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METHODOLOGIES AND APPLICATION

Multiple criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information

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Abstract In recent decades, different extensional forms of fuzzy sets have been developed. However, these multitudinous fuzzy sets are unable to deal with quantitative information better. Motivated by fuzzy linguistic approach and hesitant fuzzy sets, the hesitant fuzzy linguistic term set was introduced and it is a more reasonable set to deal with quantitative information. During the process of multiple criteria decision making, it is necessary to propose some aggregation operators to handle hesitant fuzzy linguistic information. In this paper, two aggregation operators for hesitant fuzzy linguistic term sets are introduced, which are the hesitant fuzzy linguistic Bonferroni mean operator and the weighted hesitant fuzzy linguistic Bonferroni mean operator. Correspondingly, several properties of these two aggregation operators are discussed. Finally, a practical case is shown in order to express the application of these two aggregation operators. This case mainly discusses how to choose the best hospital about conducting the whole society resource management research included in a wisdom medical health system.

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1 Introduction

When dealing with multiple criteria decision-making (MCDM) problems, people often need to evaluate alternatives based on the criteria given in advance and then make a decision according to some reasoning or decision-making methods. In these processes of decision making, the most basic and key link is that how to express the experts' evaluations or preference information. Due to the vagueness and complexity of people's cognitions, sometimes the experts cannot express their ideas exactly. Consequently, Zadeh (1965) introduced the concept of fuzzy set (FS), which can be used to represent uncertain and fuzzy information. The FS extends the characteristic function valued by 0 or 1 to a membership function, in which the function values are taken as any values in the closed interval [0,1]. Moreover, considering that the FS only includes the membership information and there does not exist a clear approach or method for us to allocate one element to the membership degree of one set, the FS has been extended into various forms from different angles, such as the intuitionistic fuzzy sets (IFSs) (Atanassov 1986), the interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov and Gargov 1989), the intuitionistic multiplicative sets (Xia et al. 2013), the type-2 fuzzy sets (Zadeh 1975), the type-n fuzzy sets (Dubois and Prade 1980) and the hesitant fuzzy sets (HFSs) (Torra 2010). All these FS extensions can only be used to represent quantitative information. However, in many practical decision-making problems, the experts cannot assess the candidate alternatives in quantitative forms, but only in qualitative form (Rodríguez et al. 2013). Zadeh

(1975) proposed the fuzzy linguistic approach which can be used to model linguistic information. Furthermore, to deal with information more reasonably and accurately, different linguistic representation models have been introduced, such as the 2-tuple fuzzy linguistic representation model (Herrera and Martínez 2000), the linguistic model based on type-2 fuzzy set (Türkşen 2002), the virtual linguistic model (Xu 2004), the proportional 2-tuple model (Wang and Hao 2006). However, if the experts vacillate in their opinions among different linguistic terms and want to utilize a more complex linguistic term to fully express their ideas, the above-mentioned fuzzy linguistic approaches cannot work. In such a case, the concept of hesitant fuzzy linguistic term set (HFLTS) was introduced (Rodríguez et al. 2012).

The HFLTS is a very useful and flexible way to represent the qualitative judgments of experts. Many scholars have devoted themselves to research on different aspects over HFLTSs. Rodríguez et al. (2012) proposed the definition and several basic operations of HFLTSs and then utilized the HFLTSs to resolve some practical linguistic MCDM problems where the experts' evaluation information is represented by linguistic terms. There are two kinds of linguistic terms (Liao et al. 2014): One is the simple linguistic terms, such as "*very complex*," "*medium*" and "*a little easy*," and the other is the linguistic expressions, such as "*between cheap and very cheap*," "*at least high*" and "*great than easy*." These complex linguistic expressions can be transformed into HFLTSs. Rodríguez et al. (2013) established a new linguistic group decision-making model, which is very close to the cognitive models of human beings on how to express their linguistic preferences by facilitating the elicitation of flexible and enriching linguistic expressions. Zhu and Xu (2014) put forward the hesitant fuzzy linguistic preference relation, which is very useful for collecting and representing the decision makers' preferences. Liao et al. (2014, 2015) and Liao and Xu (2015) investigated the basic characteristics of HFLTSs and proposed a family of distance and similarity measures, cosine distance and similarity measures, and correlation measures for HFLTSs. Afterward, they (Liao et al. 2015) proposed the hesitant fuzzy linguistic VIKOR method in qualitative MCDM. Wei et al. (2014) gave some comparison methods and aggregation theories for HFLTSs. Beg and Rashid (2013) introduced the HFL-TOPSIS method for MCDM in which the evaluation values of the experts were represented as HFLTSs. By transforming all the HFLTSs into fuzzy envelopes, Liu and Rodríguez (2014) developed another type of HFL-TOPSIS method.

As we know, the information aggregation techniques are highly important in dealing with the MCDM problems. A lot of aggregation operators have been proposed for HFLTSs, even though the study of them is still in the start stage. From the angle of optimism and pessimism, Rodríguez et al. (2012) proposed the *Min_upper* operator and the *Max_lower* operator for aggregating the hesitant fuzzy linguistic information. Wei et al. (2014) defined two aggregation operators for HFLTSs, namely the hesitant fuzzy linguistic weighted average operator and the hesitant fuzzy linguistic ordered weighted average operator, and then applied these operators as well as the comparison methods to deal with the MCDM problems. Zhang et al. (2013) introduced a series of uncertain hesitant fuzzy linguistic weighted averaging operators, such as the uncertain hesitant fuzzy linguistic ordered weighted averaging operator and the uncertain hesitant fuzzy linguistic hybrid aggregation operator. Additionally, for the extended hesitant fuzzy linguistic term sets, Wang (2015) proposed a series of aggregation operators for them, such as the extended hesitant fuzzy linguistic weighted disjunction operators, the extended hesitant fuzzy ordinal ordered weighted averaging operator and the extended hesitant fuzzy ordinal hybrid aggregation operator. Wang et al. (2014) introduced two kinds of interval-valued hesitant fuzzy linguistic prioritized aggregation operators, which were extended to a grouping prioritized situation and then applied to solve the MCDM problems. Massanet et al. (2014) presented a linguistic computational model based on discrete fuzzy numbers whose support is a subset of consecutive natural numbers. Several aggregation operators defined on the set of all discrete fuzzy numbers are then presented. Riera et al. (2015) investigated the fuzzy linguistic modeling based on discrete fuzzy numbers and applied the model to manage hesitant fuzzy linguistic information.

As Bonferroni mean (BM) (Bonferroni 1950; Yager 2009; Beliakov et al. 2010; Xia et al. 2013; Xu and Yager 2011) can capture the interrelationships among arguments, it has been applied widely in MCDM under different circumstances. Yager (2009) and Beliakov et al. (2010) proposed several generalized Bonferroni mean operators and applied them to multi-criteria aggregation. Afterward, Xia et al. (2013) developed the geometric Bonferroni means and utilized them to deal with the MCDM problems. Xu and Yager (2011) introduced the intuitionistic fuzzy Bonferroni means. Under hesitant fuzzy environment, Zhu et al. (2012) studied the Bonferroni mean (BM) combing with hesitant fuzzy information and proposed the hesitant fuzzy BM operator and the hesitant fuzzy geometric BM operator. Yu et al. (2012) developed the generalized hesitant fuzzy BM operator and applied it to multi-criteria group decision making. As we know, the previous aggregation operators do not consider the interrelationship between any two hesitant fuzzy linguistic elements (HFLEs) (the elements of the HFLTS) and are unable to deal with quantitative information better. In order to extend the BM under hesitant fuzzy linguistic environment, this paper focuses on the aggregation techniques of the BM for HFLTSs. Firstly, the hesitant fuzzy linguistic Bonferroni mean (HFLBM) operator is proposed. The HFLBM operator can combine any two HFLEs very well, and thus, we can consider all information in the process of aggregation. In addition, by developing two translation functions, it is effortless to enforce the equivalent transformation between the HFS and the HFLTS. We also define a weighted hesitant fuzzy linguistic Bonferroni mean (WHFLBM) operator, which is a more reasonable aggregation operator when each criterion has a different importance degree. As the HFLTS is very close to human's cognition, the HFLBM is very flexible in dealing with the practical MCDM problems. Finally, this paper applies the HFLBM operator to solve the practical MCDM problem involving how to choose a best hospital based on several criteria for enhancing the overall level of the hospitals.

The paper is organized as follows: Sect. 2 reviews some basic concepts. In Sect. 3, we firstly define two equivalent transformation functions between the HFLTS and the HFS and then propose a score function of the HFLTS. Afterward, we introduce the HFLBM operator and discuss its properties. The WHFLBM operator is also developed in this section. Section 4 proposes an approach to deal with the MCDM problems based on the HFLBM operator and the WHFLBM operator and then applies the HFLBM operator to solve a practical MCDM problem. The paper ends with some concluding remarks in Sect. 5.

2 Preliminaries

2.1 Hesitant fuzzy sets

Definition 2.1 (Torra 2010) Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0,1].

