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METHODOLOGIES AND APPLICATION

Soft computing-based new interval-valued hesitant fuzzy multi-criteria group assessment method with last aggregation to industrial decision problems

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Abstract Recent studies have identified that multi-criteria group analysis methods should take the concepts of risk and uncertainty into account. In some real-life situations, determining the exact values for the potential alternatives' performances and criteria weights is so difficult. To overcome with these situations, their values should be regarded as fuzzy and fuzzy intervals. In this respect, interval-valued hesitant fuzzy sets (IVHFSs) as a suitable modern fuzzy sets theory can be considered because this theory allows decision makers (DMs) to assign some interval-values membership degrees for an alternative in terms of selected criteria under a set to margin of errors. Hence, this paper proposes a novel soft computing approach, namely IVHF-MCGA, based on new interval-valued hesitant fuzzy complex proportional assessment (IVHF-COPRAS) method that can be applied in solving the multi-criteria group decision-making (MCGDM) problems under uncertainty. In this approach, preference values of potential alternatives versus the selected criteria and weights of each criterion are expressed by linguistic variables and then are transformed to interval-valued hesitant fuzzy elements (IVHFEs). In addition, an interval-valued hesitant fuzzy entropy (IVHF-entropy) method is extended to determine the criteria weights by considering the DMs' opinions about the relative importance. Also, a new interval-valued hesitant fuzzy compromise solution (IVHF-CS) method is introduced to estimate the weight of each DM in the group decision-making process along with the last aggregation for the DMs' judgments to avoid the data loss. Then, three practical applications about the robot selection, industrial site selection and rapid prototyping process selection problems are considered to explain steps of the proposed IVHF-MCGA approach and to indicate its validity and applicability. Finally, a comparative analysis between the proposed approach and fuzzy group TOPSIS method is presented based on four comparison parameters, including adequacy to changes of alternatives and criteria, agility in decision process, influence of DMs' weights and impact of first and last aggregations, to show its suitability.

Keywords Soft computing · Group compromise analysis · Complex proportional assessment · Entropy method · Interval-valued hesitant fuzzy sets (IVHFSs) · Industrial decision problems · Rapid prototyping process selection

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1 Introduction

In the last decade, industrial selection problems received extensive attention by numerous researchers in the manufacturing industry. The industrial selection problem is an important issue for manufacturing companies to enhance their performance and productivity (Mousavi et al. 2013b). Hence, an effective evaluation and decision method for selecting the best industrial alternative among several potential candidates is very consequential when a company makes a decision to choose a candidate for performing a special task among various types and models according to different spec-

ifications and factors, such as related cost, risk, complexity, demand, service level and flexibility.

In the related literature, many studies have presented methods and tools to solve the selection problems under uncertainty. In this regard, Ölçer and Odabaşi (2005) designed a fuzzy multi-attribute group decision-making model for solving the propulsion system selection problem. Li et al. (2007) proposed a grey method for solving the supplier selection problem under uncertainty. In addition, for solving the robot selection problem; for instance, Agrawal et al. (1991) considered the technique for order preference by similarity to an ideal solution (TOPSIS method); Goh (1997) considered the analytic hierarchy process (AHP) method; Malek et al. (2000) focused on a decision support system (DSS) with respect to analytical algorithms; Chu and Lin (2003) proposed a fuzzy TOPSIS method, in which the rating of an alternative among the selected criteria and the criteria weights are expressed basing on the linguistic terms; Bhangale et al. (2004) used the graphical and TOPSIS methods; Karsak (2005) presented a multi-criteria decision-making (MCDM) method based on Choquet integral; Chatterjee et al. (2010) used two Vlse Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR) and elimination and choice translating reality (ELECTRE) methods; Rao et al. (2011) proposed a decision-making method using fuzzy logic to convert the qualitative attributes into the quantitative attributes regarding objective and subjective preferences; and İç et al. (2013) introduced a DSS based on fuzzy AHP method.

For selection problems in reality for overcoming the uncertainty in classical fuzzy sets theory, it is difficult for decision makers (DMs) or experts to exactly specify their opinions as a number under an interval [0,1] (e.g., Mousavi et al. 2013b, 2015; Vahdani et al. 2014). Thus, it is more appropriate to demonstrate their opinions by an interval (Mousavi et al. 2013b, a; Zhao et al. 2014). Gorzałczany (1987) and Turksen (1986) have been firstly introduced the interval-valued fuzzy sets. This theory is widely used in real-world selection problems such as approximate reasoning (Bustince 1994; Gorzałczany 1987) and preference modeling (Türkşen and Bilgiç 1996). Yao and Yu (2004) illustrated the unknown potential alternatives effectiveness scores using the statistical data under an interval-valued fuzzy environment. Vahdani et al. (2010) and Vahdani and Hadipour (2011) developed VIKOR and ELECTRE methods based on interval-valued fuzzy sets for solving the MCDM problems, respectively.

