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FOUNDATIONS

The TL-fuzzy rough approximation operators on a lattice

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Abstract This paper mainly addresses the connection between fuzzy rough sets and lattices. Based on a complete lattice equipped with a t-norm, the concepts of TL-fuzzy lower and upper rough approximation operators induced by an L-fuzzy set on a lattice are introduced, and their basic properties are investigated. Particularly, some characterizations of TL-fuzzy ideals on distributive lattices are developed in terms of the TL-fuzzy rough approximation operators. In addition, we use these operators to define a new class of fuzzy structures, called TL-fuzzy quasi-rough ideals induced by an L-fuzzy set on a lattice, and investigate the relationships among TL-fuzzy ideals, TL-fuzzy rough ideals and TL-fuzzy quasi-rough ideals on a given lattice.

Keywords Lattice · TL-fuzzy ideal · TL-fuzzy rough approximation operator · TL-fuzzy rough ideal · TL-fuzzy quasi-rough ideal

1 Introduction

Rough set theory was born in 1982 when Pawlak (1982) defined approximate operators on a nonempty set and began to investigate their basic properties with applications to pattern recognitions, machine learning, expert systems, characteristic diagnosis and so on. The notion of rough sets is an extension of set theory, in which every subset is described in terms of two ordinary sets called lower and upper approximations, respectively, and has been successfully applied in various fields. In order to broaden application fields of the rough set theory and provide more ways for approximate reasoning, many researchers considered the connections between rough sets and algebras. Since Biswas and Nanda (1994) considered the rough structures on algebraic groups and proposed the notion of rough subgroups, many researchers have studied rough sets from algebraic points of view. Kuroki (1997) and Xiao and Zhang (2006) introduced the concepts of rough (prime) ideals of a semigroup. Davvaz (2004), Davvaz and Mahdavipour (2006) and Davvaz (2006, 2008) investigated the properties of rough approximations in rings, modules and n-ary algebraic systems, respectively. In Davvaz (2006), Leoreanu (2008) and Leoreanu and Davvaz (2008), the authors defined and analysed the roughness in algebraic hyperstructures. Recently, rough approximation operators in some ordered structures were considered by Estaji, Luo, Qi, Tantawy, Xiao, Yang, Zhou and others. The reader can refer to Estaji et al. (2012), Luo and Wang (2014), Qi and Liu (2005), Tantawy and Mustafa (2013), Xiao et al. (2012), Xiao et al. (2014), Yang and Xu (2013) and Zhou and Hu (2014) for details.

The theory of fuzzy sets, initiated by Zadeh (1965), is well known to achieve great success in various fields of science and technology. For analysing the inter-relations among fuzzy sets and rough sets, Dubois and Prade (1990, 1992) combined these two concepts and proposed the notion of fuzzy rough sets. Their work afterwards received wide attention both in practical applications (see, e.g. Jensen and Shen 2005; Wang 2003; Zhang et al. 2015) and in theoretical sides as well (see, e.g. Feng et al. 2010; Radzikowska and Kerre

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2002). However, since Dubois and Prade's fuzzy rough sets were defined based on the complements of fuzzy relations, some results in classical rough sets cannot be extended to the context of fuzzy rough sets. To improve this definition, many authors employed implicators (implication operators) to define fuzzy rough sets. For example, Morsi and Yakout (1998) studied the properties of fuzzy rough sets based on the residuated implicators which are induced by triangular norms. Radzikowska and Kerre (2005) introduced and investigated L-fuzzy rough sets based on residuated lattices. The generalized fuzzy rough approximation operators determined by fuzzy implicators were studied by Wu et al. (2005, 2013). For more information on this topic, please see Radzikowska and Kerre (2002); Wu and Zhang (2004) and Yao (1998).

Meanwhile, it was pointed out in Dubois and Prade (2001) that more efforts should be made to study new algebraic structures induced by fuzzy sets. Inspired by this viewpoint, many investigations have been carried out in order to study rough sets on fuzzy algebraic structures and finally propose new algebraic structures. The work on this topic can be classified into two groups. One group of papers took a rough algebraic structure as the universal set and then studied the fuzzy substructures on it. The study is initiated by Davvaz (2006) and then continued by Leoreanu (2008) and others. Another group of papers defined and investigated fuzzy rough sets on crisp algebraic structures. This direction can also be regarded as a directly fuzzy generalization of classical rough algebras. The examples in this direction are Li et al. (2007), Li and Yin (2007), Xiao and Zhang (2006), Yin et al. (2011), Yin et al. (2011) and Yin and Huang (2011).

It is well known that the theory of lattice, as a combination of algebraic structures and ordered structures, plays an important role in computer science, engineering and mathematics. For example, it is widely used in data mining, distributed computing, programming language semantics and machine learning (Marques and Graña 2012; Singh et al. 2016; Jin and Li 2012; Zhang and Bodenreider 2010). It also has applications in other branches of mathematics such as combinatorics, number theory and algebra (Borzooei et al. 2008; Lidl and Pilz 1998). In this paper, we intend to apply fuzzy rough sets to lattice theory. By using a residuated implicator determined by a left continuous t-norm on a complete lattice, TL-fuzzy rough upper and lower approximation operators with respect to a lattice-valued fuzzy set on a lattice are defined and studied. As examples of application in theoretical aspect, such operators are used to characterize TL-fuzzy ideals on distributive lattices. Also, we analyse the properties of TL-fuzzy rough ideals and TL-fuzzy quasi-rough ideals with respect to an L-fuzzy set on lattices and study the relations among them. The paper is in line with the fuzzification of rough sets on lattices reported in the literature, for example Estaji et al. (2012) and Xiao et al. (2012, 2014).

The rest of the paper is organized as follows: in Sect. 2, we recall some basic notions and results on lattices and L-fuzzy sets. In Sect. 3, we define and study TL-fuzzy rough approximation operators with respect to an L-fuzzy set on lattices. In Sect. 4, TL-fuzzy rough ideals and TL-fuzzy quasi-rough ideals on a lattice are given and the relations among them are investigated. We make conclusions and suggest topics for future research in the last section.

2 Preliminaries

This section reviews some basic notions and facts on lattices and L-fuzzy sets (see Birkhoff 1967; Goguen 1967; Davey and Priestley 2002) which will be needed in the paper.

Let $(L, \leq)$ be a poset. We say that L has a *bottom element* if there exists $0 \in L$ (called bottom) with the property that $0 \leq x$ for all $x \in L$. Dually, L has a *top element* if there exists $1 \in L$ such that $1 \geq x$ for all $x \in L$. It is clear that 0 (resp. 1) is unique if it exists. A poset $(L, \leq)$ is called a *lattice* if it satisfies the condition that for any α, β in L both $\alpha \vee \beta$ and $\alpha \wedge \beta$ exist in L, where $\alpha \vee \beta = \sup\{\alpha, \beta\}$ and $\alpha \wedge \beta = \inf\{\alpha, \beta\}$. A lattice L is said to be *bounded* if it has both bottom 0 and top 1. A lattice L is said to be *distributive* if for each $\alpha, \beta, \gamma \in L$, $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \beta)$. It is clear that $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \beta)$ is also valid in a distributive lattice.

Let L be a poset. A subset D of L is said to be directed if every finite subset of D has an upper bound in D. A poset is said to be a directed complete poset, or shortly DCPO, if every directed subset has a sup. A poset is called a complete lattice if every subset has both sup and inf.

Example 1 Let X be a nonempty set. A family of subsets $\mathscr{L}$ of X is called an $\bigcap$-structure if $\bigcap_{i \in I} A_i \in \mathscr{L}$ for every nonempty $\{A_i : i \in I\} \subseteq \mathscr{L}$. An $\bigcap$-structure $\mathscr{L}$ is said to be topped if $X \in \mathscr{L}$. It is easy to verify that every topped $\bigcap$-structure is a complete lattice.

Let $(L, \vee, \wedge, \leq, 0, 1)$ be a complete lattice with 0 and 1 as the bottom and the top, respectively. A triangular norm on L, or shortly a t-norm, is an increasing, associative and commutative operation $T : L^2 \to L$ satisfying the boundary condition: for any $\alpha \in L$, $\alpha T 1 = \alpha$. For $\alpha_1, \ldots, \alpha_n \in L$ $(n \geq 1)$, we write $T_{i=1}^n \alpha_i = \alpha_1 T \alpha_2 T \ldots T \alpha_n$. Let $\{\alpha_i : i \in I\}$ be a set in L and $\beta \in L$, where I is an index set. Then, we say that T is *left continuous* if $(\bigvee_{i \in I} \alpha_i) T \beta = \bigvee_{i \in I} (\alpha_i T \beta)$.

For any left continuous t-norm T, the following binary operation on L, i.e.

$$\vartheta(\alpha, \beta) := \bigvee \{\gamma \in L : \alpha T \gamma \leq \beta\}, \ \forall \alpha, \beta \in L$$

is said to be the *residuation implication* of T, or simply, the *T-residuated implication*.

For a T-residuated implication ϑ on a complete lattice, the following properties are well known (see Morsi and Yakout 1998; Wu et al. 2005): let $\alpha, \beta, \gamma, \eta, \alpha_i \in L$ $(i \in I)$, where I is an index set. Then

(R1) $\vartheta(1, \alpha) = \alpha$ and $\vartheta(\alpha, 1) = 1$;
(R2) $\alpha \leq \beta \Longrightarrow \vartheta(\alpha, \beta) = 1$;
(R3) $\alpha \leq \beta \Longleftrightarrow \vartheta(\delta, \alpha) \leq \vartheta(\delta, \beta)$ for all $\delta \in L$;
(R4) $\alpha \leq \beta \Longleftrightarrow \vartheta(\alpha, \delta) \geq \vartheta(\beta, \delta)$ for all $\delta \in L$;
(R5) $\vartheta(\alpha T \beta, \gamma) = \vartheta(\alpha, \vartheta(\beta, \gamma))$;
(R6) $\alpha \leq \vartheta(\beta, \alpha T \beta)$;
(R7) $\vartheta(\alpha, \beta) T \gamma \leq \vartheta(\alpha, \beta T \gamma)$;
(R8) $\vartheta(\alpha, \beta) T \vartheta(\gamma, \eta) \leq \vartheta(\alpha T \gamma, \beta T \eta)$;
(R9) $\vartheta(\bigvee_{i \in I} \alpha_i, \beta) = \bigwedge_{i \in I} \vartheta(\alpha_i, \beta)$;
(R10) $\vartheta(\beta, \bigwedge_{i \in I} \alpha_i) = \bigwedge_{i \in I} \vartheta(\beta, \alpha_i)$;
(R11) $\vartheta(\beta, \bigvee_{i \in I} \alpha_i) \geq \bigvee_{i \in I} \vartheta(\beta, \alpha_i)$.

Clearly, if T is a t-norm on a complete lattice L, then $(L, \vee, \wedge, \leq, T, \vartheta, 0, 1)$ forms a complete residuated lattice (see Radzikowska and Kerre 2005), where ϑ is the residuated implicator induced by T. In the sequel, unless otherwise stated, L always denotes a given complete lattice with a left continuous t-norm T and ϑ denotes the T-residuated implicator.