To be easily understood, Xia and Xu (2011) expressed the HFS by a mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\} \tag{2.1}$$

where $h_A(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set A. Additionally, $h = h_A(x)$ can be called a hesitant fuzzy element (HFE).

Definition 2.2 (Xia and Xu 2011) For a HFE h, $s(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. Moreover, for two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Definition 2.3 (Xia and Xu 2011) Some operational laws about any three HFEs h, h_1 and h_2 are given as follows:

1. $h^\lambda = \bigcup_{\gamma\in h}\{\gamma^\lambda\}$;
2. $\lambda h = \bigcup_{\gamma\in h}\{1-(1-\gamma)^\lambda\}$;
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$;
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\}$.

2.2 Fuzzy linguistic approaches

In many practical decision-making problems, it is suitable and straightforward when the decision makers express their opinions by using linguistic terms that are close to human's cognition. Therefore, the fuzzy linguistic approach (Zadeh 1975) has attracted the attention of many scholars (Herrera and Martínez 2000; Türkşen 2002; Xu 2004; Wang and Hao 2006). Herrera and Herrera-Viedma (2000) introduced an approach in which the experts' opinions are taken as the values of a linguistic variable, established by a linguistic descriptors and its semantics. In order to deal with the linguistic evaluation information, we need to translate them into a machine manipulative format. Thus, Xu (2005) proposed a subscript-symmetric additive linguistic term set, shown as:

$$S = \{s_t\,|t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\} \tag{2.2}$$

where the mid-linguistic label s_0 represents an assessment of "indifference," and the rest of them are placed symmetrically around it. $s_{-\tau}$ and s_τ are the lower and upper bounds of linguistic labels where τ is a positive integer. S satisfies the following conditions:

1. If $\alpha > \beta$, then $s_\alpha > s_\beta$;
2. The negation operator is defined as: $\text{neg}(s_\alpha) = s_{-\alpha}$, especially, $\text{neg}(s_0) = s_0$.

For example, if we let $\tau = 3$, then the linguistic term set S can be taken as:

$$\begin{aligned} S = \{&s_{-3} = \text{none}, s_{-2} = \text{very low}, s_{-1} = \text{low}, s_0 \\ &= \text{medium}, s_1 = \text{high}, s_2 = \text{very high}, s_3 = \text{perfect}\} \end{aligned} \tag{2.3}$$

The mapping between the linguistic sets and their corresponding semantics can be expressed as in Fig. 1 (Liao et al. 2014).

In addition, for the purpose of preserving all given linguistic information, Xu (2005) established a continuous linguistic term set (virtual linguistic term), which can be shown as $\bar{S} = \{s_\alpha\,|\alpha \in [-q, q]\}$, where $q\ (q > \tau)$ is a sufficiently large positive integer. Liao et al. (2014) established the mapping between virtual linguistic terms and their corresponding semantics as in Fig. 2.

For any two linguistic terms, $s_\alpha, s_\beta \in \bar{S}$ and $\lambda, \lambda_1, \lambda_2 \in$ [0.1], the following operational laws were introduced (Xu 2004):

1. $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
2. $\lambda s_\alpha = s_{\lambda\alpha}$;

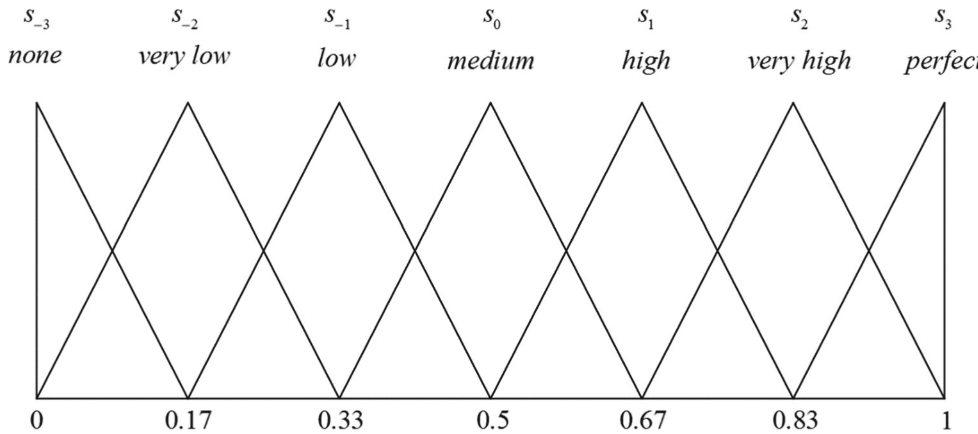

Fig. 1 Set of seven subscript-symmetric terms with its semantics

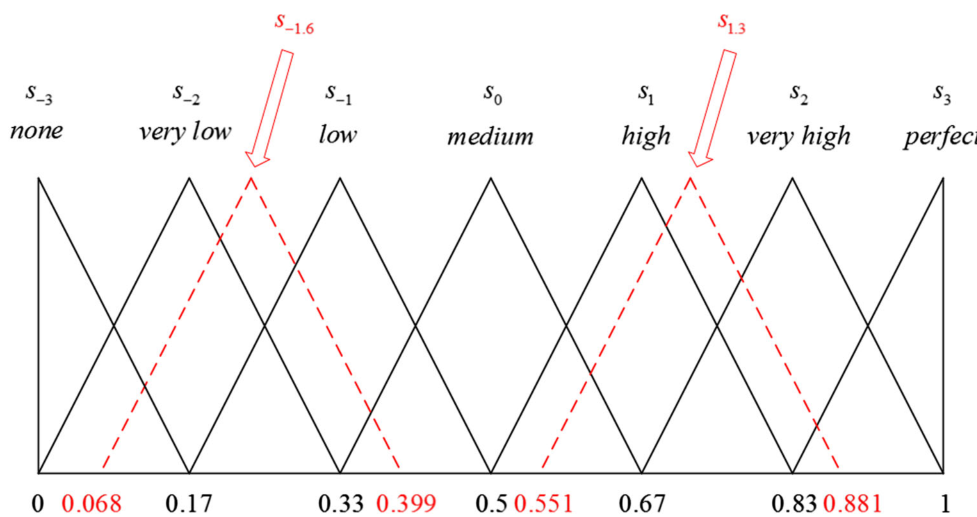

Fig. 2 Semantics of virtual linguistic terms

3. $(\lambda_1 + \lambda_2) s_\alpha = \lambda_1 s_\alpha \oplus \lambda_2 s_\alpha$;
4. $\lambda \left(s_\alpha \oplus s_\beta\right) = \lambda s_\alpha \oplus \lambda s_\beta$.

2.3 Hesitant fuzzy linguistic term set

Based on the idea of HFS and some linguistic term sets, the concept of HFLTS (Rodríguez et al. 2012) can be proposed:

Definition 2.4 (Rodríguez et al. 2012) Let $S = \{s_0, \ldots, s_\tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS), H_S, is an ordered finite subset of the consecutive linguistic terms of S.

The HFLTS can be used to elicit several linguistic values for a linguistic variable. But some shortcomings appear when we use them to make operations. For example, for a linguistic term set $S = \{s_0 =$ none, $s_1 =$ very low, $s_2 =$ low, $s_3 =$ medium, $s_4 =$ high, $s_5 =$ very high, $s_6 =$ perfect$\}$, if we operate the addition of two linguistic term, such as $s_2 \oplus s_3 = s_5$. Clearly the sum result of the linguistic terms "low" and "medium" is "very high," and this is unreasonable. But if we use the subscript-symmetric linguistic term set, shown as Eq. (2.3), we can overcome the problem easily.

Obviously, there is no any mathematical form for HFLTS in Definition 2.4. To overcome this incompleteness, Liao et al. (2015) refined a novel definition of HFLTS as follows:

Definition 2.5 (Liao et al. 2015) Let $x_i \in X$, $i = 1, 2, \ldots, N$, be fixed and $S = \{s_t \,|t = -\tau, \ldots, -1, 0, 1, , \ldots \tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS) on X, H_S, is in mathematical terms of

$$H_S = \{\langle x_i, h_S(x_i)\rangle \,|x_i \in X\} \tag{2.4}$$

where $h_S(x_i)$ is a set of some values in the linguistic term set S and can be expressed as $h_S(x_i) = \left\{s_{\phi_l}(x_i) \,\middle|s_{\phi_l}(x_i) \in S.\right.$ $\left. l = 1, \ldots, L\right\}$ with L being the number of linguistic terms in $h_S(x_i)$. $h_S(x_i)$ denotes the possible degree of the linguistic variable x_i to the linguistic term set S. For convenience,

$h_S(x_i)$ is called the hesitant fuzzy linguistic element (HFLE) and h_S is the set of all HFLEs.

2.4 Bonferroni means

Firstly, Bonferroni (1950) introduced the Bonferroni mean as follows:

Definition 2.6 (Bonferroni 1950) Let $p, \ q \geq 0$, and $a_i \ (i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers, if

$$B^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{2.5}$$

then we call $B^{p,q}$ the Bonferroni mean (BM).

