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METHODOLOGIES AND APPLICATION

Fuzzy extended filters on residuated lattices

Ninghua Gao[1] · **Qingguo Li**[1] · **Xiangnan Zhou**[1]

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Abstract The notion of fuzzy extended filters is introduced on residuated lattices, and its essential properties are investigated. By defining an operator $\rightsquigarrow$ between two arbitrary fuzzy filters in terms of fuzzy extended filters, two results are immediately obtained. We show that (1) the class of all fuzzy filters on a residuated lattice forms a complete Heyting algebra, and its classical version is equivalent to the one introduced in Kondo (Soft Comput 18(3):427–432, 2014), which is defined with respect to (crisp) generated filters of singleton sets; (2) the connection between fuzzy extended filters and fuzzy generated filters is built, with which three other classes generating complete Heyting algebras, respectively, are presented. Finally, by the aid of fuzzy t-filters, we also develop the characterization theorems of the special algebras and quotient algebras via fuzzy extended filters.

Keywords Complete Heyting algebra · Fuzzy extended filters · Fuzzy generated filters · Fuzzy t-filters · Residuated lattice

1 Introduction and preliminaries

The definition of extended filters on Rl-monoids was introduced, and its properties were considered by Haveshki and Mohamadhasani in 2012 (see Haveshki and Mohamadhasani 2012). Later on, Kondo (2014) gave a characterization theorem of the extended filters on residuated lattices. Moreover, as a generalized result in Haveshki and Mohamadhasani (2012), a description of implicative, positive implicative and fantastic filters on residuated lattices via extended filters was provided. However, Víta (2015) showed that this description can be done uniformly in terms of t-filters.

In this note, the extended filter is shifted to the fuzzy setting, and their properties and applications are discussed. We have also obtained new results, some of whose classical versions are displayed as corollaries.

In the following, we recall some fundamental definitions and results.

Definition 1 (Víta 2014) A bounded pointed commutative integral residuated lattice (abbr. residuated lattice) is a structure

$$\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1),$$

which satisfies the following conditions:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(2) $(L, \otimes, 1)$ is a commutative monoid.
(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., for any $a, b, c \in L$, $a \otimes b \leq c \Longleftrightarrow a \leq b \rightarrow c$.

A residuated lattice is called a complete residuated lattice if $(L, \vee, \wedge, 0, 1)$ is a complete lattice, and is called a Heyting algebra if $\otimes = \wedge$. For example, [0, 1] is a complete Heyting algebra.

Some other well-known algebras: MTL-algebras, BL-algebras, MV-algebras and so on, are subvarieties of residuated lattices (see Kondo 2014).

Starting now, unless otherwise stated, $\mathbf{L}$ always means a residuated lattice and L its domain. The symbol $\overline{x}$ is a formal listing of variables used in a given context. For a variety $\mathbb{B}$

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✉ Qingguo Li
liqingguoli@aliyun.com

[1] College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

of residuated lattices, we denote its subvariety, given by the equation $t = 1$, by the symbol $\mathbb{B}[t]$.

Properties of (complete) residuated lattices can be found in many papers, such as Hoo (1994), Höhle (1995), Ma and Hu (2013), Radzikowska and Kerre (2004), She and Wang (2009). We only give some properties that are used in the further text.

Proposition 1 (Haveshki and Mohamadhasani 2012; Ma and Hu 2013; Radzikowska and Kerre 2004; She and Wang 2009) *In any complete residuated lattice* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, *the following properties hold for any* $a, b, a_i, b_i, c \in L$ $(i \in I)$:

(1) $\otimes$ *is isotone in both arguments,* $\rightarrow$ *antitone in the 1st argument and isotone in the 2nd argument.*
(2) $a \leq b \rightarrow (a \otimes b)$.
(3) $a \otimes (\vee_{i\in I} b_i) = \vee_{i\in I}(a \otimes b_i)$, $a \otimes (\wedge_{i\in I} b_i) \leq \wedge_{i\in I}(a \otimes b_i)$.
(4) $a \rightarrow (\wedge_{i\in I} b_i) = \wedge_{i\in I}(a \rightarrow b_i)$, $(\vee_{i\in I} a_i) \rightarrow b = \wedge_{i\in I}(a_i \rightarrow b)$.
(5) $(a \otimes b) \rightarrow c = b \rightarrow (a \rightarrow c) = a \rightarrow (b \rightarrow c)$.
(6) $a \leq b \Longleftrightarrow a \rightarrow b = 1$, $1 \rightarrow a = a$.
(7) $a \vee (b \otimes c) \geq (a \vee b) \otimes (a \vee c)$.
(8) $a \otimes (b \rightarrow c) \leq (a \rightarrow b) \rightarrow c$, *specially,* $a \otimes (a \rightarrow b) \leq b$.

A fuzzy set of a residuated lattice $\mathbf{L}$ is a function $f : L \rightarrow [0, 1]$. Specially, for any $A \subseteq L$, the characteristic function χ_A is defined as follows:

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We denote χ_y instead of $\chi_{\{y\}}$. Put $\mathscr{F}(\mathbf{L}) = \{f \mid f$ is a fuzzy set of $\mathbf{L}\}$. Furthermore, $\overline{0}, \overline{1} \in \mathscr{F}(\mathbf{L})$ are defined as: $\overline{0}(x) = 0$, $\overline{1}(x) = 1$, $\forall x \in L$, respectively.

Definition 2 (Zhu and Xu 2010) A fuzzy set μ of $\mathbf{L}$ is a fuzzy filter on $\mathbf{L}$ if and only if for any $x, y \in L$, it satisfies the following two conditions:

(F1) if $x \leq y$, then $\mu(x) \leq \mu(y)$,
(F2) $\mu(x) \wedge \mu(y) \leq \mu(x \otimes y)$.