As a generalization of Zadeh's fuzzy sets, Goguen (1967) introduced the notion of lattice-valued fuzzy sets, or shortly L-fuzzy sets. Let X be an nonempty set. An *L-fuzzy set* on X is an arbitrary mapping $\mu : X \to L$. The family of all L-fuzzy sets on a given set X is denoted by L^X.

It should be noticed that in this research, we shall deal with roughness with respect to L-fuzzy sets on a lattice. So, both the domain and the codomain of fuzzy structures are lattices. To avoid certain confusion arising, throughout the paper, we will use X to denote the universe set and L the structure of truth values of an arbitrary L-fuzzy set.

For $A \subseteq X$ and $0 \neq \alpha \in L$, we define a pair of L-fuzzy sets α_A and α^*_A on L by

$$\alpha_A(x) = \begin{cases} \alpha, & x \in A \\ 0, & x \in X - A \end{cases} \text{ and } \alpha^*_A(x) = \begin{cases} \alpha, & x \in A \\ 1, & x \in X - A \end{cases}$$

respectively, for all $x \in X$. In particular, (i) if $\alpha = 1$, then α_A is called the *characteristic function* of A and is denoted by χ_A; (ii) if $A = \{x\}$, then α_A is called an *L-fuzzy point* with the support x and the height α on X and is denoted by x_α. An L-fuzzy point x_α is said to belong to an L-fuzzy set μ, written as $x_\alpha \in \mu$, if $\mu(x) \geq \alpha$.

Given $\mu, \nu \in L^X$, by $\mu \subseteq \nu$ we mean $\mu(x) \leq \nu(x)$ for all $x \in X$. Some new L-fuzzy sets on X are defined by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$ and $(\mu T \nu)(x) = \mu(x) T \nu(x)$, respectively, for any $x \in X$. In addition, suppose that X and Y are two nonempty sets, $\mu \in L^X$ and $\nu \in L^Y$, define $\mu \times_T \nu \in L^{X \times Y}$ as follows:

$$(\mu \times_T \nu)(x, y) := \mu(x) T \nu(y), \ \forall (x, y) \in X \times Y.$$

Definition 1 Let X be a lattice and $\mu, \nu \in L^X$. Then the L-fuzzy subsets $\mu \vee_T \nu$, $\mu \wedge_T \nu$, $\mu \vee_\vartheta \nu$ and $\mu \wedge_\vartheta \nu$ are defined, respectively, as follows: $\forall x \in X$,

(1) $(\mu \vee_T \nu)(x) = \bigvee_{x = y \vee z} \mu(x) T \nu(y)$.
(2) $(\mu \wedge_T \nu)(x) = \bigvee_{x = y \wedge z} \mu(x) T \nu(y)$.
(3) $(\mu \vee_\vartheta \nu)(x) = \bigwedge_{x = y \vee z} \vartheta(\mu(y), \nu(z))$.
(4) $(\mu \wedge_\vartheta \nu)(x) = \bigwedge_{x = y \wedge z} \vartheta(\mu(y), \nu(z))$.

The following two propositions can be easily verified from the above definition.

Proposition 1 *Let X be a lattice and $\mu, \nu, \omega \in L^X$. Then*

(1) *$\mu(x) T \nu(y) \leq (\mu \vee_T \nu)(x \vee y)$ for all $x, y \in X$.*
(2) *$\mu(x) T \nu(y) \leq (\mu \wedge_T \nu)(x \wedge y)$ for all $x, y \in X$.*
(3) *$\mu \vee_T \nu \subseteq \omega$ if and only if $\mu(x) T \nu(y) \leq \omega(x \vee y)$ for all $x, y \in X$.*
(4) *$\mu \wedge_T \nu \subseteq \omega$ if and only if $\mu(x) T \nu(y) \leq \omega(x \wedge y)$ for all $x, y \in X$.*

Proposition 2 *Let X be a lattice and $\mu, \nu, \omega \in L^X$, and let $\{\mu_i : i \in I\}$ and $\{\nu_j : j \in J\}$ be two subsets of L^X, where I and J are index sets. Then*

(1) *$\mu \vee_T \nu = \nu \vee_T \mu$ and $\mu \wedge_T \nu = \nu \wedge_T \mu$.*
(2) *$(\mu \vee_T \nu) \vee_T \omega = \mu \vee_T (\nu \vee_T \omega)$ and $(\mu \wedge_T \nu) \wedge_T \omega = \mu \wedge_T (\nu \wedge_T \omega)$.*
(3) *If $\nu \subseteq \omega$, then $\mu \vee_T \nu \subseteq \mu \vee_T \omega$ and $\mu \wedge_T \nu \subseteq \mu \wedge_T \omega$.*
(4) *If $\nu \subseteq \omega$, then $\mu \vee_\vartheta \nu \subseteq \mu \vee_\vartheta \omega$, $\nu \vee_\vartheta \mu \supseteq \omega \vee_\vartheta \mu$, $\mu \wedge_\vartheta \nu \subseteq \mu \wedge_\vartheta \omega$ and $\nu \wedge_\vartheta \mu \supseteq \omega \wedge_\vartheta \mu$.*
(5) *If X is distributive, then $\mu \wedge_T (\nu \vee_T \omega) \subseteq (\mu \wedge_T \nu) \vee_T (\mu \wedge_T \omega)$.*
(6) *$(\bigcap_{i \in I} \nu_i) \vee_T (\bigcap_{i \in J} \nu_j) \subseteq \bigcap_{i \in I} \bigcap_{i \in J} (\mu_i \vee_T \nu_j)$.*

It is well known that ideals play fundamental roles in lattice theory. In the following, we introduce the concept of TL-fuzzy ideals of a lattice and develop some of their basic properties, which can be seen as a generalization of the notion of fuzzy ideals reported in literature (see, e.g Swamy and Raju 1998).

Let X be a lattice and $\mu \in L^X$. We write $\downarrow \mu(x) = \bigvee_{x \leq y} \mu(y)$ and $\uparrow \mu(x) = \bigvee_{y \leq x} \mu(y)$, respectively. Then μ is called an *L-fuzzy lower* (resp. *L-fuzzy upper*) set if $\downarrow \mu = \mu$ (resp. $\uparrow \mu = \mu$). One may easily observe that μ is an L-fuzzy upper set if and only if $x \leq y \Rightarrow \mu(x) \leq \mu(y)$ for all $x, y \in X$. So every L-fuzzy upper set is a monotone function. Dually, every L-fuzzy lower set is an antitone function.

Definition 2 Let X be a lattice and $\mu \in L^X$. Then μ is said to be nonvoid if $\bigvee_{x\in X} \mu(x) = 1$. A nonvoid L-fuzzy set μ is called an TL-fuzzy $\vee$-semilattice on X if $\mu(x \vee y) \geq \mu(x)T\mu(y)$ for all $x, y \in X$. μ is called an TL-fuzzy ideal on X if it is both an TL-fuzzy $\vee$-semilattice and an L-fuzzy lower set on X.

Remark 1 In Definition 2, if $L = [0, 1]$ and $T = \wedge$, then the concept of TL-fuzzy ideal on a lattice is equivalent to that of fuzzy ideals given in Swamy and Raju (1998).

Lemma 1 *For any $\mu \in L^X$, $\downarrow \mu = \chi_X \wedge_T \mu$.*

Proof Let $\mu \in L^X$ and $x \in X$. We have

$$\begin{aligned}(\chi_X \wedge_T \mu)(x) &= \bigvee_{x=y\wedge z} \chi_X(y)T\mu(z)\\ &= \bigvee_{x=y\wedge z} \mu(z) \leq \bigvee_{x\leq z} \mu(z)\\ &= \downarrow \mu(x)\end{aligned}$$

and

$$\begin{aligned}(\chi_X \wedge_T \mu)(x) &= \bigvee_{x=y\wedge z} \chi_X(y)T\mu(z)\\ &= \bigvee_{x=y\wedge z} \mu(z) \geq \bigvee_{x=x\wedge z} \mu(z)\\ &= \bigvee_{x\leq z} \mu(z) =\downarrow \mu(x).\end{aligned}$$

This implies that $\downarrow \mu = \chi_X \wedge_T \mu$. □

Proposition 3 *Let X be a lattice and μ a nonvoid L-fuzzy set on X. Then μ is an TL-fuzzy ideal on X if and only if the following conditions hold:*

(1) $\mu \vee_T \mu \subseteq \mu$.
(2) $\chi_X \wedge_T \mu \subseteq \mu$.

Proof By the item (3) of Proposition 1, it is easy to see that $\mu \vee_T \mu \subseteq \mu$ if and only if μ is an TL-fuzzy $\vee$-semilattice. Since $\mu \subseteq\downarrow \mu$ always holds, it follows from Lemma 1 that $\chi_X \wedge_T \mu \subseteq \mu$ if and only if $\mu =\downarrow \mu$, i.e. μ is an L-fuzzy lower set on X. □

In the sequel, we use the symbol $\mathbf{TLFI}(X)$ to denote the set of all TL-fuzzy ideals on a lattice X. The following corollary is straightforward by Proposition 2 and 3.

Corollary 1 *Let $\mu, \nu, \mu_i \in \mathbf{TLFI}(X)$, $i \in I$, where I is an index set. Then*

(1) *$\mu T\nu \in \mathbf{TLFI}(X)$. In particular, $\bigcap_{i\in I} \mu_i \in \mathbf{TLFI}(X)$.*
(2) *If $\{\mu_i : i \in I\}$ is a directed set, then $\bigcup_{i\in I} \mu_i \in \mathbf{TLFI}(X)$.*
(3) *If X is a distributive lattice, then $\mu \vee_T \nu \in \mathbf{TLFI}(X)$.*

3 The properties of TL-fuzzy rough approximation operators on a lattice

In this section, by applying fuzzy rough sets proposed by Morsi and Yakout (1998) to lattice theory, we introduce the main concept of present paper.

Definition 3 Let X be a lattice and $\mu \in L^X$. Define two mappings $\overline{Apr_\mu}^T : L^X \to L^X$ and $\underline{Apr_{\mu}}_\vartheta : L^X \to L^X$, respectively, called TL-fuzzy upper and lower rough approximation operators with respect to μ, as follows: $\forall \nu \in L^X$ and $\forall x \in X$,

$$\overline{Apr_\mu}^T \nu(x) = \bigvee_{x\vee z=y\vee z} \mu(z)T\nu(y)$$

and

$$\underline{Apr_{\mu}}_\vartheta \nu(x) = \bigwedge_{x\vee z=y\vee z} \vartheta(\mu(z), \nu(y)).$$

For each $\nu \in L^X$, $\underline{Apr_{\mu}}_\vartheta \nu$ (resp., $\overline{Apr_\mu}^T \nu$) is called an TL-fuzzy lower (resp., upper) rough approximation of ν with respect to μ. The pair $(\overline{Apr_\mu}^T \nu, \underline{Apr_{\mu}}_\vartheta \nu)$ is called the TL-fuzzy rough set of ν with respect to μ if $\overline{Apr_\mu}^T \nu \neq \underline{Apr_{\mu}}_\vartheta \nu$.

