 10\xi_4, & if\ \xi_4 + x_1 - 2x_2 < 0. \end{cases} \end{array} \right\} \\ &= \min_{\substack{\frac{2}{3} \le x_1 \le 1 \\ x_1 + x_2 = 1}} \left\{ \begin{array}{l} \frac{1}{4}(2x_1 + 2x_2 + \xi_1) + \frac{1}{3}(2x_1 + 2x_2 + \xi_2) + \frac{1}{4}(2x_1 \\ +2x_2 + \xi_3) + \frac{1}{6}(-18 + 11x_1 - 16x_2 + 10\xi_4) \end{array} \right\} \\ &= \begin{cases} \min_x \frac{7}{2}x_1 - x_2 + 3 \\ \text{s.t.} x_1 + x_2 = 1 \\ \frac{2}{3} \le x_1 \le 1, x_2 \ge 0 \end{cases} \\ &= 5 \end{aligned}$$

whose optimal solution is $x^* = (2/3, 1/3)$. Evidently,

$$EV = WS = 7/2 \le HN = 5 = EEV.$$

Thus,

$$EVPI = HN - WS = 3/2,$$

and

$$VUS = EEV - HN = 0.$$

6 Conclusions

Based on three types of decision-making schemes, three expected value solutions to two-stage UP problems were presented; then, the concepts of two optimal indices, i.e., EVPI and VUS, were defined. Due to the complexity of calculation, the general solution method to calculate the expected value of the second-stage objective function only involving discrete uncertain variables was introduced. Two numerical examples were given to illustrate these concepts explicitly. The theoretical results obtained in this paper can provide theoretical basis for studying uncertainties and information value in decision-making process under uncertain systems.

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Compliance with ethical standards

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