In addition, Devi (2011) focused on the VIKOR method in intuitionistic fuzzy setting for solving the MCDM problems, in which rating of the candidate alternatives and the criteria weights were judged by triangular intuitionistic fuzzy sets. Vahdani et al. (2013) proposed a modified TOPSIS method based on the interval-valued fuzzy situations and the weight of criteria and rating of alternatives were expressed by linguistic terms which converted to triangular interval-valued fuzzy numbers. Liu et al. (2014) hybridized an MCDM method based on the TOPSIS with the interval 2-tuple linguistic for selection and evaluation of the robot selection problem. Wang (2015) extended an MCDM model based on the simple additive weighting method and relative preference index in which DMs assigned their judgments by triangular fuzzy numbers.

Modern fuzzy sets theory has been introduced that could help to solve the multi-criteria group decision-making (MCGDM) problems. In this regard, one of the powerful and effective fuzzy sets theories that could cope with uncertainty is interval-valued hesitant fuzzy sets (IVHFSs). This concept has been first introduced by Chen et al. (2013) by generalizing the concept of hesitant fuzzy set (HFS) and interval value form. The IVHFSs have expressed that experts or DMs could assign some membership degrees for an element under a set by interval values. Recently, some researchers have focused on this concept and solved MCGDM problems with the HFS and IVHFS. Chen et al. (2013) developed an approach to group decision making by considering the interval-valued hesitant preference relations. In their study, the opinions of each DM are considered unequal. Farhadinia (2013) focused on relationship between entropy, distance measure, and similarity measure for the HFS and IVHFS. Then, two clustering algorithms are extended under the hesitant fuzzy environment. Xu and Zhang (2013) based on the maximizing deviation method established an optimization model to determine the criteria weights. Then, they extend the TOPSIS method based on the hesitant fuzzy and interval-valued hesitant fuzzy situations. Zhang and Xu (2014) presented an interval-based programming model for MCGDM problems under hesitant fuzzy environment with incomplete preference over candidate alternatives. Li and Peng (2014) proposed some Hamacher operations under IVHFS and then extended a practical approach for selection the shale gas areas. Consequently, the gap of recent literature is shown in Table 1. This table shows that determining the DMs' weights and aggregating the DMs' judgments in the last steps are not considered in the literature as two important features for decreasing the errors and loss of data.

The review thus suggests that there is a need for a new group decision-making method for solving the selection problems and for decreasing the errors by overcoming the risk and uncertainty issues and by fulfilling the DMs' requirements. This study proposes a soft computing approach based on two new interval-valued hesitant fuzzy complex proportional assessments (IVHF-COPRAS) and interval-valued hesitant fuzzy compromise solution (IVHF-CS) methods with last aggregation, and aims to fill the gap in the selection problems and decrease the loss for the industrial selection problems. In this presented interval-valued hesitant fuzzy

Table 1 Fuzzy decision-making methods and their characteristics

Authors	Characteristics of fuzzy decision methods						
	Assigning some membership degrees	Considering the IVF approach	Group decision making	Computing decision makers' weights	Computing criteria weights	Utilizing the linguistic terms	Last aggregation
Chu and Lin (2003)			✓		✓	✓	
Byun and Lee (2005)					✓	✓	
Devi (2011)			✓		✓	✓	
Vahdani et al. (2013)		✓	✓		✓	✓	
Xu and Zhang (2013)	✓	✓			✓		
Liu et al. (2014)		✓	✓			✓	
Zhang and Xu (2014)	✓	✓	✓		✓	✓	
Wang (2015)			✓		✓	✓	
Proposed approach	✓	✓	✓	✓	✓	✓	✓

multi-criteria group assessment (IVHF-MCGA) approach under uncertain environment, some operators are needed to be extended, such as summation, multiplication, subtraction, and division that are introduced for making decisions under uncertainty. Moreover, in the group decision analysis, determining the weight of each criterion and the weight of each DM or expert is very significant issues that have been considered in the recent literature (Ayağ 2010; Yue 2011).