Remark 2 (1) In Definition 3, if both μ and ν are crisp subsets of X, then the operators $\underline{Apr_{\mu}}_\vartheta \nu$ and $\overline{Apr_\mu}^T \nu$ are equivalent to those introduced in Xiao et al. (2014).
(2) By carefully observing Definition 3, one may find that it covers Morsi and Yakout's definition. Indeed, for an L-fuzzy set μ of X, we define an L-fuzzy relation as follows: $R_\mu(x, y) := \bigvee_{x\vee z=y\vee z} \mu(z), \forall x, y \in X$. Then, by applying R_μ to Morsi and Yakout's definition, we obtain the operators $\underline{Apr_{\mu}}_\vartheta$ and $\overline{Apr_\mu}^T$ as given in Definition 3. They are more generalized fuzzy rough approximation operators since R_μ is not necessary to be an L-fuzzy equivalence relation. However, if μ is an L-fuzzy $\vee$-semilattice on X, then R_μ is an L-fuzzy equivalence relation, and thus, $\underline{Apr_{\mu}}_\vartheta \nu$ and $\overline{Apr_\mu}^T \nu$ are equivalent to those given in Morsi and Yakout (1998).

The following two theorems can be easily checked from Definition 3 and Remark 2.

Theorem 3.1 *Let X be a lattice, $\mu, \nu, \nu_i \in L^X$ ($i \in I$) and $\alpha \in L$. Then*

(1) $\overline{Apr_\mu}^T (\bigcup_{i\in I} \nu_i) = \bigcup_{i\in I} \overline{Apr_\mu}^T \nu_i$.
(2) $\overline{Apr_\mu}^T (\bigcap_{i\in I} \nu_i) \subseteq \bigcap_{i\in I} \overline{Apr_\mu}^T \nu_i$.
(3) $\underline{Apr_{\mu}}_\vartheta (\bigcap_{i\in I}) = \bigcap_{i\in I} \underline{Apr_{\mu}}_\vartheta \nu_i$.
(4) $\underline{Apr_{\mu}}_\vartheta (\bigcup_{i\in I} \nu_i) \supseteq \bigcup_{i\in I} \underline{Apr_{\mu}}_\vartheta \nu_i$.
(5) *If μ is nonvoid, then $\underline{Apr_{\mu}}_\vartheta \nu \subseteq \nu \subseteq \overline{Apr_\mu}^T \nu$.*

(6) *If μ is nonvoid, then* $\underline{Apr_{\mu}}_{\vartheta}\alpha_X = \alpha_X = \overline{Apr_{\mu}}^T\alpha_X$.

Theorem 3.2 *Let μ be an L-fuzzy $\vee$-semilattice on X. Then*

(1) $\overline{Apr_{\mu}}^T\overline{Apr_{\mu}}^T\nu = \overline{Apr_{\mu}}^T\nu$
(2) $\underline{Apr_{\mu}}_{\vartheta}\underline{Apr_{\mu}}_{\vartheta}\nu = \underline{Apr_{\mu}}_{\vartheta}\nu$.
(3) $\underline{Apr_{\mu}}_{\vartheta}\overline{Apr_{\mu}}^T\nu = \overline{Apr_{\mu}}^T\nu$.
(4) $\overline{Apr_{\mu}}^T\underline{Apr_{\mu}}_{\vartheta}\nu = \underline{Apr_{\mu}}_{\vartheta}\nu$.

It is easy to deduce from (1) of Theorem 3.1 that, for any $\nu, \omega \in L^X$, $\nu \subseteq \omega$ implies $\overline{Apr_{\mu}}^T\nu \subseteq \overline{Apr_{\mu}}^T\omega$, i.e. $\overline{Apr_{\mu}}^T$ is order preserving. By the item (1) of Theorem 3.2, if μ is an L-fuzzy $\vee$-semilattice, then $\overline{Apr_{\mu}}^T$ is idempotent. So, combining with (5) in Theorem 3.1, we conclude that if μ is an L-fuzzy $\vee$-semilattice, then $\overline{Apr_{\mu}}^T$ is a closure operator on $(L^X, \subseteq)$. Similarly, $\underline{Apr_{\mu}}_{\vartheta}$ is a kernel operator on L^X.

Theorem 3.3 *Let X be a lattice and $\mu, \nu \in L^X$. Then*

(1) $\overline{Apr_{\mu}}^T\downarrow\nu \subseteq\downarrow\overline{Apr_{\mu}}^T\nu$ *and* $\underline{Apr_{\mu}}_{\vartheta}\downarrow\nu \supseteq\downarrow\underline{Apr_{\mu}}_{\vartheta}\nu$. *Moreover, if X is a distributive lattice and μ is an L-fuzzy lower set on X, then* $\overline{Apr_{\mu}}^T\downarrow\nu =\downarrow\overline{Apr_{\mu}}^T\nu$.
(2) $\overline{Apr_{\mu}}^T\uparrow\nu \supseteq\uparrow\overline{Apr_{\mu}}^T\nu$. *Moreover, if X is a distributive lattice and μ is an L-fuzzy lower set, then* $\overline{Apr_{\mu}}^T\uparrow\nu =\uparrow\overline{Apr_{\mu}}^T\nu$ *and* $\underline{Apr_{\mu}}_{\vartheta}\uparrow\nu \supseteq\uparrow\underline{Apr_{\mu}}_{\vartheta}\nu$.

Proof (1) Let $\mu, \nu \in L^X$ and $x \in X$. Then, we have

$$
\begin{aligned}
\overline{Apr_{\mu}}^T\downarrow\nu(x) &= \bigvee_{x\vee z=y\vee z}\mu(z)T\downarrow\nu(y)\\
&= \bigvee_{x\vee z=y\vee z}\mu(z)T\bigvee_{y\le a}\nu(a)\\
&= \bigvee_{x\vee z=y\vee z}\ \bigvee_{y\vee a=a}\mu(z)T\nu(a)\\
&\le \bigvee_{(x\vee a)\vee z=a\vee z}\mu(z)T\nu(a)\\
&\le \bigvee_{a\in X}\overline{Apr_{\mu}}^T\nu(x\vee a)\\
&= \bigvee_{x\le b}\overline{Apr_{\mu}}^T\nu(b) \le\downarrow\overline{Apr_{\mu}}^T\nu(x).
\end{aligned}
$$

This implies that $\overline{Apr_{\mu}}^T\downarrow\nu \subseteq\downarrow\overline{Apr_{\mu}}^T\nu$. Next, we show that $\underline{Apr_{\mu}}_{\vartheta}\downarrow\nu \supseteq\downarrow\underline{Apr_{\mu}}_{\vartheta}\nu$. In fact, for any $x \in X$, we have

$$
\begin{aligned}
\underline{Apr_{\mu}}_{\vartheta}\downarrow\nu(x) &= \bigwedge_{x\vee z=y\vee z}\vartheta\left(\mu(z), \bigvee_{y\le a}\nu(a)\right)\\
&= \bigwedge_{x\vee z=y\vee z}\vartheta\left(\mu(z), \bigvee_{b\in X}\nu(y\vee b)\right)\\
&\ge \bigwedge_{x\vee z=y\vee z}\ \bigvee_{b\in X}\vartheta(\mu(z), \nu(y\vee b))\\
&\ge \bigvee_{x\le b}\ \bigwedge_{x\vee z=y\vee z}\vartheta(\mu(z), \nu(y\vee b))\\
&\ge \bigvee_{x\le b}\ \bigwedge_{x\vee b\vee z=y\vee b\vee z}\vartheta(\mu(z), \nu(y\vee b))\\
&= \bigvee_{x\le b}\ \bigwedge_{b\vee z=y\vee b\vee z}\vartheta(\mu(z), \nu(y\vee b))\\
&= \bigvee_{x\le b}\underline{Apr_{\mu}}_{\vartheta}\nu(b) =\downarrow\underline{Apr_{\mu}}_{\vartheta}\nu(x).
\end{aligned}
$$

That is, $\underline{Apr_{\mu}}_{\vartheta}\downarrow\nu \supseteq\downarrow\underline{Apr_{\mu}}_{\vartheta}\nu$. Moreover, if X is a distributive lattice and μ is an L-fuzzy lower set on X, then

$$
\begin{aligned}
\downarrow\overline{Apr_{\mu}}^T\nu(x) &= \bigvee_{x\le y}\overline{Apr_{\mu}}^T\nu(y) = \bigvee_{x\le y}\ \bigvee_{y\vee a=b\vee a}\mu(a)T\nu(b)\\
&\le \bigvee_{x\le y}\ \bigvee_{x\wedge(y\vee a)=x\wedge(b\vee a)}\mu(a)T\nu(b)\\
&\le \bigvee_{x\le y}\ \bigvee_{(x\wedge y)\vee(x\wedge a)=(x\wedge b)\vee(x\wedge a)}\mu(a)T\nu(b)\\
&\le \bigvee_{x\vee(x\wedge a)=(x\wedge b)\vee(x\wedge a)}\mu(a)T\nu(b)\\
&\le \bigvee_{x\vee(x\wedge a)=(x\wedge b)\vee(x\wedge a)}\mu(x\wedge a)T\downarrow\nu(x\wedge b)\\
&= \overline{Apr_{\mu}}^T\downarrow\nu(x).
\end{aligned}
$$

This means $\overline{Apr_{\mu}}^T\downarrow\nu \supseteq\downarrow\overline{Apr_{\mu}}^T\nu$, and so $\overline{Apr_{\mu}}^T\downarrow\nu =\downarrow\overline{Apr_{\mu}}^T\nu$.

The proof of (2) is similar to that of (1). □

Theorem 3.3 indicates that, for any $\mu \in L^X$, the operator $\overline{Apr_{\mu}}^T$ maps each L-fuzzy upper set to an L-fuzzy upper set and $\underline{Apr_{\mu}}_{\vartheta}$ maps each L-fuzzy lower set to an L-fuzzy lower set. Furthermore, if X is distributive and μ is an L-fuzzy lower set, then $\overline{Apr_{\mu}}^T$ maps each L-fuzzy lower set to an L-fuzzy lower set and $\underline{Apr_{\mu}}_{\vartheta}$ maps each L-fuzzy upper set to an L-fuzzy upper set.

Theorem 3.4 *Let X be a lattice and $\mu, \nu, \omega \in L^X$. Then*

(1) $\overline{Apr_{(\mu\vee_T\nu)}}^T\omega \supseteq \overline{Apr_{\mu}}^T\overline{Apr_{\nu}}^T\omega$. *Moreover, if X is distributive, then* $\overline{Apr_{(\mu\vee_T\nu)}}^T\omega = \overline{Apr_{\mu}}^T\overline{Apr_{\nu}}^T\omega$.
(2) $\underline{Apr_{(\mu\vee_T\nu)}}_{\vartheta}\omega \subseteq \underline{Apr_{\mu}}_{\vartheta}\underline{Apr_{\nu}}_{\vartheta}\omega$. *Moreover, if X is distributive, then* $\underline{Apr_{(\mu\vee_T\nu)}}_{\vartheta}\omega = \underline{Apr_{\mu}}_{\vartheta}\underline{Apr_{\nu}}_{\vartheta}\omega$.

