Soft Comput (2017) 21:1913–1922
DOI 10.1007/s00500-016-2113-2

FOUNDATIONS

Fuzzy prime and maximal filters of residuated lattices

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Published online: 22 March 2016

Abstract In this paper, given a residuated lattice M and a lattice L, we introduce the notions of L-fuzzy filter of M, L-fuzzy prime (and maximal) filter of M and give some characterizations of theses notions.

Keywords Fuzzy filter · Fuzzy prime filter · Fuzzy maximal filter

1 Introduction

Residuated lattices are algebraic structures with strong connections to mathematical logic. The algebras under investigation combine the fundamental notions of multiplication, order, and residuation and include many well-studied ordered algebraic structures. During relatively recent years, many algebras have been proposed as the semantical systems of logical systems, for example, Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, Heyting algebras, pseudo-MV-algebras, pseudo-BL-algebras, pseudo-MTL-algebras and so forth, and they are all particular cases of residuated lattices.

Filters are tools of extreme importance in studying these logical algebras and the completeness of non-classical logics. A filter is also called a deductive system. From a logical point of view, various filters correspond to various sets of provable formulae. Filters are also particularly interesting because they are closely related to congruence relations.

At present, there are two main ways to generalize the existing types of filters: the folding theory and the fuzzy sets theory.

In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. In Jun et al. (2005), Liu and Li (2005), Bakhshi (2011), fuzzy filters in MTL algebras, BL-algebras, and residuated lattices were studied, respectively. The notion of fuzzy subset was introduced by Zadeh (1965) in the sixties: a fuzzy subset of a set E is a map $f : E \to I$, where $I := [0; 1]$ is the closed unit interval of real numbers. Since then, a lot of work has been done on fuzzy mathematical structures; most authors use the above original definition of a fuzzy set. In the present work, we replace the closed interval by a suitable lattice. So, a fuzzy subset of E will be a map $\mu : E \to L$, where L is the underlying set of a lattice. It follows from our characterization of L-fuzzy filters that most of the known results with $I = [0, 1]$ are easily proved and extended to more general cases.

In Sect. 2, some basic concepts and properties are recalled. In Sect. 3, we study the notion of L-fuzzy filters and give their properties. Section 4 is devoted to the notion of fuzzy prime filter. We give a complete characterization and several properties of fuzzy prime filter. In Sect. 5, we study fuzzy maximal filters and show that a maximal fuzzy filter is a particular fuzzy maximal filter.

Communicated by A. Di Nola.

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2 Preliminaries

In this section, we collect some definitions and results which will be used in the sequel.

A pseudo-residuated lattice is a nonempty set L with five binary operations $\wedge, \vee, \otimes, \rightarrow, \rightsquigarrow$, and two constants $0, 1$ satisfying

L-1: $\mathbb{L}(L) := (L, \wedge, \vee, 0, 1)$ is a bounded lattice;
L-2: $(L, \otimes, 1)$ is a monoid;
L-3: $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ (pseudo-Residuation);
A *pseudo-RL monoid* is a pseudo-residuated lattice which satisfies the following condition:
L-4: $y \otimes (y \rightsquigarrow x) = x \wedge y = (x \rightarrow y) \otimes x$ (pseudo-Divisibility).
A *pseudo- MTL algebra* is a pseudo-residuated lattice which satisfies the following condition:
L-5: $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$ (pseudo-Prelinearity);
A *pseudo BL-algebra* is a pseudo-MTL-algebra which satisfies the pseudo-divisibility.
A *Heyting algebra* is a pseudo-residuated lattice which satisfies the following condition:
L-9: $x \wedge y = x \otimes y$.

A pseudo-RL chain is a linear pseudo-residuated lattice, that is a pseudo-residuated lattice such that its lattice order is total.

Recall that for any $x \in L$, $x^0 := 1$, and $x^{n+1} := x^n \otimes x$ for all $n \geq 0$.

For $x \in L$, we denote $\bar{x} := x \rightarrow 0$ and $\tilde{x} := x \rightsquigarrow 0$.

Proposition 1 Ciungu (2006) *Let* $(L; \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow; 0, 1)$ *be a pseudo-residuated lattice, and* $x, y, z \in L$; *then*:

(1) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$.
(2) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$; $(x \otimes y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$; $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$.
(3) $(y \rightarrow z) \otimes (x \rightarrow y) \leq x \rightarrow z$; $(x \rightsquigarrow y) \otimes (y \rightsquigarrow z) \leq x \rightsquigarrow z$.
(4) *If* $x \leq y$ *then* : $y \rightarrow z \leq x \rightarrow z$, $y \rightsquigarrow z \leq x \rightsquigarrow z$, $z \rightsquigarrow x \leq z \rightsquigarrow y$, $z \rightarrow x \leq z \rightarrow y$; $x \otimes z \leq y \otimes z$, $\bar{y} \leq \bar{x}$, $\tilde{y} \leq \tilde{x}$ *and* $z \rightarrow x \leq z \rightarrow y$.
(5) $1 \rightarrow x = x = 1 \rightsquigarrow x$; $x \rightarrow x = 1 = x \rightsquigarrow x$; $x \rightarrow 1 = 1 = x \rightsquigarrow 1$; $y \rightsquigarrow x \geq x \leq y \rightarrow x$.
(6) $\bar{x} \otimes x = 0 = x \otimes \tilde{x}$; $\bar{\tilde{\bar{x}}} = \tilde{x}$; $\tilde{\bar{\tilde{x}}} = \bar{x}$; $\bar{\tilde{x}} \geq x \leq \tilde{\bar{x}}$.
(7) $x \otimes (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \otimes y)$; $(x \rightarrow y) \otimes x \leq x \leq y \rightarrow (x \otimes y)$;
(8) $\tilde{1} = 0 = \bar{1}$; $\tilde{0} = 1 = \bar{0}$.
(9) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$; $(y \vee z) \otimes x = (y \otimes x) \vee (z \otimes x)$;
(10) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$; $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$.
(11) $x \vee y \leq ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x)$; $x \vee y \leq ((x \rightsquigarrow y) \rightarrow y) \wedge ((y \rightsquigarrow x) \rightarrow x)$.
(12) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$; $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$.
(13) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$; $y \rightsquigarrow x \leq (z \rightsquigarrow y) \rightsquigarrow (z \rightsquigarrow x)$.
(14) $x \otimes (x \rightsquigarrow y) \leq x \wedge y$; $(x \rightarrow y) \otimes x \leq x \wedge y$; $x \otimes y \leq x \wedge y$.
(15) $x \rightsquigarrow y \leq z \otimes x \rightsquigarrow z \otimes y$; $x \rightarrow y \leq x \otimes z \rightarrow y \otimes z$.