In this paper, a soft computing approach, IVHF-MCGA, is presented, in which a new IVHF-CS method is developed for estimating the relative importance of each DM under uncertainty. In addition, the classical entropy method is extended under an interval-valued hesitant fuzzy environment for specifying the weight of each criterion. In sum, the main contributions of this study are:

- Proposing a new version of the classical entropy method for the weight of each criterion under interval-valued hesitant fuzzy environment;
- Presenting a new interval-valued hesitant fuzzy weighting method based on the compromise solution for each DM or expert;
- Extending a new version of the classical COPRAS method under interval-valued hesitant fuzzy environment for the evaluation process;
- Developing a novel soft computing approach based on aggregating the DMs' judgments at end of the group decision-making process for the prevention of the data loss; and
- Introducing some IVHF operators that are used in the proposed soft computing approach.

The paper is arranged as follows: in Sect. 2, we review some basic concepts and operations for IVHFSs. Also, some operators that are considered in the proposed soft computing approach are developed. In Sect. 3, we develop two new COPRAS and compromise solution methods under the interval-valued hesitant fuzzy environment to solve the MCGDM problems. In Sect. 4, three illustrative examples are provided to show the application of the proposed IVHF-MCGA approach in selecting the most suitable candidates. In addition, the comparative analysis for proposed IVHF-MCGA approach and the fuzzy group TOPSIS method from the literature is presented in Sect. 5. Finally, some conclusions and suggestions are demonstrated in Sect. 6.

2 Preliminaries

In this section, some basic assumptions and relations for the IVHFSs are defined. In the following, the subtraction and division operations are extended for the IVHFS.

Definition 1 (Torra 2010; Torra and Narukawa 2009) Let X be a universe set, then HFS on X in terms of a function E as which applied to X returns to subset of [0, 1].

$$E = \{< x, h_E(x) > | x \in X\} \tag{1}$$

where $h_E(x)$ is defined as set of membership degrees for an element in subset of [0,1].

Definition 2 (Atanassov 1986, 1989, 2000) Let X be a reference set, then E on X is intuitionistic fuzzy set (IFS). In this respect, the membership degree and non-membership degree have been denoted by $\mu_E(x_i)$ and $\nu_E(x_i)$, respectively, such that $0 \leq \mu_E(x_i) + \nu_E(x_i) \leq 1$ for $x_i \in X$.

$$E = \langle x_i, \mu_E(x_i), \nu_E(x_i) \rangle \quad \text{for} \quad x_i \in X \tag{2}$$

Definition 3 (Chen et al. 2013) Let X be a universe set, then the IVHFS on X is demonstrated as follows:

$$\tilde{E} = \left\{\left\langle x_i, \tilde{h}_{\tilde{E}}(x_i)\right\rangle | x_i \in X,\ i = 1, 2, \ldots, n\right\} \tag{3}$$

where $\widetilde{h}_{\widetilde{E}}(x_i)$ is represented by interval membership degrees for an element $x_i \in X$ under set E. Interval-valued hesitant fuzzy element (IVHFE) is shown by $\widetilde{h}_{\widetilde{E}}(x_i)$that satisfies the relation as below:

$$\widetilde{h}_{\widetilde{E}}(x_i) = \left\{\widetilde{\gamma} | \widetilde{\gamma} \in \widetilde{h}_{\widetilde{E}}(x_i)\right\} \tag{4}$$

where $\widetilde{\gamma} = \left[\widetilde{\gamma}^L, \widetilde{\gamma}^U\right]$ is an interval number where $\widetilde{\gamma}^L$ and $\widetilde{\gamma}^U$ express the lower and upper bound of $\widetilde{\gamma}$, respectively.

Definition 4 (Chen et al. 2013). Let $\widetilde{h}, \widetilde{h}_1$and $\widetilde{h}_2$ be three IVHFEs, then the following relations are defined:

$$\widetilde{h}^c = \left\{\left[1 - \widetilde{\gamma}^U, 1 - \widetilde{\gamma}^L\right] | \widetilde{\gamma} \in \widetilde{h}\right\}; \tag{5}$$

$$\widetilde{h}_1 \cup \widetilde{h}_2 = \left\{\left[\max(\widetilde{\gamma}_1^L, \widetilde{\gamma}_2^L), \max(\widetilde{\gamma}_1^U, \widetilde{\gamma}_2^U)\right] | \widetilde{\gamma}_1 \in \widetilde{h}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2\right\}; \tag{6}$$