Denote $FFil(\mathbf{L}) = \{\mu \mid \mu$ is a fuzzy filter on $\mathbf{L}\}$ (resp. $Fil(\mathbf{L}) = \{F \mid F$ is a (crisp) filter on $\mathbf{L}\}$).

The definition of fuzzy filters on $\mathbf{L}$ can be given by many equivalent ways, for comprehensive overview see Zhu and Xu (2010).

Definition 3 (Víta 2014) A fuzzy filter μ on $\mathbf{L}$ is called a fuzzy t-filter on $\mathbf{L}$, if $\mu(t(\overline{x})) = \mu(1)$ for any $\overline{x} \in L$, where $t(\overline{x})$ is a term of the language of $\mathbf{L}$.

The fuzzy generated filter was introduced by Liu and Li (2005) on BL-algebras, Jun et al. (2005) on MTL-algebras, etc. We generalize it on residuated lattices here. Moreover, for any $\nu \in \mathscr{F}(\mathbf{L})$, $\langle \nu \rangle$ denotes the fuzzy generated filter of ν and $\langle B]$ the (crisp) generated filter of B for any $B \subseteq L$.

Definition 4 Let ν be a fuzzy set of $\mathbf{L}$. A fuzzy filter ϑ on $\mathbf{L}$ is said to be generated by ν, if $\nu \subseteq \vartheta$ and for any fuzzy filter h on $\mathbf{L}$, $\nu \subseteq h$ implies $\vartheta \subseteq h$.

Proposition 2 *Let* ν *be a fuzzy set of* $\mathbf{L}$. *Then for any* $x \in L$,

$$\langle \nu \rangle(x) = \bigvee_{a_1,\cdots,a_n \in L,\ a_1\otimes\cdots\otimes a_n \leq x} \bigwedge_{i=1}^{n} \nu(a_i).$$

Proof It is similar to the proof of Theorem 3.11 in Liu and Li (2005). □

In Liu and Li (2005), the authors have drawn the conclusion that $(FFil(\mathbf{X}), \wedge, \vee, \overline{0}, \overline{1})$ is a complete (Brouwerian) lattice, where $\mathbf{X}$ is a BL-algebra. In a similar way, we can verify that it is also a complete lattice under the framework of residuated lattices and the proof is omitted here.

Theorem 1 $(FFil(\mathbf{L}), \sqcap, \sqcup, \overline{0}, \overline{1})$ *is a complete lattice, where for any* $\{\mu_i\}_{i\in I} \subseteq FFil(\mathbf{L})$:

$$\sqcap_{i\in I}\mu_i = \bigcap_{i\in I}\mu_i, \quad \sqcup_{i\in I}\mu_i = \left\langle \bigcup_{i\in I}\mu_i \right\rangle.$$

The following content is introduced in order to discuss the lattice structures.

In a poset P, for any $S \subseteq P$, denote $S^u = \{y \in P \mid x \leq y \ (\forall x \in S)\}$.

Proposition 3 (Davey and Priestley 2002) *Let* P *be a poset such that* $\bigwedge S$ *exists in* P *for every non-empty subset* S *of* P. *Then* $\bigvee Q$ *exists for every non-empty subset* Q *of* P, *indeed,* $\bigvee Q = \bigwedge Q^u$.

Theorem 2 (Davey and Priestley 2002) *Let* P *be a non-empty poset. Then the following are equivalent:*

(1) P *is a complete lattice.*
(2) $\bigwedge S$ *exists in* P *for every subset* S *of* P.
(3) P *has a top element, and* $\bigwedge S$ *exists in* P *for every non-empty subset* S *of* P.

Definition 5 (Davey and Priestley 2002) Let P be a poset. A closure operator is a mapping $c : P \rightarrow P$ satisfying for every $a, b \in P$,

(1) $a \leq c(a)$.
(2) $a \leq b \Longrightarrow c(a) \leq c(b)$.
(3) $c(c(a)) = c(a)$.

Denote $P_c = \{x \in P \mid c(x) = x\}$.

Proposition 4 (Davey and Priestley 2002) *Let c be a closure operator on a poset P. Then*

(1) $P_c = \{c(x) \mid x \in P\}$.
(2) *P_c is a complete lattice, under the order inherited from P, such that, for every subset S of P_c:*

$$\bigwedge_{P_c} S = \bigwedge_{P} S, \quad \bigvee_{P_c} S = c\left(\bigvee_{P} S\right).$$

2 Fuzzy extended filters

The extended filter on a residuated lattice (Kondo 2014), generalized from Haveshki and Mohamadhasani (2012), is defined as:

Let F be a filter on $\mathbf{L}$ and B a subset of $\mathbf{L}$. Then $E_F(B) = \{x \in L \mid x \vee b \in F, \forall b \in B\}$ is called an extended filter associated with B, where "E" means "**e**xtended". Specially, $E_F(\{a\})$ is abbreviated as $E_F(a)$.