Proof Let $\mu, \nu, \omega \in L^X$. For any $a, z \in X$, it follows from Proposition 1 that $\mu(z) T \nu(a) \leq (\mu \vee_T \nu)(z \vee a)$. Thus, for any $x \in X$, we have

$$\begin{aligned}
\overline{Apr_\mu}^T \overline{Apr_\nu}^T \omega(x) &= \bigvee_{x \vee z = y \vee z} \mu(z) T \overline{Apr_\nu}^T \omega(y) \\
&= \bigvee_{x \vee z = y \vee z} \mu(z) T \bigvee_{y \vee a = b \vee a} \nu(a) T \omega(b) \\
&= \bigvee_{x \vee z = y \vee z} \bigvee_{y \vee a = b \vee a} \mu(z) T \nu(a) T \omega(b) \\
&\leq \bigvee_{x \vee z \vee a = b \vee z \vee a} (\mu(z) T \nu(a)) T \omega(b) \\
&\leq \bigvee_{x \vee (z \vee a) = b \vee (z \vee a)} (\mu \vee_T \nu)(z \vee a) T \omega(b) \\
&= \overline{Apr_{(\mu \vee_T \nu)}}^T \omega(x).
\end{aligned}$$

This implies $\overline{Apr_{(\mu \vee_T \nu)}}^T \omega \supseteq \overline{Apr_\mu}^T (\overline{Apr_\nu}^T \omega)$. Moreover, if X is a distributive lattice, we first prove the fact that for $x, y, a, b \in X, x \vee a \vee b = y \vee a \vee b$ implies $x \vee a = r \vee a$ and $y \vee b = r \vee b$ for some $r \in X$. In fact, if $x \vee a \vee b = y \vee a \vee b$, then we have

$$\begin{aligned}
x \vee a &= (x \vee a) \wedge (x \vee a \vee b) = (x \vee a) \wedge (y \vee a \vee b) \\
&= (x \wedge y) \vee (x \wedge a) \vee (x \wedge b) \vee (a \wedge y) \vee (a \wedge b) \vee a \\
&= (x \wedge y) \vee (x \wedge b) \vee (a \wedge y) \vee a
\end{aligned}$$

and

$$\begin{aligned}
y \vee b &= (y \vee b) \wedge (y \vee a \vee b) = (y \vee b) \wedge (x \vee a \vee b) \\
&= (x \wedge y) \vee (a \wedge y) \vee (b \wedge y) \vee (x \wedge b) \vee (a \wedge b) \vee b \\
&= (x \wedge y) \vee (a \wedge y) \vee (x \wedge b) \vee b.
\end{aligned}$$

Let $r = (x \wedge y) \vee (a \wedge y) \vee (x \wedge b)$. Then we get $x \vee a = r \vee a$ and $y \vee b = r \vee b$. Thus, for any $x \in X$, we have

$$\begin{aligned}
\overline{Apr_{(\mu \vee_T \nu)}}^T \omega(x) &= \bigvee_{x \vee z = y \vee z} (\mu \vee_T \nu)(z) T \omega(y) \\
&= \bigvee_{x \vee z = y \vee z} \bigvee_{z = a \vee b} \mu(a) T \nu(b) T \omega(y) \\
&\leq \bigvee_{x \vee a \vee b = y \vee a \vee b} \mu(a) T \nu(b) T \omega(y) \\
&\leq \bigvee_{x \vee a = r \vee a} \bigvee_{y \vee b = r \vee b} \mu(a) T \nu(b) T \omega(y) \\
&= \bigvee_{x \vee a = r \vee a} \mu(a) T \left(\bigvee_{y \vee b = r \vee b} \nu(b) T \omega(y) \right) \\
&= \bigvee_{x \vee a = r \vee a} \mu(a) T \left(\overline{Apr_\nu}^T \omega(r) \right) \\
&= \overline{Apr_\mu}^T \overline{Apr_\nu}^T \omega(x).
\end{aligned}$$

It follows that $\overline{Apr_{(\mu \vee_T \nu)}}^T \omega \subseteq \overline{Apr_\mu}^T \overline{Apr_\nu}^T \omega$, and hence $\overline{Apr_{(\mu \vee_T \nu)}}^T \omega = \overline{Apr_\mu}^T \overline{Apr_\nu}^T \omega$.

(2) It is similar to that of (1). □

Corollary 2 *Let X be a distributive lattice and $\mu, \nu, \omega \in L^X$. Then*

(1) $\overline{Apr_\mu}^T \overline{Apr_\nu}^T \omega = \overline{Apr_\nu}^T \overline{Apr_\mu}^T \omega$.
(2) $\underline{Apr_{\mu}}_{\vartheta} \underline{Apr_{\nu}}_{\vartheta} \omega = \underline{Apr_{\nu}}_{\vartheta} \underline{Apr_{\mu}}_{\vartheta} \omega$.

Proof Straightforward by Theorem 3.4. □

Theorem 3.5 *Let X be a lattice and $\mu, \nu, \omega \in L^X$. Then*

(1) *If μ is an TL-fuzzy $\vee$-semilattice on X, then $\overline{Apr_\mu}^T \nu \vee_T \overline{Apr_\mu}^T \omega \subseteq \overline{Apr_\mu}^T (\nu \vee_T \omega)$.*
(2) *If X is a distributive lattice, then $\overline{Apr_\mu}^T \nu \vee_T \overline{Apr_\mu}^T \omega \supseteq \overline{Apr_\mu}^T (\nu \vee_T \omega)$;*
$\underline{Apr_{\mu}}_{\vartheta} \nu \vee_T \underline{Apr_{\mu}}_{\vartheta} \omega \subseteq \underline{Apr_{\mu}}_{\vartheta} (\nu \vee_T \omega)$.
(3) *If X is a distributive lattice and μ is an TL-fuzzy $\vee$-semilattice on X, then $\overline{Apr_\mu}^T \nu \wedge_T \overline{Apr_\mu}^T \omega \subseteq \overline{Apr_\mu}^T (\nu \wedge_T \omega)$.*
(4) *If X is a distributive lattice and μ is an L-fuzzy upper set on X, then $\overline{Apr_\mu}^T \nu \wedge_T \overline{Apr_\mu}^T \omega \supseteq \overline{Apr_\mu}^T (\nu \wedge_T \omega)$;*
$\underline{Apr_{\mu}}_{\vartheta} \nu \wedge_T \underline{Apr_{\mu}}_{\vartheta} \omega \subseteq \underline{Apr_{\mu}}_{\vartheta} (\nu \wedge_T \omega)$.

Proof We only prove (1) and (4). The others can be proved similarly.

(1) Let μ be an TL-fuzzy $\vee$-semilattice on X and $x \in X$. Then, we have

$$\begin{aligned}
&(\overline{Apr_\mu}^T \nu) \vee_T (\overline{Apr_\mu}^T \omega)(x) \\
&= \bigvee_{x = y \vee z} \overline{Apr_\mu}^T \nu(y) T \overline{Apr_\mu}^T \omega(z) \\
&= \bigvee_{x = y \vee z} \left(\bigvee_{y \vee a = b \vee a} \mu(a) T \nu(b) \right) T \left(\bigvee_{z \vee c = d \vee c} \mu(c) T \omega(d) \right) \\
&= \bigvee_{x = y \vee z} \bigvee_{y \vee a = b \vee a} \bigvee_{z \vee c = d \vee c} \mu(a) T \nu(b) T \mu(c) T \omega(d) \\
&\leq \bigvee_{x \vee a \vee c = b \vee d \vee a \vee c} \mu(a) T \mu(c) T \nu(b) T \omega(d) \\
&\leq \bigvee_{x \vee a \vee c = b \vee d \vee a \vee c} \mu(a \vee c) T (\nu \vee_T \omega)(b \vee d) \\
&= \overline{Apr_\mu}^T (\nu \vee_T \omega)(x).
\end{aligned}$$

This implies $\overline{Apr_\mu}^T \nu \vee_T \overline{Apr_\mu}^T \omega \subseteq \overline{Apr_\mu}^T (\nu \vee_T \omega)$.

(4) If X is a distributive lattice, then we have the assertion that, for $x, y, z, a, b \in X, x \vee z = y \vee z$ and $y = a \wedge b$ imply that $x = r \wedge t, r \vee p = a \vee p, t \vee q = b \vee q, z \leq p$ and $z \leq q$

for some $r, t, p, q \in X$. In fact, if $x \vee z = y \vee z$ and $y = a \wedge b$, then $x \vee z = (a \wedge b) \vee z = (a \vee z) \wedge (b \vee z)$. By the absorption law, we have $x = x \wedge (x \vee z) = [x \wedge (a \vee z)] \wedge [x \wedge (b \vee z)]$. Let $r = x \wedge (a \vee z)$, $t = x \wedge (b \vee z)$, $p = a \vee z$ and $q = b \vee z$. Then we get $r \vee p = a \vee p$, $t \vee q = b \vee q$, $z \leq p$ and $z \leq q$. Moreover, since μ is an L-fuzzy upper set on X, it follows from $z \leq p$ and $z \leq p$ that $\mu(z) \leq \mu(p) T \mu(q)$. Thus, for any $x \in X$,

$$\begin{aligned}
&\overline{Apr_\mu}^T (\nu \wedge_T \omega)(x) \\
&\quad = \bigvee_{x \vee z = y \vee z} \mu(z) T (\nu \wedge_T \omega)(y) \\
&\quad = \bigvee_{x \vee z = y \vee z} \mu(z) T \left(\bigvee_{y = a \wedge b} \nu(a) T \omega(b) \right) \\
&\quad = \bigvee_{x \vee z = y \vee z} \bigvee_{y = a \wedge b} \mu(z) T \nu(a) T \omega(b) \\
&\quad \leq \bigvee_{x = r \wedge t} \bigvee_{r \vee p = a \vee p} \bigvee_{t \vee q = b \vee q} \mu(p) T \nu(a) T \mu(q) T \omega(b) \\
&\quad = \bigvee_{x = r \wedge t} T \left(\bigvee_{r \vee p = a \vee p} \mu(p) T \nu(a) \right) T \\
&\qquad \times \left(\bigvee_{t \vee q = b \vee q} \mu(q) T \omega(b) \right) \\
&\quad = \bigvee_{x = r \wedge t} \left(\overline{Apr_\mu}^T \nu(r) \right) T \left(\overline{Apr_\mu}^T \omega(t) \right) \\
&\quad = (\overline{Apr_\mu}^T \nu \wedge_T \overline{Apr_\mu}^T \omega)(x).
\end{aligned}$$

