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FOUNDATIONS

On congruences of weak lattices

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Abstract We characterize when an equivalence relation on the base set of a weak lattice $\mathbf{L} = (L, \sqcup, \sqcap)$ becomes a congruence on $\mathbf{L}$ provided it has convex classes. We show that an equivalence relation on L is a congruence on $\mathbf{L}$ if it satisfies the substitution property for comparable elements. Conditions under which congruence classes are convex are studied. If one fundamental operation of $\mathbf{L}$ is commutative then $\mathbf{L}$ is congruence distributive and all congruences of $\mathbf{L}$ have convex classes.

Keywords Weak lattice · Congruence · Majority term

It was recognized by the authors in their recent paper (Chajda and Länger 2013) that for some non-classical logics, their underlying ordered set is not a lattice but it still bears some properties of lattices. For example, for BCK-algebras satisfying the double negation law the underlying ordered set satisfies several interesting axioms which were collected in the definition of a so-called weak lattice. Hence, the subject of our investigation is not only a certain generalization of the concept of a lattice but a really existing structure which comes from some logical systems. The aim of the present paper is to study conditions under which an equivalence relation on the base set of a weak lattice becomes a congruence. A similar problem for lattices was investigated and solved in Dorfer (1995). In this paper Dorfer found a characterization of congruences on lattices without using lattice operations. This motivated us to do a similar job for weak lattices. In fact, we already used a similar approach in our previous paper (Chajda et al. 2012) for congruences on directoids and on directoids with involutions, respectively. The main difference now is that in the case of weak lattices, congruence classes need not be convex. Hence, we tried to find conditions under which congruence classes are convex or, alternatively, we tried to find conditions which do not ask convexity of congruence classes. We get several examples showing that our conditions are sufficient but not necessary and several examples showing that under our conditions the weak lattices are not trivial, i.e. neither lattices nor λ-lattices.

Definition 1 A *weak lattice* is an algebra $\mathbf{L} = (L, \sqcup, \sqcap)$ of type (2, 2) satisfying the following conditions for all $x, y, z \in L$:

(i) $x \sqcup y = y$ and $y \sqcup x = x$ together imply $x = y$, and $x \sqcap y = x$ and $y \sqcap x = y$ together imply $x = y$.
(ii) $x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y) = (x \sqcup y) \sqcup y = x \sqcup y$ and $(x \sqcap y) \sqcap x = (x \sqcap y) \sqcap y = x \sqcap (x \sqcap y) = x \sqcap y$.
(iii) $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$ and $(z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y)$.
(iv) $x \sqcap (x \sqcup y) = x$ and $(x \sqcap y) \sqcup y = y$.

Let us note that the original definition from Chajda and Länger (2013) contains one more axiom, namely $x \sqcup x =$

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$x \sqcap x = x$. However, this axiom turns out to be redundant since it follows from the remaining axioms as pointed in (ii) of Theorem 3.

Lemma 2 *If* $\mathbf{L} = (L, \sqcup, \sqcap)$ *is a weak lattice and* $a, b \in L$ *then* $a \sqcup b = b$ *if and only if* $a \sqcap b = a$.

Proof If $a \sqcup b = b$ then $a \sqcap b = a \sqcap (a \sqcup b) = a$ and if $a \sqcap b = a$ then $a \sqcup b = (a \sqcap b) \sqcup b = b$. □

At first, we prove some basic properties of weak lattices which can be derived directly from the axioms. It is remarkable that, contrary to the case of λ-lattices [see e.g. Snášel (1997) for this concept, see also Chajda and Länger (2011) and Karásek (1996)], weak lattices satisfy one-side monotonicity for the operations ⊔ and ⊓.

Theorem 3 *An algebra* $\mathbf{L} = (L, \sqcup, \sqcap)$ *of type* (2, 2) *is a weak lattice if and only if* (i) *and* (ii) *hold*:

(i) $(x \sqcup y) \sqcup y = x \sqcup y$ *and* $x \sqcap (x \sqcap y) = x \sqcap y$ *for all* $x, y \in L$.
(ii) *There exists a partial order relation* $\leq$ *on* L *such that for all* $x, y, z \in L$ *conditions* (a) *and* (b) *hold:*
   (a) $x \leq y$ *implies* $x \sqcup y = y$, $x \sqcap y = x$, $x \sqcup z \leq y \sqcup z$ *and* $z \sqcap x \leq z \sqcap y$.
   (b) $x \sqcap y \leq x$, $y \leq x \sqcup y$.

