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FOUNDATIONS

Fuzzy objects in spaces with fuzzy partitions

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Abstract A theory of fuzzy objects is derived in the category $SpaceFP$ of spaces with fuzzy partitions, which generalize classical fuzzy sets and extensional maps in sets with similarity relations. It is proved that fuzzy objects in $SpaceFP$ can be characterized by some morphisms in the category of sets with similarity relations. A powerset object functor $\mathcal{F}$ in the category $SpaceFP$ is introduced and it is proved that $\mathcal{F}$ defines a $CSLAT$-powerset theory in the sense of Rodabaugh.

Keywords Space with a fuzzy partition · Category of spaces with fuzzy partitions · Fuzzy objects in sets with fuzzy partitions · Powerset theory

1 Introduction

In almost all branches of mathematics the notion of a powerset and powerset operator in classical set theory is one of the most useful and exploited tools. Recall that given a set X, there exists the set $\mathcal{P}(X) = \{S : S \subseteq X\}$, called the powerset of X and such that every map $f : X \to Y$ can be extended to the forward powerset operator $f^{\to} : \mathcal{P}(X) \to \mathcal{P}(Y)$ and backward powerset operator $f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X)$, such that

$$f^{\to}(S) = f(S), \quad f^{\leftarrow}(T)=f^{-1}(T)=\{x \in X : f(x) \in T\}.$$

The powerset structures are widely used in algebra, logic, topology and also in computer science, for illustrative examples of possible applications in see, e.g., the introductory part of the paper of Solovyov (2011). A classical set theory can be considered to be a special part of fuzzy set theory, introduced by Zadeh (1965). A fuzzy set in a set A with values in the interval $I = [0, 1]$ is defined as a map $A \to I$ and it is then natural that an investigation of powerset objects I^X of fuzzy sets was of interest. The first approach was done again by Zadeh (1965), who defined I^X as a new powerset object instead of $\mathcal{P}(X)$ and introduced new powerset operators $f_Z^{\to} : I^X \to I^Y$ and $f_Z^{\leftarrow} : I^Y \to I^X$, such that for $s \in I^X, t \in I^Y, y \in Y$,

$$f_Z^{\to}(s)(y) = \bigvee_{x, f(x)=y} s(x), \quad f_Z^{\leftarrow}(t) = t \circ f.$$

A lot of papers were published about Zadeh's extension and its generalizations, see, e.g., Gerla and Scarpati (1998), Yager (1996) and Nguyen (1978). Zadeh's extension (which could be considered as an extension of a forward powerset operator $f^{\to}$) was intensively studied by Rodabaugh (1997), especially the relation between $f^{\to}$ and $f_Z^{\to}$.

Rodabaugh (2007) introduced *powerset theory* as a special structure describing powerset objects. A slight modification of that structure defined in a category $\mathbf{K}$, is represented by a system $\mathbf{P} = (P, \to, \leftarrow, V, \eta)$, where $P : |\mathbf{K}| \to CSLAT$ is a powerset generator (where $CSLAT$ is the category of complete $\vee$-semilattices), $\to$ is a forward powerset operator, such that for each $f : X \to Y$ in $\mathbf{K}$, $f_{\mathbf{P}}^{\to} : P(X) \to P(Y)$ in $CSLAT$, $\leftarrow$ is a backward powerset operator, such that

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$f_{\mathbf{P}}^{\leftarrow} : P(Y) \to P(X)$ in $CSLAT$, $V : \mathbf{K} \to Set$ is a concrete functor and $\eta_X : V(X) \to P(X)$ is a map for each object X, such that for each morphism $f : A \to B$ in a category $\mathbf{K}$, $f_{\mathbf{P}}^{\rightarrow} \circ \eta_A = \eta_B \circ V(f)$. Moreover backward and forward powerset operator should define a Galois connection. He proved that both classical and Zadeh's powerset operators define a powerset theory.

Since the original Zadeh's paper was published, the notion of "fuzzy set" has been changed significantly and it is now more general. The first important modification concerns the value set: instead of real number interval $I = [0, 1]$, more general lattice structures Q are considered. Among these lattice structures, complete residuated lattices play important role, (see e.g., Perfilijeva and Močkoř 1999), in some terminology *unital and commutative quantale*, (see Rosenthal 1990), i.e., a structure $Q = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that $(L, \wedge, \vee)$ is a complete lattice, $(L, \otimes, 1)$ is a commutative monoid with operation $\otimes$ isotone in both arguments and $\to$ is a binary operation which is adjoint with respect to $\otimes$, i.e.,

$$\alpha \otimes \beta \le \gamma \text{ iff } \alpha \le \beta \to \gamma.$$

A well-known example is the Łukasiewicz algebra $Ł = ([0, 1], \vee, \wedge, \otimes, \to_Ł, 0, 1)$, where

$$a \otimes b = 0 \vee (a + b - 1)$$
$$a \to_Ł b = 1 \wedge (1 - a + b).$$

Further classical fuzzy sets (or even fuzzy sets with values in residuated lattice Q) were originally defined on sets. But any set A can be considered as a couple $(A, =)$, where $=$ is a standard equality relation defined on A. It is then natural instead of the crisp equality relation $=$, to consider some more "fuzzy" equality relation defined on A, which is called a *similarity relation*. Hence instead of a classical set A as a basic set and a fuzzy set $s : A \to Q$, we can use a set with similarity relation (A, δ) (called a Q-set) and a map $s : (A, \delta) \to Q$. Such a map then represents some new "fuzzy object" in (A, δ). Instead of maps $A \to Q$, or $(A, \delta) \to Q$, *morphisms* in some categories can be used. An example of such category is the category of Q-sets as objects and naturally defined morphisms. A morphism $f : (A, \delta) \to (B, \gamma)$ in the category Set(Q) is a map $f : A \to B$ such that $\gamma(f(x), f(y)) \ge \delta(x, y)$ for all $x, y \in A$. It is then natural to speak about a *fuzzy object* $(A, \delta) \to (Q, \leftrightarrow)$ in the category Set(Q), instead of a "fuzzy set," where $\leftrightarrow$ is the biresiduation operation in Q defined by $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$. These fuzzy objects generalize classical fuzzy sets $A \to Q$ and in facts, a fuzzy objects $(A, \delta) \to (Q, \leftrightarrow)$ is nothing else than *extensional map* in a Q-set (see, e.g., Močkoř 2012).

Using these new fuzzy objects, powerset structures of fuzzy objects were also investigated. In Močkoř (2016), it was proved the following theorem.

Theorem 1.1 *There exists a $CSLAT$-powerset theory (in the sense of Rodabaugh)* $\mathbf{F} = (F, \to, \leftarrow, V, \eta)$*, such that*

(1) *$F(A, \delta)$ is the set of all fuzzy objects in (A, δ),*
(2) *for any morphism $f : (A, \delta) \to (B, \gamma)$,*
   (i) *$s \in F(A, \delta), b \in B$, $f_F^{\rightarrow}(s)(b) = \bigvee_{x \in A} s(x) \otimes \gamma(f(x), b)$,*
   (ii) *$t \in F(B, \gamma), a \in A$, $f_F^{\leftarrow}(t)(a) = t \circ f(a)$,*
(3) *$V : Set(Q) \to Set$ is a forgetfull functor,*
(4) *$\eta_{(A,\delta)} : V(A, \delta) \to F(A, \delta)$ is defined by $\eta_{(A,\delta)}(a)(x) = \delta(a, x)$, $a, x \in A$.*

Theorem 1.1 then represents a generalization of classical and Zadeh's powerset theories.

Since the introduction of fuzzy sets in the paper of Zadeh (1965), very rapid and extensive development of methods, tools and techniques using this concept appeared. In most of these methods and tools, a very significant role is played by the concept of fuzzy relations defined on a set X, i.e., $R : X \times X \to [0, 1]$, or, more generally, $R : X \times X \to Q$, where Q is an appropriate ordered structure. Fuzzy relations not only allow us to extend the most structures known from classical sets to the environment of fuzzy sets, but also as a transcription of fuzzy IF-THEN rules allows us to express the behavior of dynamic systems, which are influenced by some elements of uncertainty. In addition to this, explicit use of fuzzy relations, fuzzy relation contributed to a very intensive development of new areas of fuzzy mathematics, recently, *fuzzy transform* (F-transform) as a method successfully used in signal and image processing (Martino 2008), compression (Perfilieva 2006a), numerical solutions of ordinary and partial differential equations (Khastan to appear; Štěpnička and Valašek 2005), data analysis (Perfilieva et al. 2008) and many other applications. Basic structure for F-transform is a *space with a fuzzy partition*, which was introduced by Perfilieva et al. (2015). Roughly speaking, a space with a fuzzy partition is a couple $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a system of Q-valued fuzzy sets in X, such that cores of these fuzzy sets are a partition of X. If $(X, \mathcal{A})$ is a space with a fuzzy partition $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, then fuzzy transforms (upper and lower) are special maps $F^\uparrow, F^\downarrow : Q^X \to Q^\Lambda$, which fuzzify the precise values of independent variable by a closeness relation, and precise values of dependent variables as averages to an approximate values (see, e.g., Perfilieva et al. 2015). In the paper, Močkoř (to appear), we introduced the category $SpaceFP$ of spaces with fuzzy partitions and we proved that there are some functorial relationships between some subcategory of the category $SpaceFP$ and the category of Kuratowski closure (interior, respectively) operators and the category of approximation spaces.

Hence it seems that spaces with fuzzy partitions can be used as a basic category for investigation not only of

F-transform, but also of Kuratowski closure and interior operators and approximation spaces.

In the paper, we deal with *fuzzy objects* in our basic category of spaces with fuzzy partition. In all previous definitions of fuzzy objects in the category Set, or in the category Set(Q) of Q-sets, fuzzy objects are defined as morphisms $A \to \mathbf{Q}$ in a corresponding category $\mathbf{K}$, where A is an object of $\mathbf{K}$ and $\mathbf{Q}$ is a special object derived from Q. In that way, fuzzy sets in Set are only maps (=morphisms) $A \to Q$, fuzzy objects in Set(Q) are morphisms $(A, \delta) \to (Q, \leftrightarrow)$. It is then natural to define fuzzy objects in the category $SpaceFP$ in a similar way, i.e., as morphisms $(X, \mathcal{A}) \to (Q, \mathcal{Q})$, where $\mathcal{Q}$ is an appropriate partition in Q. The definition of fuzzy objects in $SpaceFP$ is introduced in Sect. 3, where we also show that such fuzzy objects are natural generalizations of classical fuzzy sets and fuzzy objects in the category Set(Q). In the category Set(Q), any map $f : A \to Q$ can be extended to a fuzzy object $\widehat{f} : (A, \delta) \to (Q, \leftrightarrow)$. In Sect. 3 we prove an analogical property for fuzzy objects in the category $SpaceFP$. In fact, we prove that fuzzy objects in the category $SpaceFP$ correspond, in some sense, fuzzy objects in the category Set(Q). We also prove that fuzzy objects in $SpaceFP$ have properties of local fix points f in F-transforms, i.e., $F^{\uparrow}(f)(\lambda) = F^{\downarrow}(f)(\lambda) = f(z)$, where $z \in core(A_\lambda)$.

In Sect. 4, we introduce powerset objects $\mathcal{F}(A, \mathcal{A}) = ((Q, \mathcal{Q})^{(A,\mathcal{A})}, \leq)$ in the category $SpaceFP$ and as the main result of the paper, we show that these powerset objects define $CSLAT$-powerset theory in the sense of Rodabaugh (2007). This $CSLAT$-powerset theory then comprises $CSLAT$-powerset theories of classical fuzzy sets and fuzzy objects in the category Set(Q).

2 Preliminaries

In the paper, by Q we denote a complete residuated lattice (see e.g., Perfilijeva and Močkoř 1999), in some terminology *unital and commutative quantale*, (see Rosenthal 1990), i.e., a structure $Q = (L, \wedge, \vee, \otimes, \to, 0_Q, 1_Q)$ such that $(L, \wedge, \vee)$ is a complete lattice, $(L, \otimes, 1_Q)$ is a commutative monoid with operation $\otimes$ isotone in both arguments and $\to$ is a binary operation which is adjoint with respect to $\otimes$, i.e.,

$$\alpha \otimes \beta \leq \gamma \text{ iff } \alpha \leq \beta \to \gamma.$$

We begin with the definition of Q-valued fuzzy partition, as it was introduced in Perfilieva et al. (2015). Recall that a *core* of a (Q-valued) fuzzy set $f : X \to Q$ is defined by $core(f) = \{x \in X : f(x) = 1_Q\}$. A normal ($Q$-)valued fuzzy set f in a set X is such that there exists $x \in X$, such that $f(x) = 1_Q$. A fuzzy relation R in X is then a fuzzy set $R : X \times X \to Q$.

An F-transform in a form introduced by Perfilieva et al. (2015) is based on the so called fuzzy partitions on the crisp set.

Definition 2.1 (Perfilieva et al. 2015) Let X be a set. A system $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ of normal Q-valued fuzzy sets in X is a *fuzzy partition* of X, if $\{core(A_\lambda) : \lambda \in \Lambda\}$ is a partition of X. A pair $(X, \mathcal{A})$ is called a space with a fuzzy partition.

In the paper Močkoř (to appear), we introduced the following definition of the category $SpaceFP$ of spaces with fuzzy partitions.