Several properties of the BM can be shown as follows:

1. $B^{p,q}(0, 0, \ldots, 0) = 0$;
2. $B^{p,q}(a, a, \ldots, a) = a$, if $a_i = a$, for all i;
3. $B^{p,q}(a_1, a_2, \ldots, a_n) \geq B^{p,q}(d_1, d_2, \ldots, d_n)$, i.e., $B^{p,q}$ is monotonic, if $a_i \geq d_i$, for all i;
4. $\min\{a_i\} \leq B^{p,q}(a_1, a_2, \ldots, a_n) \leq \max\{a_i\}$.

Additionally, Zhu and Xu (2013) developed a hesitant fuzzy Bonferroni mean (HFBM) operator, which combines the BM and hesitant fuzzy information represented by HFSs.

Definition 2.7 (Zhu and Xu 2013) Let $h_i \ (i = 1, 2, \ldots, n)$ be a collection of HFEs. For any $p, q > 0$, if

$$\mathrm{HFB}^{p,q}(h_1, h_2, \ldots, h_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(h_i^p \otimes h_j^q \right) \right) \right)^{\frac{1}{p+q}} \tag{2.6}$$

then we call $\mathrm{HFB}^{p,q}$ a hesitant fuzzy Bonferroni mean (HFBM).

3 Bonferroni means for hesitant fuzzy linguistic terms

3.1 Equivalent transformation functions

Zhang and Qi (2013) defined two equivalent transformation functions between the HFLTS and the HFS as shown in Definition 3.1:

Definition 3.1 (Zhang and Qi 2013) Let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ be a finite and totally ordered discrete linguistic label set, $h_S = \{s_\alpha \,|\alpha \in [0, g-1]\}$ be a HFLE, and $H = \{\gamma \,|\gamma \in [0, 1]\}$ be a HFS. The linguistic variable s_α that expresses the equivalent information to the membership degree γ is obtained with the following function f:

$$f : [0, g-1] \to [0, 1], \quad f(s_\alpha) = \frac{\alpha}{g-1} = \gamma$$

Similarly, the membership degree γ that expresses the equivalent information to the linguistic variable s_α is obtained with the following function f^{-1}:

$$f^{-1} : [0, 1] \to [0, g-1], \quad f^{-1}(\gamma) = s_{\gamma \times (g-1)} = s_\alpha$$

In this paper, we mainly discuss the subscript-symmetric additive linguistic term set, denoted by $S = \{s_t \,|t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$, which is different from the totally ordered discrete linguistic label set. Then, we need to define two novel transformation functions between the HFLTS and the HFS:

Definition 3.2 Let $S = \{s_t \,|t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a finite and totally ordered discrete linguistic term set, $h_S = \{s_t \,|t \in [-\tau, \tau]\}$ be a HFLE, and $H = \{\gamma \,|\gamma \in [0, 1]\}$ be a HFS. Then, the linguistic variable s_t that expresses the equivalent information to the membership degree γ is obtained with the following function g:

$$g : [-\tau, \tau] \to [0, 1], \quad g(s_t) = \frac{t}{2\tau} + \frac{1}{2} = \gamma$$

Besides, we can get this function as:

$$g : [-\tau, \tau] \to [0, 1],$$
$$g(h_S) = \left\{ g(s_t) = \frac{t}{2\tau} + \frac{1}{2} \,|t \in [-\tau, \tau] \right\} = h_\gamma$$

Additionally, the membership degree γ that expresses the equivalent information to the linguistic variable s_t is obtained with the following function g^{-1}:

$$g^{-1} : [0, 1] \to [-\tau, \tau], \quad g^{-1}(\gamma) = s_{(2\gamma-1)\tau} = s_t$$

Similar to the analyses above, we get

$$g^{-1} : [0, 1] \to [-\tau, \tau],$$
$$g^{-1}(h_\gamma) = \left\{ g^{-1}(\gamma) = s_{(2\gamma-1)\tau} \,|\gamma \in [0, 1] \right\} = h_S$$

Furthermore, in order to compare any two HFLEs, in the following we define the corresponding score function based on the transformation functions in Definition 3.2:

Definition 3.3 Let $h_S = \{s_t \,|t \in [-\tau, \tau]\}$ be a HFLE, then

$$s\left(h_S\right) = \frac{1}{l}\sum_{i=1}^{l} {}_{s_t \in h_S} g\left(s_t\right)$$

can be called the score function of h_S, where l is the number of the elements of h_S. Therefore,

1. If $s\left(h_{S_1}\right) < s\left(h_{S_2}\right)$, then h_{S_1} is smaller than h_{S_2}, denoted by $h_{S_1} \prec h_{S_2}$.
2. If $s\left(h_{S_1}\right) = s\left(h_{S_2}\right)$, then h_{S_1} is equal to h_{S_2}, denoted by $h_{S_1} = h_{S_2}$.

3.2 Hesitant fuzzy linguistic Bonferroni means operator

Based on these two equivalent transformation functions described in Definition 3.2, we can develop an aggregation operator for hesitant fuzzy linguistic information:

Definition 3.4 Let h_{S_i} $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, For any $p, q > 0$, if

$$\mathrm{HFLB}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(g\left(h_{S_i}\right)\right)^p \otimes \left(g\left(h_{S_j}\right)\right)^q\right)\right)\right)^{\frac{1}{p+q}}\right) \tag{3.1}$$

then we call $\mathrm{HFLB}^{p,q}$ a hesitant fuzzy linguistic Bonferroni mean (HFLBM) operator.

Clearly, based on the equivalent transformation function g defined in Definition 3.2, the $g\left(h_{S_i}\right)$ and $g\left(h_{S_j}\right)$ can be translated into two HFSs h_i and h_j:

$$\mathrm{HFLB}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_i)^p \otimes \left(h_j\right)^q\right)\right)^{\frac{1}{p+q}}\right) \tag{3.2}$$

Obviously, $\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_i)^p \otimes \left(h_j\right)^q\right)\right)^{\frac{1}{p+q}}$ is the HFBM operator, and we can calculate the result of the HFBM operator, which is a HFS by the operations on HFSs. Finally, we use the other equivalent transformation function g^{-1} to transform the HFS into the HFLE. So the aggregated result of Eq. (3.1) must be a HFLE.

Theorem 3.1 *Let $p, q > 0$, and h_{S_i} $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, then the aggregated value by using the HFLBM is a HFLE, and*

$$\mathrm{HFLB}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = \bigcup_{\xi_{i,j}\in\sigma_{i,j}, i\neq j} \times \left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_{i,j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\} \tag{3.3}$$

where $\sigma_{i,j} = (h_i)^p \otimes \left(h_j\right)^q$ reflects the interrelationship between h_{S_i} and h_{S_j}, $i, j = 1, 2, \ldots, n;\ i \neq j$.

Proof Firstly, we can make the equivalent transformation between the HFLEs h_{S_i} $(i = 1, 2, \ldots, n)$ and the HFEs h_i $(i = 1, 2, \ldots, n)$ by Definition 3.2, namely Eq. (3.1) can be translated into Eq. (3.2). Then, we can get

$$\sigma_{i,j} = (h_i)^p \otimes \left(h_j\right)^q = \bigcup_{\xi_{i,j}\in\sigma_{i,j}}\left\{\xi_{i,j}\right\} = \bigcup_{\gamma_i\in h_i,\gamma_j\in h_j}\left\{\gamma_i^p\gamma_j^q\right\} \tag{3.4}$$

Thus, Eq. (3.1) can be written as

$$\begin{aligned}
&\mathrm{HFLB}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right)\\
&= g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\\
&= g^{-1}\left(\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{1}{n(n-1)}\sigma_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\\
&= g^{-1}\left(\left(\bigcup_{\xi_{i,j}\in\sigma_{i,j}}\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_{i,j}\right)^{\frac{1}{n(n-1)}}\right\}\right)^{\frac{1}{p+q}}\right)\\
&= \bigcup_{\xi_{i,j}\in\sigma_{i,j}, i\neq j}\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_{i,j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\}
\end{aligned}$$

which completes the proof. □

An example is given below to show the calculation process of the HFLBM operator:

Example 3.1 Let $S = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a subscript-symmetric additive linguistic term set and $\tau = 3$,

$h_{S_1} = \{s_{-1}, s_0\}$ and $h_{S_2} = \{s_0, s_1, s_2\}$ be two HFLEs. Then, we can aggregate them by using the HFLBM operator.