Proof Let $x, y, z \in L$.

First assume $\mathbf{L}$ to be a weak lattice. Then (i) holds. To prove (ii) define a binary relation $\leq$ on L by $x \leq y$ if $x \sqcup y = y$. According to Lemma 2, $x \leq y$ is equivalent to $x \sqcap y = x$.

$x \sqcup x = (x \sqcap (x \sqcup x)) \sqcup x = ((x \sqcap (x \sqcup x)) \sqcap x) \sqcup x = x$ implies $x \leq x$.

$x \leq y \leq x$ implies $x \sqcup y = y$ and $y \sqcup x = x$ whence $x = y$.

$x \leq y \leq z$ implies $x \sqcup z = (x \sqcup z) \sqcup z = (x \sqcup z) \sqcup (y \sqcup z) = (x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z = z$ and hence $x \leq z$.

This shows that $\leq$ is a partial order. Now let us verify the conditions in (a).

$x \leq y$ implies $x \sqcup y = y$ by the definition of $\leq$.

$x \leq y$ implies $x \sqcap y = x$ according to Lemma 2.

$x \leq y$ implies $(x \sqcup z) \sqcup (y \sqcup z) = (x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z$ according to the definition of $\leq$ and (iii) of Definition 1 and hence $x \sqcup z \leq y \sqcup z$ according to the definition of $\leq$.

$x \leq y$ implies $(z \sqcap x) \sqcap (z \sqcap y) = (z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y) = z \sqcap x$ according to the definition of $\leq$, Lemma 2 and (iii) of Definition 1 and hence $z \sqcap x \leq z \sqcap y$ according to the definition of $\leq$ and Lemma 2.

Now let us check the conditions in (b).

$(x \sqcap y) \sqcap x = x \sqcap y$ implies $x \sqcap y \leq x$.

$(x \sqcap y) \sqcap y = x \sqcap y$ implies $x \sqcap y \leq y$.

$x \sqcup (x \sqcup y) = x \sqcup y$ implies $x \leq x \sqcup y$.

$y \sqcup (x \sqcup y) = x \sqcup y$ implies $y \leq x \sqcup y$.

Conversely, assume (i) and (ii).

$x \sqcup y = y$ and $y \sqcup x = x$ imply $x \leq x \sqcup y = y \leq y \sqcup x = x$ and hence $x = y$.

$x \sqcap y = x$ and $y \sqcap x = y$ imply $x = x \sqcap y \leq y = y \sqcap x \leq x$ and hence $x = y$.

$x \leq x \sqcup y$ implies $x \sqcup (x \sqcup y) = x \sqcup y$.

$y \leq x \sqcup y$ implies $y \sqcup (x \sqcup y) = x \sqcup y$.

$x \sqcap y \leq x$ implies $(x \sqcap y) \sqcap x = x \sqcap y$.

$x \sqcap y \leq y$ implies $(x \sqcap y) \sqcap y = x \sqcap y$.

$x \leq x \sqcup y$ implies $x \sqcup z \leq (x \sqcup y) \sqcup z$ and hence $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

$x \sqcap y \leq y$ implies $z \sqcap (x \sqcap y) \leq z \sqcap y$ and hence $(z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y)$.

$x \leq x \sqcup y$ implies $x \sqcap (x \sqcup y) = x$.

$x \sqcap y \leq y$ implies $(x \sqcap y) \sqcup y = y$. □

Remark 4 Let $\mathbf{L} = (L, \sqcup, \sqcap)$ be a weak lattice. Then the induced partial order relation $\leq$ on L mentioned in Theorem 3 is uniquely determined, namely, for arbitrary $a, b \in L$ we have $a \leq b$ if and only if $a \sqcup b = b$ if and only if $a \sqcap b = a$. Moreover, $\mathbf{L}$ satisfies the identities $y \sqcap (x \sqcup y) = y$ and $(x \sqcap y) \sqcup x = x$.

Definition 5 A *congruence* on a weak lattice $\mathbf{L} = (L, \sqcup, \sqcap)$ is an equivalence relation Θ on L such that $(x, y), (z, u) \in \Theta$ implies $(x \sqcup z, y \sqcup u), (x \sqcap z, y \sqcap u) \in \Theta$.