$$\widetilde{h}_1 \cap \widetilde{h}_2 = \left\{\left[\min(\widetilde{\gamma}_1^L, \widetilde{\gamma}_2^L), \min(\widetilde{\gamma}_1^U, \widetilde{\gamma}_2^U)\right] | \widetilde{\gamma}_1 \in \widetilde{\gamma}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2\right\}; \tag{7}$$

$$\widetilde{h}^\lambda = \left\{\left[(\widetilde{\gamma}^L)^\lambda, (\widetilde{\gamma}^U)^\lambda\right] | \widetilde{\gamma} \in \widetilde{h}\right\}, \quad \lambda > 0; \tag{8}$$

$$\lambda\widetilde{h} = \left\{\left[1 - (1 - \widetilde{\gamma}^L)^\lambda, 1 - (1 - \widetilde{\gamma}^U)^\lambda\right] | \widetilde{\gamma} \in \widetilde{h}\right\}, \quad \lambda > 0; \tag{9}$$

$$\widetilde{h}_1 \oplus \widetilde{h}_2 = \left\{\left[\widetilde{\gamma}_1^L + \widetilde{\gamma}_2^L - \widetilde{\gamma}_1^L\widetilde{\gamma}_2^L, \widetilde{\gamma}_1^U + \widetilde{\gamma}_2^U - \widetilde{\gamma}_1^U\widetilde{\gamma}_2^U\right] | \widetilde{\gamma}_1 \in \widetilde{h}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2\right\} \tag{10}$$

$$\widetilde{h}_1 \otimes \widetilde{h}_2 = \left\{\left[\widetilde{\gamma}_1^L\widetilde{\gamma}_2^L, \widetilde{\gamma}_1^U\widetilde{\gamma}_2^U\right] | \widetilde{\gamma}_1 \in \widetilde{h}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2\right\} \tag{11}$$

Definition 5 Let $E = \left\{\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n\right\}$ be a collection of IVHFEs, then we propose the following extended operations based on Definition 4:

$$\overset{n}{\underset{i=1}{\oplus}} \widetilde{h}_i = \cup_{\widetilde{\gamma}_1 \in \widetilde{h}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2, \ldots, \widetilde{\gamma}_n \in \widetilde{h}_n} \left\{\left[1 - \prod_{i=1}^{n}(1 - \widetilde{\gamma}_i^L), 1 - \prod_{i=1}^{n}(1 - \widetilde{\gamma}_i^U)\right]\right\} \tag{12}$$

$$\widetilde{h}_1 \otimes \widetilde{h}_2 \otimes \cdots \otimes \widetilde{h}_n = \cup_{\widetilde{\gamma}_1 \in \widetilde{h}_1, \widetilde{\gamma}_2 \in \widetilde{h}_2, \ldots, \widetilde{\gamma}_n \in \widetilde{h}_n} \left\{\left[\prod_{i=1}^{n}\widetilde{\gamma}_i^L, \prod_{i=1}^{n}\widetilde{\gamma}_i^U\right]\right\} \tag{13}$$

Theorem 1 *Let* $\widetilde{h}_1 = \left[\widetilde{\gamma}_1^L, \widetilde{\gamma}_1^U\right], \widetilde{h}_2 = \left[\widetilde{\gamma}_2^L, \widetilde{\gamma}_2^U\right], \widetilde{h}_3 = \left[\widetilde{\gamma}_3^L, \widetilde{\gamma}_3^U\right]$ *be three IVHFEs; then, we have*

$$\tilde{h}_1^c \oplus \tilde{h}_2^c \oplus \tilde{h}_3^c = (\tilde{h}_1^{\otimes}\tilde{h}_2^{\otimes}\tilde{h}_3)^c \tag{14}$$

$$\tilde{h}_1^c \otimes \tilde{h}_2^c \otimes \tilde{h}_3^c = (\tilde{h}_1 \oplus \tilde{h}_2 \oplus \tilde{h}_3)^c \tag{15}$$