That is, $\overline{Apr_\mu}^T \nu \wedge_T \overline{Apr_\mu}^T \omega \supseteq \overline{Apr_\mu}^T (\nu \wedge_T \omega)$. Next, we show $\underline{Apr_\mu}_\vartheta \nu \wedge_T \underline{Apr_\mu}_\vartheta \omega \subseteq \underline{Apr_\mu}_\vartheta (\nu \wedge_T \omega)$. Since X is a distributive lattice, we have the assertion that, for any $x, y, a, b \in X$, $(x \wedge y) \vee a = b \vee a$ implies $b = r \wedge t$, $r \vee (x \vee a) = x \vee a$ and $t \vee (y \vee a) = y \vee a$ for some $r, t \in X$. In fact, if $(x \wedge y) \vee a = b \vee a$, then $b = b \wedge (b \vee a) = b \wedge [(x \wedge y) \vee a] = [b \wedge (x \vee a)] \wedge [b \wedge (y \vee a)]$. Let $r = b \wedge (x \vee a)$ and $t = b \wedge (y \vee a)$. Then $r \vee (x \vee a) = x \vee a$ and $t \vee (y \vee a) = y \vee a$ hold. Meanwhile, since μ is an L-fuzzy upper set on X, it is clear that $\mu(a) \leq \mu(x \vee a) T \mu(y \vee a)$. Thus, for any $x, y \in X$, we have

$$\begin{aligned}
&\underline{Apr_\mu}_\vartheta (\nu \wedge_T \omega)(x \wedge y) \\
&\quad = \bigwedge_{(x \wedge y) \vee a = b \vee a} \vartheta(\mu(a), (\nu \wedge_T \omega)(b)) \\
&\quad = \bigwedge_{(x \wedge y) \vee a = b \vee a} \vartheta \left(\mu(a), \bigvee_{b = c \wedge d} \nu(c) T \omega(d) \right) \\
&\quad \geq \bigwedge_{r \vee (x \vee a) = x \vee a} \bigwedge_{t \vee (y \vee a) = y \vee a} \vartheta \left(\mu(a), \bigvee_{b = c \wedge d} \nu(c) T \omega(d) \right) \\
&\quad \geq \bigwedge_{r \vee (x \vee a) = x \vee a} \bigwedge_{t \vee (y \vee a) = y \vee a} \vartheta\, (\mu(x \vee a) T \mu(y \vee a), \nu(r) T \omega(t)) \\
&\quad \geq \bigwedge_{r \vee (x \vee a) = x \vee a} \bigwedge_{t \vee (y \vee a) = y \vee a} \vartheta(\mu(x \vee a), \nu(r)) T \vartheta(\mu(y \vee a), \omega(t)) \\
&\quad \geq \left(\bigwedge_{r \vee (x \vee a) = x \vee (x \vee a)} \vartheta(\mu(x \vee a), \nu(r)) \right) T \\
&\qquad \times \left(\bigwedge_{t \vee (y \vee a) = y \vee (y \vee a)} \vartheta(\mu(y \vee a), \omega(t)) \right) \\
&\quad = \left(\underline{Apr_\mu}_\vartheta \nu(x) \right) T \left(\underline{Apr_\mu}_\vartheta \omega(y) \right).
\end{aligned}$$

Since x, y, z are arbitrary elements of X, it follows from Proposition 1 that $\underline{Apr_\mu}_\vartheta \nu \wedge_T \underline{Apr_\mu}_\vartheta \omega \subseteq \underline{Apr_\mu}_\vartheta (\nu \wedge_T \omega)$. □

Lemma 2 *Let X be a lattice and $\mu, \nu, \omega \in L^X$. If $\mu \subseteq \nu$, then $\overline{Apr_\mu}^T \omega \subseteq \overline{Apr_\nu}^T \omega$ and $\underline{Apr_\mu}_\vartheta \omega \supseteq \underline{Apr_\nu}_\vartheta \omega$. Specially, if X has the bottom $\perp$, then $\overline{Apr_{\kappa_{\{\perp\}}}}^T \mu = \mu = \underline{Apr_{\kappa_{\{\perp\}}}}_\vartheta \mu$.*

Proof Straightforward by Definition 3. □

The following theorem comes directly from Lemma 2.

Theorem 3.6 *Let X be a lattice and $\mu, \mu_i, \omega \in L^X$, $i \in I$, where I is an index set. Then*

(1) $\overline{Apr_{\bigcap_{i \in I} \mu_i}}^T \mu \subseteq \bigcap_{i \in I} \overline{Apr_{\mu_i}}^T \mu$ *and* $\underline{Apr_{\bigcap_{i \in I} \mu_i}}_\vartheta \mu \supseteq \bigcup_{i \in I} \underline{Apr_{\mu_i}}_\vartheta \mu$.
(2) $\overline{Apr_{\bigcup_{i \in I} \mu_i}}^T \mu = \bigcup_{i \in I} \overline{Apr_{\mu_i}}^T \mu$ *and* $\underline{Apr_{(\bigcup_{i \in I} \mu_i)}}_\vartheta \mu = \bigcap_{i \in I} \underline{Apr_{\mu_i}}_\vartheta \mu$.

Next, let X and Y be two lattices, define $\vee$ and $\wedge$ coordinatewise on $X \times Y$ as follows: for $(x, y), (a, b) \in X \times Y$,

$$\begin{aligned}
(x, y) \vee (a, b) = (x \vee a, y \vee b), (x, y) \wedge (a, b) \\
= (x \wedge a, y \wedge b).
\end{aligned}$$

Then $(X \times Y, \vee, \wedge)$ is a lattice, called the *product lattice* of X and Y. It is routine to verify that if $X \times Y$ is a product lattice and $(x, y), (a, b) \in X \times Y$, then $(x, y) \leq (a, b)$ if and only if $x \leq a$ and $y \leq b$. For the L-fuzzy approximation operators on a product lattice, we have the following theorem.

Theorem 3.7 *Let X and Y be lattices, $\mu_1, \nu_1 \in L^X$, $\mu_2, \nu_2 \in L^Y$, and μ_1 and μ_2 be TL-fuzzy ideals of X and Y, respectively. Then*

(1) $\overline{Apr_{\mu_1 \times \mu_2}}^T (\nu_1 \times \nu_2) = \overline{Apr_{\mu_1}}^T \nu_1 \times \overline{Apr_{\mu_2}}^T \nu_2$.
(2) $\underline{Apr_{\mu_1 \times \mu_2}}_\vartheta (\nu_1 \times \nu_2) \supseteq \underline{Apr_{\mu_1}}_\vartheta \nu_1 \times \underline{Apr_{\mu_2}}_\vartheta \nu_2$.

Proof It is similar to that of Theorem 3.16 in Li and Yin (2007). □

As we have seen in the previous discussion, if μ is an TL-fuzzy ideal, then the TL-fuzzy approximation operators induced by μ, i.e. $\underline{Apr_{\mu}}_{\vartheta}$ and $\overline{Apr_{\mu}}^{T}$, on a distributive lattice have many nice properties. Then, a natural question raises: How can we restrict the TL-fuzzy upper and lower approximation operators induced by an L-fuzzy subset so that this L-fuzzy subset is an TL-fuzzy ideal?

In what follows, we will give an answer to this question. In fact, we will obtain some methods to characterize TL-fuzzy ideals of a distributive lattice in terms of TL-fuzzy approximation operators.

Theorem 3.8 *Let X be a distributive lattice with bottom $\perp$ and μ a nonvoid L-fuzzy set on X. Then μ is an TL-fuzzy ideal on X if and only if the following conditions hold:*

(1) $\overline{Apr_{\mu}}^{T}\nu = \mu \vee_T \nu$ *for every L-fuzzy lower set ν on X.*
(2) $\overline{Apr_{\mu}}^{T}\,\overline{Apr_{\mu}}^{T}\nu = \overline{Apr_{\mu}}^{T}\nu$ *for all $\nu \in L^X$.*

Proof ($\Longrightarrow$) Let μ be an TL-fuzzy ideal on X. For every L-fuzzy lower set ν on X and $x \in X$, we have

$$\overline{Apr_{\mu}}^{T}\nu(x) = \bigvee_{x\vee z=y\vee z} \mu(z)T\nu(y) \geq \bigvee_{x=y\vee z} \mu(z)T\nu(y) = (\mu \vee_T \nu)(x),$$

which means that $\overline{Apr_{\mu}}^{T}\nu \supseteq \mu \vee_T \nu$. On the other hand, since X is a distributive lattice, and both μ and ν are L-fuzzy lower sets of X, we have

$$\begin{aligned}\overline{Apr_{\mu}}^{T}\nu(x) &= \bigvee_{x\vee z=y\vee z} \mu(z)T\nu(y)\\ &\leq \bigvee_{x\wedge(x\vee z)=x\wedge(y\vee z)} \mu(z)T\nu(y)\\ &= \bigvee_{x=(x\wedge y)\vee(x\wedge z))} \mu(z)T\nu(y)\\ &\leq \bigvee_{x=(x\wedge y)\vee(x\wedge z)} \mu(x\wedge z)T\nu(x\wedge y)\\ &= (\mu \vee_T \nu)(x)\end{aligned}$$

i.e. $\overline{Apr}_{\mu}^{T}\nu \subseteq \mu \vee_T \nu$, and hence $\overline{Apr}_{\mu}^{T}\nu = \mu \vee_T \nu$. Now, we have proved that (1) holds. By Theorem 3.2, it is straightforward that (2) holds.

($\Longleftarrow$) Assume that (1), (2) hold and let μ be an L-fuzzy subset on X. Clearly, $\chi_{\{\perp\}}$ is an L-fuzzy lower set on X ($\perp$ is the bottom of X). Meanwhile, for any $x \in X$, we have

$$\begin{aligned}\overline{Apr_{\mu}}^{T}\chi_{\{\perp\}}(x) &= \bigvee_{x\vee z=y\vee z} \mu(z)T\chi_{\{\perp\}}(y) = \bigvee_{x\vee z=\perp\vee z} \mu(z)\\ &= \bigvee_{x\leq z} \mu(z) = \downarrow \mu(x)\end{aligned}$$

and

$$\begin{aligned}(\mu \vee_T \chi_{\{\perp\}})(x) &= \bigvee_{x=a\vee b} \mu(a)T\chi_{\{\perp\}}(b) = \bigvee_{x=a\vee\perp} \mu(a)\\ &= \bigvee_{x=a} \mu(a) = \mu(x).\end{aligned}$$

Thus, (1) implies that $\downarrow \mu = \mu$, i.e. μ is an L-fuzzy lower set on X. Next, we prove that μ is an TL-fuzzy $\vee$-semilattice on X. For $x \in X$, we have

$$\begin{aligned}\overline{Apr_{\mu}}^{T}\,\overline{Apr_{\mu}}^{T}\chi_{\{\perp\}}(x) &= \bigvee_{x\vee z=y\vee z} \mu(z)T\overline{Apr_{\mu}}^{T}\chi_{\{\perp\}}(y)\\ &= \bigvee_{x\vee z=y\vee z} \mu(z)T \bigvee_{y\vee a=b\vee a} \mu(a)T\chi_{\{\perp\}}(b)\\ &= \bigvee_{x\vee z=y\vee z} \mu(z)T \bigvee_{y\vee a=\perp\vee a} \mu(a)\\ &= \bigvee_{x\vee z=y\vee z} \mu(z)T\downarrow\mu(y)\\ &\geq \bigvee_{x=y\vee z} \mu(z)T\mu(y) = (\mu \vee_T \mu)(x).\end{aligned}$$

Similarly, we have $\overline{Apr_{\mu}}^{T}\chi_{\{\perp\}}(x) = \mu(x)$. Thus, (2) implies that $\mu \vee_T \mu \subseteq \mu$. So, μ is an TL-fuzzy ideal on X. □

Theorem 3.9 *Let X be a distributive lattice with bottom $\perp$ and μ a nonvoid L-fuzzy set on X. Then μ is an TL-fuzzy ideal on X if and only if the following conditions hold:*

(1) $\underline{Apr_{\mu}}_{\vartheta}\nu = \mu \vee_{\vartheta} \nu$ for every L-fuzzy upper set ν on X.
(2) $\underline{Apr_{\mu}}_{\vartheta}(\underline{Apr_{\mu}}_{\vartheta}\nu) = \underline{Apr_{\mu}}_{\vartheta}\nu$ for all $\nu \in L^X$.