A pseudo-residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ will often be referred to by its support set L, and will be called a residuated lattice in short, unless otherwise stated.

Note Let $a \in L$.

a is $\wedge$-prime if for all $x, y \in L$, $x \wedge y \leq a$ implies ($x \leq a$ or $y \leq a$).

a is $\vee$-prime if for all $x, y \in L$, $x \vee y \geq a$ implies ($x \geq a$ or $y \geq a$).

From Proposition 1(9), it is easily verified that any co-atom of L is $\wedge$-prime.

It is well known that the class of residuated lattices is a variety (Jipsen and Tsinakis 2002).

Recall that a nonempty subset F of $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ is called a *filter* if it satisfies the following properties:

(F1): For every $x, y \in F$, $x \otimes y \in F$.

(F2): For every $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$.

A filter F is said to be proper if $F \subsetneq L$; it is non-trivial if $\{1\} \subsetneq F \subsetneq L$.

A filter F is said to be commutative if for every $x, y \in L$, $x \rightsquigarrow y \in F$ iff $x \rightarrow y \in F$.

We denote by $\langle A \rangle$ the filter of L generated by A, that is, the smallest filter containing A; and $\langle a \rangle$ is the filter generated by $\{a\}$. Let *Fil(L)* denote the set of all filters of L. The following result is easily obtained:

Proposition 2 *Let L be a residuated lattice and F be a non empty subset of L.*

(i) *F is filter iff $1 \in F$ and for all $x, y \in L$, $x \in F$ and $x \rightarrow y \in F$ implies $y \in F$ iff $1 \in F$ and for all $x, y \in L$, $x \in F$ and $x \rightsquigarrow y \in F$ implies $y \in F$.*
(ii) *If F is a filter, then $x, y \in F$ iff $x \otimes y \in F$ iff $x \wedge y \in F$.*
(iii) *If $A \subseteq L$, then $\langle A \rangle = \{x \in L; a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_n \leq x$ for some $n \geq 1$ and $a_1, a_2, \ldots, a_n \in A\}$.*
(iv) *$\langle a \rangle = \{x \in L; a^n \leq x$ for some $n \geq 1\}$.*
(v) *If F is a filter and $x \in L$, then $\langle F \cup \{x\} \rangle = \{u \in L; (a_1 \otimes x^{n_1}) \otimes (a_2 \otimes x^{n_2}) \otimes (a_3 \otimes x^{n_3}) \otimes \cdots \otimes (a_m \otimes x^{n_m}) \leq u$ for some $m \geq 1$, $n_1, n_2, n_3, \ldots, n_m \geq 0$ and $a_1, a_2, \ldots, a_n \in F\}$.*

A lattice $(L, \wedge, \vee)$ will be called completely meet-distributive if it is complete and the identity $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$ holds in L, for all $x \in L$ and $\{y_i; i \in I\} \subseteq L$.

If (F_i) is a family of filters of L, we define $\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i$, and $\bigvee_{i \in I} F_i := \langle \bigcup_{i \in I} F_i \rangle$.

It is well known that $(Fil(L); \wedge, \vee)$ is a completely meet distributive lattice.

Example 3 Let $L = \{0, a, b, c, d, 1\}$ be a lattice such that $0 < a, b < c < d < 1$, a and b are not comparable. Define the operations $\otimes$, $\rightarrow$ and $\rightsquigarrow$ by the three tables below. Then L is a residuated lattice which is not a pseudo-MTL algebra since $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = c \neq 1$.

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	c	c	c	c	1	1
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$F = \{1\}$; $F_1 = \{1, d\}$ are the only proper filters of L.

Example 4 Let $L = \{0, a, b, c, d, e, 1\}$ be a lattice such that $0 < a < b, c < d < e < 1$, b and c are not comparable. Define the operations $\otimes$, $\rightsquigarrow$ and $\rightarrow$ by the three tables below. L is a residuated lattice which is not a pseudo-MTL algebra since $(b \rightsquigarrow c) \vee (c \rightsquigarrow b) = e \neq 1$.

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	b	c	d	e	e
1	0	a	b	c	d	e	1

$\rightsquigarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	e	1	1	1
e	0	a	b	c	d	1	1
1	0	a	b	c	d	e	1

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$F_1 = \{1\}$; $F_2 = \{1, e\}$; $F_3 = \{1, e, d, c\}$; $F_4 = \{1, e, d, c, b, a\}$ are the proper filters of L.

Example 5 Let $L = \{0, a, b, c, d, 1\}$ be a lattice such that $0 < a < b, c < d < 1$, b and c are not comparable. Define the operations $\otimes$, $\rightarrow$ and $\rightsquigarrow$ by the three tables below. Then L is a residuated lattice which is not a pseudo-MTL algebra, since $(b \rightarrow c) \vee (c \rightarrow b) = d \neq 1$.

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	a	a
b	0	0	b	0	b	b
c	0	a	a	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	1	1	1
b	0	c	1	c	1	1
c	b	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	c	c	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$F_1 = \{1\}$, $F_2 = \{1, d\}$, $F_3 = \{1, d, c\}$, and $F_4 = \{1, d, b\}$ are the proper filters of L.

Example 6 Let $L = \{0, a, b, c, 1\}$ be a lattice such that $0 < a < b, c < 1$, b and c are not comparable. Define the operations $\otimes$, $\rightarrow$ and $\rightsquigarrow$ by the three tables below. Then L is a pseudo-residuated lattice which is not a pseudo-BL algebra, since $(b \rightarrow c) \otimes b = c \otimes b = 0 \neq a = b \wedge c$.

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

$F_1 = \{1\}$, $F_2 = \{1, c\}$, $F_3 = \{1, b\}$ are the proper filters of L.

Example 7 Let $L = \{0, a, b, c, 1\}$ be a lattice such that $0 < a < b < c < 1$. Define the operations $\otimes$, $\rightarrow$ and $\rightsquigarrow$ by the three tables below. Then L is a residuated lattice which is not a pseudo-BL algebra since $(c \rightarrow b) \otimes c = c \otimes c = a \neq b = b \wedge c$.

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

$F_1 = \{1\}$ is the only proper filter of L.