Proof For three IVHFEs $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$, we have

$$\tilde{h}_1^{\oplus}\tilde{h}_2^{\oplus}\tilde{h}_3^{=} \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \tilde{\gamma}_3 \in \tilde{h}_3} \left\{\left[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L + \tilde{\gamma}_3^L - \tilde{\gamma}_1^L\tilde{\gamma}_2^L - \tilde{\gamma}_1^L\tilde{\gamma}_3^L - \tilde{\gamma}_2^L\tilde{\gamma}_3^L + \tilde{\gamma}_1^L \times \tilde{\gamma}_2^L\tilde{\gamma}_3^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U + \tilde{\gamma}_3^U - \tilde{\gamma}_1^U\tilde{\gamma}_2^U - \tilde{\gamma}_1^U\tilde{\gamma}_3^U - \tilde{\gamma}_2^U\tilde{\gamma}_3^U + \tilde{\gamma}_1^U\tilde{\gamma}_2^U\tilde{\gamma}_3^U\right]\right\} \tag{16}$$

$$\tilde{h}_1^c \oplus \tilde{h}_2^c \oplus \tilde{h}_3^c = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \tilde{\gamma}_3 \in \tilde{h}_3} \left\{\left[1 - \tilde{\gamma}_1^L.\tilde{\gamma}_2^L.\tilde{\gamma}_3^L, 1 - \tilde{\gamma}_1^U.\tilde{\gamma}_2^U.\tilde{\gamma}_3^U\right]\right\} = (\tilde{h}_1^{\otimes}\tilde{h}_2^{\otimes}\tilde{h}_3)^c$$

$$\begin{aligned}
&\tilde{h}_1^{\otimes}\tilde{h}_2^{\otimes}\tilde{h}_3^{=} \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \tilde{\gamma}_3 \in \tilde{h}_3} \left\{\left[\tilde{\gamma}_1^L\tilde{\gamma}_2^L\tilde{\gamma}_3^L, \tilde{\gamma}_1^U\tilde{\gamma}_2^U\tilde{\gamma}_3^U\right]\right\} \\
&\quad \times \tilde{h}_1^c \otimes \tilde{h}_2^c \otimes \tilde{h}_3^c = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \tilde{\gamma}_3 \in \tilde{h}_3} \\
&\quad \left\{\left[(1 - \tilde{\gamma}_1^L)(1 - \tilde{\gamma}_2^L)(1 - \tilde{\gamma}_3^L), (1 - \tilde{\gamma}_1^U)(1 - \tilde{\gamma}_2^U)(1 - \tilde{\gamma}_3^U)\right]\right\} \\
&\quad = \left[1 - \tilde{\gamma}_1^L - \tilde{\gamma}_2^L - \tilde{\gamma}_3^L + \tilde{\gamma}_1^L\tilde{\gamma}_2^L + \tilde{\gamma}_1^L\tilde{\gamma}_3^L + \tilde{\gamma}_2^L\tilde{\gamma}_3^L - \tilde{\gamma}_1^L\tilde{\gamma}_2^L\tilde{\gamma}_3^L,\right. \\
&\quad \left. 1 - \tilde{\gamma}_1^U - \tilde{\gamma}_2^U - \tilde{\gamma}_3^U + \tilde{\gamma}_1^U\tilde{\gamma}_2^U + \tilde{\gamma}_1^U\tilde{\gamma}_3^U + \tilde{\gamma}_2^U\tilde{\gamma}_3^U - \tilde{\gamma}_1^U\tilde{\gamma}_2^U\tilde{\gamma}_3^U\right] \\
&\quad = (\tilde{h}_1 \oplus \tilde{h}_2 \oplus \tilde{h}_3)^c
\end{aligned} \tag{17}$$

The generalized of relation (16) is represented, equivalent to Eq. (12), as follows:

$$\begin{aligned}
&\tilde{h}_1 \oplus \tilde{h}_2 \oplus \cdots \oplus \tilde{h}_n^{=} \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \ldots, \tilde{\gamma}_n \in \tilde{h}_n} \\
&\left\{\left(\sum_{i=1}^{n}\tilde{\gamma}_i^L - \sum_{\substack{i=1 \\ \forall j=i+1,\ldots,n}}^{n-1}\tilde{\gamma}_i^L.\tilde{\gamma}_j^L + \sum_{\substack{i=1 \\ \forall j=i+1,\ldots,n-1 \\ \forall k=j+1,\ldots,n}}^{n-2}\tilde{\gamma}_i^L.\tilde{\gamma}_j^L.\tilde{\gamma}_k^L - \cdots\right.\right. \\
&\left.\left. + \prod_{i=1}^{n}\tilde{\gamma}_i^L, \sum_{i=1}^{n}\tilde{\gamma}_i^U - \sum_{\substack{i=1 \\ \forall j=i+1,\ldots,n}}^{n-1}\tilde{\gamma}_i^U.\tilde{\gamma}_j^U + \sum_{\substack{i=1 \\ \forall j=i+1,\ldots,n-1 \\ \forall k=j+1,\ldots,n}}^{n-2}\tilde{\gamma}_i^U.\tilde{\gamma}_j^U.\tilde{\gamma}_k^U - \cdots + \prod_{i=1}^{n}\tilde{\gamma}_i^U\right)\right\} \\
&\forall i \neq j \neq k
\end{aligned} \tag{18}$$