Definition 2.2 (Močkoř, to appear) The category $SpaceFP$ is defined by

1. Fuzzy partitions $(X, \mathcal{A})$, as objects,
2. Morphisms $(g, \sigma) : (X, \{A_\lambda : \lambda \in \Lambda\}) \to (Y, \{B_\omega : \omega \in \Omega\})$, such that
   (a) $g : X \to Y$ is a map,
   (b) $\sigma : \Lambda \to \Omega$ is a map,
   (c) $\forall \lambda \in \Lambda, \quad g_Z^{\to}(A_\lambda) \subseteq B_{\sigma(\lambda)}$.
3. The composition of morphisms in $SpaceFP$ is defined by $(h, \tau) \circ (g, \sigma) = (h \circ g, \tau \circ \sigma)$.

Instead of $g_Z^{\to}$ we will use only $g^{\to}$.

Recall that a set with similarity relation (or Q-set) is a couple (A, δ), where $\delta : A \times A \to Q$ is a map such that

(a) $(\forall x \in A) \quad \delta(x, x) = 1_Q$,
(b) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,
(c) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$ (generalized transitivity).

We will use the category Set(Q) with Q-sets as objects and with morphisms $f : (A, \delta) \to (B, \gamma)$ defined as a map $f : A \to B$, such that $\gamma(f(x), f(y)) \geq \delta(x, y)$, for all $x, y \in A$. The category Set(Q) has its origin in Wyler's category introduced in Wyler (1995), and in a more general way it was developed by Höhle (2007). Within morphisms in the category Set(Q), we will be specially interested in morphisms $(A, \delta) \to (Q, \leftrightarrow)$, which are called *fuzzy objects* in a corresponding category, or *extensional sets*. Let $F(A, \delta) = ((Q, \leftrightarrow)^{(A,\delta)}, \leq)$ be the ordered set of all such fuzzy objects in Set(Q), ordered point-wise. Then $F(A, \delta)$ is a complete $\bigvee$-semilattice and if $CSLAT$ is the category of all complete $\bigvee$-semilattices with $\bigvee$-preserving maps as morphisms, then $F : \text{Set}(Q) \to CSLAT$ is a covariant functor, such that for any morphism $f : (A, \delta) \to (B, \gamma)$,

$$s \in F(A,\delta), b \in B, \quad f_F^{\rightarrow}(s)(b) = F(f)(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(f(x), b).$$

In many papers (see, e.g., Höhle 2007) for any Q-set (A,δ), the following extensional map $\widehat{}: \{s : s : A \to Q \text{ is a map}\} \to F(A,\delta)$ was introduced, such that for any map $s : A \to Q$, $\widehat{s}(a) = \bigvee_{x\in A} \delta(a,x) \otimes s(x)$, for every $a \in A, \alpha \in Q$. Then $\widehat{s} \in F(A,\delta)$.

3 Fuzzy objects in spaces with fuzzy partitions

We show firstly that Q-sets can be considered to be spaces with specially defined fuzzy partitions.

Proposition 3.1 *There exists a full and faithfull functor I : $Set(Q) \to SpaceFP$, which is also injective on objects.*

Proof Let $(X,\delta) \in |\mathrm{Set}(Q)|$. Let us define a binary relation $\equiv_\delta$ by

$$(\forall x, y \in X) \quad x \equiv_\delta y \leftrightarrow \delta(x,y) = 1.$$

It is clear that $\equiv_\delta$ is an equivalence relation. Let X/δ be the partition of a set X by the relation $\equiv_\delta$. We set

$$\mathcal{C}_{X,\delta} = \{C_{X,\mathbf{a}} : \mathbf{a} \in X/\delta\},$$
$$(\forall x \in X) \quad C_{X,\mathbf{a}}(x) = \delta(x,y), \text{ for any } y \in \mathbf{a}.$$

The definition is correct. In fact, if $y' \in \mathbf{a}$, then $\delta(y,y') = 1_Q$ and it follows that $\delta(x,y) = \delta(x,y')$. The fuzzy partition $\mathcal{C}_{X,\delta}$ in a set X will be called to be *defined by a similarity relation* δ. Let the object function of the functor I be defined by $I(X,\delta) = (X, \mathcal{C}_{X,\delta})$, and let $f : (X,\delta) \to (Y,\gamma)$ be a morphism in $\mathrm{Set}(Q)$. We set

$$I(f) = (f,\sigma), \quad \sigma : X/\delta \to Y/\gamma,$$

such that for any $\mathbf{a} \in X/\delta$,

$$\sigma(\mathbf{a}) = \mathbf{b} \in Y/\gamma \Leftrightarrow (x \in \mathbf{a} \Rightarrow f(x) \in \mathbf{b}).$$

The definition is correct. In fact, let $x' \in \mathbf{a}$, then we have $1_Q = \delta(x,x') \le \gamma(f(x), f(x'))$ and $f(x') \in \mathbf{b}$. Then (f,σ) is a morphism in SpaceFP. In fact, let $\mathbf{a} \in X/\delta$, then

$$f^{\rightarrow}(C_{X,\mathbf{a}})(y) = \bigvee_{f(x)=y} C_{X,\mathbf{a}}(x) = \bigvee_{f(x)=y} \delta(x,z) \le \gamma(y, f(z)) = C_{Y,\sigma(\mathbf{a})}(y),$$

where $z \in \mathbf{a}$ is an arbitrary element. It is clear that I : $\mathrm{Set}(Q) \to SpaceFP$ is a faithfull functor. Moreover we show that I is injective on objects of $\mathrm{Set}(Q)$. In fact, let $I(X,\delta) = I(Y,\gamma)$, i.e., $(X, \mathcal{C}_{X,\delta}) = (Y, \mathcal{C}_{Y,\gamma})$. Then we have $X = Y$ and $\mathcal{C}_{X,\delta} = \mathcal{C}_{X,\gamma}$. Let elements of $\mathcal{C}_{X,\gamma}$ be denoted by $C_{X,\mathbf{a}}$, and elements of $\mathcal{C}_{Y,\gamma}$ by $D_{Y,\mathbf{b}}$, where $\mathbf{a}, \mathbf{b} \in X/\gamma = X/\delta$. Then there exists a bijection $\sigma : X/\gamma \to X/\delta$ such that $C_{X,\mathbf{a}} = D_{X,\sigma(\mathbf{a})}$, for any $\mathbf{a} \in X/\gamma$. Let $x \in X$, then for any $y \in \mathbf{a}$ and any $z \in \sigma(\mathbf{a})$, we have

$$C_{X,\mathbf{a}}(x) = \delta(x,y) = \gamma(x,z) = D_{X,\sigma(\mathbf{a})}(x).$$

Then for any $x \in \mathbf{a}$ we have $1_Q = \delta(x,y) = \gamma(x,z)$ and it follows that $x \in \sigma(\mathbf{a})$. Since $\mathbf{a}, \sigma(\mathbf{a})$ are elements in a partition, it follows that $\sigma(\mathbf{a}) = \mathbf{a}$ and $C_{X,\mathbf{a}} = D_{X,\mathbf{a}}$. Let $x, y \in X$, then there exists $\mathbf{a} \in X/\delta$ such that $y \in \mathbf{a}$. Hence we have

$$\delta(x,y) = C_{X,\mathbf{a}}(x) = D_{X,\mathbf{a}}(x) = \gamma(x,y),$$

and $(X,\delta) = (Y,\gamma)$. We show finally that $I(\mathrm{Set}(Q))$ is a full subcategory in $SpaceFP$. Let $(X,\delta), (Y,\gamma) \in |\mathrm{Set}(Q)|$ and let $(f,\sigma) : (X, \mathcal{C}_{X,\delta}) \to (Y, \mathcal{C}_{Y,\gamma})$ be a morphism in $SpaceFP$. Let $x, x' \in X$. Then for $\mathbf{a} \in X/\delta$, such that $x' \in \mathbf{a}$, we have $f(x') \in f(\mathbf{a}) \subseteq \sigma(\mathbf{a})$ and for $y = f(x)$ we obtain

$$\delta(x,x') \le f^{\rightarrow}(C_{X,\mathbf{a}}(y) = \bigvee_{z, f(z)=y} \delta(z,x') \le C_{Y,\gamma}(f(x)) = \gamma(f(x), f(x')).$$

Hence $f : (X,\delta) \to (Y,\gamma)$ is a morphism in $\mathrm{Set}(Q)$ and $I(f) = (f,\sigma)$. □

In the following proposition, we prove conditions, under which a fuzzy partition is defined by a similarity relation.

Proposition 3.2 *Let $(X, \mathcal{A})$ be a space with a fuzzy partition, $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$. Then the following statements are equivalent.*

1. *There exists a similarity relation δ on X, such that $\mathcal{A} = \mathcal{C}_{X,\delta}$.*
2. *The following condition holds:*

$$(\forall \lambda, \omega \in \Lambda, x \in core(A_\omega), z \in X)$$
$$A_\lambda(x) \otimes A_\lambda(z) \le A_\omega(z),$$

Proof (1) ⇒ (2) This part is only a simple verification of properties of $\mathcal{C}_{X,\delta}$.

(2) ⇒ (1) Let $x \in core(A_\lambda)$, $y \in core(A_\omega)$. From the condition 2., it follows that $A_\lambda(y) = A_\omega(x)$. We set

$$\delta(x,y) = A_\lambda(y) = A_\omega(x) = \delta(y,x).$$

Then δ is a similarity relation on X and $(X, \mathcal{A}) = (X, \mathcal{C}_{X,\delta})$. In fact, let $\sigma : \Lambda \to X/\delta$ be defined by

$$\lambda \in \Lambda, \quad x \in core(A_\lambda) \Rightarrow \sigma(\lambda) = \mathbf{x} \in X/\delta,$$

where $\mathbf{x} \in X/\delta$ is such that $x \in \mathbf{x}$. The definition of σ is correct and σ is a bijection, as it can be proved simply. Then for any $\lambda \in \Lambda$, $y \in core(A_\lambda)$ and $x \in X$, we have $A_\lambda(x) = \delta(x, y) = C_{\mathbf{y}}(x) = C_{\sigma(\lambda)}(x)$. Hence $\mathcal{A} = \mathcal{C}_{X,\delta}$. □

As we mentioned in Introduction, Q-valued fuzzy objects in the category *Set* of classical sets or in the category Set(Q) of Q-sets, can be considered to be morphisms $\mathbf{A} \to \mathbf{Q}$ in these categories, where $\mathbf{A}$ is an underlying object in that category and $\mathbf{Q}$ is a special object representing valued structure in that category. In the category *Set*, $\mathbf{Q}$ is the underlying set of a lattice Q and in the category Set(Q), $\mathbf{Q}$ is the Q-set $(Q, \leftrightarrow)$. From that point of view, in the category *SpaceFP*, we can also consider special morphisms $(A, \mathcal{A}) \to (Q, \mathcal{Q})$, where $(Q, \mathcal{Q})$ would be the space Q with some fuzzy partition $\mathcal{Q}$. And these special morphisms then could represent Q-valued fuzzy objects in the category *SpaceFP*. Since $\leftrightarrow$ is a similarity relation in the set Q, for a construction of a fuzzy partition $\mathcal{Q}$ we can use the Q-set $(Q, \leftrightarrow)$ and the result of Proposition 3.1.

Let us mention that for the equivalence relation $\equiv_\leftrightarrow$ defined on Q, we have $Q/\leftrightarrow = \{\{\alpha\} : \alpha \in Q\} \cong Q$. Let Q_α be a fuzzy set in Q defined by

$$Q_\alpha(\beta) = \alpha \leftrightarrow \beta.$$

Then $core(Q_\alpha) = \{\alpha\}$ and by $\mathcal{Q}$ we denote the fuzzy partition $\mathcal{C}_{Q,\leftrightarrow}$ defined by Q-set $(Q, \leftrightarrow)$, according to Proposition 3.1, i.e., $\mathcal{Q} = \{Q_\alpha : \alpha \in Q\}$. Hence we can consider the pair $(Q, \mathcal{Q})$ to be the set Q with the fuzzy partition $\mathcal{Q}$.

Definition 3.1 A (Q-valued) fuzzy object in a space with a fuzzy partition $(X, \mathcal{A})$ is a morphism $(f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{Q})$ in the category *SpaceFP*. By $(Q, \mathcal{Q})^{(X,\mathcal{A})}$ we denote the set of all fuzzy objects in $(X, \mathcal{A})$.

To be more explicit, $(f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{Q})$ is a fuzzy object, where $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ if

1. $f : X \to Q$ and $\sigma : \Lambda \to Q$ are maps,
2. $(\forall \lambda \in \Lambda, \alpha \in Q)\quad \bigvee_{x, f(x)=\alpha} A_\lambda(x) \le Q_{\sigma(\lambda)}(\alpha) = \alpha \leftrightarrow \sigma(\lambda)$.

In the following proposition, we show that any fuzzy object (f, σ) in a space with a fuzzy partition $(X, \mathcal{A})$ can be uniquely determined by either of the maps $f : X \to Q$ or $\sigma : \Lambda \to Q$.

Proposition 3.3 *Let* $(X, \mathcal{A}) \in |SpaceFP|$, $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$.