Definition 8 A proper filter F is said to be

(i) Prime if for all $x, y \in L$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.
(ii) Prime of the second kind if for all $x, y \in L$, $(x \rightarrow y \in F$ or $y \rightarrow x \in F)$ and $(x \rightsquigarrow y \in F$ or $y \rightsquigarrow x \in F)$.
(iii) Prime of the third kind if for all $x, y \in L$, $[(x \rightarrow y) \vee (y \rightarrow x)] \wedge [(x \rightsquigarrow y) \vee (y \rightsquigarrow x)] \in F$.
(iv) Maximal if it is contained in no proper filter of L.

Remark 9 Let L be a residuated lattice.

(i) Prime filters of the second kind are prime filters. The converse is true if L is a pseudo-MTL algebra.
(ii) Prime filters of the second kind are prime filters of the third kind. The converse is true if the filter is also prime.

We denote by $Spec(L)$ the set of prime filters of L and $Spec_2(L)$ the set of prime filters of the second kind of L.

Definition 10 A subset S of L is a $\vee$-closed system in L if $1 \notin S$ and for all $x, y \in S$, $x \vee y \in S$.
Filters and ideals of lattices have the usual meanings.

Like in the case of commutative residuated lattice, we also have the following three results (which were established in Kadji et al. (submitted)):

Remark 11 (i) Prime filter Theorem: Let F be a filter of L and $\emptyset \neq S$ be a $\vee$-closed system (or a lattice ideal) in L

such that $F \cap S = \emptyset$. Then, there is a prime filter P of L such that $F \subseteq P$ and $P \cap S = \emptyset$.

(ii) Any maximal filter of L is prime.

(iii) If F is a proper filter of L, there is a maximal filter G such that $F \subseteq G$.

Definition 12 A residuated lattice L is said to satisfy the prime condition of the second kind if for all $F \in \mathrm{Fil(L)}$ and S a $\vee$-closed system such that $F \cap S = \emptyset$, there is $P \in Spec_2(L)$ such that $P \cap S = \emptyset$ and $F \subseteq P$.

From the definition, we get the following result:

Lemma 13 *If L satisfies the prime condition of the second kind, then*:

(i) *For all $a \neq 1$, there is $P \in Spec_2(L)$ such that $a \notin P$;*

(ii) *Any proper filter of L can be extended to a prime filter of the second kind.*

Note that if L is a RL chain, then $Spec_2(L) = Spec(L)$. From this and Remark 11, it is clear that any RL-chain satisfies the prime condition of the second kind.

Example 14 The residuated lattice of Example 6 satisfies the prime condition of the second kind; but the residuated lattice of Example 4 does not, because $1 \neq d$ and there is no prime filter of the second kind F of L such that $d \notin F$, since $\{1, e\} \notin Spec_2(L)$.

Proposition 15 Bakhshi (2013) *Let F be a filter of L. Then F is a prime filter iff for any filters H and G, $H \cap G \subseteq F$ implies $H \subseteq F$ or $G \subseteq F$.*

Let F be a commutative filter of L. The well-known binary relation $\equiv_F$ defined on L by $x \equiv_F y$ if and only if $x \to y$ and $y \to x \in F$, is a congruence on L. The quotient structure L/F is also a pseudo-residuated lattice where $x/F \wedge y/F = (x \wedge y)/F$; $x/F \vee y/F = (x \vee y)/F$; $x/F \otimes y/F = (x \otimes y)/F$; $x/F \to y/F = (x \to y)/F$, and $(x \rightsquigarrow y)/F = x/F \rightsquigarrow y/F$.
Note that $L/\{1\} \cong L$.

Given a commutative filter F of L, and $a \in L$, the congruence class of a in L/F will sometime be denoted by a_F.

Definition 16 A residuated lattice L is said to be

(i) Local if it has a unique maximal filter.

(ii) Locally finite if for all $x \in L$, $x \neq 1$ implies that there is $n \geq 1$ such that $x^n = 0$.

(iii) Integral if for all $x, y \in L$, $x \otimes y = 0$ implies $x = 0$ or $y = 0$.

So L is locally finite iff $\{1\}$ is the only proper filter of L (see Example 7).

3 Fuzzy filter

We now introduce fuzzy subsets with values in a lattice. From now on, $(M, \wedge, \vee, \otimes, \to, \rightsquigarrow; 0, 1)$ is a residuated lattice and $(L, \wedge, \vee; 0, 1)$ is a complete lattice.

Definition 17 An L-fuzzy subset of M is a function $\mu : M \to L$.
If $\alpha \in L$, the α-cut of μ is the set $\mu_\alpha := \{x \in M : \mu(x) \geq \alpha\}$.

Note that if $\alpha, \beta \in Im(\mu)$, then $\mu_\alpha = \mu_\beta$ implies $\alpha = \beta$, and $\alpha < \beta$ implies $\mu_\alpha \supsetneq \mu_\beta$.

An L-fuzzy subset $\mu : M \to L$ will simply be called a fuzzy subset of M.

For any $\alpha \in L$, let c_α denote the constant map $M \to L$ with value α.

Definition 18 A fuzzy subset μ of M is called a fuzzy lattice ideal of M if for all $\alpha \in L$, μ_α is either empty or a lattice ideal of M.

For any fuzzy subsets δ, η, we define $\delta \leq \eta$ if and only if $\delta(x) \leq \eta(x)$ for all $x \in M$.

Remark 19 Let L^M be the set of fuzzy subsets of M, and consider the structure $(L^M, \wedge, \vee, c_0; c_1)$, where the operations are defined componentwise, that is

$$\begin{cases} (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) \\ (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) \\ c_0(x) = 0 \\ c_1(x) = 1 \end{cases}$$

Then $(L^M, \wedge, \vee; c_0, c_1)$ is a complete lattice.

Note that if $(L; \wedge, \vee, \otimes, \to, \rightsquigarrow, 0, 1)$ is also a residuated lattice, we may define the other binary operations on L^M componentwise; i.e., for all $\lambda, \mu \in L^M$ and $x \in M$

$$\begin{cases} (\lambda \otimes \mu)(x) \ = \lambda(x) \otimes \mu(x) \\ (\lambda \to \mu)(x) = \lambda(x) \to \mu(x) \\ (\lambda \rightsquigarrow \mu)(x) = \lambda(x) \rightsquigarrow \mu(x) \end{cases}$$

Then $(L^M, \wedge, \vee, \otimes, \to, \rightsquigarrow; c_0, c_1)$ is a residuated lattice (since the class of residuated lattices is a variety).