If $p = 1$ and $q = 0$, then

$$\begin{aligned}
h_1 &= g\left(h_{S_1}\right) = \left\{ g\left(s_t\right) = \frac{t}{2\times 3} + \frac{1}{2} \,|t \in [-3, 3] \right\} \\
&= \left\{ \frac{-1}{2\times 3} + \frac{1}{2}, \frac{0}{2\times 3} + \frac{1}{2} \right\} = \left\{ \frac{1}{3}, \frac{1}{2} \right\} \\
h_1 &= g\left(h_{S_2}\right) = \left\{ g\left(s_t\right) = \frac{t}{2\times 3} + \frac{1}{2} \,|t \in [-3, 3] \right\} \\
&= \left\{ \frac{0}{2\times 3} + \frac{1}{2}, \frac{1}{2\times 3} + \frac{1}{2}, \frac{2}{2\times 3} + \frac{1}{2} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right\} \\
\sigma_{1,2} &= (h_1)^1 \otimes (h_2)^0 = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j} \left\{ \gamma_i^p \gamma_j^q \right\} \\
&= \left\{ \frac{1}{3}, \frac{1}{2} \right\} \sigma_{2,1} = (h_2)^1 \otimes (h_1)^0 \\
&= \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j} \left\{ \gamma_i^p \gamma_j^q \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right\}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{HFLB}^{1,0}\left(h_{S_1}, h_{S_2}\right) \\
&= \bigcup_{\xi_{i,j} \in \sigma_{i,j}, i\neq j} \left\{ g^{-1}\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{2} \left(1-\xi_{i,j}\right)^{\frac{1}{n(n-1)}} \right) \right\} \\
&= g^{-1}\left(1 - \left(1-\sigma_{1,2}\right)^{\frac{1}{2}} \left(1-\sigma_{2,1}\right)^{\frac{1}{2}} \right) \\
&= g^{-1}\left(\{0.4226, 0.5, 0.5286, 0.5918, 0.6667, 0.7113\}\right) \\
&= \{s_{-0.46}, s_0, s_{0.17}, s_{0.55}, s_1, s_{1.27}\}
\end{aligned}$$

Obviously, we can change the values of p and q based on the actual requirements. In addition, we can further investigate the properties of the HFLBM operator, such as monotonicity, boundedness and commutativity.

Theorem 3.2 (Monotonicity). *Let $h_{S_a} = \left\{h_{S_{a_1}}, h_{S_{a_2}}, \ldots, h_{S_{a_n}}\right\}$ and $h_{S_b} = \left\{h_{S_{b_1}}, h_{S_{b_2}}, \ldots, h_{S_{b_n}}\right\}$ be two collections of HFLEs. If for any $s_{t_{a_i}} \in h_{S_{a_i}}$ and $s_{t_{b_i}} \in h_{S_{b_i}}$, we have $s_{t_{a_i}} \leq s_{t_{b_i}}$ for any i, then*

$$\begin{aligned}
&\mathrm{HFLB}^{p,q}\left(h_{S_{a_1}}, h_{S_{a_2}}, \ldots, h_{S_{a_n}}\right) \\
&\leq \mathrm{HFLB}^{p,q}\left(h_{S_{b_1}}, h_{S_{b_2}}, \ldots, h_{S_{b_2}}\right)
\end{aligned} \tag{3.5}$$

Proof Based on Definition 3.2, we can transform all HFLEs into HFSs, namely

$$g: [-\tau, \tau] \to [0, 1]\,,\; g\left(s_{t_{\theta_i}}\right) = \frac{t_{\theta_i}}{2\tau} + \frac{1}{2} = \gamma_{\theta_i},$$

$$\begin{aligned}
&g: [-\tau, \tau] \to [0, 1]\,,\; g\left(h_{S_{\theta_i}}\right) \\
&= \left\{ g\left(s_{t_{\theta_i}}\right) = \frac{t_{\theta_i}}{2\tau} + \frac{1}{2} \,\middle|\, t_{\theta_i} \in [-\tau, \tau] \right\} = h_{\theta_i},
\end{aligned}$$

where $i = 1, 2, \ldots, n$ and $\theta = a$ and b. Then, for any $\gamma_{\theta_i} \in h_{\theta_i}$, we obtain $\gamma_{a_i} \leq \gamma_{b_i}$, and

$$\begin{aligned}
&\mathrm{HFLB}^{p,q}\left(h_{S_{a_1}}, h_{S_{a_2}}, \ldots, h_{S_{a_n}}\right) \\
&= g^{-}\left(HFB^{p,q}\left(h_{a_1}, h_{a_2}, \ldots, h_{a_n}\right)\right) \\
&\leq \mathrm{HFLB}^{p,q}\left(h_{S_{b_1}}, h_{S_{b_2}}, \ldots, h_{S_{b_n}}\right) \\
&= g^{-1}\left(HFB^{p,q}\left(h_{b_1}, h_{b_2}, \ldots, h_{b_n}\right)\right)
\end{aligned}$$

Similarly, for any $\gamma_{a_i} \in h_{\gamma_{a_i}}$ and $\gamma_{b_i} \in h_{\gamma_{b_i}}$, $i \neq j$, we have

$$\gamma_{a_i}\gamma_{a_j} \leq \gamma_{b_i}\gamma_{b_j} \tag{3.6}$$

So for any $\xi_{a_{i,j}} \in \sigma_{a_{i,j}}$ and $\xi_{b_{i,j}} \in \sigma_{b_{i,j}}$, $i \neq j$, we can get $\xi_{a_{i,j}} = \gamma_{a_i}^p \gamma_{a_j}^q \leq \xi_{b_{i,j}} = \gamma_{b_i}^p \gamma_{b_j}^q$. It follows

$$\begin{aligned}
&\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-\xi_{a_{i,j}}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&\leq \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-\xi_{b_{i,j}}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}
\end{aligned} \tag{3.7}$$

Based on Definition 3.4 and Theorem 3.1, we can get

$$\begin{aligned}
&\mathrm{HFLB}^{p,q}\left(h_{S_{a_1}}, h_{S_{a_2}}, \ldots, h_{S_{a_n}}\right) \\
&= g^{-1}\left(HFB^{p,q}\left(h_{a_1}, h_{a_2}, \ldots, h_{a_n}\right)\right) \\
&= g^{-1}\left(\left(\left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \sigma_{a_{i,j}} \right) \right) \right)^{\frac{1}{n(n-1)}} \right) \\
&= \bigcup_{\xi_{a_{i,j}} \in \sigma_{a_{i,j}}, i\neq j} \left\{ g^{-1}\left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-\xi_{a_{i,j}}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \right\} \\
&\leq \bigcup_{\xi_{b_{i,j}} \in \sigma_{b_{i,j}}, i\neq j} \left\{ g^{-1}\left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-\xi_{b_{i,j}}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \right\}
\end{aligned}$$

$$= g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_{b_{i,j}}\right)\right)^{\frac{1}{n(n-1)}}\right)$$
$$= g^{-1}\left(HFB^{p,q}\left(h_{b_1},h_{b_2},\ldots,h_{b_n}\right)\right)$$
$$= \mathrm{HFLB}^{p,q}\left(h_{S_{b_1}},h_{S_{b_2}},\ldots,h_{S_{b_n}}\right) \tag{3.8}$$

which completes the proof. □

Theorem 3.3 (Boundedness). *Let* h_{S_i} $(i=1,2,\ldots,n)$ *be a collection of HFLEs,* $h_{S_i}^{+}=\bigcup_{s_i\in h_{S_i}}\max\{s_i\}$ *and* $h_{S_i}^{-}=\bigcup_{s_i\in h_{S_i}}\min\{s_i\}$, $s_i^{+}\in h_{S_i}^{+}$, $s_i^{-}\in h_{S_i}^{-}$, *then*

$$h_{S_i}^{-}\le \mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)\le h_{S_i}^{+} \tag{3.9}$$

Proof Based on Definition 3.2, we can transform all HFLEs into HFSs, namely

$$g:[-\tau,\tau]\to[0,1],\quad g(s_i)=\frac{i}{2\tau}+\frac{1}{2}=\gamma_i$$

and

$$g:[-\tau,\tau]\to[0,1],\quad g\left(h_{S_i}\right)=\{g(s_i)$$
$$=\left.\frac{i}{2\tau}+\frac{1}{2}\,\right|i\in[-\tau,\tau]\Big\}=h_i$$

where $\gamma_i\in h_i$ $(i=1,2,\ldots,n)$. So there are $\gamma_i^{+}\in h_i^{+}$, $\gamma_i^{-}\in h_i^{-}$ and $\gamma_i^{-}\le\gamma_i\le\gamma_i^{+}$, for all i. Then,

$$\left(\gamma_i^{-}\right)^{p+q}\le\gamma_i^{p}\gamma_j^{q}\le\left(\gamma_i^{+}\right)^{p+q} \tag{3.10}$$

In addition, $\sigma_{i,j}=\left(h_i^{p}\otimes h_j^{q}\right)=\bigcup_{\xi_{i,j}\in\sigma_{i,j}}\{\xi_{i,j}\}=\bigcup_{\gamma_i\in h_i,\gamma_j\in h_j}\left\{\gamma_i^{p}\gamma_j^{q}\right\}$. Then, we have

$$\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})\right)^{\frac{1}{n(n-1)}}\ge\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_i^{+}\right)^{p+q}\right)\right)^{\frac{1}{n(n-1)}}$$
$$=1-\left(\gamma_i^{+}\right)^{p+q} \tag{3.11}$$

$$\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})\right)^{\frac{1}{n(n-1)}}\le\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_i^{-}\right)^{p+q}\right)\right)^{\frac{1}{n(n-1)}}$$
$$=1-\left(\gamma_i^{-}\right)^{p+q} \tag{3.12}$$