1. *Let $\sigma : \Lambda \to Q$ be a map. Then the following statements are equivalent:*
   (a) *There exists the unique map $f : X \to Q$ such that $(f, \sigma) \in (Q, \mathcal{Q})^{(X,\mathcal{A})}$,*
   (b) *For all $\lambda, \lambda' \in \Lambda$ and any $x \in core(A_\lambda)$, $A_{\lambda'}(x) \le \sigma(\lambda) \leftrightarrow \sigma(\lambda')$ holds.*
2. *Let $f : X \to Q$ be a map. Then the following statements are equivalent.*
   (a) *There exists the unique map $\sigma : \Lambda \to Q$, such that $(f, \sigma) \in (Q, \mathcal{Q})^{(X,\mathcal{A})}$,*
   (b) *For any $\lambda \in \Lambda$ and $x \in core(A_\lambda)$, $x' \in X$, $A_\lambda(x') \le f(x) \leftrightarrow f(x')$ holds.*

Proof (1) $(a) \Rightarrow (b)$. Let $\sigma : \Lambda \to Q$ be a map and let $f : X \to Q$ be such that (f, σ) be a fuzzy object in $(X, \mathcal{A})$. From the definition of fuzzy objects it follows that $f(core(A_\lambda)) \subseteq core(Q_{\sigma(\lambda)}) = \{\sigma(\lambda)\}$ and $f(x) = \sigma(\lambda)$, for all $x \in core(A_\lambda)$. Let $\lambda \in \Lambda, \alpha \in Q$ and let $x \in X$ be such that $f(x) = \alpha$. If $x \in core(A_{\lambda'})$, then we have $A_\lambda(x) \le Q_{\sigma(\lambda)}(\alpha) = \sigma(\lambda) \leftrightarrow \alpha = \sigma(\lambda) \leftrightarrow f(x) = \sigma(\lambda) \leftrightarrow \sigma(\lambda')$, and the condition (b) holds.

1) $(b) \Rightarrow (a)$. Let $x \in X, x \in core(A_\lambda)$. We set $f(x) = \sigma(\lambda)$ and we show that (f, σ) is a fuzzy object. Let $\lambda \in \Lambda, \alpha \in Q$. Then for any $x \in core(A_{\lambda'})$, $f(x) = \alpha$, we have $A_\lambda(x) \le \sigma(\lambda) \leftrightarrow \alpha = \sigma(\lambda) \leftrightarrow f(x) = \sigma(\lambda) \leftrightarrow \sigma(\lambda')$ and the condition (a) holds. The uniqueness of f is clear.

2) $(a) \Rightarrow (b)$. Let (f, σ) be a fuzzy object for some $\sigma : \Lambda \to Q$. Let $\lambda \in \Lambda, x \in core(A_\lambda), x' \in X$. Since $f(core(A_\lambda)) \subseteq \{\sigma(\lambda)\}$, we have $f(x) = \sigma(\lambda)$. We put $\alpha = f(x')$. Then we have $A_\lambda(x') \le Q_{\sigma(\lambda)}(\alpha) = \alpha \leftrightarrow \sigma(\lambda) = f(x') \leftrightarrow f(x)$, and (b) holds.

2) $(b) \Rightarrow (a)$. Let $\lambda \in \Lambda, x \in core(A_\lambda)$. We set $\sigma(\lambda) = f(x) \in Q$. The definition is correct. In fact, let $x, x' \in core(A_\lambda)$, then from (b) it follows that $1_Q = A_\lambda(x') \le f(x) \leftrightarrow f(x')$ and we obtain $f(x) = f(x')$. Let $\lambda \in \Lambda, \alpha \in Q, x \in X$ be such that $f(x) = \alpha$. Let $x' \in core(A_\lambda)$. Then from (b) it follows that $A_\lambda(x) \le f(x') \leftrightarrow f(x) = \sigma(\lambda) \leftrightarrow \alpha = Q_{\sigma(\lambda)}(\alpha)$. Therefore (f, σ) is a fuzzy object. The uniqueness of σ is clear. □

We will use the following notation. If $f : X \to Q$ is a map which satisfies the condition 2)(b) from previous proposition, by $[f]$ we denote the unique map $\Lambda \to Q$ such that $(f, [f])$ is a fuzzy object in $(X, \mathcal{A})$. Similarly, if $\sigma : \Lambda \to Q$ is a map satisfying the condition 1)(b), then by $|\sigma|$ we denote the unique map $X \to Q$ such that $(|\sigma|, \sigma)$ is a fuzzy object in $(X, \mathcal{A})$. In that case we say that f (σ, respectively) *defines a fuzzy object* $(f, [f])$ $((|\sigma|, \sigma)$, respectively).

Let $(X, \mathcal{A})$ be a space with a fuzzy partition, $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$. We set

$$\mathcal{F}(X, \mathcal{A}) = \{(t, \rho) : (t, \rho) \text{ is a fuzzy object in } (X, \mathcal{A})\}, \tag{1}$$

$$\mathcal{F}_1(X, \mathcal{A}) = \{t | t : X \to Q \text{ defines a fuzzy object in } (X, \mathcal{A})\}, \tag{2}$$

$$\mathcal{F}_2(X, \mathcal{A}) = \{\tau | \tau : \Lambda \to Q \text{ defines a fuzzy object in } (X, \mathcal{A}). \tag{3}$$

Then according to previous Proposition 3.3, we have

$$\begin{aligned}\mathcal{F}(X, \mathcal{A}) &= \{(t, [t]) : t \in \mathcal{F}_1(X, \mathcal{A})\}\\ &= \{(|\sigma|, \sigma) : \sigma \in \mathcal{F}_2(X, \mathcal{A})\},\end{aligned} \tag{4}$$

and the following corollary holds.

Corollary 3.1 *For any space with a fuzzy partition $(X, \mathcal{A} = \{A_\lambda : \lambda \in \Lambda\})$ there exist maps*

$$\begin{aligned}&[.] : \mathcal{F}_1(X, \mathcal{A}) \to \mathcal{F}_2(X, \mathcal{A}), \quad |.| : \mathcal{F}_2(X, \mathcal{A}) \to \mathcal{F}_1(X, \mathcal{A}),\\ &\quad s \in \mathcal{F}_1(X, \mathcal{A}), \tau \in \mathcal{F}_2(X, \mathcal{A}), \quad [s](\lambda) = s(x), |\tau|(x)\\ &= \tau(\lambda) \ \Leftrightarrow x \in core(A_\lambda).\end{aligned}$$

On a set $\mathcal{F}(X, \mathcal{A})$ an ordering can defined such that $(t, \rho) \leq (s, \tau) \Leftrightarrow t \leq s, \rho \leq \tau$. From the previous Proposition 3.3 it follows that $(t, \rho) \leq (s, \tau)$ iff $t \leq s$ or $\rho \leq \tau$. Analogical ordering can be also defined on sets $\mathcal{F}_1(A, \mathcal{A})$ and $\mathcal{F}_2(A, \mathcal{A})$.

Moreover we have

Lemma 3.1 *$(\mathcal{F}(X, \mathcal{A}), \leq)$ is a complete lattice.*

Proof Let $\{(t_i, \rho_i) : i \in I\} \subseteq \mathcal{F}(X, \mathcal{A})$. We set $T = \bigvee_{i \in I} t_i, t = \bigwedge_{i \in I} t_i$ in a lattice Q^X. Then according to Proposition 3.3, T and t uniquely define fuzzy object in $(X, \mathcal{A})$. In fact, let $\lambda \in \Lambda, x \in core(A_\lambda)$ and $x' \in X$. Then we have

$$\begin{aligned}&T(x) \leftrightarrow T(x') = \left(\bigvee_i t_i(x)\right) \leftrightarrow \left(\bigvee_i t_i(x')\right)\\ &\geq \bigwedge_i (t_i(x) \leftrightarrow t_i(x')) \geq A_\lambda(x'),\\ &t(x) \leftrightarrow t(x') = \left(\bigwedge_i t_i(x)\right) \leftrightarrow \left(\bigwedge_i t_i(x')\right)\\ &\geq \bigwedge_i (t_i(x) \leftrightarrow t_i(x')) \geq A_\lambda(x').\end{aligned}$$

Hence $(T, [T]) = \bigvee_i (t_i, \rho_i)$, $(t, [t]) = \bigwedge_i (t_i, \rho_i)$ in $(\mathcal{F}(X, \mathcal{A}), \leq)$. □

In the following examples, we show that the notion of a fuzzy object in a space with fuzzy partition extends classical notions of a fuzzy set in a set and also of an extensional map in a set with similarity relation.

Example 3.1 Let $\mathcal{A} = \{\{x\} : x \in X\}$ and let $(f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{Q})$ be a morphism in $SpaceFP$. Hence $f, \sigma : X \to Q$ are maps and we have $\{f(x)\} = f(core(\{x\})) \subseteq core(Q_{\sigma(x)}) = \{\sigma(x)\}$. Therefore $f = \sigma$ and it is clear that the property from the definition of morphisms in $SpaceFP$ holds automatically. Therefore $(f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{Q})$ is a morphism iff $f : X \to Q$ is a map. Hence $(Q, \mathcal{Q})^{(X, \mathcal{A})} = \{(f, f) : f \in Q^X\}$ and $\mathcal{F}_1(X, \mathcal{A}) = Q^X$ is the set of all fuzzy sets in a set X. □

Example 3.2 Let $(X, \delta) \in |\mathrm{Set}(Q)|$ and let $(f, \sigma) : (X, \mathcal{C}_{X,\delta}) \to (Q, \mathcal{Q})$ be a morphism in $SpaceFP$. Since $\mathcal{Q}$ is the fuzzy partition defined by the similarity relation $\leftrightarrow$ and, according to Proposition 3.1, $I(\mathrm{Set}(Q))$ is a full subcategory in $SpaceFP$, it follows that $f : (X, \delta) \to (Q, \leftrightarrow)$ is a morphism in $\mathrm{Set}(Q)$. Hence $(Q, \mathcal{Q})^{(X, \mathcal{C}_{X,\delta})} = \{(f, [f]) : f$ is an extensional map in $(X, \delta)\}$ and $\mathcal{F}_1(X, \mathcal{A}) = (Q, \leftrightarrow)^{(X,\delta)}$. □

Example 3.3 Let X be a set and let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, where $A_\lambda \subseteq X$ be crisp sets. Let $(f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{Q})$ be a morphism in $SpaceFP$. Then $f(A_\lambda) = f(core(A_\lambda)) \subseteq core(Q_{\sigma(\lambda)}) = \{\sigma(\lambda)\}$ and it follows that $f(A_\lambda) = \{\sigma(\lambda)\}$. The condition from the definition of morphisms in $SpaceFP$ is satisfied automatically and we obtain that $(Q, \mathcal{Q})^{(X, \mathcal{A})} \cong Q^\Lambda$, which represents special fuzzy sets, constant on elements of a partition $\mathcal{A}$. □

If a function $f : X \to Q$ defines a fuzzy object in a space with a fuzzy partition $(X, \mathcal{A})$, then for any $\lambda \in \Lambda$, f is a constant function on a $core(A_\lambda)$, i.e., $f(core(A_\lambda)) = \{[f](\lambda)\}$. In the next example we show that this property is a necessary but not a sufficient condition to define a fuzzy object.

Example 3.4 Let $X = [0, 1]$, Q be a Łukasiewicz algebra and let fuzzy sets A_1, A_2, A_3 in X be such that

1. $core(A_1) = [0, 1/3)$, $core(A_2) = [1/3, 2/3)$, $core(A_3) = [2/3, 1]$,
2. there exists $x_0 \in [1/3, 2/3)$, such that $A_3(x_0) > 0.5$.

Let $\mathcal{A} = \{A_i : i \in \{1, 2, 3\}\}$ and let a function $f : X \to Q$ be defined such that

$$f([0, 1/3)) = 0.5,\ f([1/3, 2/3)) = 0.7,\ f([2/3, 1]) = 0.2.$$

If f should define a fuzzy object in $(X, \mathcal{A})$, then for indexes $2, 3$ and any $x \in core(A_3)$, we should have $A_3(x_0) \leq f(x) \leftrightarrow f(x_0) = 0.2 \leftrightarrow 0.7 = 0.5$, a contradiction with $A_3(x_0) > 0.5$. Hence f does not define a fuzzy object. □

As we have mentioned in Sect. 2, if (A, δ) is a Q-set and $s : A \to Q$ is a map, then s can be extended to an extensional set $\widehat{s} : (A, \delta) \to (Q, \leftrightarrow)$, such that $\widehat{s}$ is the smallest extensional set in (A, δ), such that $\widehat{s} \geq s$. In the next theorem we prove that analogical result holds also for spaces with fuzzy partitions. Let $(X, \mathcal{A})$ be a space with a fuzzy partition. We prove that any map $t : X \to Q$ (or $\sigma : \Lambda \to Q$, respectively) can be extended in the smallest way to the map $\widetilde{t} : X \to Q$ (or $\widetilde{\sigma} : \Lambda \to Q$, respectively), which defines a fuzzy object.

Theorem 3.1 *Let $(X, \mathcal{A})$ be a space with a fuzzy partition. Then there exist maps*

$$Q^X \to \mathcal{F}_1(X, \mathcal{A}), \quad t \mapsto \widetilde{t},$$
$$Q^\Lambda \to \mathcal{F}_2(X, \mathcal{A}), \quad \xi \mapsto \widetilde{\xi},$$

with the following properties.