Lemma 6 *If* $\mathbf{L} = (L, \sqcup, \sqcap)$ *is a weak lattice,* $a \in L$, $\Theta \in \mathrm{Con}\mathbf{L}$ *and* $b \in [a]\Theta$ *then* $a \sqcup b, b \sqcup a, a \sqcap b, b \sqcap a \in [a]\Theta$.

Proof We have $a \sqcup b, b \sqcup a \in [a \sqcup a]\Theta = [a]\Theta$ and $a \sqcap b, b \sqcap a \in [a \sqcap a]\Theta = [a]\Theta$. □

We are now ready to prove that, similarly as for lattices, an equivalence relation on a weak lattice is a congruence if it satisfies the substitution properties for comparable elements.

Theorem 7 *An equivalence relation* Θ *on the base set of a weak lattice* $\mathbf{L} = (L, \sqcup, \sqcap)$ *is a congruence on* $\mathbf{L}$ *if and only if for all* $x, y, z \in L$ *conditions* (i) *and* (ii) *hold*:

(i) $(x, y) \in \Theta$ *implies* $(x, x \sqcup y), (y, x \sqcup y) \in \Theta$.
(ii) $x \leq y$ *and* $(x, y) \in \Theta$ *imply* $(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y), (x \sqcap z, y \sqcap z), (z \sqcap x, z \sqcap y) \in \Theta$.

These properties imply that for any $(x, y), (z, u) \in \Theta$ *we have* $(x \sqcup z, y \sqcup u), (x \sqcap z, y \sqcap u) \in \Theta$ *showing that* $\Theta \in \mathrm{Con}\mathbf{L}$.

Proof If $\Theta \in \mathrm{Con}\mathbf{L}$ then (i) and (ii) clearly hold. Conversely, assume (i) and (ii). Let $(x, y) \in \Theta$ and $z \in L$. Because of (i), we have $(x, x \sqcup y), (y, x \sqcup y) \in \Theta$. Applying (ii) we obtain

$$(x \sqcup z, (x \sqcup y) \sqcup z), (y \sqcup z, (x \sqcup y) \sqcup z),$$
$$(z \sqcup x, z \sqcup (x \sqcup y)), (z \sqcup y, z \sqcup (x \sqcup y)) \in \Theta$$

and

$$(x \sqcap z, (x \sqcup y) \sqcap z), (y \sqcap z, (x \sqcup y) \sqcap z), (z \sqcap x, z \sqcap (x \sqcup y)), (z \sqcap y, z \sqcap (x \sqcup y)) \in \Theta$$

which implies

$$(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y), (x \sqcap z, y \sqcap z), (z \sqcap x, z \sqcap y) \in \Theta$$

showing $\Theta \in \mathrm{Con}\mathbf{L}$. □

The first step for characterizing congruences on weak lattices in a way similar to that of Dorfer (1995) is to list their properties with respect to the induced order.

Theorem 8 *For a congruence Θ on a weak lattice $\mathbf{L} = (L, \sqcup, \sqcap)$ the following conditions hold for arbitrary $x, y, z \in L$:*

(i) *$x \le y$, $(x, y) \in \Theta$ and $z \in L$ imply that there exists*

some $x_1 \in [x \sqcup z]\Theta$ with $x_1 \ge y \sqcup z$,
some $x_2 \in [y \sqcup z]\Theta$ with $x_2 \ge x \sqcup z$,
some $x_3 \in [z \sqcup x]\Theta$ with $x_3 \ge z \sqcup y$,
some $x_4 \in [z \sqcup y]\Theta$ with $x_4 \ge z \sqcup x$,
some $x_5 \in [x \sqcap z]\Theta$ with $x_5 \le y \sqcap z$,
some $x_6 \in [y \sqcap z]\Theta$ with $x_6 \le x \sqcap z$,
some $x_7 \in [z \sqcap x]\Theta$ with $x_7 \le z \sqcap y$ and
some $x_8 \in [z \sqcap y]\Theta$ with $x_8 \le z \sqcap x$.

(ii) *$x \le y$ and $z \in [x]\Theta$ imply that there exists some $u \in [y]\Theta$ with $u \ge z$.*
(iii) *$x \le y$ and $z \in [y]\Theta$ imply that there exists some $u \in [x]\Theta$ with $u \le z$.*

Proof Let $a, b, c \in L$ and assume $a \le b$.