Thus, the proof of theorem 1 is complete □

Definition 6 (Liao and Xu 2014) The subtraction and division operations for the HFS are defined by considering the operations of the IFS and regarding the correlation between IFS and HFS as follows:

$$\tilde{h}_1 - \tilde{h}_2 = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \left\{ \begin{array}{ll} \frac{\tilde{\gamma}_1 - \tilde{\gamma}_2}{1-\tilde{\gamma}_2} & \text{if } \tilde{\gamma}_1 \geq \tilde{\gamma}_2 \text{ and } \tilde{\gamma}_2 \neq 1; \\ 0 & \text{otherwise} \end{array} \right\} \tag{19}$$

$$\frac{\tilde{h}_1}{\tilde{h}_2} = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \left\{ \begin{array}{ll} \frac{\tilde{\gamma}_1}{\tilde{\gamma}_2} & \text{if } \tilde{\gamma}_1 \leq \tilde{\gamma}_2 \text{ and } \tilde{\gamma}_2 \neq 0; \\ 1 & \text{otherwise} \end{array} \right\} \tag{20}$$

Definition 7 We extend the subtraction and division operators based on definition 5 under IVHFSs as follows:

$$\tilde{h}_1 - \tilde{h}_2 = \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2} \left\{ \begin{array}{ll} \left[\frac{\tilde{\gamma}_1^l - \tilde{\gamma}_2^l}{1-\tilde{\gamma}_2^l}, \frac{\tilde{\gamma}_1^u - \tilde{\gamma}_2^u}{1-\tilde{\gamma}_2^u} \right] & \text{if } \tilde{\gamma}_1^l \geq \tilde{\gamma}_2^l, \tilde{\gamma}_1^u \geq \tilde{\gamma}_2^u \text{ and } \tilde{\gamma}_2^l, \tilde{\gamma}_2^u \neq 1; \\ 0 & \text{otherwise} \end{array} \right\} \tag{21}$$

$$\frac{\tilde{h}_1}{\tilde{h}_2} = \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2} \left\{ \begin{array}{ll} \left[\frac{\tilde{\gamma}_1^l}{\tilde{\gamma}_2^l}, \frac{\tilde{\gamma}_1^u}{\tilde{\gamma}_2^u} \right] & \text{if } \tilde{\gamma}_1^l \leq \tilde{\gamma}_2^l, \tilde{\gamma}_1^u \leq \tilde{\gamma}_2^u, \ \tilde{\gamma}_1^l . \tilde{\gamma}_2^u \leq \tilde{\gamma}_2^l . \tilde{\gamma}_1^u \text{ and } \tilde{\gamma}_2^l, \tilde{\gamma}_2^u \neq 0; \\ 1 & \text{otherwise} \end{array} \right\} \tag{22}$$

Theorem 2 *Let h_1 and h_2 be two IVHFEs, then*

$$(\tilde{h}_1 - \tilde{h}_2) \oplus \tilde{h}_2 = \tilde{h}_1 \tag{23}$$

$$\left(\frac{\tilde{h}_1}{\tilde{h}_2}\right) \otimes \tilde{h}_2 = \tilde{h}_1 \tag{24}$$