(1) *There exist a similarity relation $\rho_{X,\mathcal{A}}$ in Λ and a similarity relation $\delta_{X,\mathcal{A}}$ in X, such that*

$$x \in X, \quad \widetilde{t}(x) = \bigvee_{z \in A} t(z) \otimes \delta_{X,\mathcal{A}}(z, x),$$
$$\lambda \in \Lambda, \quad \widetilde{\xi}(\lambda) = \bigvee_{\omega \in \Lambda} \xi(\omega) \otimes \rho_{X,\mathcal{A}}(\omega, \lambda),$$

(2) *$\widetilde{t}$ ($\widetilde{\xi}$, respectively) defines a fuzzy object in $(X, \mathcal{A})$ and $\widetilde{t} \geq t$ ($\widetilde{\xi} \geq \xi$, respectively),*
(3) *if t (ξ, respectively) defines a fuzzy object in $(X, \mathcal{A})$, then $\widetilde{t} = t$ ($\widetilde{\xi} = \xi$, respectively),*
(4) *$\widetilde{t} = \bigwedge\{g : g$ defines a fuzzy object in $(X, \mathcal{A}), g \geq t\}$,*
(5) *$\widetilde{\xi} = \bigwedge\{\psi : \psi$ defines a fuzzy object in $(X, \mathcal{A}), \psi \geq \xi\}$.*

Proof Let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ and let $t : X \to Q, \xi : \Lambda \to Q$ be maps. We define a fuzzy binary relation $\pi : \Lambda \times \Lambda \to Q$ by

$$\pi(\lambda, \omega) = \bigvee_{x \in core(A_\omega)} A_\lambda(x) \vee \bigvee_{x \in core(A_\lambda)} A_\omega(x).$$

Then π is a symmetric and reflective fuzzy relation, and we can consider the smallest fuzzy transitive closure $\rho = \rho_{X,\mathcal{A}}$ of π, i.e., $\rho_{X,\mathcal{A}}$ is the smallest fuzzy relation, such that for any $\lambda, \omega, \alpha \in \Lambda$, we have

$$\rho_{(X,\mathcal{A})}(\lambda, \omega) \otimes \rho_{X,\mathcal{A}}(\omega, \alpha) \leq \rho_{X,\mathcal{A}}(\lambda, \alpha),$$
$$\pi(\lambda, \omega) \leq \rho_{X,\mathcal{A}}(\lambda, \omega).$$

In that case, $\rho_{X,\mathcal{A}}$ is a similarity relation in a set Λ. For details of the construction of the smallest fuzzy transitive closure see, e.g., Garmendia (2009).

Ad (1), (2). Let $\widehat{\xi}$ be an extension of ξ to an extensional map in a Q-set $(\Lambda, \rho_{X,\mathcal{A}})$, i.e.,

$$\widehat{\xi}(\lambda) = \bigvee_{\omega \in \Lambda} \xi(\omega) \otimes \rho_{X,\mathcal{A}}(\omega, \lambda) \geq \xi(\lambda).$$

In that case we have $\widehat{\xi}(\lambda) \otimes \rho_{X,\mathcal{A}}(\lambda, \omega) \leq \widehat{\xi}(\omega)$, and we show that $\widehat{\xi}$ defines a fuzzy object in $(X, \mathcal{A})$. In fact, let $\lambda \in \Lambda, x \in core(A_\omega)$. Then we have

$$\widehat{\xi}(\lambda) \otimes \pi(\lambda, \omega) \leq \widehat{\xi}(\lambda) \otimes \rho_{X,\mathcal{A}}(\lambda, \omega) \leq \widehat{\xi}(\omega),$$
$$\widehat{\xi}(\omega) \otimes \pi(\lambda, \omega) \leq \widehat{\xi}(\omega) \otimes \rho_{X,\mathcal{A}}(\lambda, \omega) \leq \widehat{\xi}(\lambda),$$

and it follows that

$$\widehat{\xi}(\lambda) \leftrightarrow \widehat{\xi}(\omega) \geq \pi(\lambda, \omega) \geq A_\lambda(x).$$

Hence $\widehat{\xi}$ defines a fuzzy object in $(X, \mathcal{A})$ and we put $\widetilde{\xi} := \widehat{\xi}$.

Now let a binary fuzzy relation $\delta_{X,\mathcal{A}}$ in X be defined by

$$a, b \in X, \quad \delta_{X,\mathcal{A}}(a, b) := \rho_{X,\mathcal{A}}(\alpha, \beta)$$
$$\Leftrightarrow a \in core(A_\alpha), b \in core(A_\beta).$$

Since $\rho_{X,\mathcal{A}}$ is a similarity relation in Λ, from Corollary 3.1 it follows that $\delta_{X,\mathcal{A}}$ is a similarity relation in a set X. Then we can construct an extension $\widehat{t}$ of t to an extensional map in a Q-set $(X, \delta_{X,\mathcal{A}})$, i.e.,

$$\widehat{t}(x) = \bigvee_{z \in A} t(z) \otimes \delta_{X,\mathcal{A}}(x, z).$$

We show that $\widehat{t}$ defines a fuzzy object in $(X, \mathcal{A})$. In fact, let $\lambda \in \Lambda, x \in core(A_\lambda)$ and $x' \in X, x' \in core(A_\omega)$. Then we have

$$\widehat{t}(x) \otimes \pi(\lambda, \omega) \leq \widehat{t}(x) \otimes \rho_{X,\mathcal{A}}(\lambda, \omega)$$
$$= \widehat{t}(x) \otimes \delta_{X,\mathcal{A}}(x, x') \leq \widehat{t}(x'),$$
$$\widehat{t}(x') \otimes \pi(\lambda, \omega) \leq \widehat{t}(x') \otimes \rho_{X,\mathcal{A}}(\lambda, \omega)$$
$$= \widehat{t}(x') \otimes \delta_{X,\mathcal{A}}(x, x') \leq \widehat{t}(x),$$

and it follows that

$$\widehat{t}(x) \leftrightarrow \widehat{t}(x') \geq \pi(\lambda, \omega) \geq A_\lambda(x).$$

Hence $\widehat{t}$ defines a fuzzy object in $(X, \mathcal{A})$ and we put $\widetilde{t} := \widehat{t}$. It follows that properties (1) and (2) hold for $\widetilde{t}$ and $\widetilde{\xi}$.

Properties (3),(4),(5) follow directly from analogical properties of extensional maps in Q-sets $(X, \delta_{X,\mathcal{A}})$ and $(\Lambda, \rho_{X,\mathcal{A}})$ (see, e.g., Höhle 2007), and these proofs will be omitted. □

From Example 3.2 and previous Theorem 3.1, part 4., it follows the corollary.

Corollary 3.2 *Let (X, δ) be a Q-set and let $t : X \to Q$ be a map. Then for a space with a fuzzy partition $(X, \mathcal{C}_{X,\delta})$, we have $\widetilde{t} = \widehat{t}$, where $\widehat{t}$ is an extension of t to an extensional map in a Q-set (X, δ).*

In the proof of Theorem 3.1, we defined similarity relations $\rho_{X,\mathcal{A}}$ and $\delta_{X,\mathcal{A}}$. Corresponding Q-sets $(\Lambda, \rho_{X,\mathcal{A}})$ and $(X, \delta_{X,\mathcal{A}})$ are closely connected with the space with fuzzy partition $(X, \mathcal{A})$. In fact, we have

Proposition 3.4 *There exist functors*

$$SpaceFP \underset{G}{\overset{H}{\rightrightarrows}} Set(Q),$$

such that for any space with a fuzzy partition $(X, \mathcal{A} = \{A_\lambda : \lambda \in \Lambda\})$,

1. $G(X, \mathcal{A}) = (\Lambda, \rho_{X,\mathcal{A}})$, $\quad H(X, \mathcal{A}) = (X, \delta_{X,\mathcal{A}})$,
2. *For any Q-set (X, δ), $G.I(X, \delta) = (X/\delta, \underline{\delta})$, where $\underline{\delta}(\mathbf{a}, \mathbf{b}) = \delta(a, b)$, for any $a \in \mathbf{a}, b \in \mathbf{b}$.*

Proof (1) Let $(f, \sigma) : (X, \mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}) \to (Y, \mathcal{B} = \{B_\omega : \omega \in \Omega\})$ be a morphism in the category $SpaceFP$. Then $\sigma : (\Lambda, \rho_{X,\mathcal{A}}) \to (\Omega, \rho_{Y,\mathcal{B}})$ is a morphism in the category Set(Q) and we set $G(f, \sigma) = \sigma$. In fact, let for any space $(X, \mathcal{A})$ with a fuzzy partition, a fuzzy relation $\pi_{X,\mathcal{A}}$ be defined by

$$\pi_{X,\mathcal{A}}(\lambda, \omega) = \bigvee_{x \in core(A_\omega)} A_\lambda(x) \vee \bigvee_{x \in core(A_\lambda)} A_\omega(x).$$

Since (f, σ) is a morphism, we have $f(core(A_\lambda)) \subseteq core(B_{\sigma(\lambda)})$ and $A_\lambda(x) \le B_{\sigma(\lambda)}(f(x))$, for any $x \in X$. Then for any $\alpha, \beta \in \Lambda$, we have

$$\begin{aligned}\pi_{Y,\mathcal{B}}(\sigma(\alpha), \sigma(\lambda)) \ge & \bigvee_{y \in f(core(A_\alpha))} B_{\sigma(\alpha)}(y) \vee \bigvee_{y \in f(core(A_\beta))} \\ & \times B_{\sigma(\beta)}(y) \ge \pi_{X,\mathcal{A}}(\alpha, \beta).\end{aligned}$$

If θ is a similarity relation in Ω, such that $\theta \ge \pi_{Y,\mathcal{B}}$, then it is clear that $\zeta = \theta \circ (\sigma \times \sigma)$ is a similarity relation in Λ, such that $\zeta \ge \pi_{X,\mathcal{A}}$. Hence from the definition of $\rho_{X,\mathcal{A}}$ presented in the proof of Theorem 3.1, it follows that

$$\begin{aligned}&\rho_{X,\mathcal{A}}(\alpha, \beta) \\ &\quad = \bigwedge\{\zeta(\alpha, \beta) : \zeta \text{ is a similarity in } \Lambda, \zeta \ge \pi_{X,\mathcal{A}}\} \\ &\quad \le \bigwedge\{\theta \circ (\sigma \times \sigma)(\alpha, \beta)) : \theta \text{ is a similarity in } \Omega, \theta \ge \pi_{Y,\mathcal{B}}\} \\ &\quad = \rho_{Y,\mathcal{B}}(\sigma(\alpha), \sigma(\beta)),\end{aligned}$$

and σ is a morphism in Set(Q). Hence G is a functor. Further we show that $f : (X, \delta_{X,\mathcal{A}}) \to (Y, \delta_{Y,\mathcal{B}})$ is a morphism in the category Set(Q). In fact, let $x, y \in X$ and let $x \in core(A_\lambda)$, $y \in core(A_\omega)$. Since σ is a morphism in Set(Q) and $f(x) \in f(core(A_\lambda)) \subseteq core(B_{\sigma(\lambda)})$ and analogously $f(y) \in core(B_{\sigma(\omega)})$, we obtain

$$\delta_{X,\mathcal{A}}(x, y) = \rho_{X,\mathcal{A}}(\lambda, \omega) \le \rho_{Y,\mathcal{B}}(\sigma(\lambda), \sigma(\omega)) = \delta_{Y,\mathcal{B}}(f(x), f(y)).$$

Hence f is a morphism and we can put $H(f, \sigma) = f$. It is clear that H is a functor.

(2) Let (X, δ) be a Q-set. We need to prove that $G.I(X, \delta) = G(X, \mathcal{C}_{X,\delta}) = (X, \delta)$. We have $G(X, \mathcal{C}_{X,\delta}) = (X/\delta, \rho_{X,\mathcal{C}_{X,\delta}})$, where $\rho_{X,\mathcal{C}_{X,\delta}}$ is the smallest similarity relation in X/δ, which is greater or equal to $\pi_{X,\mathcal{C}_{X,\delta}}$. We have

$$\begin{aligned}&\mathbf{a}, \mathbf{b} \in X/\delta, \quad \pi_{X,\mathcal{C}_{X,\delta}}(\mathbf{a}, \mathbf{b}) \\ &\quad = \bigvee_{x \in core(C_{X,\mathbf{b}})} C_{X,\mathbf{a}}(x) \vee \bigvee_{z \in core(C_{X,\mathbf{a}})} C_{X,\mathbf{b}}(z) \\ &\quad = \bigvee_{x, \delta(x,b) = 1_Q} \delta(x, a) \vee \bigvee_{z, \delta(z,a) = 1_Q} \delta(z, b) \\ &\quad = \delta(a, b) \vee \delta(b, a) = \delta(a, b),\end{aligned}$$

for any $a \in \mathbf{a}, b \in \mathbf{b}$. Hence $\pi_{X,\mathcal{C}_{X,\delta}}$ is a similarity relation and it follows that $\underline{\delta}(\mathbf{a}, \mathbf{b}) = \rho_{X,\mathcal{C}_{X,\delta}}(\mathbf{a}, \mathbf{b}) = \delta(a, b)$, where $a \in \mathbf{a}, b \in \mathbf{b}$. □

In the following we use the functor F, which was introduced in the Sect. 2.