(i) If $(a, b) \in \Theta$ then

$b \sqcup c \in [a \sqcup c]\Theta, b \sqcup c \ge b \sqcup c,$
$a \sqcup c \in [b \sqcup c]\Theta, a \sqcup c \ge a \sqcup c,$
$c \sqcup b \in [c \sqcup a]\Theta, c \sqcup b \ge c \sqcup b,$
$c \sqcup a \in [c \sqcup b]\Theta, c \sqcup a \ge c \sqcup a,$
$b \sqcap c \in [a \sqcap c]\Theta, b \sqcap c \le b \sqcap c,$
$a \sqcap c \in [b \sqcap c]\Theta, a \sqcap c \le a \sqcap c,$
$c \sqcap b \in [c \sqcap a]\Theta, c \sqcap b \le c \sqcap b$ and
$c \sqcap a \in [c \sqcap b]\Theta, c \sqcap a \le c \sqcap a.$

(ii) If $c \in [a]\Theta$ then $c \sqcup b \in [a \sqcup b]\Theta = [b]\Theta$ and $c \sqcup b \ge c$.
(iii) If $c \in [b]\Theta$ then $a \sqcap c \in [a \sqcap b]\Theta = [a]\Theta$ and $a \sqcap c \le c$.

□

Unfortunately, congruences on weak lattices need not have convex classes as shown by the following example. However, we can state an easy sufficient condition which implies convexity of congruence classes.

Example 9 Consider the algebra $\mathbf{L} = (L, \sqcup, \sqcap)$ of type (2, 2) with $L = \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of all integers),

$$x \sqcup y = \begin{cases} y & \text{if } x \le y \\ x & \text{if } x > y \text{ and } x + y \text{ is even} \\ x + 1 & \text{if } x > y \text{ and } x + y \text{ is odd} \end{cases}$$

and

$$x \sqcap y = \begin{cases} x & \text{if } x \le y \\ y & \text{if } x > y \text{ and } x + y \text{ is even} \\ y - 1 & \text{if } x > y \text{ and } x + y \text{ is odd} \end{cases}$$

($x, y \in \mathbb{Z}$). Let $a, b, c \in \mathbb{Z}$. First, by using Theorem 3, we will check that $\mathbf{L}$ is a weak lattice.

If $a \le b$ then $(a \sqcup b) \sqcup b = b \sqcup b = b = a \sqcup b$ and $a \sqcap (a \sqcap b) = a \sqcap a = a = a \sqcap b$.

If $a > b$ and $a + b$ is even then $(a \sqcup b) \sqcup b = a \sqcup b$ and $a \sqcap (a \sqcap b) = a \sqcap b$.

If $a > b$ and $a + b$ is odd then $(a \sqcup b) \sqcup b = (a + 1) \sqcup b = a + 1 = a \sqcup b$ and $a \sqcap (a \sqcap b) = a \sqcap (b - 1) = b - 1 = a \sqcap b$.

If $a \le b$ then $a \sqcup b = b$ and $a \sqcap b = a$.

If $c \le a < b$ then $a \sqcup c \le a + 1 \le b \le b \sqcup c$ and $c \sqcap a = c = c \sqcap b$.

If $a \le c \le b$ then $a \sqcup c = c \le b \le b \sqcup c$ and $c \sqcap a \le a \le c = c \sqcap b$.

If $a < b \le c$ then $a \sqcup c = c = b \sqcup c$ and $c \sqcap a \le a \le b - 1 \le c \sqcap b$.

Obviously, $a \sqcap b \le a, b \le a \sqcup b$.

According to Theorem 3, $\mathbf{L}$ is a weak lattice. Put $\Theta := \{(x, y) \in \mathbb{Z}^2 \mid x + y \text{ is even}\}$. Then Θ is an equivalence relation on $\mathbb{Z}$ and $(a \sqcup b) + b$ and $(a \sqcap b) + a$ are even. Now let $(a, b) \in \Theta$. Then $a + b$ is even. Now we have

$(a \sqcup c) + (b \sqcup c) = ((a \sqcup c) + c) + ((b \sqcup c) + c) - 2c$ is even and hence $(a \sqcup c, b \sqcup c) \in \Theta$,

$(c \sqcup a) + (c \sqcup b) = ((c \sqcup a) + a) + ((c \sqcup b) + b) - (a + b)$ is even and hence $(c \sqcup a, c \sqcup b) \in \Theta$,

$(a \sqcap c) + (b \sqcap c) = ((a \sqcap c) + a) + ((b \sqcap c) + b) - (a + b)$ is even and hence $(a \sqcap c, b \sqcap c) \in \Theta$ and

$(c \sqcap a) + (c \sqcap b) = ((c \sqcap a) + c) + ((c \sqcap b) + c) - 2c$ is even and hence $(c \sqcap a, c \sqcap b) \in \Theta$.