Therefore

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\ge\left(1-\left(1-\left(\gamma_i^{+}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}=\gamma_i^{+} \tag{3.13}$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\le\left(1-\left(1-\left(\gamma_i^{-}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}=\gamma_i^{-} \tag{3.14}$$

Additionally, from Definition 3.4 and Theorem 3.1, we can get

$$\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)=g^{-1}\left(HFB^{p,q}\left(h_i,h_2,\ldots,h_n\right)\right)$$
$$=\bigcup_{\xi_{i,j}\in\sigma_{i,j},i\neq j}\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\}$$
$$\ge\bigcup_{\xi_{i,j}^{-}\in\sigma_{i,j}^{-}}\left\{g^{-1}\left(\left(1-\left(1-\xi_{i,j}^{-}\right)\right)^{\frac{1}{p+q}}\right)\right\}$$
$$=\bigcup_{\xi_{i,j}^{-}\in\sigma_{i,j}^{-}}\left\{g^{-1}\left(\left(\xi_{i,j}^{-}\right)^{\frac{1}{p+q}}\right)\right\}$$
$$=\bigcup_{\gamma_i^{-}\in h_i^{-}}\left\{g^{-1}\left(\gamma_i^{-}\right)\right\}=h_{S_i}^{-} \tag{3.15}$$

$$\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)=g^{-1}\left(\mathrm{HFB}^{p,q}\left(h_i,h_2,\ldots,h_n\right)\right)$$
$$=\bigcup_{\xi_{i,j}\in\sigma_{i,j},i\neq j}\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\xi_{i,j})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\}$$
$$\le\bigcup_{\xi_{i,j}^{+}\in\sigma_{i,j}^{+}}\left\{g^{-1}\left(\left(1-\left(1-\xi_{i,j}^{+}\right)\right)\right)^{\frac{1}{p+q}}\right\}$$
$$=\bigcup_{\xi_{i,j}^{+}\in\sigma_{i,j}^{+}}\left\{g^{-1}\left(\left(\xi_{i,j}^{+}\right)^{\frac{1}{p+q}}\right)\right\}$$
$$=\bigcup_{\gamma_i^{+}\in h_i^{+}}\left\{g^{-1}\left(\gamma_i^{+}\right)\right\}=h_{S_i}^{+} \tag{3.16}$$

Then, $h_{S_i}^{-}\le\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)\le h_{S_i}^{+}$. This completes the proof.

Theorem 3.4 (Commutativity). *Let* h_{S_i} $(i=1,2,\ldots,n)$ *be a collection of HFLEs, and* $\left(\ddot{h}_{S_1},\ddot{h}_{S_2},\ldots,\ddot{h}_{S_n}\right)$ *be any permutation of* $\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)$, *then*

$$\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)=\mathrm{HFLB}^{p,q}\left(\ddot{h}_{S_1},\ddot{h}_{S_2},\ldots,\ddot{h}_{S_n}\right) \tag{3.17}$$

Proof Based on Definition 3.4 and Theorem 3.1, the HFLEs h_{S_i} $(i=1,2,\ldots,n)$ can be changed into the HFEs h_i $(i=1,2,\ldots,n)$ equivalently. Then,

$$\begin{aligned}&\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)\\&=g^{-1}\left(\mathrm{HFB}^{p,q}\left(h_1,h_2,\ldots,h_n\right)\right)\\&=g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\sigma_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\\&\mathrm{HFLB}^{p,q}\left(\ddot{h}_{S_1},\ddot{h}_{S_2},\ldots,\ddot{h}_{S_n}\right)\\&=g^{-1}\left(\mathrm{HFB}^{p,q}\left(\ddot{h}_1,\ddot{h}_2,\ldots,\ddot{h}_n\right)\right)\\&=g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\ddot{\sigma}_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\end{aligned}$$

where $\sigma_{i,j}=h_i^p\otimes h_j^q$ and $\ddot{\sigma}_{i,j}=\ddot{h}_i^p\otimes\ddot{h}_j^q$ $(i,j=1,2,\ldots,n;\ i\neq j)$. As we know, the results of $\sigma_{i,j}$ and $\ddot{\sigma}_{i,j}$ are equivalent when we change the places of h_i $(i=1,2,\ldots,n)$. Therefore, we can get

$$\begin{aligned}&\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)\\&=g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\sigma_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\\&=\mathrm{HFLB}^{p,q}\left(\ddot{h}_{S_1},\ddot{h}_{S_2},\ldots,\ddot{h}_{S_n}\right)\\&=g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\ddot{\sigma}_{i,j}\right)\right)^{\frac{1}{p+q}}\right)\end{aligned}$$

which completes the proof. □

In Eqs. (3.1) and (3.2), the advantage of HFLBM is that it can capture the interrelationship between HFLEs. However, considering the interrelationship between any two HFLEs, such as h_{S_i} and h_{S_j}, we need to introduce a concept called "bonding satisfaction" factor on the HFS, which was defined (see Zhu and Xu 2013) and can be shown as follows:

Definition 3.5 (Zhu and Xu 2013) Let $p,q\geq 0$, and h_i $(i=1,2,\ldots,n)$ be a collection of HFEs, then

$$\tau_{i,j}=\left(h_i^p\otimes h_j^q\right)\oplus\left(h_j^p\otimes h_i^q\right) \tag{3.18}$$

can be considered as the "bonding satisfaction" factor used as a calculation unit, capturing the connection between h_i and h_j, $i,j=1,2,\ldots,n;\ i\neq j$.

Similarly, it is easy to develop a novel "bonding satisfaction" factor for HFLEs. For any two HFLEs h_{S_i} and h_{S_j}, and let $p,q>0$, then

$$\vartheta_{i,j}=\left(\left(g\left(h_{S_i}\right)\right)^p\otimes\left(g\left(h_{S_j}\right)\right)^q\right)\oplus\left(\left(g\left(h_{S_j}\right)\right)^p\otimes\left(g\left(h_{S_i}\right)\right)^q\right) \tag{3.19}$$

can be called the novel "bonding satisfaction" factor of HFLEs.

Utilizing this novel "bonding satisfaction" factor of HFLEs, Eq. (3.1) can be developed into

$$\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)=g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\vartheta_{i,j}\right)\right)^{\frac{1}{p+q}}\right) \tag{3.20}$$

where $\vartheta_{i,j}=\left(\left(g\left(h_{S_i}\right)\right)^p\otimes\left(g\left(h_{S_j}\right)\right)^q\right)\oplus\left(\left(g\left(h_{S_j}\right)\right)^p\otimes\left(g\left(h_{S_i}\right)\right)^q\right)$.

Clearly, $g\left(h_{S_i}\right)$ and $g\left(h_{S_j}\right)$ can be translated into two HFSs h_i and h_j respectively. Therefore, in the same way, Eq. (3.3) can be changed into

$$\begin{aligned}\mathrm{HFLB}^{p,q}\left(h_{S_1},h_{S_2},\ldots,h_{S_n}\right)=&\bigcup_{\kappa_{i,j}\in\vartheta_{i,j},i\neq j}\\&\times\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\kappa_{i,j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\}\end{aligned} \tag{3.21}$$

where $\vartheta_{i,j}=\left(h_i^p\otimes h_j^q\right)\oplus\left(h_j^p\otimes h_i^q\right)$ $(i,j=1,2,\ldots,n;\ i<j)$.

Next, for the HFLBM operator, several special cases based on the different values of p and q can be shown as follows:

1. If $q \to 0$, then Eqs. (3.1) and (3.3) can be reduced into

$$\begin{aligned}\mathrm{HFLB}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) &= g^{-1}\left(\left(\frac{1}{n}\left(\bigoplus_{i=1}^{n}\left(g\left(h_{S_i}\right)\right)^{p}\right)\right)^{\frac{1}{p}}\right)\\ &= \bigcup_{\gamma_i \in h_i}\left\{g^{-1}\left(\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right)\right\}\end{aligned} \tag{3.22}$$

where h_i $(i = 1, 2, \ldots, n)$ is a collection of HFEs developed from the HFLEs h_{S_i} $(i = 1, 2, \ldots, n)$. We can call Eq. (3.22) the generalized hesitant fuzzy linguistic mean (GHFLM) operator.

2. If $p = q = 1$, then Eqs. (3.1) and (3.3) can be developed into

$$\begin{aligned}\mathrm{HFLB}^{1,1}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) &= g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} g\left(h_{S_i}\right)\otimes g\left(h_{S_j}\right)\right)\right)^{\frac{1}{2}}\right)\\ &= \bigcup_{\xi_{b_{i,j}} \in \sigma_{b_{i,j}}, i\neq j}\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_{b_{i,j}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)\right\}\\ &= \bigcup_{\kappa_{i,j} \in \vartheta_{i,j}, i\neq j}\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\kappa_{i,j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)\right\}\end{aligned} \tag{3.23}$$

where $\xi_{b_{i,j}} = h_i^p \otimes h_j^q$ and $\vartheta_{i,j} = \left(h_i^p \otimes h_j^q\right) \oplus \left(h_j^p \otimes h_i^q\right)$ $(i, j = 1, 2, \ldots, n;\ i < j)$. Eq. (3.23) can be called the hesitant fuzzy linguistic interrelated square mean (HFLSM) operator.