According to the notation, we propose the concept of the fuzzy extended filter on $\mathbf{L}$. For a fuzzy filter μ on $\mathbf{L}$ and a fuzzy set ν of $\mathbf{L}$, $\varepsilon_\mu(\nu) \in \mathscr{F}(\mathbf{L})$ defined by

$$(\forall x \in L) \;\; \varepsilon_\mu(\nu)(x) = \bigwedge_{b \in L} (\nu(b) \rightarrow \mu(x \vee b))$$

is called a fuzzy extended filter on $\mathbf{L}$ associated with ν. Specially,

$$\begin{aligned}(\forall x, y \in L) \;\; \varepsilon_\mu(\chi_y)(x) &= \bigwedge_{b \in L} \left(\chi_y(b) \rightarrow \mu(x \vee b)\right) \\ &= \mu(x \vee y).\end{aligned}$$

Remark 1 Let μ be a fuzzy filter on $\mathbf{L}$ and ν a fuzzy set of $\mathbf{L}$. Then for any $x \in L$,

$$\varepsilon_\mu(\nu)(x) = \bigwedge_{b \in L} \left(\nu(b) \rightarrow \varepsilon_\mu(\chi_x)(b)\right).$$

We give the crisp version of Remark 1, which plays a key role in the proof of Theorem 7, and one can examine it easily.

Remark 2 Let F be a filter on $\mathbf{L}$ and B a subset of $\mathbf{L}$. Then $E_F(B) = \{x \in L \mid B \subseteq E_F(x)\}$.

Theorem 3 *Let μ be a fuzzy filter on $\mathbf{L}$ and ν a fuzzy set of $\mathbf{L}$. Then*

(1) $\varepsilon_\mu(\nu) \in FFil(\mathbf{L})$.
(2) $\mu \subseteq \varepsilon_\mu(\nu)$.

Proof (1) For any $x, y \in L$, we have

(i) Assuming that $x \leq y$, (F1) implies

$$\begin{aligned}\varepsilon_\mu(\nu)(x) &= \bigwedge_{b \in L} (\nu(b) \rightarrow \mu(x \vee b)) \\ &\leq \bigwedge_{b \in L} (\nu(b) \rightarrow \mu(y \vee b)) \\ &= \varepsilon_\mu(\nu)(y).\end{aligned}$$

(ii) μ is a fuzzy filter, which implies

$$\begin{aligned}&\varepsilon_\mu(\nu)(x) \wedge \varepsilon_\mu(\nu)(y) \\ &= \bigwedge_{b \in L} (\nu(b) \rightarrow \mu(x \vee b)) \wedge \bigwedge_{m \in L} (\nu(m) \rightarrow \mu(y \vee m)) \\ &\leq \bigwedge_{b \in L} (\nu(b) \rightarrow \mu(x \vee b)) \wedge (\nu(b) \rightarrow \mu(y \vee b)) \\ &\quad \textit{(by (4) in Proposition 1)} \\ &= \bigwedge_{b \in L} \nu(b) \rightarrow (\mu(x \vee b) \wedge \mu(y \vee b)) \\ &\quad \textit{(by (F2) and (1) in Proposition 1)} \\ &\leq \bigwedge_{b \in L} \nu(b) \rightarrow \mu((x \vee b) \otimes (y \vee b)) \\ &\quad \textit{(by (7) in Proposition 1)} \\ &\leq \bigwedge_{b \in L} \nu(b) \rightarrow \mu((x \otimes y) \vee b) \\ &= \varepsilon_\mu(\nu)(x \otimes y).\end{aligned}$$

Thus, $\varepsilon_\mu(\nu) \in FFil(\mathbf{L})$.

(2) For any $x \in L$, since μ is a fuzzy filter,

$$\begin{aligned}\varepsilon_\mu(\nu)(x) &= \bigwedge_{b \in L} \nu(b) \rightarrow \mu(x \vee b) \\ &\geq \bigwedge_{b \in L} \nu(b) \rightarrow \mu(x) \\ &\geq \bigwedge_{b \in L} (1 \rightarrow \mu(x)) \\ &= \mu(x),\end{aligned}$$

i.e., $\mu \subseteq \varepsilon_\mu(\nu)$. □

Proposition 5 *Let μ, μ_1, μ_2 be fuzzy filters on $\mathbf{L}$ and $\nu, \nu_1, \nu_2, \omega$ fuzzy sets of $\mathbf{L}$. We have*

(1) *if $\nu_1 \subseteq \nu_2$ then $\varepsilon_\mu(\nu_2) \subseteq \varepsilon_\mu(\nu_1)$.*
(2) *if $\mu_1 \subseteq \mu_2$ then $\varepsilon_{\mu_1}(\nu) \subseteq \varepsilon_{\mu_2}(\nu)$.*
(3) $\nu \subseteq \varepsilon_\mu\left(\varepsilon_\mu(\nu)\right)$.
(4) $\varepsilon_\mu(\nu) = \varepsilon_\mu(\langle \nu \rangle)$.
(5) $\varepsilon_{\varepsilon_\mu(\nu)}(\omega) = \varepsilon_{\varepsilon_\mu(\omega)}(\nu)$.
(6) $\varepsilon_\mu(\nu) = \varepsilon_\mu\left(\varepsilon_\mu\left(\varepsilon_\mu(\nu)\right)\right)$.
(7) $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) = \varepsilon_\mu(\nu)$.

(8) $\bigcap_{i\in I} \varepsilon_\mu(\nu_i) = \varepsilon_\mu\left(\bigcup_{i\in I} \nu_i\right)$.

Proof (1) For any $x \in L$,

$$\begin{aligned}\varepsilon_\mu(\nu_2)(x) &= \bigwedge_{b\in L} \nu_2(b) \to \mu(x\vee b)\\ &\leq \bigwedge_{b\in L} \nu_1(b) \to \mu(x\vee b)\\ &= \varepsilon_\mu(\nu_1)(x),\end{aligned}$$

i.e., $\varepsilon_\mu(\nu_2) \subseteq \varepsilon_\mu(\nu_1)$.