From now on, $(L, \wedge, \vee)$ is completely meet distributive; thus, so is $(L^M, \wedge, \vee)$.

Definition 20 A fuzzy subset μ of M is called a fuzzy filter of M if for all $\alpha \in L$, μ_α is either empty or a filter of M.

It is easy to see that for all $\alpha \in L$, c_α is a fuzzy filter of M (called a constant fuzzy filter). A fuzzy filter is said to be proper if it is non-constant.

Theorem 21 Bakhshi (2011) *A fuzzy subset μ of M is a fuzzy filter of M iff for all $x, y \in M$, $\mu(x \otimes y) = \mu(x) \wedge \mu(y)$ and ($x \leq y$ implies $\mu(x) \leq \mu(y)$).*

Example 22 (i) Let L and M be the residuated lattices of Examples 3 and 4, respectively; consider the fuzzy subsets υ and μ of M defined by

$$\upsilon(x) = \begin{cases} 1 & \text{if } x \in \{1, e\} \\ b & \text{if not} \end{cases} \qquad \mu(x) = \begin{cases} 0 & \text{if } x \in \{0, a\} \\ b & \text{if } x = b \\ c & \text{if } x \in \{c, d, e\} \\ 1 & \text{if } x = 1 \end{cases}$$

Then υ is a fuzzy filter of M; but μ is not a fuzzy filter of M because $\mu(b \otimes c) = \mu(a) = 0 \neq b = \mu(b) \wedge \mu(c)$.

(ii) Let G be a filter of M, and $\alpha < \beta$ in L. Define the map $(G)_\alpha^\beta$ as follows:

$$(G)_\alpha^\beta(x) = \begin{cases} \beta & \text{if } x \in G \\ \alpha & \text{otherwise} \end{cases}$$

Then, $(G)_\alpha^\beta$ is a fuzzy filter of M; and it is non-constant *iff* G is a proper filter of M.

(iii) Let μ be a fuzzy filter of M, and $\beta \in L$; then $\mu \vee c_\beta$ is a fuzzy filter of M.

Proposition 23 (i) *If μ is a fuzzy filter, then for all $x, y \in M$, $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$ and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.*

(ii) *Let $\theta : M \to L$ be a fuzzy subset. Then the following assertions are equivalent*:

(ii)$_1$ *θ is a fuzzy filter.*

(ii)$_2$ *for all $x, y \in M$, $\theta(x \to y) \wedge \theta(x) \leq \theta(y)$ and $\theta(x) \leq \theta(1)$.*

(ii)$_3$ *for all $x, y \in M$, $\theta(x \rightsquigarrow y) \wedge \theta(x) \leq \theta(y)$ and $\theta(x) \leq \theta(1)$.*

Proof (i) is obvious.

(ii)$_1$ $\Rightarrow$ (ii)$_2$: Since $x \leq 1$, we have $\theta(x) \leq \theta(1)$. In addition, since $(x \to y) \otimes x \leq x \wedge y$, it follows that $\theta(x \to y) \wedge \theta(x) = \theta((x \to y) \otimes x) \leq \theta(x \wedge y) \leq \theta(y)$.

(ii)$_2$ $\Rightarrow$ (ii)$_1$: Let $x, y \in M$. If $x \leq y$; then $\theta((x \wedge y) \to y) \wedge \theta(x \wedge y) \leq \theta(y)$, that is $\theta(x) \leq \theta(y)$ $(*)$. In addition, since $y \leq x \to x \otimes y$, it follows by $(*)$ that $\theta(x) \wedge \theta(y) \leq \theta(x) \wedge \theta(x \to x \otimes y) \leq \theta(x \otimes y)$.

(ii)$_1$ $\Leftrightarrow$ (ii)$_3$: Similar to (ii)$_1$ $\Leftrightarrow$ (ii)$_2$. □

Let $\mathcal{F}uFil(M)$ denote the set of fuzzy filters of M.

Definition 24 A fuzzy filter μ is said to be commutative if for all $x, y \in M$, $\mu(x \rightsquigarrow y) = \mu(1)$ if and only if $\mu(x \to y) = \mu(1)$.

It is easy to see that μ is a commutative fuzzy filter iff $\mu_{\mu(1)}$ is a commutative filter of M. In particular, if G is a proper filter of M and $\alpha < \beta$ in L, then $(G)_\alpha^\beta$ is a commutative fuzzy filter iff G is a commutative filter.

Definition 25 Let f be a fuzzy subset of M. A fuzzy filter g of M is said to be generated by f if $f \leq g$ and for any fuzzy filter h of M, $f \leq h$ implies $g \leq h$. The fuzzy filter generated by f will be denoted by $\hat{f}$.

Theorem 26 Tonga (2011) *Let $(L, \wedge, \vee; 0, 1)$ be completely meet distributive. If $f : M \to L$ is a fuzzy subset of M, then $\hat{f}(x) = \bigvee\{\alpha \in L; x \in \langle f_\alpha \rangle\}$.*

For $\mu, \nu \in \mathcal{F}uFil(M)$, we define $\mu \sqcup \nu := \mu \hat{\vee} \nu$. So, for a family $\{f_i;\ i \in I\}$ of fuzzy filters, $\bigsqcup\{f_i;\ i \in I\}$ is the fuzzy filter generated by the fuzzy set $\bigvee\{f_i;\ i \in I\}$. Clearly, the meet of a family of fuzzy filters is a fuzzy filter; so, $(\mathcal{F}uFil(M); \wedge, \sqcup, c_0, c_1)$ is a complete lattice.

Problem 1 Is it possible that $\mathcal{F}uFil(M)$ be a completely meet distributive lattice?

A class **F** of filters of M is said to be closed under extension if for every filters G, H of M such that $G \subseteq H$, $G \in$ **F** implies $H \in$ **F**.

A class C of fuzzy filters is said to be closed under extension if for all $\theta, \vartheta \in$ FuFil(M) such that $\theta(1) = \vartheta(1)$, ($\theta \in C$ and $\theta \leq \vartheta$) implies $\vartheta \in C$.

It may be required that the filters (resp., the fuzzy filters) in question satisfy a particular condition. For instance, the set of prime filters of the second kind is closed under extension of proper filters.

4 Fuzzy prime filters

The set of fuzzy filters of M is a lattice. In this section, we first introduce the notion of *fuzzy prime* filter and then look at its prime elements.