Proof ($\Longrightarrow$) Let μ be an TL-fuzzy ideal on X. Then, for any L-fuzzy upper set ν on X and $x \in X$, we have

$$\begin{aligned}\underline{Apr_{\mu}}_{\vartheta}\nu(x) &= \bigwedge_{x\vee z=y\vee z} \vartheta(\mu(z), \nu(y))\\ &\leq \bigwedge_{x=y\vee z} \vartheta(\mu(z), \nu(y))\\ &= (\mu \vee_{\vartheta} \nu)(x),\end{aligned}$$

i.e. $\underline{Apr_{\mu}}_{\vartheta}\nu \subseteq \mu \vee_{\vartheta} \nu$. On the other hand, by the hypothesis, X is a distributive lattice, μ and ν are an L-fuzzy lower and an L-fuzzy upper set on X, respectively. Thus, we have

$$\begin{aligned}\underline{Apr_{\mu_\vartheta}}\nu(x) &= \bigwedge_{x\vee z=y\vee z} \vartheta(\mu(z),\nu(y))\\ &\geq \bigwedge_{x\wedge(x\vee z)=x\wedge(y\vee z)} \vartheta(\mu(z),\nu(y))\\ &= \bigwedge_{x=(x\wedge y)\vee(x\wedge z))} \vartheta(\mu(z),\nu(y))\\ &\geq \bigwedge_{x=(x\wedge y)\vee(x\wedge z)} \vartheta(\mu(x\wedge z),\nu(x\wedge y))\\ &= (\mu\vee_\vartheta \nu)(x).\end{aligned}$$

This implies $\underline{Apr_{\mu_\vartheta}}\nu \supseteq \mu\vee_\vartheta \nu$. Therefore, (1) holds. It follows from Theorem 3.2 that (2) holds.

($\Longleftarrow$) Assume that (1), (2) hold and let μ be an L-fuzzy subset on X. Clearly, for any $\alpha\in L$, $\alpha^*_{\{\perp\}}$ is an L-fuzzy upper set on X. Meanwhile, for any $x\in X$, we have

$$\begin{aligned}\underline{Apr_{\mu_\vartheta}}\alpha^*_{\{\perp\}}(x) &= \bigwedge_{x\vee z=y\vee z} \vartheta(\mu(z),\alpha^*_{\{\perp\}}(y))\\ &= \bigwedge_{x\vee z=\perp\vee z} \vartheta(\mu(z),\alpha) = \bigwedge_{x\vee z=z} \vartheta(\mu(z),\alpha)\\ &= \vartheta\left(\bigvee_{x\vee z=z} \mu(z),\alpha\right) = \vartheta(\downarrow\mu(x),\alpha)\end{aligned}$$

and

$$\begin{aligned}(\mu\vee_\vartheta \alpha^*_{\{\perp\}})(x) &= \bigwedge_{x=y\vee z} \vartheta(\mu(y),\alpha^*_{\{\perp\}})\\ &= \bigwedge_{x=y\vee\perp} \vartheta(\mu(y),\alpha) = \vartheta(\mu(x),\alpha).\end{aligned}$$

Then it follows from (1) and (R4) that $\downarrow\mu=\mu$, i.e. μ is an L-fuzzy lower set on X. Next, we prove that μ is an TL-fuzzy $\vee$-semilattice on X. Let $x\in X$. Then

$$\begin{aligned}&\underline{Apr_\mu}\,\underline{Apr_\mu}\alpha^*_{\{\perp\}}(x)\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\left(\mu(z),\underline{Apr_\mu}\alpha^*_{\{\perp\}}(y)\right)\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\left(\mu(z),\bigwedge_{y\vee a=b\vee a} \vartheta(\mu(a),\alpha^*_{\{\perp\}}(b))\right)\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\left(\mu(z),\bigwedge_{y\vee a=\perp\vee a} \vartheta(\mu(a),\alpha)\right)\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\left(\mu(z),\vartheta\left(\bigvee_{y\leq a} \mu(a),\alpha\right)\right)\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\,(\mu(z),\vartheta\,(\downarrow\mu(y),\alpha))\\ &= \bigwedge_{x\vee z=y\vee z} \vartheta\,(\mu(z)T\mu(y),\alpha)\\ &= \bigwedge_{x=y\vee z} \vartheta\,(\mu(z)T\mu(y),\alpha)\\ &= \vartheta\left(\bigvee_{x=y\vee z} \mu(z)T\mu(y),\alpha\right)\\ &= \vartheta((\mu\vee_T\mu)(x),\alpha).\end{aligned}$$

Similarly, we have $\underline{Apr_\mu}\alpha^*_{\{\perp\}}(x)=\vartheta(\mu(x),\alpha)$. Thus, (2) and (R4) derive that $\mu\vee_T\mu\subseteq\mu$. Therefore μ is an TL-fuzzy ideal on X. □

Remark 3 The item (1) in Theorem 3.8 (resp. Theorem 3.9) provides the computation of $\overline{Apr_\mu}^T\nu$ (resp., $\underline{Apr_{\mu_\vartheta}}\nu$) when μ is an TL-fuzzy ideal and ν is an L-fuzzy upper (resp., L-fuzzy lower) set on X.

4 TL-fuzzy (quasi-)rough ideals on a lattice

Pawlak (1982) introduced the concepts of rough membership between a point and a subset and rough inclusion relation between two sets. In this section, we extend these concepts to L-fuzzy environment, and further, use them to investigate TL-fuzzy rough ideals and TL-fuzzy quasi-rough ideals on lattices, respectively.

Let $x\in X$, $\alpha\in L-\{0\}$ and $\mu,\nu,\omega\in L^X$. We say that:

(i) x_α strongly belongs to ν with respect to μ, denoted by $x_\alpha\underline{\in}_\mu\nu$, if $x_\alpha\in\underline{Apr_{\mu_\vartheta}}\nu$;
(ii) x_α weakly belongs to ν with respect to μ, denoted by $x_\alpha\overline{\in}_\mu\nu$, if $x_\alpha\in\overline{Apr_\mu}^T\nu$;
(iii) ν is roughly lower-included in ω with respect to μ, denoted by $\nu\underline{\sqsubseteq}_\mu\omega$, if $\underline{Apr_{\mu_\vartheta}}\nu\subseteq\underline{Apr_{\mu_\vartheta}}\omega$;
(iv) ν is roughly upper-included in ω with respect to μ, denoted by $\nu\overline{\sqsubseteq}_\mu\omega$, if $\overline{Apr_\mu}^T\nu\subseteq\overline{Apr_\mu}^T\omega$.

It is easy to check that, for any $\mu,\nu,\omega\in L^X$,

(1) $\nu\underline{\sqsubseteq}_\mu\omega$ if and only if, for any $x\in X$ and $\alpha\in L-\{0\}$, $x_\alpha\underline{\in}_\mu\nu\Rightarrow x_\alpha\underline{\in}_\mu\omega$.
(2) $\nu\overline{\sqsubseteq}_\mu\omega$ if and only if, for any $x\in X$ and $\alpha\in L-\{0\}$, $x_\alpha\overline{\in}_\mu\nu\Rightarrow x_\alpha\overline{\in}_\mu\omega$.

We next introduce the notion of TL-fuzzy rough ideals on a lattice, which can be viewed as a fuzzy generalization of rough ideals on lattices proposed by Xiao et al. (2012).

Definition 4 Let X be a lattice and $\mu,\nu\in L^X$. Then ν is called an TL-fuzzy lower (reps. upper) rough ideal with respect to μ on X if $\underline{Apr_{\mu_\vartheta}}\nu$ (resp. $\overline{Apr_\mu}^T\nu$) is an TL-fuzzy ideal on X. Further, ν is called an TL-fuzzy rough ideal with respect to μ on X if both $\underline{Apr_{\mu_\vartheta}}\nu$ and $\overline{Apr_\mu}^T\nu$ are TL-fuzzy ideals on X.

In the sequel, we denote by $\mathbf{TLFLRI}_\mu(X)$ the set of all TL-fuzzy lower rough ideal with respect to μ on X and by $\mathbf{TLFURI}_\mu(X)$ the set of all TL-fuzzy upper rough ideal with respect to μ on X.

Theorem 4.1 *Let X be a lattice and $\mu, \nu \in L^X$. Then ν is an TL-fuzzy lower rough ideal with respect to μ on X if and only if the following properties hold:*

(1) *For any $x, y \in X$ and $\alpha, \beta \in L - \{0\}$, if $x_\alpha \underline{\in}_\mu \nu$ and $y_\beta \underline{\in}_\mu \nu$, then $(x \vee y)_{T(\alpha,\beta)} \underline{\in}_\mu \nu$.*
(2) *For any $x, y \in X$ and $\alpha \in L - \{0\}$, if $y_\alpha \underline{\in}_\mu \nu$, then $(x \wedge y)_\alpha \underline{\in}_\mu \nu$.*

Proof ($\Longleftarrow$) Suppose that ν is an TL-fuzzy lower rough ideal with respect to μ on X. For $x, y \in X$ and $\alpha, \beta \in L - \{0\}$, if $x_\alpha \underline{\in}_\mu \nu$ and $y_\beta \underline{\in}_\mu \nu$, then $x_\alpha \in \underline{Apr}_{\mu_\vartheta} \nu$ and $y_\beta \in \underline{Apr}_{\mu_\vartheta} \nu$, which imply that $(\underline{Apr}_{\mu_\vartheta} \nu)(x) \geq \alpha$ and $(\underline{Apr}_{\mu_\vartheta} \nu)(y) \geq \beta$. Since $\underline{Apr}_{\mu_\vartheta} \nu$ is an TL-fuzzy ideal on X, by Proposition 3, we have $\underline{Apr}_{\mu_\vartheta} \nu \vee_T \underline{Apr}_{\mu_\vartheta} \nu \subseteq \underline{Apr}_{\mu_\vartheta} \nu$. Then, by Proposition 1, $(\underline{Apr}_{\mu_\vartheta} \nu)(x \vee y) \geq (\underline{Apr}_{\mu_\vartheta} \nu)(x) T (\underline{Apr}_{\mu_\vartheta} \nu)(y)) \geq \alpha T \beta$, which implies $(x \vee y)_{\alpha T \beta} \underline{\in}_\mu \nu$, i.e. (1) holds. In a similar way, we can prove that (2) holds.