(2) For any $x \in L$,

$$\begin{aligned}\varepsilon_{\mu_1}(\nu)(x) &= \bigwedge_{b\in L} \nu(b) \to \mu_1(x\vee b)\\ &\leq \bigwedge_{b\in L} \nu(b) \to \mu_2(x\vee b)\\ &= \varepsilon_{\mu_2}(\nu)(x),\end{aligned}$$

i.e., $\varepsilon_{\mu_1}(\nu) \subseteq \varepsilon_{\mu_2}(\nu)$.

(3) For any $x \in L$,

$$\begin{aligned}\varepsilon_\mu\left(\varepsilon_\mu(\nu)\right)(x) &= \bigwedge_{b\in L} \varepsilon_\mu(\nu)(b) \to \mu(x\vee b)\\ &= \bigwedge_{b\in L}\left(\bigwedge_{c\in L} \nu(c) \to \mu(b\vee c)\right) \to \mu(x\vee b)\\ &\geq \bigwedge_{b\in L}(\nu(x) \to \mu(b\vee x)) \to \mu(x\vee b)\\ &\quad \textit{(by (8) in Proposition 1)}\\ &\geq \bigwedge_{b\in L} \nu(x)\otimes(\mu(b\vee x)\to\mu(x\vee b))\\ &= \nu(x),\end{aligned}$$

hence, $\nu \subseteq \varepsilon_\mu(\varepsilon_\mu(\nu))$.

(4) $\nu \subseteq \langle\nu\rangle$ implies $\varepsilon_\mu(\nu) \supseteq \varepsilon_\mu(\langle\nu\rangle)$ by (1). For any $b, x \in L$, we obtain

$$\begin{aligned}&\langle\nu\rangle(b)\otimes\varepsilon_\mu(\nu)(x)\\ &= \left(\bigvee_{a_1,\cdots,a_n\in L,\ a_1\otimes\cdots\otimes a_n\leq b}\ \bigwedge_{i=1}^{n} \nu(a_i)\right)\otimes\varepsilon_\mu(\nu)(x)\\ &\leq \left(\bigvee_{a_1,\cdots,a_n\in L,\ x\vee(a_1\otimes\cdots\otimes a_n)\leq x\vee b}\ \bigwedge_{i=1}^{n} \nu(a_i)\right)\otimes\varepsilon_\mu(\nu)(x)\\ &\quad \textit{(by (7) and (1) in Proposition 1)}\\ &\leq \left(\bigvee_{a_1,\cdots,a_n\in L,\ (x\vee a_1)\otimes\cdots\otimes(x\vee a_n)\leq x\vee b}\ \bigwedge_{i=1}^{n} \nu(a_i)\right)\otimes\varepsilon_\mu(\nu)(x)\\ &\leq \left(\bigvee_{\substack{a_1,\cdots,a_n\in L,\\ \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\leq\mu(x\vee b)}}\ \bigwedge_{i=1}^{n} \nu(a_i)\right)\otimes\left(\bigwedge_{c\in L}\nu(c)\to\mu(x\vee c)\right)\\ &\quad \textit{(by (3) in Proposition 1)}\\ &\leq \bigvee_{\substack{a_1,\cdots,a_n\in L,\\ \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\leq\mu(x\vee b)}}\ \bigwedge_{i=1}^{n}\left(\nu(a_i)\otimes\left(\bigwedge_{c\in L}\nu(c)\to\mu(x\vee c)\right)\right)\\ &\leq \bigvee_{\substack{a_1,\cdots,a_n\in L,\\ \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\leq\mu(x\vee b)}}\ \bigwedge_{i=1}^{n}(\nu(a_i)\otimes(\nu(a_i)\to\mu(x\vee a_i)))\\ &\quad \textit{(by (8) in Proposition 1)}\\ &= \bigvee_{\substack{a_1,\cdots,a_n\in L,\\ \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\leq\mu(x\vee b)}}\ \bigwedge_{i=1}^{n}\mu(x\vee a_i)\\ &\quad \textit{(by (F2))}\\ &\leq \bigvee_{\substack{a_1,\cdots,a_n\in L,\\ \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\leq\mu(x\vee b)}} \mu((x\vee a_1)\otimes\cdots\otimes(x\vee a_n))\\ &\leq \mu(x\vee b),\end{aligned}$$

i.e., for any $b, x \in L$, $\langle\nu\rangle(b) \to \mu(x\vee b) \geq \varepsilon_\mu(\nu)(x)$. Thus, for any $x \in L$,

$$\varepsilon_\mu(\langle\nu\rangle)(x) = \bigwedge_{b\in L}\langle\nu\rangle(b)\to\mu(x\vee b) \geq \varepsilon_\mu(\nu)(x),$$

which implies $\varepsilon_\mu(\nu) \subseteq \varepsilon_\mu(\langle\nu\rangle)$.

(5) For any $x \in L$, we get

$$\begin{aligned}&\varepsilon_{\varepsilon_\mu(\nu)}(\omega)(x)\\ &= \bigwedge_{b\in L}\omega(b)\to\varepsilon_\mu(\nu)(x\vee b)\\ &= \bigwedge_{b\in L}\left(\omega(b)\to\bigwedge_{c\in L}(\nu(c)\to\mu(x\vee b\vee c))\right)\\ &= \bigwedge_{b\in L}\bigwedge_{c\in L}(\omega(b)\to(\nu(c)\to\mu(x\vee b\vee c)))\\ &\quad \textit{(by (5) in Proposition 1)}\\ &= \bigwedge_{b\in L}\bigwedge_{c\in L}(\nu(c)\to(\omega(b)\to\mu(x\vee b\vee c)))\\ &= \bigwedge_{c\in L}\left(\nu(c)\to\bigwedge_{b\in L}(\omega(b)\to\mu(x\vee b\vee c))\right)\\ &= \bigwedge_{c\in L}\nu(c)\to\varepsilon_\mu(\omega)(x\vee c)\\ &= \varepsilon_{\varepsilon_\mu(\omega)}(\nu)(x),\end{aligned}$$

therefore, $\varepsilon_{\varepsilon_\mu(\nu)}(\omega) = \varepsilon_{\varepsilon_\mu(\omega)}(\nu)$.