3. If $p = 1$ and $q \to 0$, then Eqs. (3.1) and (3.3) can be transformed into the hesitant fuzzy linguistic averaging (HFLA) operator:

$$\begin{aligned}\mathrm{HFLB}^{1,0}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) &= g^{-1}\left(\frac{1}{n}\left(\bigoplus_{i=1}^{n} g\left(h_{S_i}\right)\right)\right)\\ &= \bigcup_{\gamma_i \in h_i}\left\{g^{-1}\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\frac{1}{n}}\right)\right\}\end{aligned} \tag{3.24}$$

4. If $p = 2$ and $q \to 0$, then Eqs. (3.1) and (3.3) can be transformed into the hesitant fuzzy linguistic square mean (HFLSM) operator:

$$\begin{aligned}\mathrm{HFLB}^{2,0}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) &= g^{-1}\left(\left(\frac{1}{n}\left(\bigoplus_{i=1}^{n}\left(g\left(h_{S_i}\right)\right)^{2}\right)\right)^{\frac{1}{2}}\right)\\ &= \bigcup_{\gamma_i \in h_i}\left\{g^{-1}\left(\left(1-\prod_{i=1}^{n}\left(1-\gamma_i^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right)\right\}\end{aligned} \tag{3.25}$$

3.3 Weighted hesitant fuzzy linguistic Bonferroni means operator

When dealing with some MCDM problems, it is necessary to consider the importance degrees of different criteria. Thus, in the following, we develop a weighted hesitant fuzzy linguistic Bonferroni means operator:

Definition 3.6 Let $p, q > 0$ and h_{S_i} $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vector of them, where $\omega_i \in [0, 1]$, and $\sum_{i=1}^{n} \omega_i = 1$. If

$$\mathrm{WHFLB}_{\omega}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(g\left(\omega_i h_{S_i}\right)\right)^{p}\otimes\left(g\left(\omega_j h_{S_j}\right)\right)^{q}\right)\right)\right)^{\frac{1}{p+q}}\right) \tag{3.26}$$

then the $\mathrm{WHFLB}_{\omega}^{p,q}$ can be called a weighted hesitant fuzzy linguistic Bonferroni mean (WHFLBM) operator.

For $\left(g\left(\omega_i h_{S_i}\right)\right)^{p} \otimes \left(g\left(\omega_j h_{S_j}\right)\right)^{q}$, let $g\left(\omega_i h_{S_i}\right) = h_i^{\omega_i}$, then $\sigma_{i,j}^{\omega} = \left(h_i^{\omega_i}\right)^{p} \otimes \left(h_j^{\omega_j}\right)^{q}$, so we can extend Theorem 3.4 into Theorem 3.5 based on Eq. (3.26):

Theorem 3.5 *Let $p, q > 0$ and h_{S_i} $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vector of them, where $\omega_i \in [0, 1]$, and $\sum_{i=1}^{n} \omega_i = 1$, the aggregated value of the WHFLBM operator is a HFLE, and*

$$\mathrm{WHFLB}_{\omega}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = \bigcup_{\xi_{i,j}^{\omega} \in \sigma_{i,j}^{\omega}, i\neq j} \times\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_{i,j}^{\omega}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\} \tag{3.27}$$

Table 1 Hesitant fuzzy linguistic decision matrix

	c_1	c_2	$\dots$	c_n
y_1	$h_{S_{11}}$	$h_{S_{12}}$	$\dots$	$h_{S_{1n}}$
y_2	$h_{S_{21}}$	$h_{S_{22}}$	$\dots$	$h_{S_{2n}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_m	$h_{S_{m1}}$	$h_{S_{m2}}$	$\dots$	$h_{S_{mn}}$

where $\sigma_{i,j}^{\omega} = \left(h_i^{\omega_i}\right)^p \otimes \left(h_j^{\omega_j}\right)^q$ *reflects the interrelationship between* $h_i^{\omega_i}$ *and* $h_j^{\omega_j}$ $\quad (i, j = 1, 2, \dots, n;\ i \neq j)$.

4 The application of the HFLBM and WHFLBM operators

Based on the HFLBM and WHFLBM operators, in the following, an approach is developed to deal with the MCDM problems with hesitant fuzzy linguistic information:

Step 1. For a MCDM problem, let $A = \{a_1, a_2, \dots, a_m\}$ be a set of m alternatives, and $C = \{c_1, c_2, \dots, c_n\}$ be a set of n criteria. The weight vector of these criteria is $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, satisfying $\omega_j > 0$, $j = 1, 2, \dots, n$, and $\sum_{j=1}^{n} \omega_j = 1$. During the process of decision making, the decision makers provide all their evaluation values that the alternative a_i satisfies the criterion c_j represented by the HFLEs $h_{S_{ij}} = \bigcup_{s_{ij} \in h_{S_{ij}}} \{s_{ij}\}$ $\quad (i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$. All the HFLEs are contained in the hesitant fuzzy linguistic decision matrix $H = \left(h_{S_{ij}}\right)_{m \times n}$ shown as in Table 1.

As we know, there exist two kinds of criteria: the benefit-type criteria and the cost-type criteria. Generally, we need to translate the cost-type criteria into the benefit-type criteria for handling the decision-making problems more reasonable. Therefore, we can transform the hesitant fuzzy linguistic decision matrix $H = \left(h_{S_{ij}}\right)_{m \times n}$ into a normalized matrix $K = \left(k_{S_{ij}}\right)_{m \times n}$, where

$$k_{S_{ij}} = \bigcup_{s_{ij} \in k_{S_{ij}}} = \begin{cases} \{s_{ij}\}, & \text{for benefit criterion } c_j \\ \{1 - s_{ij}\}, & \text{for } \cos t \text{ criterion } c_j \end{cases},$$

$$i = 1, 2, \dots, m;\ j = 1, 2, \dots, n$$

Step 2 Utilize the HFLBM or WHFLBM operator to aggregate all HFLEs $k_{S_{ij}}$ $(j = 1, 2, \dots, n)$ in the i-th line of the normalized matrix $K = \left(k_{S_{ij}}\right)_{m \times n}$; then, we can get the overall performance values k_i (or k_i') corresponding to the alternative a_i:

$$k_i = \text{WHFLB}^{p,q}\left(k_{S_{i1}}, k_{S_{i2}}, \dots, k_{S_{in}}\right),$$
$$k_i' = \text{WHFLB}_{\omega}^{p,q}\left(k_{S_{i1}}, k_{S_{i2}}, \dots, k_{S_{in}}\right)$$

Step 3 Calculate the score values $s(k_i)$ $(i = 1, 2, \dots, n)$ of all the performance values k_i (or k_i') $\quad (i = 1, 2, \dots, n)$ based on Definition 3.3. Then, the ranking order of all the score values can be obtained, and the optimal alternative can also be selected.

A practice case study is given below to illustrate the application of the HFLBM and WHFLBM operators:

Example 4.1 The environmental pollution in China is more and more serious, especially since 2013 when plenty of the extreme weather conditions, such as haze, are appearing. The haze leads the super-scale of outdoor air particulate matter (PM2.5) to 95.5 % in 74 cities on air quality detection. Almost one-fifth reasons of infecting lung cancer are related to the atmosphere pollution. The deteriorating environment puts forward new challenges to people on the optimal allocation of limited healthcare resources in China. Furthermore, we are also faced with amounts of unsolved traditional healthcare problems in current stage, such as the population aging, the increasing healthcare costs and benefit reduced. With the limited medical resources, China has to face these tough problems and challenges that are how to optimize the resource allocation and enhance the benefit about resource input and output. Therefore, we need to change the old healthcare system and establish a novel personalized healthcare system. Considering the relationship between environment and medical health, we need to conduct the whole society resource management research included in a wisdom medical health system. In recent years, some domestic hospitals in China have started to do such type of work.

To do so, we should consider three main criteria to determine which hospital is the best one. Firstly, we should consider the environmental factor of medical and health service, denoted by C_1, which includes two sub-criteria: medical and health service demand factors (C_{11}) and demand forecasting (C_{12}). The next criterion is personalized diagnosis and treatment decision-making optimization, denoted by C_2. There are three sub-criteria which are the prognosis of disease prevention and decisions (C_{21}), disease diagnosis decision making (C_{22}) and the basic treatment decisions (C_{23}). The final criterion (C_3) is the social resource allocation optimization under the pattern of wisdom medical and health services, whose sub-criteria are monomer hospital internal resource optimizing configuration (C_{31}) and collaborative optimization configuration between all level medical institutions (C_{32}).