Corollary 3.3 *Let $(X, \mathcal{A})$ be a space with a fuzzy partition $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$. Then*

$$\mathcal{F}_1(X, \mathcal{A}) = F(X, \delta_{X,\mathcal{A}}), \quad \mathcal{F}_2(X, \mathcal{A}) = F(\Lambda, \rho_{X,\mathcal{A}}).$$

Proof Let σ defines a fuzzy object, $\sigma \in \mathcal{F}_2(X, \mathcal{A})$. Then for any $\lambda, \omega \in \Lambda$, we have $\theta(\lambda, \omega) := \sigma(\lambda) \leftrightarrow \sigma(\omega) \ge \pi_{X,\mathcal{A}}(\lambda, \omega)$, where we use a notation from the proof of Proposition 3.4. Since θ is a similarity relation in Λ, we have $\sigma(\lambda) \leftrightarrow \sigma(\omega) \ge \rho_{X,\mathcal{A}}$, and it follows that σ is an extensional map in $(\Lambda, \rho_{X,\mathcal{A}})$. The converse implication can be done similarly. The proof for $\mathcal{F}_1$ can be done analogously. □

It should be observed that if (X, δ) is a Q-set, then $(X, \mathcal{C}_{X,\delta})$ is a space with a fuzzy partition (see Proposition 3.1) and we obtain $G(X, \mathcal{C}_{X,\delta}) = (Q, \leftrightarrow)$. In fact, for any $\alpha, \beta \in Q$, we have $\pi_{X,\mathcal{A}}(\alpha, \beta) = \bigvee_{\gamma \in core(Q_\beta)} Q_\alpha(\gamma) \vee \bigvee_{\gamma \in core(Q_\alpha)} Q_\beta(\gamma) = \alpha \leftrightarrow \beta$, and it is a similarity relation. Hence we have $\rho_{X,\mathcal{C}_{X,\delta}}(\alpha, \beta) = \alpha \leftrightarrow \beta$.

Let X be a set with a fuzzy preorder relation R. Then according to Tiwari and Singh (2013), a fuzzy set $f : X \to Q$ is called an *upper set* of (X, R), if $f(x) \otimes R(x, y) \le f(y)$, for all $x, y \in X$. In the next example we show that for any set with a fuzzy preorder relation (X, R), where $core(R)$ is an equivalence relation, there exists a space with a fuzzy partition $(X, \mathcal{A})$, such that any map $f : X \to Q$ which defines a fuzzy object in $(X, \mathcal{A})$, is an upper set in (X, R).

Example 3.5 Let X be a set with a fuzzy preorder relation R, such that $core(R)$ is an equivalence relation. A pair (X, R) is then called a space with a fuzzy preorder relation.

Let $\Lambda = X/core(R)$ and let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, where for any $\lambda \in \Lambda$, A_λ be a fuzzy set in X defined by $A_\lambda(x) = R(x, y)$, for any $y \in \lambda$. It can be proved simply that this definition is correct. It is clear that $(X, \mathcal{A})$ is a space with a fuzzy partition and $core(A_\lambda) = \lambda$. Now let $f : X \to Q$ defines a fuzzy object in $(X, \mathcal{A})$, i.e., $f \in \mathcal{F}_1(X, \mathcal{A})$. According to Corollary 3.3, $f(x) \otimes \delta_{X,\mathcal{A}}(x, y) \le f(y)$ holds for any $x, y \in X$. Using the notation from the proof of Theorem 3.1, for $x, y \in X$, $x \in \lambda$, $y \in \omega$, we obtain

$$\delta_{X,\mathcal{A}}(x, y) = \rho_{X,\mathcal{A}}(\lambda, \omega) \ge \bigvee_{z\in\omega} A_\lambda(z) \vee \bigvee_{t\in\lambda} A_\omega(t) = \bigvee_{z\in\omega} R(z, x) \vee \bigvee_{t\in\lambda} R(t, y) \ge R(x, y).$$

Hence we obtain $f(y) \ge f(x) \otimes \delta_{X,\mathcal{A}}(x, y) \ge f(x) \otimes R(x, y)$, and f is an upper set in (X, R). □

As we have mentioned in the introduction, spaces with fuzzy partitions are basic structures for fuzzy transforms (F-transforms). If $(X, \mathcal{A})$ is a space with a fuzzy partition $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, then fuzzy transforms (upper and lower) are special maps $F^\uparrow, F^\downarrow : Q^X \to Q^\Lambda$, which fuzzify the precise values of independent variable by a closeness relation, and precise values of dependent variables as averages to an approximate values (see, e.g., Perfilieva 2006b). More precisely, if $f \in Q^X$, then

$$F^\uparrow(f)(\lambda) = \bigvee_{x\in X} f(x) \otimes A_\lambda(x),$$
$$F^\downarrow(f)(\lambda) = \bigwedge_{x\in X} (A_\lambda(x) \to f(x)).$$

In the next proposition, we show that fuzzy objects in $(X, \mathcal{A})$ are, in some sense, locally fix points of these F-transforms.

Proposition 3.5 *Let $(X, \mathcal{A})$ be a space with a fuzzy partition. Then for any $f \in \mathcal{F}_1(X, \mathcal{A})$, we have*

$$(\forall \lambda \in \Lambda) \quad F^\uparrow(f)(\lambda) = F^\downarrow(f)(\lambda) = f(z), \quad z \in core(A_\lambda).$$

Proof Let $\lambda \in \Lambda$ and $z \in core(A_\lambda)$. For any $x \in X$, by λ_x we denote an element of Λ, such that $x \in core(A_{\lambda_x})$. From the proof of Theorem 3.1, it follows that

$$\rho_{X,\mathcal{A}}(\lambda, \lambda_x) \ge \bigvee_{u\in core(A_{\lambda_x})} A_\lambda(u) \vee \bigvee_{v\in core(A_\lambda)} A_{\lambda_x}(v) \ge A_\lambda(x), \quad x \in core(A_{\lambda_x}).$$

Using the notation from Corollary 3.1 and a fact, that $[f]$ is an extensional map in a Q-set $(\Lambda, \rho_{X,\mathcal{A}})$ (see Corollary 3.3), we obtain

$$F^\uparrow(f)(\lambda) = \bigvee_{x\in X} f(x) \otimes A_\lambda(x) = \bigvee_{x\in X} [f](\lambda_x) \otimes A_\lambda(x) \le \bigvee_{x\in X} [f](\lambda_x) \otimes \rho_{X,\mathcal{A}}(\lambda, \lambda_x) \le [f](\lambda) = f(z).$$

On the other hand, we have

$$F^\uparrow(f)(\lambda) = \bigvee_{x\in X} [f](\lambda_x) \otimes A_\lambda(x) \ge \bigvee_{x\in core(A_\lambda)} [f](\lambda_x) = [f](\lambda) = f(z).$$

Now using above mentioned properties of $[f]$ and $\rho_{X,\mathcal{A}}$, we obtain

$$F^\downarrow(f)(\lambda) = \bigwedge_{x\in X} A_\lambda(x) \to f(x) = \bigwedge_{x\in X} A_\lambda(x) \to [f](\lambda_x) \ge \bigwedge_{x\in X} \rho_{X,\mathcal{A}}(\lambda, \lambda_x) \to [f](\lambda_x) \ge \bigwedge_{x\in X} [f](\lambda) = f(z).$$

On the other hand, we have $F^\downarrow(f)(\lambda) \le A_\lambda(z) \to [f](\lambda_z) = 1_Q \to f(z) = f(z)$, and the proposition is proved. □

In the paper Močkoř (to appear), we introduced functors $\mathcal{F}^\uparrow : SpaceFP \to$ **KClo** and $\mathcal{F}^\downarrow : SpaceFP \to$ **KInt** from the category $SpaceFP$ to the category of Kuratowski closure operators **KClo** and Kuratowski interior operators **KInt**, respectively. In the following example we show that fuzzy objects in $(X, \mathcal{A})$ are fix points in both closure and interior spaces $\mathcal{F}^\uparrow(X, \mathcal{A})$ and $\mathcal{F}^\downarrow(X, \mathcal{A})$.

Example 3.6 According to Močkoř (to appear); Theorem 4.1 and Theorem 4.2, closure and interior spaces $\mathcal{F}^\uparrow(X, \mathcal{A})$ and $\mathcal{F}^\downarrow(X, \mathcal{A})$ are defined by

$$\mathcal{F}^\uparrow(X, \mathcal{A}) = ((Q^X)^\Lambda, c), \quad \mathcal{F}^\downarrow(X, \mathcal{A}) = ((Q^X)^\Lambda, i),$$

where $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ and for $\mathbf{f} = (f_\lambda)_\lambda \in (Q^X)^\Lambda$, $c(\mathbf{f}) = (c_\lambda(f_\lambda))$, $i(\mathbf{f}) = (i_\lambda(f_\lambda))$,

$$c_\lambda(f_\lambda)(x) = \begin{cases} F^\uparrow(f_\lambda) \\ f(x) \end{cases} \quad i_\lambda(f_\lambda)(x) = \begin{cases} F^\downarrow(f_\lambda) & \text{iff } x \in core(A_\lambda), \\ f_\lambda(x) & \text{otherwise.} \end{cases}$$

Let $f \in \mathcal{F}_1(X, \mathcal{A})$ and let $\mathbf{f} = (f_\lambda)_\lambda \in (Q^X)^\Lambda$, where $f_\lambda = f$ for any $\lambda \in \Lambda$. Then according to Proposition 3.5, we have $c(\mathbf{f}) = i(\mathbf{f}) = \mathbf{f}$. □

4 Powerset objects in spaces with fuzzy partitions

In the next part, we will deal with the powerset object

$$\mathcal{F}(X, \mathcal{A}) = ((Q, \mathcal{Q})^{(X,\mathcal{A})}, \leq)$$

of a space with a fuzzy partition with ordering defined in Lemma 3.1. Our goal is to prove that, analogically as powerset objects of classical fuzzy sets or powerset objects of fuzzy objects in sets with similarity relations, $\mathcal{F}(X, \mathcal{A})$ satisfies conditions of a general definition of powerset objects, presented by Rodabaugh (2007), i.e.,

Definition 4.1 (Rodabaugh 2007) Let $\mathbf{K}$ be a category and let $CSLAT$ be the category of complete $\vee$-semilattices with $\vee$-preserving maps as morphisms. Then $\mathbf{P} = (P, \to, \leftarrow, V, \eta)$ is called $CSLAT$**-powerset theory in K**, if

1. $P : |\mathbf{K}| \to |CSLAT|$ is an object-mapping,
2. for each $f : A \to B$ in $\mathbf{K}$, there exists $f_{\mathbf{P}}^{\to} : P(A) \to P(B)$ in $CSLAT$,
3. for each $f : A \to B$ in $\mathbf{K}$, there exists $f_{\mathbf{P}}^{\leftarrow} : P(B) \to P(A)$ in $CSLAT$,
4. $(f_{\mathbf{P}}^{\to}, f_{\mathbf{P}}^{\leftarrow})$ is a Galois connection,
5. There exists a concrete functor $V : \mathbf{K} \to Set$, such that η determines in § for each $A \in \mathbf{K}$ a mapping $\eta_A : V(A) \to P(A)$,
6. For each $f : A \to B$ in $\mathbf{K}$, $f_{\mathbf{P}}^{\to} \circ \eta_A = \eta_B \circ V(f)$.

We want to show firstly that the object functions $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$ defined by (1), (2) and (3), define functors $SpaceFP \to CSLAT$. To do it, we repeat the definition of the powerset object functor $F : \mathrm{Set}(Q) \to CSLAT$, such that for any Q-set (X, δ),

$$F(X, \delta) = ((Q, \leftrightarrow)^{(X,\delta)}, \leq)$$

is the ordered set of all extensional maps $X \to Q$, i.e., morphisms $(X, \delta) \to (Q, \leftrightarrow)$ in the category $\mathrm{Set}(Q)$. If $f : (X, \delta) \to (Y, \gamma)$ is a morphism in the category $\mathrm{Set}(Q)$, then $f_F^{\to} = F(f) : F(X, \delta) \to F(Y, \gamma)$ is defined by $f_F^{\to}(s)(y) = \bigvee_{x \in X} s(x) \otimes \gamma(f(x), y)$, for $s \in F(X, \delta)$, $y \in Y$. For more information about the functor F see, e.g., Močkoř (2016).

Recall that if $f : L \to M$, $g : M \to L$ are isotone maps between pre-ordered sets, then $f \vdash g$, provided that for any $a \in L, b \in M$, $a \leq g(b) \Leftrightarrow f(a) \leq b$. It is clear that $f \vdash g$ iff (f, g) is a Galois connection, i.e., $f \circ g \leq 1_M$, $g \circ f \geq 1_L$. From a general lattice theory the following Adjoint Functor Theorem is well known (see, e.g., Rodabaugh 1997).

Theorem 4.1 *Let L, M be partially ordered sets such that L has arbitrary $\bigvee$ and let $f^{\to} : L \to M$ be a map which preserves arbitrary $\bigvee$. Then the map $f^{\leftarrow} : M \to L$ defined by*

$$(\forall y \in M) \quad f^{\leftarrow}(y) = \bigvee_{\{x \in L,\, f^{\to}(x) \leq y\}} x,$$

is the unique map $M \to L$ such that

1. *$(f^{\to}, f^{\leftarrow})$ is a Galois connection,*
2. *$f^{\leftarrow}$ preserves all meets in M.*

We will frequently deal with the following situation. Let $\mathbf{K}$ be a category and let $P : \mathbf{K} \to CSLAT$ be a covariant functor. It follows that for any morphism $f : A \to B$, $P(f)$ is a map preserving all sup. Instead of $P(f)$, we use $f_P^{\to}$.