This shows $\Theta \in \mathrm{Con}\mathbf{L}$. Obviously, Θ has two classes, namely the set of even integers and that of odd integers and both are not convex.

Theorem 10 *Every congruence on a weak lattice* $\mathbf{L} = (L, \sqcup, \sqcap)$ *satisfying the identity*

$$(x \sqcup y) \sqcup x = x \sqcup y \tag{1}$$

has convex classes.

Proof If $a \in L$, $\Theta \in \mathrm{Con}\mathbf{L}$, $b, c \in [a]\Theta$, $d \in L$ and $b \leq d \leq c$ then

$$d = b \sqcup d \in [c \sqcup d]\Theta = [(d \sqcup c) \sqcup d]\Theta$$
$$= [d \sqcup c]\Theta = [c]\Theta = [a]\Theta.$$

□

Remark 11 By duality, an analogous result holds for the identity $y \sqcap (x \sqcap y) = x \sqcap y$.

That this condition is not necessary for the convexity of the congruence classes follows from the following

Example 12 Each congruence on the weak lattice $\mathbf{L} = (L, \sqcup, \sqcap)$ with $L = \{0, a, b, 1\}$ and

$\sqcup$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	0	b	b
1	0	a	b	1

has convex classes since $0 < a < b < 1$ with respect to the induced order and since $\Theta = \{0, a\}^2 \cup \{b, 1\}^2$ is the only non-trivial congruence on $\mathbf{L}$, but

$$(a \sqcup b) \sqcup a = b \sqcup a = 1 \neq b = a \sqcup b$$

contradicting (1).

However, there exist weak lattices that are not lattices, but satisfy identity (1) as shown by the following

Example 13 The weak lattice $\mathbf{L} = (L, \sqcup, \sqcap)$ with $L = \{0, a, 1\}$ and

$\sqcup$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$\sqcap$	0	a	1
0	0	0	0
a	0	a	a
1	0	0	1

is not a lattice since $\sqcap$ is not commutative, but it satisfies (1).

Theorem 14 *Let* $\mathbf{L} = (L, \sqcup, \sqcap)$ *be a weak lattice and* $\Theta \in \mathrm{Con}\mathbf{L}$. *Then* $\mathbf{L}/\Theta$ *is a weak lattice if and only if* Θ *has convex classes.*

Proof Let $a, b, c, d \in L$.

If $\mathbf{L}/\Theta$ is a weak lattice, $b, c \in [a]\Theta$ and $d \in [b, c]$ then

$$[a]\Theta \sqcup [d]\Theta = [b]\Theta \sqcup [d]\Theta = [b \sqcup d]\Theta = [d]\Theta$$

and

$$[d]\Theta \sqcup [a]\Theta = [d]\Theta \sqcup [c]\Theta = [d \sqcup c]\Theta = [c]\Theta = [a]\Theta$$

and hence $[a]\Theta = [d]\Theta$ which implies $d \in [a]\Theta$.

Conversely, assume Θ to have convex classes. Since $\mathbf{L}/\Theta$ satisfies all identities holding in $\mathbf{L}$, it satisfies (ii)–(iv) of Definition 1. Now assume $[a]\Theta \sqcup [b]\Theta = [b]\Theta$ and $[b]\Theta \sqcup [a]\Theta = [a]\Theta$. Then $[a \sqcup b]\Theta = [a]\Theta \sqcup [b]\Theta = [b]\Theta$ and, analogously, $[b \sqcup a]\Theta = [a]\Theta$. This means $b \;\Theta\; (a \sqcup b)$. Thus also $(a \sqcap b) \;\Theta\; (a \sqcap (a \sqcup b))$. Since $a \sqcap (a \sqcup b) = a$ according to (iv) of Definition 1 and, by the second equality in $\mathbf{L}/\Theta$, also $a \;\Theta\; (b \sqcup a)$, we conclude $(a \sqcap b) \;\Theta\; (a \sqcap (a \sqcup b)) = a \;\Theta\; (b \sqcup a)$. Since $(a \sqcap b) \leq b \leq (b \sqcup a)$ according to Theorem 3 and since Θ has convex classes we obtain $a \;\Theta\; b$ and hence $[a]\Theta = [b]\Theta$. The second assertion follows by duality. □

The following example shows that it is not exceptional that a weak lattice has congruences with convex classes.