4.1 Fuzzy prime filter

Definition 27 Let $\mu : M \to L$ be a proper fuzzy filter of M. μ is a fuzzy prime filter of M if for any $\alpha \in L$, μ_α is a prime filter whenever it is proper.

Lemma 28 *Let μ be a proper fuzzy filter of M; then μ is fuzzy prime iff $Im(\mu)$ is a chain, and $\mu(x \vee y) = max\,\{\mu(x), \mu(y)\}$ for all $x, y \in M$.*

Proof Let μ be a non-constant fuzzy filter of M.

Assume that μ is fuzzy prime; then $x \vee y \in \mu_{\mu(x \vee y)}$, so $\mu_{\mu(x \vee y)}$ is M or a prime filter; thus $x \in \mu_{\mu(x \vee y)}$ or $y \in \mu_{\mu(x \vee y)}$; this implies $\mu(x) \geq \mu(x \vee y)$ or $\mu(y) \geq \mu(x \vee y)$; so $\mu(x) \geq \mu(y)$ or $\mu(x) \leq \mu(y)$.

Conversely, assume that the conclusion holds. Let $\alpha \in L$ such that μ_α is a proper filter, and $x, y \in M$ such that $x \vee y \in \mu_\alpha$. That is $\mu(x \vee y) \geq \alpha$; since $\mu(x \vee y) \in \{\mu(x), \mu(y)\}$, we obtain $x \in \mu_\alpha$ or $y \in \mu_\alpha$. So μ_α is a prime filter of M. □

Remark 29 (i) Let $\alpha < \beta$ in L, and G be a proper filter of M. Then $(G)_\alpha^\beta$ is a fuzzy prime filter of M iff G is a prime filter of M.
(ii) The notion of fuzzy prime filter given in Swamy and Swamy (1988) is not the same as ours; rather, it corresponds to what we call *prime fuzzy filter* in the last part of this section.

Definition 30 An element $\alpha \in L$ satisfies the chain property on $X \subseteq L$ if it is comparable with all elements of X.

Theorem 31 (Fuzzy prime filter Theorem)
Let μ be a proper fuzzy filter of M with $\mu(1) < 1$, and $\alpha \in L$. Suppose that there is a fuzzy lattice ideal δ of M such that $\mu \wedge \delta < c_\alpha$ and α satisfies the chain property on $Im(\mu) \cup Im(\delta)$. Then there is a fuzzy prime filter η of M such that $\mu \leq \eta$ and $\eta \wedge \delta \leq c_\alpha$.

Proof Let $\beta = \mu(1)$; since μ is non-constant, $\mu_\beta = \mu_{\mu(1)}$ is a proper filter of M. We look at the following three cases, relative to the α-cuts of δ and μ:
Case 1: $\delta_\alpha = \emptyset$. Then $\delta < c_\alpha$. From Remark 11(i), taking $S = \{0\}$, there is a prime filter F of M such that $\mu_\beta \subseteq F$. Then $\eta := (F)_\beta^1$ is a fuzzy prime filter of M, with $\mu \leq \eta$; and $\eta \wedge \delta \leq c_\alpha$.
Case 2: $\mu_\alpha = \emptyset$. Then $\mu < c_\alpha$, and $\beta < \alpha$. If F is the prime filter given in case 1, and $\eta := (F)_\beta^\alpha$, we have $\mu < \eta \leq c_\alpha$, and $\eta \wedge \delta \leq c_\alpha$.
Case 3: $\delta_\alpha \neq \emptyset \neq \mu_\alpha$. Then δ_α is a lattice ideal of M, and $\alpha \leq \beta$.
$\mu \wedge \delta < c_\alpha$ implies $\mu_\alpha \cap \delta_\alpha = \emptyset$; so $\mu_\alpha \neq M$, and μ_α is a proper filter of M. By Remark 11(i), there is a prime filter G of M such that $\mu_\alpha \subseteq G$ and $G \cap \delta_\alpha = \emptyset$. Let $\eta := (G)_\alpha^1$; it is a fuzzy prime filter, and it remains to show that $\mu < \eta$ and $\eta \wedge \delta \leq c_\alpha$. For each $x \in M$,
$x \in G$ implies $x \notin \delta_\alpha$; so $\mu(x) \leq 1 = \eta(x)$, and $\delta(x) < \alpha$; so $(\eta \wedge \delta)(x) < \alpha$;
$x \notin G$ implies $x \notin \mu_\alpha$; so $\mu(x) < \alpha = \eta(x)$, and $(\eta \wedge \delta)(x) \leq \alpha$.
So, in any case, $\mu(x) \leq \eta(x)$ and $(\eta \wedge \delta)(x) \leq c_\alpha(x)$. □

Definition 32 A fuzzy filter μ is called a fuzzy boolean filter if for all $x \in M$, $\mu(x \vee \overline{x}) = \mu(1) = \mu(x \vee \widetilde{x})$

Proposition 33 *Let μ be a fuzzy filter of M. If μ is fuzzy boolean and fuzzy prime, then $Im(\mu) = \{\mu(1), \mu(0)\}$.*

Proof For every $x \in M$, we have $\mu(x) \vee \mu(\overline{x}) = \mu(1) = \mu(x) \vee \mu(\widetilde{x})$, and $\mu(x) \wedge \mu(\overline{x}) = \mu(0) = \mu(x) \wedge \mu(\widetilde{x})$; since $Im(\mu)$ is a chain, we obtain $\mu(x) = \mu(1)$ or $\mu(\overline{x}) = \mu(1) = \mu(\widetilde{x})$; it follows that $\mu(x) \in \{\mu(1), \mu(0)\}$. □

4.2 Fuzzy prime filter of the second kind

Definition 34 A proper fuzzy filter θ is called a fuzzy prime filter of the second kind if for all $x, y \in M$, $(\theta(x \to y) = \theta(1)$ or $\theta(y \to x) = \theta(1))$ and $(\theta(x \rightsquigarrow y) = \theta(1)$ or $\theta(y \rightsquigarrow x) = \theta(1))$.