(6) It follows from (1) and (3).

(7) It is sufficient to prove $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) \subseteq \varepsilon_\mu(\nu)$, for any $x \in L$,

$$\begin{aligned}&\left(\varepsilon_{\varepsilon_\mu(\nu)}(\nu)\right)(x)\\ &= \bigwedge_{b\in L}\left(\nu(b)\to\varepsilon_\mu(\nu)(x\vee b)\right)\end{aligned}$$

$$= \bigwedge_{b\in L}\left(\nu(b) \to \left(\bigwedge_{c\in L} \nu(c) \to \mu(x \vee b \vee c)\right)\right)$$
$$\leq \bigwedge_{b\in L} (\nu(b) \to (\nu(b) \to \mu(x \vee b \vee b)))$$
$$= \bigwedge_{b\in L} (\nu(b) \to (\nu(b) \to \mu(x \vee b)))$$
(by (5) in Proposition 1)
$$= \bigwedge_{b\in L} ((\nu(b) \wedge \nu(b)) \to \mu(x \vee b))$$
$$= \bigwedge_{b\in L}(\nu(b) \to \mu(x \vee b))$$
$$= \varepsilon_\mu(\nu),$$

i.e., $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) = \varepsilon_\mu(\nu)$.

(8) For any $x \in L$,

$$\varepsilon_\mu\left(\bigcup_{i\in I} \nu_i\right)(x) = \bigwedge_{b\in L}\left(\bigvee_{i\in I} \nu_i(b) \to \mu(x \vee b)\right)$$
(by (4) in Proposition 1)
$$= \bigwedge_{i\in I}\left(\bigwedge_{b\in L} \nu_i(b) \to \mu(x \vee b)\right)$$
$$= \bigcap_{i\in I} \varepsilon_\mu(\nu_i),$$

i.e., $\bigcap_{i\in I} \varepsilon_\mu(\nu_i) = \varepsilon_\mu\left(\bigcup_{i\in I} \nu_i\right)$. □

Remark 3 Let ν be a fuzzy set of **L**. Then

$$\langle\nu\rangle \subseteq \bigcap_{\mu\in FFil(\mathbf{L})} \varepsilon_\mu\left(\varepsilon_\mu(\nu)\right)$$

according to (3) and (4) in Proposition 5. However, its inverse is not always true.

The following theorem indicates that any fuzzy filter on **L** can be characterized by all its fuzzy extended filters.

Theorem 4 *Let μ be a fuzzy filter on* **L**. *Then*

$$\mu = \bigcap_{\nu\in\mathscr{F}(\mathbf{L})} \varepsilon_\mu(\nu).$$

Proof Obviously, $\mu \subseteq \bigcap_{\nu\in\mathscr{F}(\mathbf{L})} \varepsilon_\mu(\nu)$ by (2) in Theorem 3. On the other hand, for any $x \in L$, we have

$$\left(\bigcap_{\nu\in\mathscr{F}(\mathbf{L})} \varepsilon_\mu(\nu)\right)(x) = \bigwedge_{\nu\in\mathscr{F}(\mathbf{L})} \varepsilon_\mu(\nu)(x)$$
$$\leq \varepsilon_\mu(\chi_x)(x)$$
$$= \mu(x \vee x)$$
$$= \mu(x).$$

Thus, $\mu = \bigcap_{\nu\in\mathscr{F}(\mathbf{L})} \varepsilon_\mu(\nu)$. □

Furthermore, we give the crisp form of Theorem 4 as a corollary.

Corollary 1 *Let F be a filter on* **L**. *Then* $F = \bigcap_{B\subseteq L} E_F(B)$.

Proof It is trivial that $F \subseteq \bigcap_{B\subseteq L} E_F(B)$ from Theorem 3.1 in Haveshki and Mohamadhasani (2012). Conversely, if $x \in \bigcap_{B\subseteq L} E_F(B)$, then $x \in E_F(x)$, i.e., $x \in F$. □

3 Application of the fuzzy extended filter in studying lattice structures

As mentioned in Theorem 1, all fuzzy filters on a residuated lattice (resp. a special residuated lattice, e.g., a BL-algebra) generate a complete lattice (resp. a special complete lattice, e.g., a complete Brouwerian lattice). In this section, we prove that it is also a Heyting algebra.

Theorem 5 $(FFil(\mathbf{L}), \sqcap, \sqcup, \rightsquigarrow, \bar{0}, \bar{1})$ *is a complete Heyting algebra, where for any* $\{\mu_i\}_{i\in I} \subseteq FFil(\mathbf{L})$, $\sqcap, \sqcup$ *are defined as in Theorem 1 and for any* $\mu, \vartheta \in FFil(\mathbf{L})$:

$$\mu \rightsquigarrow \vartheta = \varepsilon_\vartheta(\mu).$$

Proof It is sufficient to prove that for any $\mu, \vartheta, \psi \in FFil(\mathbf{L})$, $\mu \sqcap \vartheta \subseteq \psi \Longleftrightarrow \mu \subseteq \vartheta \rightsquigarrow \psi$.