By using Theorem 4.1, for any morphism $f : A \to B$ in $\mathbf{K}$, there exists the map $f_P^{\leftarrow} : P(B) \to P(A)$ defined by

$$(\forall Y \in P(B)) \quad f_P^{\leftarrow}(Y) = \bigvee_{\{X \in P(A) : f_P^{\to}(X) \leq Y\}} X. \tag{5}$$

It is then clear that $f_P^{\leftarrow} : P(B) \to P(A)$ preserves all existing meets and $(f_P^{\to}, f_P^{\leftarrow})$ is a Galois connection. *If a functor P will be given, then by $f_P^{\leftarrow}$ we will understand the map defined by (5) from $P(f) = f_P^{\to}$.*

Theorem 4.2 (Extension principle for fuzzy objects) *For any $(X, \mathcal{A}) \in |SpaceFP|$, $\mathcal{F}_1(X, \mathcal{A})$, $\mathcal{F}_2(X, \mathcal{A})$ and $\mathcal{F}(X, \mathcal{A})$ define object functions of functors $SpaceFP \to CSLAT$.*

Proof Let $(f, \sigma) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism in $SpaceFP$. Then according to Proposition 3.4,

$$\sigma = G(f, \sigma) : G(X, \mathcal{A}) = (\Lambda, \rho_{X,\mathcal{A}}) \to (\Omega, \rho_{Y,\mathcal{B}}) = G(Y, \mathcal{B})$$

is a morphism in the category $\mathrm{Set}(Q)$ and by using the functor $F : \mathrm{Set}(Q) \to CSLAT$, we obtain a morphism

$$\sigma_F^{\to} := F(\sigma) = FG(f, \sigma) : F(\Lambda, \rho_{X,\mathcal{A}}) \to F(\Omega, \rho_{Y,\mathcal{B}}).$$

According to Corollary 3.3, we have $\mathcal{F}_2(X, \mathcal{A}) = F(\Lambda, \rho_{X,\mathcal{A}})$ and we can define

$$(f, \sigma)_{\mathcal{F}_2}^{\to} = \mathcal{F}_2(f, \sigma) := FG(f, \sigma) = \sigma_F^{\to}.$$

It means that if $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$, then for any $\tau \in \mathcal{F}_2(X, \mathcal{A})$, $\omega \in \Omega$, $(f, \sigma)_{\mathcal{F}_2}^{\to}(\tau)$ is a function $\Omega \to Q$ defined by

$$(f, \sigma)_{\mathcal{F}_2}^{\to}(\tau)(\omega) = \bigvee_{\lambda \in \Lambda} \tau(\lambda) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega).$$

Since F, G are functors, $\mathcal{F}_2 = FG$ is also a functor $SpaceFP \to CSLAT$.

Now we define a functor $\mathcal{F}_1$, i.e., we define a function

$$(f,\sigma)^{\rightarrow}_{\mathcal{F}_1} = \mathcal{F}_1(f,\sigma) : \mathcal{F}_1(X,\mathcal{A}) \to \mathcal{F}_1(Y,\mathcal{B}).$$

According to Proposition 3.4, $f = H(f,\sigma) : H(X,\mathcal{A}) = (X,\delta_{X,\mathcal{A}}) \to (Y,\delta_{Y,\mathcal{B}}) = H(Y,\mathcal{B})$ is a morphism in the category Set(Q) and, again by using the functor F, we can define

$$(f,\sigma)^{\rightarrow}_{\mathcal{F}_1} = \mathcal{F}_1(f,\sigma) = FH(f,\sigma) = f^{\rightarrow}_F.$$

It means that if $t \in \mathcal{F}_1(X,\mathcal{A})$, then we have

$$y \in Y, \quad (f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(t)(y) = \bigvee_{x\in X} t(x) \otimes \delta_{Y,\mathcal{B}}(f(x),y).$$

Hence $\mathcal{F}_1 = FH$ is a functor.

Now we define the functor $\mathcal{F}$. Let $(t,\tau) \in \mathcal{F}(X,\mathcal{A})$ be a fuzzy object. Then we set

$$(f,\sigma)^{\rightarrow}_{\mathcal{F}}(t,\tau) = \mathcal{F}(f,\sigma)(t,\tau) := (\mathcal{F}_1(f,\sigma)(t), \mathcal{F}_2(f,\sigma)(\tau)).$$

Using the definition of morphisms composition in the category $SpaceFP$, it is cleat that $\mathcal{F}$ is a functor. □

Proposition 4.1 *There exist mutually inverse natural transformations*

$$\mathcal{F}_1 \underset{|.|}{\overset{[.]}{\rightleftarrows}} \mathcal{F}_2.$$

Proof Let $(X,\mathcal{A})$ be a space with a fuzzy partition. We have $\mathcal{F}_1 = FH, \mathcal{F}_2 = FG$. Then for $t \in FH(X,\mathcal{A})$, $g \in G(X,\mathcal{A})$, we put $[.]_{(X,\mathcal{A})}(t) = [t]$, $|.|_{(X,\mathcal{A})}(g) = |g|$, where we use a notation from Corollary 3.1. Then for any $\lambda \in \Lambda$, $x \in X$, $x \in core(A_\lambda)$, we have

$$|.|_{(X,\mathcal{A})} \circ [.]_{(X,\mathcal{A})}(t)(x) = |[t]|(x) = [t](\lambda) = t(x),$$
$$[.]_{(X,\mathcal{A})} \circ |.|_{(X,\mathcal{A})}(g)(\lambda) = [|g|](\lambda) = |g|(x) = g(\lambda),$$

and it follows that these maps are mutually inverse. We show that $[.]$ is a natural transformation. Let $(f,\sigma) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a morphism in $SpaceFP$. Then the following diagram commutes:

$$\begin{array}{ccc} FH(X,\mathcal{A}) = F(X,\delta_{X,\mathcal{A}}) & \xrightarrow{[.]_{(X,\mathcal{A})}} & F(\Lambda,\rho_{X,\mathcal{A}}) = FG(X,\mathcal{A}) \\ {\scriptstyle f^{\rightarrow}_F = FH(f,\sigma)}\downarrow & & \downarrow{\scriptstyle FG(f,\sigma)=\sigma^{\rightarrow}_F} \\ FH(Y,\mathcal{B}) = F(Y,\delta_{Y,\mathcal{B}}) & \xrightarrow{[.]_{(Y,\mathcal{B})}} & F(\Omega,\rho_{Y,\mathcal{B}}) = FG(Y,\mathcal{B}). \end{array}$$

In fact, let $t \in FH(X,\mathcal{A})$, $\omega \in \Omega$ and $y \in core(B_\omega)$. Then we have

$$\begin{aligned} \sigma^{\rightarrow}_F[.]_{(X,\mathcal{A})}(t)(\omega) &= \sigma^{\rightarrow}_F([t])(\omega) \\ &= \bigvee_{\lambda\in\Lambda} [t](\lambda) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega) \\ &= \bigvee_{x\in X} t(x) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega) \\ &= \bigvee_{x\in X} t(x) \otimes \delta_{Y,\mathcal{B}}(f(x),y) = f^{\rightarrow}_F(t)(y) \\ &= [f^{\rightarrow}_F(t)](\omega) = [.]_{Y,\mathcal{B}} f^{\rightarrow}_F(t)(\omega). \end{aligned}$$

Similarly it can be proved that $|.|$ is a natural transformation. □

It should be observed that from Proposition 4.1 and Corollary 3.1, it follows for any morphism $(f,\sigma) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ in $SpaceFP$,

$$t \in \mathcal{F}_1(X,\mathcal{A}), \quad \mathcal{F}_1(f,\sigma)(t) = |\mathcal{F}_2(f,\sigma)([t])|. \tag{6}$$

Let $f : (X,\delta) \to (Y,\gamma)$ be a morphism in the category Set(Q). According to Proposition 3.1, any (X,δ) defines a space with a fuzzy partition $(X,\mathcal{C}_{X,\delta})$. Then according to Example 3.2, we have $F(X,\delta) = \mathcal{F}_1(X,\mathcal{C}_{X,\delta})$ and any fuzzy set t in (X,δ), i.e., morphisms $(X,\delta) \to (Q,\leftrightarrow)$ in the category Set(Q), can be extended to the fuzzy set in a Q-set (Y,γ) in two ways: either by extended Zadeh's principle in the category Set(Q), or by extension principle in the category $SpaceFP$, i.e., either by $f^{\rightarrow}_F : F(X,\delta) \to F(Y,\delta)$, or $f^{\rightarrow}_{\mathcal{F}_1} : \mathcal{F}_1(X,\mathcal{C}_{X,\delta}) \to \mathcal{F}_1(Y,\mathcal{C}_{Y,\gamma})$. Hence we can consider the diagram

$$\begin{array}{ccc} F(X,\delta) & \xrightarrow{f^{\rightarrow}_F} & F(Y,\gamma) \\ \| & & \| \\ \mathcal{F}_1(X,\mathcal{C}_{X,\delta}) & \xrightarrow{f^{\rightarrow}_{\mathcal{F}_1}} & \mathcal{F}_1(Y,\mathcal{C}_{Y,\gamma}). \end{array}$$

In the next theorem, we prove these extensions are identical, i.e., the extension principle in the category $SpaceFP$ equals to the extension principle in the category Set(Q).

Proposition 4.2 *The following diagram of functors commutes.*

$$\begin{array}{ccc} Set(Q) & \overset{I}{\hookrightarrow} & SpaceFP \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \mathcal{F}_1} & \\ & CSLAT & \end{array}$$

Proof For object functions of these functors we have $\mathcal{F}_1 I(X,\delta) = F(X,\delta)$. Let $f : (X,\delta) \to (Y,\gamma)$ be a morphism in Set(Q). Then according to Proposition 3.1, $I(f) = (f, \sigma) : (X, \mathcal{C}_{X,\delta}) \to (Y, \mathcal{C}_{Y,\gamma})$ is such that

$$\mathcal{C}_{X,\delta} = \{C_{X,\mathbf{a}} : \mathbf{a} \in X/\delta\}, \quad \mathcal{C}_{Y,\gamma} = \{C_{Y,\mathbf{b}} : \mathbf{b} \in Y/\gamma\},$$
$$\sigma : X/\delta \to Y/\gamma, \quad \sigma(\mathbf{a}) = \mathbf{b} \Leftrightarrow (x \in \mathbf{a} \Rightarrow f(x) \in \mathbf{b}),$$
$$C_{X,\mathbf{a}}(x) = \delta(x, z), z \in \mathbf{a}.$$

We need to prove $(f, \sigma)^{\rightarrow}_{\mathcal{F}_1} = \mathcal{F}_1 I(f) = F(f) = f^{\rightarrow}_F$. Let $t \in F(X, \delta)$. Then we have

$$y \in Y, \quad f^{\rightarrow}_F(t)(y) = \bigvee_{x \in X} t(x) \otimes \gamma(f(x), y).$$

We need to calculate $(f, \sigma)^{\rightarrow}_{\mathcal{F}_1}(t)(y) = \mathcal{F}_1(f, \sigma)(t)(y)$. According to relation (6) and Proposition 3.4, for $\mathbf{b} \in Y/\gamma$ such that $y \in \mathbf{b}$, we have

$$\begin{aligned}\mathcal{F}_1(f, \sigma)(t)(y) &= |\mathcal{F}_2(f, \sigma)([t])|(y) = \mathcal{F}_2(f, \sigma)([t])(\mathbf{b}) \\ &= FG(f, \sigma)([t])(\mathbf{b}) = F(\sigma)([t])(\mathbf{b}) \\ &= \sigma^{\rightarrow}_F([t])(\mathbf{b}) = \bigvee_{\mathbf{a} \in X/\delta} [t](\mathbf{a}) \otimes \underline{\gamma}(\sigma(\mathbf{a}), \mathbf{b}) \\ &= \bigvee_{x \in \mathbf{a} \in X/\delta} t(x) \otimes \underline{\gamma}(\mathbf{f}(\mathbf{x}), \mathbf{b}) \\ &= \bigvee_{x \in X} t(x) \otimes \gamma(f(x), y) = f^{\rightarrow}_F(t)(y).\end{aligned}$$

Hence we received $\mathcal{F}_1 I(f) = F(f)$ and the diagram of functors commutes. □

The principal goal of the paper is to solve the question, if the powerset objects and powerset operators $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of new fuzzy objects in the category *SpaceFP* presented above have similar properties to those of classical fuzzy objects $s : X \to Q$ or to fuzzy objects in the category Set(Q). We want to show that all these fuzzy objects have powerset structures which are powerset theories in the category *SpaceFP*, in the sense of Rodabaugh (2007). For classical Zadeh's powerset theory $\mathbf{Z}$ and classical powerset theory $\mathbf{P}$ in sets, there exists a strong relation between these two theories, which can be represented as some homomorphism $\mathbf{P} \to \mathbf{Z}$. We show that analogously for these new powerset theories $\mathbf{F}$ there exist "new classical" powerset theories $\mathbf{R}$ and a homomorphism $\mathbf{R} \to \mathbf{F}$.

We introduce firstly concrete functors T_1, T_2 and T in the category *SpaceFP*, which are an analogy of the classical functor $X \mapsto (2^X, \subseteq)$ in the category *Set*. The functors $T_i : SpaceFP \to CSLAT$ are defined for objects $(X, \mathcal{A})$ and morphisms (f, σ) by

$$T_1(X, \mathcal{A}) = (2^X, \subseteq), \quad (\forall A \subseteq X)(f, \sigma)^{\rightarrow}_{T_1}(A) = f(A) \quad (7)$$
$$T_2(X, \mathcal{A}) = (2^\Lambda, \subseteq), \quad (\forall \Phi \subseteq \Lambda)(f, \sigma)^{\rightarrow}_{T_2}(\Phi) = \sigma(\Phi). \quad (8)$$

The functor T will be defined lately.