Example 15 If $(L, \leq, 0, 1)$ is a bounded poset and

$$x \sqcup y := \begin{cases} y & \text{if } x \leq y \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad x \sqcap y := \begin{cases} x & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

then $\mathbf{L} = (L, \sqcup, \sqcap)$ is a weak lattice whose congruences have convex classes since $a \in L$, $\Theta \in \mathrm{Con}\mathbf{L}$, $b, c \in [a]\Theta$, $d \in L$ and $b \leq d < c$ together imply

$$d = b \sqcup d \in [c \sqcup d]\Theta = [1]\Theta = [c \sqcup b]\Theta = [b \sqcup b]\Theta$$
$$= [b]\Theta = [a]\Theta.$$

Lemma 16 *For a weak lattice* $\mathbf{L} = (L, \sqcup, \sqcap)$ *and a congruence* Θ *on* $\mathbf{L}$ *the following hold*:

(i) *If* 0 *is the smallest element of* $(L, \leq)$ *then* $[0]\Theta$ *is convex.*
(ii) *If* 1 *is the greatest element of* $(L, \leq)$ *then* $[1]\Theta$ *is convex.*

Proof (i) If $a, b \in [0]\Theta$, $c \in L$ and $a \leq c \leq b$ then $c = c \sqcap b \in [c \sqcap 0]\Theta = [0]\Theta$.
(ii) If $a, b \in [1]\Theta$, $c \in L$ and $a \leq c \leq b$ then $c = a \sqcup c \in [1 \sqcup c]\Theta = [1]\Theta$.

□

Corollary 17 *From Lemma 16 it follows that all congruences on weak lattices with at most four elements have convex classes.*

Assuming convexity of congruence classes, we can now prove that conditions (i)–(iii) of Theorem 8 together with (i) of Theorem 7 are also sufficient for an equivalence relation on a weak lattice to be a congruence. These conditions correspond to those in Dorfer (1995).

Theorem 18 *Let* $\mathbf{L} = (L, \sqcup, \sqcap)$ *be a weak lattice and* Θ *an equivalence relation on* L *and assume that* (i)–(iii) *of Theorem* 8 *are satisfied, that (i) of Theorem* 7 *holds and that*

(iv) Θ *has convex classes.*

Then $\Theta \in \operatorname{Con}\mathbf{L}$.

Proof Let $a, b, c \in L$ and assume $a \le b$ and $(a, b) \in \Theta$. According to (i) there exists

some $a_1 \in [a \sqcup c]\Theta$ with $a_1 \ge b \sqcup c$,
some $a_2 \in [b \sqcup c]\Theta$ with $a_2 \ge a \sqcup c$,
some $a_3 \in [c \sqcup a]\Theta$ with $a_3 \ge c \sqcup b$,
some $a_4 \in [c \sqcup b]\Theta$ with $a_4 \ge c \sqcup a$,
some $a_5 \in [a \sqcap c]\Theta$ with $a_5 \le b \sqcap c$,
some $a_6 \in [b \sqcap c]\Theta$ with $a_6 \le a \sqcap c$,
some $a_7 \in [c \sqcap a]\Theta$ with $a_7 \le c \sqcap b$ and
some $a_8 \in [c \sqcap b]\Theta$ with $a_8 \le c \sqcap a$.

Since $b \sqcup c \le a_1$ and $a \sqcup c \in [a_1]\Theta$ there exists some $d \in [b \sqcup c]\Theta$ with $d \le a \sqcup c$ according to (iii). Now $d \le a \sqcup c \le a_2$ and $d, a_2 \in [b \sqcup c]\Theta$ which implies $a \sqcup c \in [b \sqcup c]\Theta$ according to (iv), i.e. $(a \sqcup c, b \sqcup c) \in \Theta$.