($\Longrightarrow$) For any $x \in L$, we get

$$(\vartheta \rightsquigarrow \psi)(x) = \varepsilon_\psi(\vartheta)(x)$$
$$= \bigwedge_{b\in L}(\vartheta(b) \to \psi(x \vee b))$$
$$\geq \bigwedge_{b\in L}(\vartheta(b) \to (\mu \cap \vartheta)(x \vee b))$$
$$= \bigwedge_{b\in L}(\vartheta(b) \to (\mu(x \vee b) \wedge \vartheta(x \vee b)))$$
$$\geq \bigwedge_{b\in L}(\vartheta(x \vee b) \to (\mu(x \vee b) \wedge \vartheta(x \vee b)))$$
(by (2) in Proposition 1)
$$\geq \bigwedge_{b\in L} \mu(x \vee b)$$
$$\geq \mu(x),$$

i.e., $\mu \subseteq \vartheta \rightsquigarrow \psi$.

($\Longleftarrow$) For any $x \in L$, we have

$$\begin{aligned}(\mu \sqcap \vartheta)(x) &= \mu(x) \wedge \vartheta(x)\\ &\leq (\vartheta \rightsquigarrow \psi)(x) \wedge \vartheta(x)\\ &= \varepsilon_{\psi}(\vartheta)(x) \wedge \vartheta(x)\\ &= \bigwedge_{b \in L}(\vartheta(b) \rightarrow \psi(x \vee b)) \wedge \vartheta(x)\\ &\leq (\vartheta(x) \rightarrow \psi(x \vee x)) \wedge \vartheta(x)\\ &= (\vartheta(x) \rightarrow \psi(x)) \wedge \vartheta(x)\\ &= (\vartheta(x) \rightarrow \psi(x)) \otimes \vartheta(x)\\ &\quad \textit{(by (8) in Proposition 1)}\\ &\leq \psi(x),\end{aligned}$$

i.e., $\mu \sqcap \vartheta \subseteq \psi$. □

The following corollary shows the crisp version of Theorem 5.

Corollary 2 *$(Fil(\mathbf{L}), \wedge, \vee, \hookrightarrow, \{1\}, L)$ is a complete Heyting algebra, where for any $\{F_i\}_{i \in I} \subseteq Fil(L)$, $F, G \in Fil(L)$:*

$$\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i, \ \bigvee_{i \in I} F_i = \left\langle \bigcup_{i \in I} F_i \right], \ F \hookrightarrow G = E_G(F).$$

Proof It is well known that $(Fil(\mathbf{L}), \wedge, \vee)$ is a complete lattice. We only illustrate that for any $F, G, H \in Fil(\mathbf{L})$, $F \wedge G \subseteq H \Longleftrightarrow F \subseteq G \hookrightarrow H$.

($\Longrightarrow$) Let $x \in F$. Then for any $b \in G$, we have $x \vee b \in F$ and $x \vee b \in G$, i.e., $x \vee b \in F \cap G$ by the property of filters, thus $x \in E_{F \cap G}(G) \subseteq E_H(G) = G \hookrightarrow H$ from 2. of Theorem 3.5 in Haveshki and Mohamadhasani (2012).

($\Longleftarrow$) Assuming that $x \in F \wedge G$. Then $x \in (G \hookrightarrow H) \cap G = E_H(G) \cap G$, i.e., for any $b \in G$, $x \vee b \in H$ and $x \in G$, pick $b = x$, we have $x \in H$. □

Now, we further study the relationship between fuzzy extended filters and fuzzy generated filters.

Theorem 6 *Let μ be a fuzzy filter on $\mathbf{L}$ and ν a fuzzy set of $\mathbf{L}$. Then $\varepsilon_{\mu}(\nu) = \langle \nu \rangle \rightsquigarrow \mu$.*

Proof $\varepsilon_{\mu}(\nu) = \varepsilon_{\mu}(\langle \nu \rangle) = \langle \nu \rangle \rightsquigarrow \mu$ by Theorem 5 and (4) in Proposition 5. □

Corollary 3 *Let F be a filter on $\mathbf{L}$ and B a subset of $\mathbf{L}$. Then $E_F(B) = \langle B] \hookrightarrow F$.*

Proof From 8. of Theorem 3.5 in Haveshki and Mohamadhasani (2012) and Corollary 2, we have $E_F(B) = E_F(\langle B])$ $= \langle B] \hookrightarrow F$. □

In Kondo (2014), the author presented that $(Fil(\mathbf{L}), \wedge, \vee, \rightarrowtail, \{1\}, L)$ is a complete Heyting algebra, where for any $F, G \in Fil(\mathbf{L})$, "$F \rightarrowtail G$" is defined by

$$F \rightarrowtail G = \{x \in L \mid F \cap \langle x] \subseteq G\}. \tag{*}$$

The question naturally arises whether it conflicts with the result in Corollary 2 or not? It is shown that the answer is negative by the following theorem.

Theorem 7 *Let F, G be filters on $\mathbf{L}$. Then $F \hookrightarrow G$ defined in Corollary 2 is equal to $F \rightarrowtail G$ defined by Eq. (∗).*

Proof In the residuated lattice $(Fil(\mathbf{L}), \wedge, \vee, \hookrightarrow, \{1\}, L)$, $(\wedge, \hookrightarrow)$ is an adjoint pair, which implies that for any $x \in L$, $F \subseteq \langle x] \hookrightarrow G \Longleftrightarrow F \wedge \langle x] \subseteq G \Longleftrightarrow F \cap \langle x] \subseteq G$. Then combining with Remark 2 and Corollaries 2, 3, we have $F \hookrightarrow G = E_G(F) = \{x \in L \mid F \subseteq E_G(x)\} = \{x \in L \mid F \subseteq \langle x] \hookrightarrow G\} = \{x \in L \mid F \cap \langle x] \subseteq G\} = F \rightarrowtail G$. □

From Corollary 3 and Theorem 7, for any $F \in Fil(\mathbf{L})$, $B \subseteq L$, $E_F(B) = \langle B] \rightarrowtail F$ holds. This is the characterization of (crisp) extended filters in Kondo (2014).