Now we consider four hospitals in China, namely the West China Hospital of Sichuan University (A_1), the Huashan Hospital of Fudan University (A_2), the Union Medical College Hospital (A_3) and the Chinese PLA General Hospital (A_4). Based on the three main criteria and several sub-criteria, we can use the HFLTSs to evaluate every criterion

Table 2 Evaluation values of four hospitals

	C_1	C_2	C_3
A_1	$\langle x_{11}, \{s_0, s_1\}\rangle$	$\langle x_{12}, \{s_2\}\rangle$	$\langle x_{13}, \{s_{-1}, s_0\}\rangle$
A_2	$\langle x_{21}, \{s_1\}\rangle$	$\langle x_{22}, \{s_{-1}, s_0\}\rangle$	$\langle x_{23}, \{s_2\}\rangle$
A_3	$\langle x_{31}, \{s_1\}\rangle$	$\langle x_{32}, \{s_1, s_2\}\rangle$	$\langle x_{33}, \{s_2, s_3\}\rangle$
A_4	$\langle x_{41}, \{s_2\}\rangle$	$\langle x_{42}, \{s_0, s_1\}\rangle$	$\langle x_{43}, \{s_1\}\rangle$

of these hospitals. The invited experts give their evaluations, which constitute the hesitant fuzzy linguistic decision matrix $K = \left(k_{S_{ij}}\right)_{m\times n}$ as shown in Table 2.

Firstly, utilizing the approach based on the HFLBM operator to make the decision. The first step has been introduced as above, so we calculate the aggregated values from Step 2.

Step 2. Suppose that $p = 1$ and $q = 1$, then utilizing the HFLBM operator to aggregate all HFLEs $k_{S_{ij}}$ $(j = 1, 2, 3, 4)$ of the i-th line included in the matrix $K = \left(k_{S_{ij}}\right)_{m\times n}$, we get the overall performance values k_i $(i = 1, 2, \ldots, n)$ corresponding to the alternatives y_i $(i = 1, 2, \ldots, n)$. When $i = 1$, then

$$\begin{aligned} k_1 &= \mathrm{HFLB}^{1,1}\left(h_{S_{11}}, h_{S_{12}}, h_{S_{13}}\right) = \bigcup_{\xi_{i,j}\in\sigma_{i,j}, i\neq j} \\ &\quad\times\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{3}\left(1-\xi_{i,j}\right)^{\frac{1}{6}}\right)^{\frac{1}{2}}\right)\right\} \\ &= g^{-1}\{0.5223, 0.5262, 0.0.5283, \ldots, 0.6267, \\ &\qquad 0.6295, 0.6356\} \\ &= \{s_{0.13}, s_{0.16}, s_{0.17}, \ldots, s_{0.76}, s_{0.78}, s_{0.81}\} \end{aligned}$$

Similarly, the others can be calculated as:

$$\begin{aligned} k_2 &= \mathrm{HFLB}^{1,1}\left(h_{S_{21}}, h_{S_{22}}, h_{S_{23}}\right) \\ &= g^{-1}\{0.6085, 0.6215, 0.6263, 0.0.6339, 0.6384, \\ &\qquad 0.6430, 0.6501, 0.6545, 0.6655\} \\ &= \{s_{0.65}, s_{0.73}, s_{0.76}, s_{0.80}, s_{0.83}, s_{0.86}, s_{0.90}, s_{0.93}, s_{0.99}\} \\ k_3 &= \mathrm{HFLB}^{1,1}\left(h_{S_{31}}, h_{S_{32}}, h_{S_{33}}\right) \\ &= g^{-1}\{0.7219, 0.7339, 0.7373, \ldots, 0.8284, \\ &\qquad 0.8334, 0.8419\} \\ &= \{s_{1.28}, s_{1.40}, s_{1.42}, \ldots, s_{1.97}, s_{2.00}, s_{2.05}\} \\ k_4 &= \mathrm{HFLB}^{1,1}\left(h_{S_{41}}, h_{S_{42}}, h_{S_{43}}\right) \\ &= g^{-1}\{0.6654, 0.6779, 0.6837, 0.6897, 0.6953, \\ &\qquad 0.7008, 0.7064, 0.7116, 0.7219\} \\ &= \{s_{0.99}, s_{1.07}, s_{1.10}, s_{1.14}, s_{1.17}, s_{1.20}, s_{1.24}, s_{1.27}, s_{1.33}\} \end{aligned}$$

Step 3. Calculate the score values of k_i $(i = 1, 2, 3, 4)$, which are $s(k_1) = 0.5820$, $s(k_2) = 0.6380$, $s(k_3) = 0.7808$ and $s(k_4) = 0.6948$, respectively. Then, the ranking order is: $s(k_3) \succ s(k_4) \succ s(k_2) \succ s(k_1)$. So the best hospital is Union Medical College Hospital.

In the above calculation process, we only consider the interrelationship of each criteria and do not consider their weights. So the importance degrees of these criteria are not reflected. Next, we add the weight vector $\omega = (0.3, 0.2, 0.5)$ into these three criteria; then, we can obtain the aggregation values by Eq. (3.27) as follows:

Step 2. We also let the $p = 1$ and $q = 1$, then

$$\begin{aligned} k_1' &= \mathrm{WHFLB}_{\omega}^{1,1}\left(h_{S_{11}}, h_{S_{12}}, h_{S_{13}}\right) = \bigcup_{\xi^{\omega}_{i,j}\in\sigma^{\omega}_{i,j}, i\neq j} \\ &\quad\times\left\{g^{-1}\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{3}\left(1-\xi^{\omega}_{i,j}\right)^{\frac{1}{6}}\right)^{\frac{1}{2}}\right)\right\} \\ &= g^{-1}\{0.4967, 0.5001, 0.5034, \ldots, 0.6138, \\ &\qquad 0.6159, 0.6193\} \\ &= \{s_{-0.02}, s_0, s_{0.02}, \ldots, s_{0.68}, s_{0.70}, s_{0.72}\} \\ k_2' &= \mathrm{WHFLB}_{\omega}^{1,1}\left(h_{S_{21}}, h_{S_{22}}, h_{S_{23}}\right) \\ &= \{0.6584, 0.6603, 0.6620, \ldots, 0.7487, 0.7499, 0.7512\} \\ &= \{s_{0.95}, s_{0.96}, s_{0.97}, \ldots, s_{1.49}, s_{1.50}, s_{1.51}\} \\ k_3' &= \mathrm{WHFLB}_{\omega}^{1,1}\left(h_{S_{31}}, h_{S_{32}}, h_{S_{33}}\right) \\ &= \{0.5824, 0.5848, 0.5856, \ldots, 0.6182, 0.6196, 0.6204\} \\ &= \{s_{0.49}, s_{0.51}, s_{0.52}, \ldots, s_{0.71}, s_{0.72}, s_{0.722}\} \\ k_4' &= \mathrm{WHFLB}_{\omega}^{1,1}\left(h_{S_{41}}, h_{S_{42}}, h_{S_{43}}\right) \\ &= \{0.5608, 0.5635, 0.5637, \ldots, 0.5692, 0.5694, 0.5721\} \\ &= \{s_{0.36}, s_{0.38}, s_{0.382}, \ldots, s_{0.415}, s_{0.416}, s_{0.433}\} \end{aligned}$$

Step 3. Similarly, by calculating the score values of k_i' $(i = 1, 2, 3, 4)$, i.e., $s\left(k_1'\right) = 0.5446$, $s\left(k_2'\right) = 0.7065$, $s\left(k_3'\right) = 0.6016$, and $s\left(k_4'\right) = 0.5665$. Comparing the sizes of these four score values of k_i' $(i = 1, 2, 3, 4)$, the ranking order is: $s(k_2) \succ s(k_3) \succ s(k_4) \succ s(k_1)$. So the best hospital is Huashan Hospital of Fudan University.

For these two aggregation and selection operations about the MCDM problem, we make the following analyses:

1. Clearly, the optimal selection results about these two kinds of aggregation method are different. The main reason is that these two kinds of methods utilize two different aggregation operators. The second computational method adopts the WHFLBM operator which considers

the effect of the weights, while the HFLBM operator does not consider the weights of criteria. As we know, it is impossible that every criterion has the same importance degree in real life, so it is necessary for us to give weight to each criterion. Therefore, the WHFLBM operator is more reasonable and accurate than the HFLBM operator.

2. For these two aggregation operators, both of them have the variable numbers p and q. In fact, when changing the values of p and q, the importance degrees of the corresponding base numbers also must be changed, so the result of $\sigma_{i,j} = h_i^p \otimes h_j^q$ should be different. In the above example, the values of p and q are $p = 1$ and $q = 1$. Of course, we can take many different values for p and q according to the actual circumstances.
3. Both of the two aggregation operators have one common characteristic, that is, they can consider the interrelationship between two criteria. So we do not ignore any information when combining any two criteria in the process of decision making.