($\Longleftarrow$) Suppose that (1) and (2) hold. For any $x, y \in X$, set $\alpha = \underline{Apr}_{\mu_\vartheta} \nu(x)$ and $\beta = \underline{Apr}_{\mu_\vartheta} \nu(y)$. We consider the following two cases.

Case 1: $\alpha = 0$ or $\beta = 0$. In this case, it is clear that $(\underline{Apr}_{\mu_\vartheta} \nu)(x) T (\underline{Apr}_{\mu_\vartheta} \nu)(y) = 0 \leq (\underline{Apr}_{\mu_\vartheta} \nu)(x \vee y)$.

Case 2: $\alpha \neq 0 \neq \beta$. In this case, we have $x_\alpha \in \underline{Apr}_{\mu_\vartheta} \nu$ and $y_\beta \in \underline{Apr}_{\mu_\vartheta} \nu$, that is, $x_\alpha \underline{\in}_\mu \nu$ and $y_\beta \underline{\in}_\mu \nu$. Consequently, by (1), we have $(x \vee y)_{T(\alpha,\beta)} \underline{\in}_\mu \nu$. Hence, $(\underline{Apr}_{\mu_\vartheta} \nu)(x) T (\underline{Apr}_{\mu_\vartheta} \nu)(y) = \alpha T \beta \leq (\underline{Apr}_{\mu_\vartheta} \nu)(x \vee y)$.

As a consequence, for any $x, y \in X$, it always holds that $(\underline{Apr}_{\mu_\vartheta} \nu)(x) T (\underline{Apr}_{\mu_\vartheta} \nu)(y) \leq (\underline{Apr}_{\mu_\vartheta} \nu)(x \vee y)$. By Proposition 1, we have $\underline{Apr}_{\mu_\vartheta} \nu \vee_T \underline{Apr}_{\mu_\vartheta} \nu \subseteq \underline{Apr}_{\mu_\vartheta} \nu$. Similarly, we can prove that $\chi_X \wedge_T \underline{Apr}_{\mu_\vartheta} \nu \subseteq \underline{Apr}_{\mu_\vartheta} \nu$. So, it follows from Proposition 3 that $\underline{Apr}_{\mu_\vartheta} \nu$ is an TL-fuzzy ideal on X, i.e. ν is an TL-fuzzy lower rough ideal on X. □

Theorem 4.2 *Let X be a lattice and $\mu, \nu \in L^X$. Then ν is an TL-fuzzy upper rough ideal with respect to μ on X if and only if the following properties hold:*

(1) *For any $x, y \in X$ and $\alpha, \beta \in L - \{0\}$, if $x_\alpha \overline{\in}_\mu \nu$ and $y_\beta \overline{\in}_\mu \nu$, then $(x \vee y)_{T(\alpha,\beta)} \overline{\in}_\mu \nu$.*
(2) *For any $x, y \in X$ and $\alpha \in L - \{0\}$, if $y_\alpha \overline{\in}_\mu \nu$, then $(x \wedge y)_\alpha \overline{\in}_\mu \nu$.*

Proof It is similar to that of Theorem 4.1. □

In what follows, inspired by Proposition 3, we develop the concept of TL-fuzzy quasi-rough ideals on a lattice.

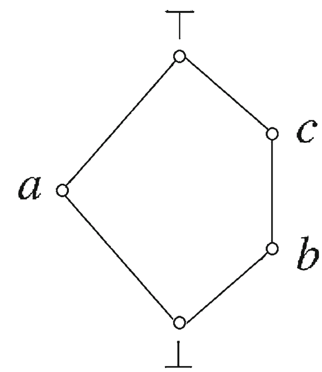

Fig. 1 The lattice in Example 2

Definition 5 Let X be a lattice and $\mu, \nu \in L^X$. Then ν is called a TL-fuzzy lower (resp. upper) quasi-rough ideal with respect to μ on X if the following properties hold:

(QRI1) $\nu \vee_T \nu \underline{\sqsubseteq}_\mu \nu$ (resp. $\nu \vee_T \nu \overline{\sqsubseteq}_\mu \nu$);
(QRI2) $\chi_X \wedge_T \nu \underline{\sqsubseteq}_\mu \nu$ (resp. $\chi_X \wedge_T \nu \overline{\sqsubseteq}_\mu \nu$).

Further, ν is called an TL-fuzzy quasi-rough ideal with respect to μ on X if it is both an TL-fuzzy lower and TL-fuzzy upper quasi-rough ideal with respect to μ on X.

We denote by $\mathbf{TLFLQRI}_\mu(X)$ the set of all TL-fuzzy lower quasi-rough ideals with respect to μ on X and by $\mathbf{TLFUQRI}_\mu(X)$ the set of all TL-fuzzy upper quasi-rough ideals with respect to μ on X.

Next, we discuss the relationships among TL-fuzzy ideals, TL-fuzzy rough ideals and TL-fuzzy quasi-rough ideals on a lattice.

Theorem 4.3 *Let X be a lattice and μ a nonvoid L-fuzzy set on X. If ν is an TL-fuzzy ideal on X, then ν is an TL-fuzzy quasi-rough ideal with respect to μ on X.*

Proof Straightforward by Proposition 3 and Theorem 3.1. □

However, the converse of Theorem 4.3 is not true in general, as shown in the following example.

Example 2 Let $X = \{\bot, a, b, c, \top\}$ be a lattice as shown in Fig. 1, $L = [0, 1]$ and $T = \wedge$. Define μ and ν be L-fuzzy sets on X such that $\mu = \frac{1}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{\top}$ and $\nu = \frac{1}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{0.5}{\top}$. Then $\nu \vee_T \nu = \frac{1}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{\top}$ and $\chi_X \wedge_T \mu = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0.5}{\top}$. It follows that

(1) $\overline{Apr_\mu}^T (\nu \vee_T \nu) = \overline{Apr_\mu}^T (\chi_X \wedge_T \nu) = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{\top} = \overline{Apr_\mu}^T \nu$.
(2) $\underline{Apr}_{\mu_\vartheta} (\nu \vee_T \nu) = \underline{Apr}_{\mu_\vartheta} (\chi_X \wedge_T \nu) = \frac{1}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{\top} = \underline{Apr}_{\mu_\vartheta} \nu$.

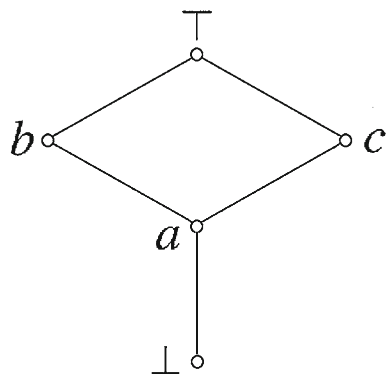

Fig. 2 The lattice in Example 3

Consequently, ν is an TL-fuzzy quasi-rough ideal with respect to μ on X. But ν is not an TL-fuzzy ideal on X, since $\nu(a \vee c) = \nu(\top) = 0.5 < 1 = \nu(a) T \nu(c)$.

Theorem 4.4 *Let X be a distributive lattice and $\mu \in L^X$. Then every TL-fuzzy lower quasi-rough ideal with respect to μ is an TL-fuzzy lower rough ideal with respect to μ on X.*

Proof Let ν be an TL-fuzzy lower quasi-rough ideal with respect to μ on X. Since X is a distributive lattice, by Theorem 3.5(2) and Definition 5, we have

$$\underline{Apr_{\mu_\vartheta}}\nu \vee_T \underline{Apr_{\mu_\vartheta}}\nu \subseteq \underline{Apr_{\mu_\vartheta}}(\nu \vee_T \nu) \subseteq \underline{Apr_{\mu_\vartheta}}\nu.$$

Meanwhile, it follows from Lemma 1 and Theorem 3.3 that

$$\chi_X \wedge_T \underline{Apr_{\mu_\vartheta}}\nu = \downarrow \underline{Apr_{\mu_\vartheta}}\nu \subseteq \underline{Apr_{\mu_\vartheta}} \downarrow \nu$$
$$= \underline{Apr_{\mu_\vartheta}}(\chi_X \wedge_T \nu) \subseteq \underline{Apr_{\mu_\vartheta}}\nu.$$

Thus, Proposition 3 implies that $\underline{Apr_{\mu_\vartheta}}\nu$ is an TL-fuzzy ideal on X, that is, ν is an TL-fuzzy lower rough ideal with respect to μ on X. □

Theorem 4.5 *Let X be a distributive lattice and μ an TL-fuzzy ideal on X. Then every TL-fuzzy upper quasi-rough ideal with respect to μ is an TL-fuzzy upper rough ideal with respect to μ on X.*

Proof It is similar to that of Theorem 4.4. □

The following example shows that the converse of Theorems 4.4 and 4.5 does not hold in general.

Example 3 Let $X = \{\bot, a, b, c, \top\}$ be a distributive lattice as shown in Fig. 2, $L = [0, 1]$ and $T = \wedge$. Define μ, ν_1 and ν_2 be L-fuzzy subsets on X such that $\mu = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{0.5}{c} + \frac{0.5}{\top}$, $\nu_1 = \frac{1}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{0.5}{\top}$ and $\nu_2 = \frac{0.5}{\bot} + \frac{1}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{1}{\top}$. Then

(1) $\underline{Apr_{\mu_\vartheta}}\nu_1 = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{0.5}{c} + \frac{0.5}{\top}$ is an TL-fuzzy ideal on X. So, ν_1 is an TL-fuzzy lower rough ideal with respect to μ on X. But ν_1 is not an TL-fuzzy lower quasi-rough ideal with respect to μ on X, since $\underline{Apr_{\mu_\vartheta}}(\nu_1 \vee_T \nu_1) = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{\top} \nsubseteq \underline{Apr_{\mu_\vartheta}}\nu_1$

(2) $\overline{Apr_\mu}^T \nu_2 = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{0.5}{c} + \frac{1}{\top}$ is an TL-fuzzy ideal on P. So, ν_2 is an TL-fuzzy upper rough ideal with respect to μ on X. But ν_2 is not an TL-fuzzy upper quasi-rough ideal with respect to μ on X, since $\overline{Apr_\mu}^T (\chi_X \wedge_T \nu_2) = \frac{1}{\bot} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{\top} \nsubseteq \overline{Apr_\mu}^T \nu_2$

Combining Theorems 4.3–4.5, we conclude that if X is a distributive lattice and $\mu \in \mathbf{TLFI}(X)$, then

$$\mathbf{TLFI}(X) \subsetneqq \mathbf{TLFLQRI}_\mu(X) \subsetneqq \mathbf{TLFLRI}_\mu(X)$$

and

$$\mathbf{TLFI}(X) \subsetneqq \mathbf{TLFUQRI}_\mu(X) \subsetneqq \mathbf{TLFURI}_\mu(X).$$

Finally, we analyse the properties of the ordered structures consisting of TL-fuzzy (quasi-)rough ideals on lattices.