Proposition 35 *The following conditions are equivalent for a proper fuzzy filter μ of M:*

(i) *μ is a fuzzy prime filter of the second kind of M.*
(ii) *For every $\alpha \in L$, if μ_α is a proper filter, then it is a prime filter of the second kind.*
(iii) *$\mu_{\mu(1)}$ is a prime filter of the second kind.*

Proof (i) $\Rightarrow$ (ii): Assume that μ is a fuzzy prime filter of the second kind of M. Let μ_α be a proper α-cut of μ. Since μ_α is a filter, we have $1 \in \mu_\alpha$, so $\mu(1) \geq \alpha$. In addition, for all $x, y \in M$, $(\mu(x \to y) = \mu(1) \geq \alpha$ or $\mu(y \to x) = \mu(1) \geq \alpha)$ and $(\mu(x \rightsquigarrow y) = \mu(1) \geq \alpha$ or $\mu(y \rightsquigarrow x) = \mu(1) \geq \alpha)$. That is $(x \to y \in \mu_\alpha$ or $y \to x \in \mu_\alpha)$ and $(x \rightsquigarrow y \in \mu_\alpha$ or $y \rightsquigarrow x \in \mu_\alpha)$.
(ii) $\Rightarrow$ (iii): Obviously, since μ non-constant implies $\mu_{\mu(1)}$ is a proper filter.
(iii) $\Rightarrow$ (i): This is a consequence of the definitions. □

The following remark is a direct consequence of the definition:

Remark 36 (i) If M is a RL-chain, then any proper fuzzy filter of M is a fuzzy prime filter of the second kind.
(ii) For $\alpha \neq 1$ in L and G a proper filter of M, $(G)_\alpha^1$ is a fuzzy prime filter of the second kind if and only if G is a prime filter of the second kind of M.
(iii) M is a RL-chain iff $(\{1\})_\alpha^\beta$ is a fuzzy prime filter of the second kind for any $\alpha < \beta$ in L.
(iv) μ is a commutative fuzzy prime filter of the second kind of M iff $M/\mu_{\mu(1)}$ is a RL chain.

Proposition 37 *The class of fuzzy prime filters of the second kind of M is closed under extension of proper fuzzy filters.*

Proof Let μ be a fuzzy prime filter of the second kind of M and δ be a non-constant fuzzy filter of M such that $\mu \leq \delta$ and $\mu(1) = \delta(1)$. So for all $x, y \in M$, $(\delta(x \to y) \geq \mu(x \to y) = \mu(1) = \delta(1)$ or $\delta(y \to x) \geq \mu(y \to x) = \mu(1) = \delta(1))$ and $(\delta(x \rightsquigarrow y) \geq \mu(x \rightsquigarrow y) = \mu(1) = \delta(1)$ or $\delta(y \rightsquigarrow x) \geq \mu(y \rightsquigarrow x) = \mu(1) = \delta(1))$. That is $(\delta(x \to y) = \delta(1)$ or $\delta(y \to x) = \delta(1))$ and $(\delta(x \rightsquigarrow y) = \delta(1)$ or $\delta(y \rightsquigarrow x) = \delta(1))$. □

Corollary 38 *Let μ be a fuzzy prime filter of the second kind, and $\alpha \in L$ with $\alpha < \mu(1)$. Assume that $\mu(1)$ is $\vee$-irreducible, or there is $x \in M$ such that $\alpha \vee \mu(x) < \mu(1)$. Then, $\mu \vee c_\alpha$ is a fuzzy prime filter of the second kind.*

Proof From Example 22, $\mu \vee c_\alpha$ is a fuzzy filter. Now, $\mu \leq \mu \vee c_\alpha$, and $(\mu \vee c_\alpha)(1) = \mu(1)$; so we only have to prove that $\mu \vee c_\alpha$ is a non-constant fuzzy subset.

But $(\mu \vee c_\alpha)(1) = \mu(1) \vee \alpha = \mu(1) > \mu(x) \vee \alpha \geq \mu(0) \vee \alpha = (\mu \vee c_\alpha)(0)$. □

Theorem 39 *Let M be a residuated lattice which satisfies the prime condition of the second kind. Let μ be a non-constant fuzzy filter of M such that $\mu(1)$ is completely $\vee$-irreducible, or $\mu(1) < 1$. Then, there is a fuzzy prime filter of the second kind δ such that $\mu \leq \delta$*

Proof Since μ is a proper fuzzy filter, $\mu_{\mu(1)}$ is a proper filter and by Lemma 13, there is a prime filter of the second kind F such that $\mu_{\mu(1)} \subseteq F$. Let $\alpha := \bigvee\{\mu(x); x \in M\backslash F\}$; then $\alpha < \mu(1)$. Let $\beta := \mu(x)$ for some $x \notin F$; then $(F)^1_\beta$ is a fuzzy prime filter of the second kind, and so is $\delta := (F)^1_\beta \vee c_\alpha$ by Corollary 38. Clearly, $\mu \leq \delta$. □

Theorem 40 *Let M be a residuated lattice which satisfies the prime condition of the second kind. Let μ be a proper fuzzy filter of M, $\alpha \in L$ such that $\alpha < \mu(1)$, and α is $\wedge$-irreducible. Let η be a fuzzy subset of M which satisfies $\eta(x \vee y) = \eta(x) \wedge \eta(y)$ and $\eta \wedge \mu \leq c_\alpha$. Suppose that α satisfies the chain property on $Im(\mu) \cup Im(\eta)$. Then there is a fuzzy prime filter of the second kind θ such that $\mu \leq \theta$ and $\eta \wedge \theta \leq c_\alpha$.*

Proof Let $S := \{x \in M; \eta(x) > \alpha\}$, $F := \{x \in M; \mu(x) > \alpha\}$. Then S is a $\vee$-closed subset of M, and F is a filter of M. Since $\eta \wedge \mu \leq c_\alpha$, we have $S \cap F = \emptyset$. So, there is a prime filter of the second kind P of M such that $F \subseteq P$ and $S \cap P = \emptyset$. Let $\theta := (P)^1_\alpha = (P)^1_0 \vee c_\alpha$. Then θ is a fuzzy prime filter of the second kind, by Corollary 38. If $x \in P$, then $\theta(x) = 1 \geq \mu(x)$, and if $x \notin P$, then $x \notin F$; thus $\mu(x) \leq \alpha = \theta(x)$. Hence, $\mu \leq \theta$.

In addition, if $x \in S$, then $x \notin P$, so $\theta(x) = \alpha$ and thus $(\eta \wedge \theta)(x) = \eta(x) \wedge \theta(x) = \alpha$. If $x \notin S$, then $\eta(x) \leq \alpha$ and thus $(\eta \wedge \theta)(x) = \eta(x) \wedge \theta(x) \leq \eta(x) \leq \alpha$. Therefore, $\eta \wedge \theta \leq c_\alpha$. □

4.3 Prime fuzzy filter

We now look for prime elements of the lattice $(\mathcal{F}uFil(M); \wedge, \sqcup; c_0, c_1)$.