Since $c \sqcup b \le a_3$ and $c \sqcup a \in [a_3]\Theta$ there exists some $e \in [c \sqcup b]\Theta$ with $e \le c \sqcup a$ according to (iii). Now $e \le c \sqcup a \le a_4$ and $e, a_4 \in [c \sqcup b]\Theta$ which implies $c \sqcup a \in [c \sqcup b]\Theta$ according to (iv), i.e. $(c \sqcup a, c \sqcup b) \in \Theta$.

Since $a_5 \le b \sqcap c$ and $a \sqcap c \in [a_5]\Theta$ there exists some $f \in [b \sqcap c]\Theta$ with $f \ge a \sqcap c$ according to (ii). Now $a_6 \le a \sqcap c \le f$ and $a_6, f \in [b \sqcap c]\Theta$ which implies $a \sqcap c \in [b \sqcap c]\Theta$ according to (iv), i.e. $(a \sqcap c, b \sqcap c) \in \Theta$.

Since $a_7 \le c \sqcap b$ and $c \sqcap a \in [a_7]\Theta$ there exists some $g \in [c \sqcap b]\Theta$ with $g \ge c \sqcap a$ according to (ii). Now $a_8 \le c \sqcap a \le g$ and $a_8, g \in [c \sqcap b]\Theta$ which implies $c \sqcap a \in [c \sqcap b]\Theta$ according to (iv), i.e. $(c \sqcap a, c \sqcap b) \in \Theta$.

According to Theorem 7, $\Theta \in \operatorname{Con}\mathbf{L}$. □

It was pointed out in Chajda and Länger (2013) that a weak lattice becomes a λ-lattice if and only if both fundamental operations are commutative. As shown in Example 13, there exist weak lattices where only one fundamental operation is commutative and hence they are neither lattices nor λ-lattices. However, the class of these weak lattices has several interesting properties as stated in the following theorem.

Theorem 19 *The class* $\mathcal{V}$ *of weak lattices satisfying the identity* $x \sqcup y = y \sqcup x$ *is a congruence distributive variety and every congruence on a member of* $\mathcal{V}$ *has convex classes.*

Proof Let $\mathbf{L} = (L, \sqcup, \sqcap) \in \mathcal{V}$ and $a, b \in L$.

If $a \sqcup b = b$ and $b \sqcup a = a$ then $a = b \sqcup a = a \sqcup b = b$.

If $a \sqcap b = a$ and $b \sqcap a = b$ then according to Lemma 2 $a \sqcup b = b$ and $b \sqcup a = a$ which implies $a = b \sqcup a = a \sqcup b = b$.

Hence, the first condition of Definition 1 follows from the remaining axioms which shows that $\mathcal{V}$ is a variety. Since

$$m(x, y, z) := ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (y \sqcap z)$$

satisfies

$$\begin{aligned} m(x, x, y) &= ((x \sqcap x) \sqcup (x \sqcap y)) \sqcup (x \sqcap y) \\ &= (x \sqcup (x \sqcap y)) \sqcup (x \sqcap y) \\ &= ((x \sqcap y) \sqcup x) \sqcup (x \sqcap y) = x \sqcup (x \sqcap y) \\ &= (x \sqcap y) \sqcup x = x, \\ m(x, y, x) &= ((x \sqcap y) \sqcup (x \sqcap x)) \sqcup (y \sqcap x) \\ &= ((x \sqcap y) \sqcup x) \sqcup (y \sqcap x) = x \sqcup (y \sqcap x) = \\ &= (y \sqcap x) \sqcup x = x \quad \text{and} \\ m(y, x, x) &= ((y \sqcap x) \sqcup (y \sqcap x)) \sqcup (x \sqcap x) = (y \sqcap x) \sqcup x = x \end{aligned}$$

in $\mathcal{V}$, m is a majority term in $\mathcal{V}$ and hence $\mathcal{V}$ is congruence distributive. Finally, if $\mathbf{L} \in \mathcal{V}$ and $\Theta \in \operatorname{Con}\mathbf{L}$ then $\mathbf{L}/\Theta \in \mathcal{V}$ which by Theorem 14 implies Θ has convex classes. □

Remark 20 By duality, an analogous result holds for the identity $x \sqcap y = y \sqcap x$.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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