Theorem 4.6 *Let X be a lattice and μ a nonvoid L-fuzzy set on X. Then $(\mathbf{TLFLRI}_\mu(X), \subseteq)$ is a* DCPO.

Proof Let $\{\nu_i : i \in I\} \subseteq \mathbf{TLFURI}_\mu(X)$ be directed. Then $\overline{Apr_\mu}^T \nu_i$ is an TL-fuzzy ideal on X for every $i \in I$ and, clearly, $\{\overline{Apr_\mu}^T \nu_i : i \in I\}$ is also a directed set. By Theorem 3.1 and Corollary 1, $\overline{Apr_\mu}^T (\bigcup_{i \in I} \nu_i) = \bigcup_{i \in I} \overline{Apr_\mu}^T \nu_i$ is an TL-fuzzy ideal on X. This implies that $\bigcup_{i \in I} \nu_i \in \mathbf{TLFURI}_\mu(X)$. Obviously, $\bigcup_{i \in I} \nu_i$ is the sup of $\{\nu_i : i \in I\} \subseteq \mathbf{TLFURI}_\mu(X)$. Thus, $\mathbf{TLFURI}_\mu(X)$ is a DCPO. □

Theorem 4.7 *Let X be a lattice and μ a nonvoid L-fuzzy set on X. Then $(\mathbf{TLFLRI}_\mu(X), \subseteq)$ is a complete lattice.*

Proof It is similar to that of Theorem 4.6. □

Lemma 3 *Let X be a distributive lattice and μ an TL-fuzzy $\vee$-semilattice on X. If $\nu, \omega \in \mathbf{TLFUQRI}_\mu(X)$, then $\nu \vee_T \omega \in \mathbf{TLFUQRI}_\mu(X)$.*

Proof Let ν and ω be two TL-fuzzy upper quasi-rough ideals with respect to μ on X. Then, we have $\nu \vee_T \nu \overline{\sqsubseteq}_\mu \nu$ and $\chi_X \wedge_T \nu \overline{\sqsubseteq}_\mu \nu$, or equivalently, $\overline{Apr_\mu}^T (\nu \vee_T \nu) \subseteq \overline{Apr_\mu}^T \nu$ and $\overline{Apr_\mu}^T (\chi_X \wedge_T \nu) \subseteq \overline{Apr_\mu}^T \nu$. Since X is a distributive lattice and μ is an TL-fuzzy $\vee$-semilattice on X, it follows from Theorem 3.5, Proposition 2 and Lemma 2 that

$$\overline{Apr_\mu}^T ((\nu \vee_T \omega) \vee_T (\nu \vee_T \omega))$$
$$= \overline{Apr_\mu}^T ((\nu \vee_T \nu) \vee_T (\omega \vee_T \omega))$$
$$= \overline{Apr_\mu}^T (\nu \vee_T \nu) \vee_T \overline{Apr_\mu}^T (\omega \vee_T \omega)$$
$$\subseteq \overline{Apr_\mu}^T \nu \vee_T \overline{Apr_\mu}^T \omega$$
$$= \overline{Apr_\mu}^T (\nu \vee_T \omega).$$

and, similarly, we have

$$\begin{aligned}
&\overline{Apr_\mu}^T(\chi_X \wedge_T (\nu \vee_T \omega))\\
&\quad\subseteq \overline{Apr_\mu}^T((\chi_X \wedge_T \nu) \vee_T (\chi_X \wedge_T \omega))\\
&\quad= \overline{Apr_\mu}^T(\chi_X \wedge_T \nu) \vee_T \overline{Apr_\mu}^T(\chi_X \wedge_T \omega)\\
&\quad\subseteq \overline{Apr_\mu}^T \nu \vee_T \overline{Apr_\mu}^T \omega\\
&\quad= \overline{Apr_\mu}^T(\nu \vee_T \omega).
\end{aligned}$$

Therefore, $\nu \vee_T \omega$ is an TL-fuzzy upper quasi-rough ideal with respect to μ on X, i.e. $\nu \vee_T \omega \in \mathbf{TLFUQRI}_\mu(X)$. □

Theorem 4.8 *Let X be a distributive lattice with the bottom $\perp$ and μ be an TL-fuzzy $\vee$-semilattice on X. Assume that $\mathbf{TLFUQRI}_\mu(X)_\perp$ denotes the set of all TL-fuzzy upper quasi-rough ideals with respect to μ on X satisfying the property that $\nu(\perp) = 1$ for all $\nu \in \mathbf{TLFUQRI}_\mu(X)_\perp$. Then $(\mathbf{TLFUQRI}_\mu(P)_\perp, \vee_T, \odot)$ is a lattice (ordered by $\subseteq$), where $\nu \odot \omega = (\vee_T)_{i\in I}\{\eta_i \in \mathbf{TLFUQRI}_\mu(P)_\perp : \eta_i \subseteq \nu, \eta_i \subseteq \omega\}$ for all $\nu, \omega \in \mathbf{TLFUQRI}_\mu(X)_\perp$.*

Proof Let $\nu, \omega \in \mathbf{TLFUQRI}_\mu(X)_\perp$. Then we have $\nu, \omega \in \mathbf{TLFUQRI}_\mu(X)$ and $\nu(\perp) = \omega(\perp) = 1$. By Lemma 3, $\nu \vee_T \omega \in \mathbf{TLFUQRI}_\mu(X)$, and it is not difficult to verify that $(\nu \vee_T \omega)(\perp) = 1$. So we get $\nu \vee_T \omega \in \mathbf{TLFUQRI}_\mu(X)_\perp$. Next, we prove that $\nu \vee_T \omega$ is the least upper bound of ν and ω. Since $\nu(\perp) = \omega(\perp) = 1$, we have $\nu \subseteq \nu \vee_T \omega$ and $\omega \subseteq \nu \vee_T \omega$. Let $\lambda \in \mathbf{TLFUQRI}_\mu(X)_\perp$ such that $\nu \subseteq \lambda$ and $\omega \subseteq \lambda$. Then by Proposition 2 and Lemma 3, we have $\nu \vee_T \omega \subseteq \lambda \vee_T \lambda \subseteq \lambda$. Hence $\nu \vee_T \omega$ is the least upper bound of ν and ω. Similarly, we can prove $\nu \odot \omega \in \mathbf{TLFUQRI}_\mu(X)_\perp$ and that it is the greatest lower bound of ν and ω. □

Theorem 4.9 *Let X be a distributive lattice and μ an TL-fuzzy $\vee$-semilattice on X. Then $(\mathbf{TLFLQRI}_\mu(X), \subseteq)$ is a complete lattice.*

Proof Let $\{\nu_i \mid i \in I\} \subseteq \mathbf{TLFLQRI}_\mu(X)$. Then, we have $\nu_i \vee_T \nu_i \sqsubseteq_\mu \nu_i$ and $\chi_X \wedge_T \nu_i \sqsubseteq_\mu \nu_i$ for every $i \in I$, which are equivalent to $\underline{Apr_{\mu}}_\vartheta(\nu_i \vee_T \nu_i) \subseteq \underline{Apr_{\mu}}_\vartheta \nu_i$ and $\underline{Apr_{\mu}}_\vartheta(\chi_X \wedge_T \nu_i) \subseteq \underline{Apr_{\mu}}_\vartheta \nu_i$ for every $i \in I$. Since μ is an TL-fuzzy $\vee$-semilattice on X, by Proposition 2 and Theorem 3.1, we have

$$\begin{aligned}
&\underline{Apr_{\mu}}_\vartheta\left(\left(\bigcap_{i\in I}\nu_i\right) \vee_T \left(\bigcap_{i\in I}\nu_i\right)\right)\\
&\quad\subseteq \underline{Apr_{\mu}}_\vartheta\left(\bigcap_{i\in I}(\nu_i \vee_T \nu_i)\right)\\
&\quad= \bigcap_{i\in I}\underline{Apr_{\mu}}_\vartheta(\nu_i \vee_T \nu_i)\\
&\quad\subseteq \bigcap_{i\in I}\underline{Apr_{\mu}}_\vartheta \nu_i\\
&\quad= \underline{Apr_{\mu}}_\vartheta\left(\bigcap_{i\in I}\nu_i\right)
\end{aligned}$$

and

$$\begin{aligned}
&\underline{Apr_{\mu}}_\vartheta\left(\chi_X \wedge_T \left(\bigcap_{i\in I}\nu_i\right)\right)\\
&\quad\subseteq \underline{Apr_{\mu}}_\vartheta\left(\bigcap_{i\in I}(\chi_X \wedge_T \nu_i)\right)\\
&\quad= \bigcap_{i\in I}\underline{Apr_{\mu}}_\vartheta(\chi_X \wedge_T \nu_i)\\
&\quad\subseteq \bigcap_{i\in I}\underline{Apr_{\mu}}_\vartheta \nu_i\\
&\quad= \underline{Apr_{\mu}}_\vartheta\left(\bigcap_{i\in I}\nu_i\right).
\end{aligned}$$

This means that $(\bigcap_{i\in I}\nu_i) \vee_T (\bigcap_{i\in I}\nu_i) \sqsubseteq_\mu \bigcap_{i\in I}\nu_i$ and $\chi_X \wedge_T (\bigcap_{i\in I}\nu_i) \sqsubseteq_\mu \bigcap_{i\in I}\nu_i$, and hence, $\bigcap_{i\in I}\nu_i \in \mathbf{TLFLQRI}_\mu(X)$. In addition, it is clear that $\chi_X \in \mathbf{TLFLQRI}_\mu(X)$ and χ_X is the top element in $\mathbf{TLFLQRI}_\mu(X)$. Thus, $\mathbf{TLFLQRI}_\mu(X)$ is a topped $\bigcap$-structure, and hence, by Example 1, is a complete lattice. □

5 Conclusions

Fuzzy sets, rough sets and lattice theory have applications across a wide variety of fields. From these aspects, we introduced in this paper the concepts of TL-fuzzy upper and lower approximation operators on a lattice and investigated their basic properties, which extended some notions and results introduced in Xiao et al. (2012, 2014) in the framework of lattice-valued fuzzy set theory. However, we are concerned more with the usefulness of approximation operators compared to Xiao et al. (2012, 2014). For example, we considered the characterizations of TL-fuzzy ideals on a distributive lattice in terms of TL-fuzzy upper and lower approximation operators. Also, by using these operators, we defined and studied a new class of fuzzy structures, called TL-fuzzy quasi-rough ideals based on L-fuzzy sets. The results presented in this paper can hopefully provide more insights into fuzzy rough sets on lattices.

There are still several problems left for further study. For example,

(1) A distributive lattice plays an important role in our discussion. Then it is interesting to consider whether the distributivity can be weakened or not.

(2) From Theorems 4.4, 4.5 and Example 3, it is easily seen that the concept of TL-fuzzy quasi-rough ideals is different from that of TL-fuzzy rough ideals even if the underlying lattice is distributive. Then a natural question is raised: Under which conditions will these two concepts coincide with each other?

We will consider the above questions in the future work.

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Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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