In the following, we discuss one previous aggregation operator, the hesitant fuzzy linguistic weighted averaging (HFLWA) operator (Zhang and Qi 2013), to make some comparisons with the WHFLBM operator and the HFLBM operator. The definition of the HFLWA operator is shown as follows:

Definition 4.1 (Zhang and Qi 2013) Let $h_j^{\bar{S}}$ $(j = 1, 2, \ldots, n)$ be a collection of HFLEs. A hesitant fuzzy linguistic weighted averaging (HFLWA) operator is a mapping $\bar{S}^n \to \bar{S}$, where $\bar{S} = \left\{s_{\alpha_j} \left| \alpha_j \in [0, g-1]\right.\right\}$ such that

$$\mathrm{HFLWA}\left(h_1^{\bar{S}}, h_2^{\bar{S}}, \ldots, h_j^{\bar{S}}, \ldots, h_n^{\bar{S}}\right) = \bigoplus_{j=1}^{n}\left(\omega_j h_j^{\bar{S}}\right)$$
$$= f\left(\bigcup_{s_{\alpha_1} \in h_1^{\bar{S}}, s_{\alpha_2} \in h_2^{\bar{S}}, \ldots, s_{\alpha_n} \in h_n^{\bar{S}}} \left\{1 - \prod_{j=1}^{n}\left(1 - f^{-1}\left(s_{\alpha_j}\right)\right)^{\omega_j}\right\}\right) \tag{4.1}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $h_j^{\bar{S}}$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

The HFLWA operator is based on the concept of HFLTS (Rodríguez et al. 2012), which is different from what we discuss in this paper. The linguistic term set in Definition 4.1 can be denoted by $S = \{s_0, \ldots, s_\tau\}$, while this paper mainly discusses the subscript-symmetric additive linguistic term set, which can be shown as $S = \{s_t \,|t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$. However, these two kinds of linguistic term sets can be made by equivalent transformation, so it is necessary to transform the HFLEs included in Table 2 into another form for utilizing the HFLWA operator better.

Table 3 Transformed hesitant fuzzy linguistic decision matrix

	C_1	C_2	C_3
A_1	$\langle x_{11}, \{s_3, s_4\}\rangle$	$\langle x_{12}, \{s_5\}\rangle$	$\langle x_{13}, \{s_2, s_3\}\rangle$
A_2	$\langle x_{21}, \{s_4\}\rangle$	$\langle x_{22}, \{s_2, s_3\}\rangle$	$\langle x_{23}, \{s_5\}\rangle$
A_3	$\langle x_{31}, \{s_4\}\rangle$	$\langle x_{32}, \{s_4, s_5\}\rangle$	$\langle x_{33}, \{s_5, s_6\}\rangle$
A_4	$\langle x_{41}, \{s_5\}\rangle$	$\langle x_{42}, \{s_3, s_4\}\rangle$	$\langle x_{43}, \{s_4\}\rangle$

When obtaining all the aggregation values by using the HFLWA operator, it is important to compare them, and thus, the next method is necessary:

Definition 4.2 (Zhang and Qi 2013) For a HFLE h_S, $s\left(h_S\right) = \frac{1}{n}\left(\sum_{j=1}^{n} {}_{s_{\alpha_j} \in h_S}\, f\left(s_{\alpha_j}\right)\right)$ is called the score function of h_S, where n is the number of the elements in h_S,

1. If $s\left(h_S^1\right) < s\left(h_S^2\right)$, then h_S^1 is smaller than h_S^2, denoted by $h_S^1 \prec h_S^2$;
2. If $s\left(h_S^1\right) = s\left(h_S^2\right)$, then h_S^1 is equal to h_S^2, denoted by $h_S^1 = h_S^2$.

Successively, we can utilize the HFLWA operator to aggregate the HFLEs included in Table 3 and decide which hospital is optimal.

Step 1. Utilize the HFLWA operator to aggregate the HFLEs included in Table 3:

$$k_1'' = \mathrm{HFLWA}\left(h_1^{\bar{S}}, h_2^{\bar{S}}, h_3^{\bar{S}}, h_3^{\bar{S}}\right) = \bigoplus_{j=1}^{4}\left(\omega_j h_j^{\bar{S}}\right)$$
$$= f^{-1}\left(\bigcup_{s_{\alpha_1} \in h_1^{\bar{S}}, s_{\alpha_2} \in h_2^{\bar{S}}, \ldots, s_{\alpha_n} \in h_n^{\bar{S}}} \left\{1 - \prod_{j=1}^{n}\left(1 - f\left(s_{\alpha_j}\right)\right)^{\omega_j}\right\}\right)$$
$$= \{0.5633, 0.5896, 0.5987, 0.6446\} = \{s_{3.22}, s_{3.54}, s_{3.59}, s_{3.87}\}$$
$$k_2'' = \{0.7293, 0.7444\} = \{s_{4.38}, s_{4.47}\}$$
$$k_3'' = \{0.7643, 0.7948, 1\} = \{s_{4.59}, s_{4.77}, s_6\}$$
$$k_4'' = \{0.7064, 0.7293\} = \{s_{4.24}, s_{4.38}\}.$$

Step 2. Utilizing the score function introduced in Definition 4.2, we get the score values of these four alternatives: $s\left(k_1''\right) = 0.5991$, $s\left(k_2''\right) = 0.7369$, $s\left(k_1''\right) = 0.8530$, and $s\left(k_1''\right) = 0.7179$.

Step 3. Based on the score values obtained in Step 2, we can get the ranking of them: $s\left(k_3''\right) \succ s\left(k_2''\right) \succ s\left(k_4''\right) \succ s\left(k_1''\right)$. So the Union Medical College Hospital is the optimal choice.

Clearly, we can find that these two examples have different results by utilizing three different aggregation operators. Some advantages and drawbacks about the HFLBM operator and the WHFLBM operator can be summarized as follows:

1. Firstly, we utilize three different aggregation operators to aggregate the given linguistic information. They have different forms and their focuses are different. The

HFLBM operator does not consider the weight of each criterion, which is the main drawback. The HFLWA operator considers the importance degree of each criterion and gives the weights of them, but it is just a simple weighted average operator and it does not consider the interrelationships among the criteria. However, the WHFLBM operator not only considers the significance of the weights of all criteria, but also considers the interrelationships among the criteria, that is to say, the WHFLBM is more reasonable to deal with such a situation.

2. The linguistic term sets sometimes are very important in decision making. In these two examples, we have utilized two kinds of linguistic term sets. One is the subscript-symmetric additive linguistic term set, which was introduced by Yu et al. (2012), and we have mainly used it in this paper. The other one can be denoted by $S = \{s_0, \ldots, s_\tau\}$, which has many shortcomings as it is unreasonable when calculating the additive operations among different linguistic terms. For example, if one linguistic term set is

$$S = \{s_0 = worst,\ s_1 = very\ bad,\ s_2 = bad,\ s_3 = medium,\ s_4 = good,\ s_5 = very\ good,\ s_6 = best\}$$

then $s_2 + s_4 = s_6$, that is to say, the result of the "bad" adding the "good" is "best." Obviously, it is unreasonable. Therefore, the subscript-symmetric additive linguistic term set is more convincing. Hence, the HFLBM operator and the WHFLBM operator are better than the HFLWA operator.

We have mainly discussed the advantages of the HFLBM operator and the WHFLBM operator. However, there also exists one main drawback about these two operators. During the process of aggregations, we have to omit some data because too many data are calculated, such as k_1, k_1' and so on. Obviously, the final result will be inaccurate if we omit some data. So in the future work, it needs to develop some methods to avoid this issue.

5 Conclusions

In this paper, we have introduced two novel aggregation operators, i.e., the HFLBM operator and the WHFLBM operator. Firstly, we have defined two kinds of equivalent transformation functions, which can be used to do the transformation between the HFLTS and the HFS. Afterward, we have introduced the HFLBM operator and discussed its score function and properties. Next, considering the weight information of each criterion, the WHFLBM has been established. Finally, we have applied these two aggregation operators to develop an approach to MCDM with hesitant fuzzy linguistic information and made a practical case study concerning how to choose the best hospital about conducting the whole society resource management research included in a wisdom medical health system. Several comparisons have been analyzed to show the advantages and disadvantages of the aggregation operators and the MCDM approach. In the future, we will focus on establishing the decision support system based on our MCDM approach and give its application in real life decision making with hesitant fuzzy linguistic information.

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Compliance with ethical standards

Conflict of interest Xunjie Gou, Zeshui Xu and Huchang Liao declare that they no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

Appendix

See Table 4.

Table 4 Abbreviations and their full names

Abbreviation	Full name
MCDM	Multiple criteria decision making
FSs	Fuzzy sets
IFSs	Intuitionistic fuzzy sets
IVIFNs	Interval-valued intuitionistic fuzzy sets
HFSs	Hesitant fuzzy sets
HFEs	Hesitant fuzzy elements
HFLTSs	Hesitant fuzzy linguistic term sets
VIKOR	Vlsekriterijumska optimizacija i kompromisno resenje in serbian, meaning multi-criteria optimization and compromise solution)
HFL-TOPSIS	Hesitant fuzzy linguistic technique for order preference by similarity to an ideal solution
BM	Bonferroni mean
HFLEs	Hesitant fuzzy linguistic elements
HFLBM	Hesitant fuzzy linguistic Bonferroni mean
WHFLBM	Weighted hesitant fuzzy linguistic Bonferroni mean

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