Proposition 4.3 *There exist the following natural transformations:*

(2) $\eta_1 : T_1 \to \mathcal{F}_1$,
(3) $\eta_2 : T_2 \to \mathcal{F}_2$.

Proof Let $(f, \sigma) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism in the category *SpaceFP* and let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$.

We defined firstly a natural transformation η_2. Let for any $\Phi \subseteq \Lambda$, $\chi_\Lambda(\Phi) : \Lambda \to Q$ be a characteristic map of a subset Φ in a set Λ. Then according to the proof of Theorem 3.1 and Corollary 3.3, we can define

$$\eta_{2,(X,\mathcal{A})} : T_2(X, \mathcal{A}) \to \mathcal{F}_2(X, \mathcal{A}) = F(\Lambda, \rho_{X,\mathcal{A}}),$$
$$\Phi \subseteq \Lambda, \quad \eta_{2,(X,\mathcal{A})}(\Phi) = \widehat{\chi_\Lambda(\Phi)} \in \mathcal{F}_2(X, \mathcal{A}).$$

Then η_2 is a natural transformation. In fact, we prove that the following diagram commutes.

$$\begin{array}{ccc} T_2(X, \mathcal{A}) & \xrightarrow{\eta_{2,(X,\mathcal{A})}} & \mathcal{F}_2(X, \mathcal{A}) \\ {\scriptstyle (f,\sigma)^{\rightarrow}_{T_2}} \downarrow & & \downarrow {\scriptstyle (f,\sigma)^{\rightarrow}_{\mathcal{F}_2}} \\ T_2(Y, \mathcal{B}) & \xrightarrow{\eta_{2,(Y,\mathcal{B})}} & \mathcal{F}_2(Y, \mathcal{B}). \end{array}$$

Let $\Phi \in T_2(X, \mathcal{A})$. Then according to the proof of Theorem 3.1, for any $\omega \in \Omega$, we have

$$\begin{aligned}&(f, \sigma)^{\rightarrow}_{\mathcal{F}_2}.\eta_{2,(X,\mathcal{A})}(\Phi)(\omega) = (f, \sigma)^{\rightarrow}_{\mathcal{F}_2}.\widehat{\chi_\Lambda(\Phi)}(\omega) \\ &= \sigma^{\rightarrow}_F(\widehat{\chi_\Lambda(\Phi)})(\omega) \\ &= \bigvee_{\lambda \in \Lambda} \widehat{\chi_\Lambda(\Phi)}(\lambda) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega) \\ &= \bigvee_{\lambda \in \Lambda} \bigvee_{\lambda' \in \Lambda} \chi_\Lambda(\Phi)(\lambda') \otimes \rho_{X,\mathcal{A}}(\lambda, \lambda') \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega) \\ &= \bigvee_{\lambda \in \Lambda} \bigvee_{\lambda' \in \Phi} \rho_{X,\mathcal{A}}(\lambda, \lambda') \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega) = (*).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\eta_{2,(Y,\mathcal{B})}.(f, \sigma)^{\rightarrow}_{T_2}(\Phi)(\omega) \\ &= \eta_{2,(Y,\mathcal{B})}(\sigma(\Phi))(\omega) = \chi_\Omega\widehat{(\sigma(\Phi))}(\omega) \\ &= \bigvee_{\omega' \in \Omega} \chi_\Omega(\sigma(\Phi))(\omega') \otimes \rho_{Y,\mathcal{B}}(\omega, \omega') \\ &= \bigvee_{\omega' \in \sigma(\Phi)} \rho_{Y,\mathcal{B}}(\omega, \omega') = (**).\end{aligned}$$

Then we obtain

$$(*) \leq \bigvee_{\lambda\in\Lambda}\bigvee_{\lambda'\in\Phi} \rho_{Y,\mathcal{B}}(\sigma(\lambda),\sigma(\lambda'))\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$\leq \bigvee_{\lambda'\in\Phi}\rho_{Y,\mathcal{B}}(\sigma(\lambda'),\omega) = (**),$$
$$(*) \geq \bigvee_{\lambda'\in\Phi}\rho_{X,\mathcal{A}}(\lambda',\lambda')\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda'),\omega) = (**).$$

Hence the diagram commutes and η_2 is a natural transformation.

We define a natural transformation η_1. Let $A \subseteq X$. Then $\chi_X(A) : X \to Q$ is a characteristic map of a subset A and we can define a map $\tau_{X,A} : \Lambda \to Q$ by

$$\tau_{X,A}(\lambda) = \bigvee_{x\in core(A_\lambda)}\chi_X(A)(x) = \begin{cases} 1_Q & A\cap core(A_\lambda)\neq\emptyset \\ 0_Q & \text{otherwise.}\end{cases}$$

Then we set

$$\eta_{1,(X,\mathcal{A})} : T_1(X,\mathcal{A}) \to \mathcal{F}_1(X,\mathcal{A}),$$
$$A\subseteq X,\ x\in core(A_\lambda),\quad \eta_{1,(X,\mathcal{A})}(A)(x) = \widehat{\tau_{X,A}}(\lambda).$$

We show that η_1 is a natural transformation, i.e., for a morphism (f,σ), the following diagram commutes.

$$\begin{array}{ccc} T_1(X,\mathcal{A}) & \xrightarrow{\eta_{1,(X,\mathcal{A})}} & \mathcal{F}_1(X,\mathcal{A}) \\ {\scriptstyle (f,\sigma)^{\to}_{T_1}}\downarrow & & \downarrow{\scriptstyle (f,\sigma)^{\to}_{\mathcal{F}_1}} \\ T_1(Y,\mathcal{B}) & \xrightarrow{\eta_{1,(Y,\mathcal{B})}} & \mathcal{F}_1(Y,\mathcal{B}). \end{array}$$

In fact, let $y \in core(B_\omega)$, then we have

$$\eta_{1,(Y,\mathcal{B})}.(f,\sigma)^{\to}_{T_1}(A)(y)$$
$$= \eta_{1,(Y,\mathcal{B})}(f(A))(y) = \widehat{\tau_{Y,f(A)}}(\omega) = (***),$$

where

$$\tau_{Y,f(A)}(\omega) = \bigvee_{y\in core(B_\omega)\cap f(A)}(\chi_Y(f(A))(y))$$
$$= \begin{cases} 1_Q & f(A)\cap core(B_\omega)\neq\emptyset \\ 0_Q & \text{otherwise}\end{cases}.$$

On the other hand, by using the relation (6), we have

$$(f,\sigma)^{\to}_{\mathcal{F}_1}.\eta_{1,(X,\mathcal{A})}(A)(y) = |\mathcal{F}_2(f,\sigma)([\eta_{1,(X,\mathcal{A})}(A)])|(y)$$
$$= |\sigma^{\to}_F([\eta_{1,(X,\mathcal{A})}(A)])|(y)$$
$$= \sigma^{\to}_F([\eta_{1,(X,\mathcal{A})}(A)])(\omega)$$
$$= \bigvee_{\lambda\in\Lambda}[\eta_{1,(X,\mathcal{A})}(A)](\lambda)\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$= \bigvee_{\lambda\in\Lambda}\bigvee_{x\in core(A_\lambda)}\eta_{1,(X,\mathcal{A})}(A)(x)\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$= \bigvee_{\lambda\in\lambda}\widehat{\tau_{X,A}}(\lambda)\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$= \bigvee_{\lambda\in\Lambda}\bigvee_{\lambda'\in\Lambda}\tau_{X,A}(\lambda')\otimes\rho_{X,\mathcal{A}}(\lambda,\lambda')\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$= (****).$$

Since $\{\sigma(\lambda') : A\cap core(A_{\lambda'})\neq\emptyset\} \subseteq \{\omega' : f(A)\cap core(B_{\omega'})\neq\emptyset\}$, by using properties of a similarity relation ρ and a morphism σ in the category Set(Q), we obtain

$$(****) \leq \bigvee_{\lambda\in\Lambda}\ \bigvee_{\lambda',A\cap core(A_{\lambda'})\neq\emptyset}\rho_{Y,\mathcal{B}}(\sigma(\lambda),\sigma(\lambda'))$$
$$\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega) \leq \bigvee_{\lambda',A\cap core(A_{\lambda'})\neq\emptyset}\rho_{Y,\mathcal{B}}(\sigma(\lambda'),\omega)$$
$$\leq \bigvee_{\omega',f(A)\cap core(B_{\omega'})\neq\emptyset}\rho_{Y,\mathcal{B}}(\omega',\omega) = (***).$$

On the other hand, if we set $\lambda := \lambda'$, we have

$$(****) \geq \bigvee_{\lambda',A\cap core(A_{\lambda'})\neq\emptyset}\rho_{Y,\mathcal{B}}(\sigma(\lambda'),\omega) = (***).$$

Therefore the diagram commutes and η_1 is a natural transformation.

It should be observed that η_1 could be equivalently defined also by using the similarity relation $\delta_{X,\mathcal{A}}$. In fact, $\eta_{1,(X,\mathcal{A})}$ can be defined as an extension of χ_A to an extensional map in the Q-set $(X,\delta_{X,\mathcal{A}})$, i.e., by $\eta_{1,(X,\mathcal{A})}(A)(x) = \bigvee_{z\in X}\chi_A(z)\otimes\delta_{X,\mathcal{A}}(x,z) = \bigvee_{z\in A}\delta_{X,\mathcal{A}}(x,z)$. □

We define the functor $T : SpaceFP \to CSLAT$. Let $(X,\mathcal{A})$ be a space with a fuzzy partition, then we set

$$T(X,\mathcal{A}) = \{(A,\Phi) : A\subseteq X, \Phi\subseteq\Lambda, [\eta_{1,(X,\mathcal{A})}(A)]$$
$$= \eta_{2,(X,\mathcal{A})}(\Phi)\} = \{(A,\Phi) : A\subseteq X, \Phi\subseteq\Lambda, \eta_{1,(X,\mathcal{A})}(A)$$
$$= |\eta_{2,(X,\mathcal{A})}(\Phi)|\} \subseteq T_1(X,\mathcal{A})\times T_2(X,\mathcal{A}),$$

where we use a notation from (4) and the set $T(X,\mathcal{A})$ is ordered point-wise by inclusion. Let $(f,\sigma) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a morphism in $SpaceFP$. Then a $\bigvee$-preserving map $T(f,\sigma) = (f,\sigma)^{\to}_T$ is defined by

$$(A,\Phi)\in T(X,\mathcal{A}),\quad T(f,\sigma)(A,\Phi) = (f(A),\sigma(\Phi)).$$

Proposition 4.4 *$T : SpaceFP \to CSLAT$ is a functor.*

Proof We need to prove that $T(f,\sigma)(A,\Phi) \in T(Y,\mathcal{B})$. According to Proposition 4.1, $[.] : \mathcal{F}_1 \to \mathcal{F}_2$ is a natural transformation. Using properties of $(A,\Phi)\in T(X,\mathcal{A})$, natural transformation [.], and natural transformation $\eta_1 : T_1 \to$

$\mathcal{F}_1$, we obtain

$$\begin{aligned}[\eta_{1,(Y,\mathcal{B})}(f(A))] &= [(f,\sigma)^{\rightarrow}_{\mathcal{F}_1}.\eta_{1,(X,\mathcal{A})}(A)]\\ &= (f,\sigma)^{\rightarrow}_{\mathcal{F}_2}([\eta_{1,(X,\mathcal{A})}(A)])\\ &= (f,\sigma)^{\rightarrow}_{\mathcal{F}_2}(\Phi) = \eta_{2,(Y,\mathcal{B})}(\sigma(\Phi)).\end{aligned}$$

Therefore $(f(A),\sigma(\Phi)) \in T(Y,\mathcal{B})$ and T is a functor such that $T(f,\sigma)$ preserves $\bigvee$. □

Theorem 4.3 *There exists a natural transformation*

$$\eta : T \to \mathcal{F}.$$

Proof For any $(X,\mathcal{A}) \in |SpaceFP|$, we define a map

$$\begin{aligned}&\eta_{(X,\mathcal{A})} : T(X,\mathcal{A}) \to \mathcal{F}(X,\mathcal{A}),\\ &\quad (A,\Phi) \in T(X,\mathcal{A}), \quad \eta_{(X,\mathcal{A})}(A,\Phi)\\ &\quad = (\eta_{1,(X,\mathcal{A})}(A), \eta_{2,(X,\mathcal{A})}(\Phi)).\end{aligned}$$

This definition is correct, as follows directly from (4), using the relation

$$(\eta_{1,(X,\mathcal{A})}(A), \eta_{2,(X,\mathcal{A})}(\Phi)) = (\eta_{1,(X,\mathcal{A})}(A), [\eta_{1,(X,\mathcal{A})}(A)]).$$

Using properties of natural transformations η_1, η_2, it can be proved simply that $\eta : T \to \mathcal{F}$ is also a natural transformation. □

Theorem 4.4 *(*Powerset theories in SpaceFP*) Let $V : CSLAT \to Set$ be the forgetful functor. The following statements then hold.*

(1) $\mathbf{F_1} = (\mathcal{F}_1, \to, V.T_1, \eta_1)$ *is a $CSLAT$-powerset theory in the category $SpaceFP$.*
(2) $\mathbf{F_2} = (\mathcal{F}_2, \to, V.T_2, \eta_2)$ *is a $CSLAT$-powerset theory in the category $SpaceFP$.*
(3) $\mathbf{F} = (\mathcal{F}, \to, V.T, \eta)$ *is a $CSLAT$-powerset theory in the category $SpaceFP$.*

Proof Let $(f,\sigma) : (X,\mathcal{A}) \to (Y,\mathcal{B})$ be a morphism in $SpaceFP$, $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$. From Definition 4.1 and Proposition 4.3 it follows, that we need only to define the maps $(f,\sigma)^{\leftarrow}$ for functors $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}$, and to prove that $((f,\sigma)^{\rightarrow}, (f,\sigma)^{\leftarrow})$ is a Galois connection for these functors.