Definition 41 A proper fuzzy filter μ of M is prime if for every fuzzy filters θ and σ of M, $\theta \wedge \sigma \leq \mu$ implies $\theta \leq \mu$ or $\sigma \leq \mu$.

Example 42 Let L and M be the residuated lattices of Example 3 and 4, respectively. Consider the fuzzy subset μ of M defined by Example 4

$$\mu(x) = \begin{cases} d & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

It is easy to check that μ is a prime fuzzy filter of M.

Lemma 43 *Any prime fuzzy filter μ of M is a fuzzy prime filter.*

Proof Let $\alpha \in L$ such that μ_α is a proper filter of M, and F, G be filters of M such that $F \cap G \subseteq \mu_\alpha$. Consider the fuzzy filters $(F)^\alpha_0$ and $(G)^\alpha_0$ of M; then $(F)^\alpha_0 \wedge (G)^\alpha_0 \leq \mu$. So, $(F)^\alpha_0 \leq \mu$ or $(G)^\alpha_0 \leq \mu$. But, $(F)^\alpha_0 \leq \mu$ implies $F \subseteq \mu_\alpha$, and $(G)^\alpha_0 \leq \mu$ implies $G \subseteq \mu_\alpha$. Thus μ_α is a prime filter of M; and μ is a fuzzy prime filter. □

Theorem 44 *A proper fuzzy filter μ is prime iff $\mu = (G)^1_\alpha$, where G is a prime filter of M and α is $\wedge$-prime in L.*

Proof ($\Rightarrow$): Note that $F := \mu_{\mu(1)}$ is a prime filter by the above lemma. Consider the fuzzy filters $\lambda := (F)^1_0$ and $\delta := c_{\mu(1)}$; then

$$(\lambda \wedge \delta)(x) = \begin{cases} \mu(1) & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases}$$

So $\lambda \wedge \delta \leq \mu$; and $\lambda \leq \mu$ or $\delta \leq \mu$. Since μ is non constant, we have $\delta > \mu$; thus $\lambda \leq \mu$, showing that $\mu(1) = 1$.

Now, let $a, b \notin F$, and consider the fuzzy filters f and g defined by

$$f(x) = \begin{cases} 1 & \text{if } a^n \leq x \text{ for some } n \geq 1 \\ 0 & \text{if not,} \end{cases}$$

and $g = c_{\mu(a)}$.

Then, $f \wedge g \leq \mu$; since $f \nleq \mu$, we have $g \leq \mu$. So $\mu(a) = g(a) = g(b) \leq \mu(b)$. Similarly, we show that $\mu(a) \geq \mu(b)$. So, μ takes only one value other than $1 = \mu(1)$; let us call it α; then $\mu = (F)^1_\alpha$, and it remains to show that α is $\wedge$-prime.

Suppose that β, γ are elements of L such that $\beta \wedge \gamma \leq \alpha$; then $c_\beta \wedge c_\gamma \leq \mu = (F)^1_\alpha$. So, $c_\beta \leq (F)^1_\alpha$ or $c_\gamma \leq (F)^1_\alpha$, that is, $\beta \leq \alpha$ or $\gamma \leq \alpha$.

($\Leftarrow$): Let $\mu := (G)^1_\alpha$ where G is a prime filter of M and α is $\wedge$-prime in L. Let λ, δ be fuzzy filters of M such that $\lambda \nleq \mu$ and $\delta \nleq \mu$. Then there are $a, b \notin G$ such that $\lambda(a) \nleq \alpha$ and $\gamma(b) \nleq \alpha$. Since G is a prime filter, we have $a \vee b \notin G$ and $\mu(a \vee b) = \alpha$.

Now, $\lambda(a \vee b) \nleq \alpha$ and $\gamma(a \vee b) \nleq \alpha$; thus $(\lambda \wedge \gamma)(a \vee b) \nleq \alpha = \mu(a \vee b)$, and $\lambda \wedge \gamma \nleq \mu$. So μ is a prime fuzzy filter. □

5 Fuzzy maximal filter

5.1 Maximal fuzzy filter

Definition 45 A proper fuzzy filter μ is called a maximal fuzzy filter if for every fuzzy filter θ of M, $\mu \leq \theta$ implies $\theta = c_1$ or $\theta = \mu$.

It is easy to see that the fuzzy filter of Example 42 is a maximal fuzzy filter. In fact, we have the following result:

Theorem 46 Tonga (2011) *A non-constant fuzzy filter μ is maximal iff $\mu = (G)^1_\alpha$ where G is a maximal filter of M and α is a co-atom in L.*

Remark 47 (i) The existence of at least one co-atom in L is necessary for $\mathcal{F}uFil(M)$ to have maximal elements.
(ii) From Remark 11(ii), if L is a residuated lattice, any maximal fuzzy filter $\mu : M \to L$ is a prime fuzzy filter (since any co-atom in L is $\wedge$-prime).
(iii) A maximal fuzzy filter is called "fuzzy maximal filter" in Swamy and Swamy (1988); here the latter expression has a quite different meaning, as given in what follows:

5.2 Fuzzy maximal filter

Definition 48 A proper fuzzy filter $\mu : M \to L$ is called fuzzy maximal if for each $\alpha \in L$, μ_α non-trivial (i.e., $\{1\} \subsetneq \mu_\alpha \subsetneq M$) implies μ_α is a maximal filter of M.

Example 49 Let M be the residuated lattice of Example 5 and F_2, F_3 the filters considered there, L be a lattice and α, β be two incomparable elements in L. Define the functions μ, and η from M to L by

$$\mu(x) = \begin{cases} \alpha \vee \beta & \text{if } x \in F_3 \\ \alpha & \text{if not} \end{cases} \qquad \eta(x) = \begin{cases} \alpha \vee \beta & \text{if } x \in F_2 \\ \alpha & \text{if } x = c \\ \beta & \text{if } x = b \\ \alpha \wedge \beta & \text{elsewhere} \end{cases}$$

Then, μ is a fuzzy maximal filter of M (which is not a maximal fuzzy filter if $\alpha \vee \beta \neq 1$ or α is not a co-atom); and η is not fuzzy maximal, because $\{1\} \subsetneq \eta_{\alpha\vee\beta} \subsetneq M$ but $\eta_{\alpha\vee\beta}$ is not a maximal filter of M.