Proof For two IVHFEs $\tilde{h}_1$ and $\tilde{h}_2$, we have

$$\begin{aligned}
&(\tilde{h}_1 - \tilde{h}_2) \oplus \tilde{h}_2 = \tilde{h}_1 \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^l \geq \tilde{\gamma}_2^l, \tilde{\gamma}_2^l \neq 1, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2, \tilde{\gamma}_1^u \geq \tilde{\gamma}_2^u, \tilde{\gamma}_2^u \neq 1} \\
&\quad \left\{ \frac{\tilde{\gamma}_1^l - \tilde{\gamma}_2^l}{1-\tilde{\gamma}_2^l}, \frac{\tilde{\gamma}_1^u - \tilde{\gamma}_2^u}{1-\tilde{\gamma}_2^u} \right\} \oplus \tilde{h}_2 \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^l \geq \tilde{\gamma}_2^l, \tilde{\gamma}_2^l \neq 1, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2, \tilde{\gamma}_1^u \geq \tilde{\gamma}_2^u, \tilde{\gamma}_2^u \neq 1} \\
&\quad \left\{ \frac{\tilde{\gamma}_1^l - \tilde{\gamma}_2^l}{1-\tilde{\gamma}_2^l} \right. \\
&\quad \left. + \tilde{\gamma}_2^l - \frac{\tilde{\gamma}_1^l - \tilde{\gamma}_2^l}{1-\tilde{\gamma}_2^l} \tilde{\gamma}_2^l, \frac{\tilde{\gamma}_1^u - \tilde{\gamma}_2^u}{1-\tilde{\gamma}_2^u} + \tilde{\gamma}_2^u - \frac{\tilde{\gamma}_1^u - \tilde{\gamma}_2^u}{1-\tilde{\gamma}_2^u} \tilde{\gamma}_2^u \right\} \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^l \geq \tilde{\gamma}_2^l, \tilde{\gamma}_2^l \neq 1, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2, \tilde{\gamma}_1^u \geq \tilde{\gamma}_2^u, \tilde{\gamma}_2^u \neq 1} \\
&\quad \left\{ \frac{\tilde{\gamma}_1^l (1-\tilde{\gamma}_2^l)}{1-\tilde{\gamma}_2^l}, \frac{\tilde{\gamma}_1^u (1-\tilde{\gamma}_2^u)}{1-\tilde{\gamma}_2^u} \right\} \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1} = \tilde{h}_1
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\left(\frac{\tilde{h}_1}{\tilde{h}_2}\right) \otimes \tilde{h}_2 = \tilde{h}_1 \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^l \leq \tilde{\gamma}_2^l, \tilde{\gamma}_2^l \neq 0, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2, \tilde{\gamma}_1^u \leq \tilde{\gamma}_2^u, \tilde{\gamma}_2^u \neq 0} \\
&\quad \left\{ \frac{\tilde{\gamma}_1^l}{\tilde{\gamma}_2^l}, \frac{\tilde{\gamma}_1^u}{\tilde{\gamma}_2^u} \right\} \otimes \tilde{h}_2 \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_2^l \in \tilde{h}_2, \tilde{\gamma}_1^l \leq \tilde{\gamma}_2^l, \tilde{\gamma}_2^l \neq 0, \tilde{\gamma}_1^u \in \tilde{h}_1, \tilde{\gamma}_2^u \in \tilde{h}_2, \tilde{\gamma}_1^u \leq \tilde{\gamma}_2^u, \tilde{\gamma}_2^u \neq 0} \\
&\quad \left\{ \frac{\tilde{\gamma}_1^l}{\tilde{\gamma}_2^l} . \tilde{\gamma}_2^l, \frac{\tilde{\gamma}_1^u}{\tilde{\gamma}_2^u} . \tilde{\gamma}_2^u \right\} \\
&= \cup_{\tilde{\gamma}_1^l \in \tilde{h}_1, \tilde{\gamma}_1^u \in \tilde{h}_1} \left\{ \tilde{\gamma}_1^l, \tilde{\gamma}_1^u \right\} = \tilde{h}_1
\end{aligned} \tag{26}$$

Thus, the proof of theorem 2 is complete □

Definition 8 (Farhadinia 2013) Consider $\tilde{M}$ and$\tilde{N}$ as two IVHFSs on X. The component-wise ordering and the total ordering are defined as two types of ordering of IVHFSs, respectively, as follows:

$$\begin{aligned}
&\tilde{M} \leq \tilde{N} \quad if \quad h_{\tilde{M}}^{\sigma(j)L}(x_i) \leq h_{\tilde{N}}^{\sigma(j)L}(x_i), \\
&h_{\tilde{M}}^{\sigma(j)U}(x_i) \leq h_{\tilde{N}}^{\sigma(j)U}(x_i) \\
&\quad \forall i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, l_{x_i}
\end{aligned} \tag{27}$$

$$\tilde{M} \preceq \tilde{N} \quad if \ Score(\tilde{M}) \leq Score(\tilde{N}) \tag{28}$$

$$Score(\tilde{M}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} \left[\frac{h_{\tilde{M}}^{\sigma(j)L}(x_i) + h_{\tilde{M}}^{\sigma(j)U}(x_i)}{2} \right] \right) \tag{29}$$

where h_M and h_N are IVHFSs which represented as $h_{\tilde{M}}^{\sigma(j)}(x_i) = \left[h_{\tilde{M}}^{\sigma(j)L}(x_i), h_{\tilde{M}}^{\sigma(j)U}(x_i) \right]$, $h_{\tilde{N}}^{\sigma(j)}(x_i) = \left[h_{\tilde{N}}^{\sigma(j)L}(x_i), h_{\tilde{N}}^{\sigma(j)U}(x_i) \right]$, respectively; also, $h_{\tilde{M}}^{\sigma(j)}$and $h_{\tilde{N}}^{\sigma(j)}$are the jth largest intervals in $h_{\tilde{M}}(x_i)$ and $h_{\tilde{N}}(x_i)$, respectively.