Denote

$\varepsilon_{\mu} = \{\varepsilon_{\mu}(\nu) \mid \nu \in \mathscr{F}(\mathbf{L})\}$.

$S(\nu) = \{\mu \in FFil(\mathbf{L}) \mid \varepsilon_{\mu}(\nu) = \mu\}$, $\varepsilon^{\nu} = \{\varepsilon_{\mu}(\nu) \mid \mu \in FFil(\mathbf{L})\}$.

$\varepsilon_{\mu\mu} = \{\varepsilon_{\mu}\left(\varepsilon_{\mu}(\nu)\right) \mid \nu \in \mathscr{F}(\mathbf{L})\}$, $S(\mu\mu) = \{\nu \in \mathscr{F}(\mathbf{L}) \mid \varepsilon_{\mu}\left(\varepsilon_{\mu}(\nu)\right) = \nu\}$.

$\varepsilon(\nu): FFil(\mathbf{L}) \to FFil(\mathbf{L}): \mu \mapsto \varepsilon_{\mu}(\nu)$.

$\varepsilon_{\mu}(\varepsilon_{\mu}): \mathscr{F}(\mathbf{L}) \to \mathscr{F}(\mathbf{L}): \nu \mapsto \varepsilon_{\mu}(\varepsilon_{\mu}(\nu))$.

Proposition 6 *Let μ be a fuzzy filter on $\mathbf{L}$ and μ a fuzzy set of $\mathbf{L}$. Then*

(1) *$(\varepsilon_{\mu}, \subseteq)$ is a complete lattice.*
(2) *$\varepsilon(\nu)$ is a closure operator on $FFil(\mathbf{L})$, $S(\nu) = \varepsilon^{\nu}$, $(S(\nu), \subseteq)$ is a complete lattice.*
(3) *$\varepsilon_{\mu}(\varepsilon_{\mu})$ is a closure operator on $\mathscr{F}(\mathbf{L})$, $\varepsilon_{\mu\mu} = S(\mu\mu)$, $(S(\mu\mu), \subseteq)$ is a complete lattice.*

Proof (1) Obviously, $\overline{1} = \varepsilon_{\mu}\left(\overline{0}\right) \in \varepsilon_{\mu}$, it follows from (8) in Proposition 5 and Theorem 2 that $(\varepsilon_{\mu}, \subseteq)$ is a complete lattice.

(2) $\varepsilon(\nu)$ is a closure operator on $FFil(\mathbf{L})$ by (2) in Theorem 3 and (2), (7) in Proposition 5. Then Proposition 4 implies that $S(\nu) = \varepsilon^{\nu}$ and $(S(\nu), \subseteq)$ is a complete lattice.

(3) It is straightforward by (1), (3) and (6) in Proposition 5 that $\varepsilon_{\mu}(\varepsilon_{\mu})$ is a closure operator on $\mathscr{F}(\mathbf{L})$. Similar to (2), $\varepsilon_{\mu\mu} = S(\mu\mu)$ and $(S(\mu\mu), \subseteq)$ is a complete lattice. □

With the help of Theorem 6, we further discuss the lattice structures of $(\varepsilon_{\mu}, \subseteq)$, $(S(\nu), \subseteq)$ and $(S(\mu\mu), \subseteq)$.

Theorem 8 *Let μ be a fuzzy filter on* **L** *and μ a fuzzy set of* **L**. *Then*

(1) $(\varepsilon_\mu, \subseteq)$ *is a complete Heyting algebra.*
(2) $(S(\nu), \subseteq)$ *is a complete Heyting algebra.*
(3) $(S(\mu\mu), \subseteq)$ *is a complete Heyting algebra.*

Proof (1) For any $\varepsilon_\mu(\nu_1), \varepsilon_\mu(\nu_2) \in \varepsilon_\mu$, according to Theorem 6, (5) in Proposition 1 and Theorem 5, we have

$$\begin{aligned}\varepsilon_\mu(\nu_1) \rightsquigarrow \varepsilon_\mu(\nu_2) &= \varepsilon_\mu(\nu_1) \rightsquigarrow (\langle\nu_2\rangle \rightsquigarrow \mu)\\ &= \left(\varepsilon_\mu(\nu_1) \wedge \langle\nu_2\rangle\right) \rightsquigarrow \mu\\ &= \varepsilon_\mu\left(\varepsilon_\mu(\nu_1) \cap \langle\nu_2\rangle\right) \in \varepsilon_\mu,\end{aligned}$$

then $(\varepsilon_\mu, \subseteq)$ is a complete Heyting algebra by Proposition 4 and (1) in Proposition 6.

(2) For any $\mu, \varrho \in S(\nu)$, we get

$$\begin{aligned}\mu \rightsquigarrow \varrho &= \mu \rightsquigarrow \varepsilon_\varrho(\nu) = \mu \rightsquigarrow (\langle\nu\rangle \rightsquigarrow \varrho) = \langle\nu\rangle \rightsquigarrow (\mu \rightsquigarrow \varrho)\\ &= \varepsilon_{\mu\rightsquigarrow\varrho}(\nu) \in \varepsilon^\nu = S(\nu)\end{aligned}$$

by (2) in Proposition 6, Theorem 6, (5) in Proposition 1 and Theorem 5. Then it follows from Proposition 4 and (2) in Proposition 6 that $(S(\nu), \subseteq)$ is a complete Heyting algebra.