Proposition 50 Tonga (2011) *Let $\mu : M \to L$ be a proper fuzzy filter, and $\alpha, \beta \in L$ be incomparable elements of $Im(\mu)$. Then*

(i) *μ_α and μ_β are non comparable proper filters.*
(ii) *If $\mu : M \to L$ is fuzzy maximal, μ_α and μ_β are maximal filters with $\mu_\alpha \cap \mu_\beta = \{1\}$.*

Corollary 51 *A proper fuzzy filter $\mu : M \to L$ is fuzzy maximal iff $\forall x, y, \in M$, $\mu(x) \nleq \mu(y)$ implies $x = 1$ or $\mu_{\mu(x)}$ is a maximal filter of M.*

Proof ($\Rightarrow$): Let $\alpha = \mu(x)$ and $\beta = \mu(y)$.
If α and β are incomparable, then by Proposition 50, μ_α and μ_β are maximal filters.
If α and β are comparable, then $\beta < \alpha$; so $x \neq 1$ implies $\{1\} \subsetneq \mu_\alpha \subsetneq \mu_\beta \subseteq M$, and μ_α is a maximal filter

($\Leftarrow$): Let $\alpha \in L$ such that μ_α is a non trivial filter of M. There are elements $x, y \in M$ such that $x \in \mu_\alpha$ and $x \neq 1, y \notin \mu_\alpha$. Then $\mu(x) = \alpha \nleq \mu(y)$, and $x \neq 1$, so $\mu_\alpha = \mu_{\mu(x)}$ is a maximal filter. □

From this we can give a shape to the image of a fuzzy maximal filter. First the following:

Proposition 52 *Let μ be a fuzzy maximal filter of M.*

(i) *If $x, y \in M$ such that $\mu(x) < \mu(y)$, then $\mu_{\mu(y)} = \{1\}$ or $\mu_{\mu(x)} = M$.*
(ii) *A chain in Im(μ) has no more than three elements.*

Proof (i): Obvious.
(ii) Let $\alpha \leq \beta < \gamma \leq \delta$ be a chain in $Im(\mu)$.
Since $\gamma \nleq \beta$, we obtain from the above corollary that $\mu_\gamma = \{1\}$ or μ_γ is a maximal filter.
If $\mu_\gamma = \{1\}$, then $\{1\} \subseteq \mu_\delta \subseteq \mu_\gamma = \{1\}$; so $\mu_\delta = \{1\} = \mu_\gamma$, and $\gamma = \delta$.
If $\mu_\gamma \neq \{1\}$, then μ_γ is a maximal filter of M, and $\mu_\beta = M$. Since $\mu_\beta \subseteq \mu_\alpha \subseteq M$, we obtain that $\mu_\beta = M = \mu_\alpha$, and $\alpha = \beta$. □

Note The implication in Proposition 52 (i) is not enough for μ to be fuzzy maximal, as shown by the following example:
Let M be the residuated lattice of Example 5, L be a lattice, and α, β be two incomparable elements in L. Consider the function $h : M \to L$ defined by

$$h(x) = \begin{cases} \alpha \vee \beta & \text{if } x = 1 \\ \alpha & \text{if } x \in \{c, d\} \\ \alpha \wedge \beta & \text{elsewhere} \end{cases}$$

Then h is a fuzzy filter satisfying the condition of Proposition 52 (i), but h is not fuzzy maximal.

Corollary 53 *Let μ be a fuzzy maximal filter of M. In the ordered set $(Im(\mu); \leq)$, every element of $Im(\mu)\backslash\{\mu(0), \mu(1)\}$ is both an atom and a co-atom.*

Proof Let $\alpha = \mu(0)$ and $\gamma = \mu(1)$.
If M is locally finite, $\{1\}$ is the unique (maximal) proper filter of M; so $\mu(x) = \alpha$ for every $x \neq 1$, and $Im(\mu) = \{\alpha, \gamma\}$.
If M is not locally finite and $\mu_{\mu(1)} \neq \{1\}$, then $\mu_{\mu(1)}$ is a maximal filter by Corollary 51; so $\mu(x) = \alpha$ for every $x \notin \mu_{\mu(1)}$, and $Im(\mu) = \{\alpha, \gamma\}$.
So we are remained with the case M not locally finite and $\mu_{\mu(1)} = \{1\}$. If $\mu(x) = \beta > \alpha$ for some $x \neq 1$, then $\{\alpha, \beta, \gamma\} \subseteq Im(\mu)$, and $\alpha < \beta < \gamma$ is a maximal chain in $Im(\mu)$, by Proposition 52. So, β is both an atom and a co-atom in $(Im(\mu); \leq)$. □

Now, it is clear that a maximal fuzzy filter is fuzzy maximal, and a fuzzy maximal filter μ is fuzzy prime iff $\mu_{\mu(1)}$ is a prime filter.

Problem 2 It would be nice to have suitable conditions under which the lattice (or a sublattice) of fuzzy filters is a residuated lattice.

Acknowledgements We thank the referee for the helpful comments that allow us to improve the quality of the work.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this work

References

Bakhshi M (2011) Fuzzy Boolean and prime filters in non-commutative residuated lattices. In: Proceedings of the 11th Conference on fuzzy systems, pp 159–166, Sistan and Baluchestan University

Bakhshi M (2013) Spectrum topology of a residuated lattice. Fuzzy Inf Eng 5:159–172

Ciungu LC (2006) Classes of residuated lattices. Ann Univ Craiova Math Comput Sci Ser 33:189–207. ISSN: 1223-6934

Jipsen P, Tsinakis C (2002) A survey of residuated lattices. In: Martinez J (ed) Ordered algebraic structures. Kluwer Acad. Publ., Dordrecht, pp 19–56

Jun YB, Xu Y, Zhang XH (2005) Fuzzy filters of MTL-algebras. Inf Sci 175:120–138

Kadji A, Lele C, Tonga M Some classes of pseudo-residuated lattices. Afr Mat. doi:10.1007/s13370-016-0401-8

Liu L, Li K (2005) Fuzzy filters of BL-algebras. Inf Sci 173:141–154

Swamy UM, Swamy KL (1988) Fuzzy prime ideals of rings. J Math Anal Appl 134:94–103

Tonga M (2011) Maximality on fuzzy filters of lattices. Afr Mat 22:105–114. doi:10.1007/s13370-011-0009-y

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353