(1) Let $t \in \mathcal{F}_1(Y,\mathcal{B})$. We define $(f,\sigma)^{\leftarrow}_{\mathcal{F}_1} : \mathcal{F}_1(Y,\mathcal{B}) \to \mathcal{F}_1(X,\mathcal{A})$ by

$$x \in X, \quad (f,\sigma)^{\leftarrow}_{\mathcal{F}_1}(t)(x) = (t \circ f)(x).$$

The definition is correct. In fact, we need only to verify the condition 2(b) from Proposition 3.3. Let $\lambda \in \Lambda$, $x \in core(A_\lambda)$ and $x' \in X$. Then since (f,σ) is a morphism, we obtain $f(x) \in f(core(A_\lambda)) \subseteq core(B_{\sigma(\lambda)})$, and it follows that

$$(t \circ f)(x) \leftrightarrow (t \circ f)(x') \geq B_{\sigma(\lambda)}(f(x')) \geq A_\lambda(x').$$

Hence $t \circ f$ defines a fuzzy object in the category $SpaceFP$. To prove that $((f,\sigma)^{\rightarrow}_{\mathcal{F}_1}, (f,\sigma)^{\leftarrow}_{\mathcal{F}_1})$ is the Galois connection, we prove that the relation (1) holds, i.e., we prove that

$$t \in \mathcal{F}_1(Y,\mathcal{B}), \quad (f,\sigma)^{\leftarrow}_{\mathcal{F}_1}(t) = \bigvee_{s \in \mathcal{F}_1(X,\mathcal{A}), (f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(s) \leq t} s. \tag{9}$$

In fact, let $s := t \circ f$ and $y \in core(B_\omega)$. Then according to Corollary 3.3, $[t]$ is an extensional map in a Q-set $(\Omega, \rho_{Y,\mathcal{B}})$, and according to Corollary 3.1, we have

$$\begin{aligned}&(f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(t \circ f)(y) = |(f,\sigma)^{\rightarrow}_{\mathcal{F}_2}([t \circ f])|(y)\\ &\quad = (f,\sigma)^{\rightarrow}_{\mathcal{F}_2}([t \circ f])(\omega)\\ &\quad = \bigvee_{\lambda \in \Lambda} [t \circ f](\lambda) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega)\\ &\quad = \bigvee_{\lambda \in \Lambda} t(f(x_\lambda)) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega)\\ &\quad = \bigvee_{\lambda} [t](\sigma(\lambda)) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \omega) \leq [t](\omega) = t(y),\end{aligned}$$

where $x_\lambda \in core(A_\lambda)$. Hence we have

$$\bigvee_{s \in \mathcal{F}_1(X,\mathcal{A}), (f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(s) \leq t} s \geq t \circ f.$$

On the other hand, let $s \in \mathcal{F}_1(X,\mathcal{A})$ be such that $(f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(s) \leq t$, i.e., for any $x \in core(A_\lambda)$, we have $f(x) \in core(B_{\sigma(\lambda)})$, and according to Corollary 3.1, we obtain

$$\begin{aligned}&t(f(x)) \geq (f,\sigma)^{\rightarrow}_{\mathcal{F}_1}(s)(f(x)) = |(f,\sigma)^{\rightarrow}_{\mathcal{F}_2}([s])|(f(x))\\ &\quad = (f,\sigma)^{\rightarrow}_{\mathcal{F}_2}([s])(\sigma(\lambda))\\ &\quad = \bigvee_{\alpha \in \Lambda} [s](\alpha) \otimes \rho_{Y,\mathcal{B}}(\sigma(\lambda), \sigma(\alpha)) \geq [s](\lambda) = s(x).\end{aligned}$$

Hence we have $s(x) \leq t(f(x))$ and it follows that the relation (9) holds. Therefore $((f,\sigma)^{\rightarrow}_{\mathcal{F}_1}, (f,\sigma)^{\leftarrow}_{\mathcal{F}_1})$ is a Galois connection and from Proposition 4.3, it follows that $\mathbf{F_1}$ is a $CSLAT$-powerset theory in the category $SpaceFP$.

(2) Let $\tau \in \mathcal{F}_2(Y,\mathcal{B})$. We define $(f,\sigma)^{\leftarrow}_{\mathcal{F}_2} : \mathcal{F}_2(Y,\mathcal{B}) \to \mathcal{F}_2(X,\mathcal{A})$ by

$$\lambda \in \Lambda, \quad (f,\sigma)^{\leftarrow}_{\mathcal{F}_2}(\tau)(\lambda) = (\tau \circ \sigma)(\lambda).$$

To prove that the definition is correct is similar to the proof for $(f,\sigma)^{\leftarrow}_{\mathcal{F}_1}$, and it will be omitted. Now similarly as in the previous case (1), we show that

$$\tau \in \mathcal{F}_2(Y,\mathcal{B}), \quad (f,\sigma)^{\leftarrow}_{\mathcal{F}_2}(\tau) = \bigvee_{\xi\in\mathcal{F}_2(X,\mathcal{A}),(f,\sigma)^{\rightarrow}_{\mathcal{F}_2}(\xi)\le\tau} \xi. \tag{10}$$

In fact, we put $\xi = \tau\circ\sigma$. According to Corollary 3.3, τ is an extensional map in the Q-set $(\Omega,\rho_{Y,\mathcal{B}})$ and for any $\omega\in\Omega$, we obtain

$$(f,\sigma)^{\rightarrow}_{\mathcal{F}_2}(\tau\circ\sigma)(\omega) = \bigvee_{\lambda\in\Lambda} \tau(\sigma(\lambda))\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda),\omega)$$
$$\le \tau(\omega).$$

It follows that the inequality $\le$ holds in (10). On the other hand, let $\xi\in\mathcal{F}_2(X,\mathcal{A})$ be such that $(f,\sigma)^{\rightarrow}_{\mathcal{F}_2}(\xi)\le\tau$. Then for any λ we have

$$\tau(\sigma(\lambda)) \ge \bigvee_{\lambda'\in\Lambda} \xi(\lambda')\otimes\rho_{Y,\mathcal{B}}(\sigma(\lambda'),\sigma(\lambda)) \ge \xi(\lambda),$$

and it follows the opposite inequality in (10) also holds. Therefore $((f,\sigma)^{\rightarrow}_{\mathcal{F}_2},(f,\sigma)^{\leftarrow}_{\mathcal{F}_2})$ is a Galois connection and from Proposition 4.3, it follows that $\mathbf{F_2}$ is a $CSLAT$-powerset theory in the category $SpaceFP$.

(3) Let $(t,\tau)\in\mathcal{F}(Y,\mathcal{B})$. Then we set

$$(f,\sigma)^{\leftarrow}_{\mathcal{F}}(t,\tau) = ((f,\sigma)^{\leftarrow}_{\mathcal{F}_1}(t),(f,\sigma)^{\leftarrow}_{\mathcal{F}_2}(\tau)).$$

We show firstly that $(f,\sigma)^{\leftarrow}_{\mathcal{F}}(t,\tau)\in\mathcal{F}(X,\mathcal{A})$. To do it, we need to prove that

$$([t.f]=)\quad [(f,\sigma)^{\leftarrow}_{\mathcal{F}_1}(t)] = (f,\sigma)^{\leftarrow}_{\mathcal{F}_2}(\tau) \quad (=\tau.\sigma).$$

Since (f,σ) is a morphism in $SpaceFP$ and $[t](\omega)=\tau(\omega)=t(y)$, for any $\omega\in\Omega$, $y\in core(B_\omega)$, for any $\lambda\in\Lambda$, $x\in core(A_\lambda)$ we obtain $[t](\sigma(\lambda))=\tau.\sigma(\lambda)=t(f(x))$. It follows that $(\tau.\sigma)(\lambda)=[t.f](\lambda)$ and $\tau.\sigma=[t.f]$. Since the ordering of $\mathcal{F}(X,\mathcal{A})$ is defined pointwise, $((f,\sigma)^{\rightarrow}_{\mathcal{F}},(f,\sigma)^{\leftarrow}_{\mathcal{F}})$ is a Galois connection. The result then follows directly from Theorem 4.2, Theorem 4.3 and Propositions 4.4, 4.3. □

Example 4.1 Let $(X,\mathcal{A})$ be a space with fuzzy partition from Example 3.1, i.e., $\mathcal{A}=\{\{x\}:x\in X\}$. In that case we have $\mathcal{F}(X,\mathcal{A})=\mathcal{F}_1(X,\mathcal{A})=\mathcal{F}_2(X,\mathcal{A})=Q^X$. It is then clear that

$$T(X,\mathcal{A}) = \{(A,A):A\subseteq X\}\cong T_1(X,\mathcal{A})\cong T_2(X,\mathcal{A})\cong 2^X.$$

and the powerset theory $\mathbf{F}$ is the classical Zadeh's powerset theory. □

Example 4.2 In the paper Močkoř (2016), we introduced $CSLAT$-powerset theory $\mathbf{F}_{\mathrm{Set}(Q)}=(F_{\mathrm{Set}(Q)},\rightarrow,V,\mu)$ in the category Set(Q), which is based on the powerset functor $F_{\mathrm{Set}(Q)}$ of extensional maps in a Q-set (X,δ) and forgetful functor $V:\mathrm{Set}(Q)\to CSLAT$, such that $V(X,\delta)=(2^X,\subseteq)$. According to Proposition 3.1, the category Set(Q) is a full subcategory of the category $SpaceFP$, and we can consider the restriction $\mathbf{F}/\mathrm{Set}(Q)$ of the powerset theory $\mathbf{F}$ to the subcategory Set(Q). Then we obtain $\mathbf{F}_{\mathrm{Set}(Q)}\cong\mathbf{F}/\mathrm{Set}(Q)$, i.e.,*the CSLAT-powerset theory in the category SpaceFP is a generalization of the CSLAT-powerset theory in the category* Set(Q). In fact, according to the relation (4) and Example 3.2, for any Q-set (X,δ), we have $F_{\mathrm{Set}(Q)}(X,\delta)\cong\mathcal{F}(X,\mathcal{C}_{X,\delta})$. Moreover using Proposition 3.4, it can be proved that for any $A\subseteq X$ the exists the unique subset $\Phi\subseteq X/\delta$, such that $(A,\Phi)\in T(X,\mathcal{C}_{X,\delta})$. It follows that $T(X,\mathcal{C}_{X,\delta})\cong V(X,\delta)$ and it can be proved simply that $\mathbf{F}_{\mathrm{Set}(Q)}$ is isomorphic to the restriction of $\mathbf{F}$ to the category Set(Q). □

5 Conclusions

We investigated *fuzzy objects* in the category $SpaceFP$ of spaces with fuzzy partition, which could be a basic category for F-transforms and some other construction, as closure spaces or fuzzy approximation spaces. (Q-valued) Fuzzy objects in the category $SpaceFP$ are morphisms $(X,\mathcal{A})\to(Q,\mathcal{Q})$, where $\mathcal{Q}$ is an appropriate partition in a complete residuated lattice Q, derived from the biresiduation operation $\leftrightarrow$ in Q. We show that fuzzy objects in $SpaceFP$ are natural generalizations of classical fuzzy sets in the category of sets and fuzzy objects in the category Set(Q) of sets with similarity relations. In the category Set(Q), any map $f:A\to Q$ can be extended to a fuzzy object $\widehat{f}:(A,\delta)\to(Q,\leftrightarrow)$. Using theorem describing relationships between fuzzy objects in Set(Q) and fuzzy objects in $SpaceFP$, we prove an analogical property for fuzzy objects in the category $SpaceFP$, namely, fuzzy objects in a space with a fuzzy partition $(X,\mathcal{A})$ in the category $SpaceFP$ are in 1-1 correspondence with fuzzy objects in a Q-set $(\Lambda,\rho_{X,\mathcal{A}})$ in the category Set(Q), where Λ is the index set of a fuzzy partition $\mathcal{A}$ and $\rho_{X,\mathcal{A}}$ is a similarity relation in Λ derived from $\mathcal{A}$. We also prove that fuzzy objects in $SpaceFP$ are fix points in F-transforms, i.e., $F^{\uparrow}(f)(\lambda)=F^{\downarrow}(f)(\lambda)=f(z)$, where $z\in core(A_\lambda)$.

We introduce powerset objects functor $\mathcal{F}(X,\mathcal{A})=((Q,\mathcal{Q})^{(X,\mathcal{A})},\le)$ in the category $SpaceFP$ and, as the main result of the paper, we show that these powerset objects define $CSLAT$-powerset theory in the sense of Rodabaugh (2007). This $CSLAT$-powerset theory then comprises $CSLAT$-powerset theories of classical fuzzy sets and fuzzy objects in the category Set(Q).

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Compliance with ethical standards

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