(3) For any $\nu, \omega \in S(\mu\mu)$, it follows from (3) in Proposition 6, Theorems 5, 6 and (4), (5) in Proposition 1 that

$$\begin{aligned}\nu \rightsquigarrow \omega &= \varepsilon_\mu\left(\varepsilon_\mu(\nu)\right) \rightsquigarrow \varepsilon_\mu\left(\varepsilon_\mu(\omega)\right)\\ &= \left(\varepsilon_\mu(\nu) \rightsquigarrow \mu\right) \rightsquigarrow (((\langle\omega\rangle \rightsquigarrow \mu) \rightsquigarrow \mu)\\ &= \left(\left(\varepsilon_\mu(\nu) \rightsquigarrow \mu\right) \wedge (\langle\omega\rangle \rightsquigarrow \mu)\right) \rightsquigarrow \mu\\ &= \left(\left(\varepsilon_\mu(\nu) \vee \langle\omega\rangle\right) \rightsquigarrow \mu\right) \rightsquigarrow \mu\\ &= \varepsilon_\mu\left(\varepsilon_\mu\left(\langle\varepsilon_\mu(\nu) \cup \langle\omega\rangle\rangle\right)\right) \in \varepsilon_{\mu\mu} = S(\mu\mu).\end{aligned}$$

Then $(S(\mu\mu), \subseteq)$ is a complete Heyting algebra by Proposition 4 and (3) in Proposition 6. □

Remark 4 It is straightforward that $(\varepsilon^\nu, \subseteq)$ and $(\varepsilon_{\mu\mu}, \subseteq)$ are also complete Heyting algebras from Proposition 6 and Theorem 8.

4 Application of the fuzzy extended filter in characterizing special algebras and quotient algebras

In Víta (2014), the author concluded that fuzzy t-filters can be used to characterize special algebras and quotient algebras (associate to fuzzy filters).

Theorem 9 (Víta 2014) (Equivalent characteristics) *Let $\mathbb{B}$ be a variety of residuated lattices and* $\mathbf{B} \in \mathbb{B}$. *Then the following statements are equivalent:*

(1) *Every fuzzy filter of* **B** *is a fuzzy t-filter.*
(2) χ_1 *is a fuzzy t-filter.*
(3) $\mu_{\mu(1)}$ *is a t-filter for any $\mu \in FFil(\mathbf{L})$.*
(4) $\mathbf{B} \in \mathbb{B}[t]$.

Theorem 10 (Víta 2014) (Quotient characteristics) *Let $\mathbb{B}$ be a variety of residuated lattices,* $\mathbf{B} \in \mathbb{B}$ *and μ a fuzzy filter on* **B**. *Then the fuzzy quotient L/μ belongs to $\mathbb{B}[t]$ if and only if μ is a fuzzy t-filter on* **B**.

Theorem 11 *Let μ be a fuzzy filter on* **L**. *Then μ is a fuzzy t-filter if and only if for any $\overline{x} \in L$, $\varepsilon_\mu(\chi_{t(\overline{x})}) = \overline{1}$.*

Proof ($\Longrightarrow$) μ is a fuzzy t-filter, which implies $\mu(t(\overline{x})) = \mu(1)$. Then for any $y \in L$, $\varepsilon_\mu(\chi_{t(\overline{x})})(y) = \mu(y \vee t(\overline{x})) \geq \mu(t(\overline{x})) = 1$.

($\Longleftarrow$) If $\varepsilon_\mu(\chi_{t(x)}) = \overline{1}$ for any $x \in L$, then $\varepsilon_\mu(\chi_{t(\overline{x})})(0) = \mu(0 \vee t(\overline{x})) = \mu(t(\overline{x})) = 1$, i.e., μ is a fuzzy t-filter. □

Theorem 11 states that fuzzy t-filters can be characterized by fuzzy extended filters. Thus, we can apply fuzzy extended filters to characterize special algebras and quotient algebras.

Theorem 12 (New equivalent characteristics). *Let $\mathbb{B}$ be a variety of residuated lattices and* $\mathbf{B} \in \mathbb{B}$. *Then the following statements are equivalent:*

(1) *For every fuzzy filter μ on* **L**, $\varepsilon_\mu(\chi_{t(\overline{x})}) = \overline{1}$ *for any $\overline{x} \in L$.*
(2) $\varepsilon_{\chi_1}(\chi_{t(\overline{x})}) = \overline{1}$ *for any $\overline{x} \in L$.*
(3) $E_{\mu_{\mu(1)}}(t(\overline{x})) = L$ *for any $\mu \in FFil(\mathbf{L}), \overline{x} \in L$.*
(4) $\mathbf{B} \in \mathbb{B}[t]$.

Proof It follows from Theorems 9, 11. □

Theorem 13 (New quotient characteristics) *Let $\mathbb{B}$ be a variety of residuated lattices and* $\mathbf{B} \in \mathbb{B}$. *Let μ be a fuzzy filter on* **B**. *Then the fuzzy quotient L/μ belongs to $\mathbb{B}[t]$ if and only if $\varepsilon_\mu(\chi_{t(\overline{x})}) = \overline{1}$ for any $\overline{x} \in L$.*

Proof It can be easily obtained by Theorems 10, 11. □

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