Assumptions The number of intervals in various IVHFEs may be different. Assume that the number of intervals in $h_{\tilde{M}}(x)$ is denoted by $l(h_{\tilde{M}}(x))$. In this respect, two assumptions are made as follows:

(A1) In each $h_{\tilde{M}}(x)$, all the elements have to be arranged in increasing order, and

(A2) If $l(h_{\tilde{M}}(x)) \neq l(h_{\tilde{N}}(x))$, then $l_x = \max\{l(h_{\tilde{M}}(x)), l(h_{\tilde{N}}(x))\}$ for some $x \in X$. Two IVHFEs such as $h_{\tilde{M}}(x)$and $h_{\tilde{N}}(x)$ should have the same length when compared. If $h_{\tilde{M}}(x)$ has less elements than in$h_{\tilde{N}}(x)$, then $h_{\tilde{M}}(x)$ should be extended based on an optimistic approach, in which value it has to be added$(h_{\tilde{M}}(x))$.

Definition 9 (Chen et al. 2013) The interval-valued hesitant fuzzy Euclidean distance measure and the interval-valued hesitant fuzzy hamming distance measure are defined, respectively, as follows:

$$d(\tilde{h}_M, \tilde{h}_N) = \sqrt{\frac{1}{2l_{x_i}} \sum_{j=1}^{l_{x_i}} \left(\left| \tilde{h}_M^{\sigma(j)L}(x_i) - \tilde{h}_N^{\sigma(j)L}(x_i) \right|^2 + \left| \tilde{h}_M^{\sigma(j)U}(x_i) - \tilde{h}_N^{\sigma(j)U}(x_i) \right|^2 \right)} \tag{30}$$

$$d(\tilde{h}_M, \tilde{h}_N) = \frac{1}{2l_{x_i}} \sum_{j=1}^{l_{x_i}} \left(\left| \tilde{h}_M^{\sigma(j)L}(x_i) - \tilde{h}_N^{\sigma(j)L}(x_i) \right| + \left| \tilde{h}_M^{\sigma(j)U}(x_i) - \tilde{h}_N^{\sigma(j)U}(x_i) \right| \right) \tag{31}$$

Definition 10 (Chen et al. 2013) The interval-valued hesitant fuzzy geometric (IVHFG) relation is defined below:

$$IVHFG(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \left(\bigotimes_{j=1}^{n} (\tilde{h}_j)^{\frac{1}{n}} \right) = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \ldots, \tilde{\gamma}_n \in \tilde{h}_n} \left\{ \left[\prod_{j=1}^{n} (\tilde{\gamma}_j^L)^{\frac{1}{n}}, \prod_{j=1}^{n} (\tilde{\gamma}_j^U)^{\frac{1}{n}} \right] \right\} \tag{32}$$

Definition 11 (Chen et al. 2013) The interval-valued fuzzy hesitant weighted geometric (IVHFWG) relation is expressed below:

$$IVHFWG(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \left(\bigotimes_{j=1}^{n} (\tilde{h}_j)^{w_j} \right) = \cup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \ldots, \tilde{\gamma}_n \in \tilde{h}_n} \left\{ \left[\prod_{j=1}^{n} (\tilde{\gamma}_j^L)^{w_j}, \prod_{j=1}^{n} (\tilde{\gamma}_j^U)^{w_j} \right] \right\} \tag{33}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ are the weight vector of $\tilde{h}_j (j = 1, 2, \ldots, n)$ and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

Definition 12 (Zhang et al. 2014) In MCGDM problems, there are two kinds of attributes such as benefit and cost types. In this respect, the cost attributes' values are transformed to benefit attributes' values; i.e., the interval-valued hesitant fuzzy decision matrix $(\tilde{H} = (\tilde{h}_{ij})_{m \times n})$ is normalized $(\tilde{B} = (\tilde{b}_{ij})_{m \times n})$ based on the following relations:

$$\tilde{b}_{ij} = \begin{cases} \left\{ \left[\tilde{\gamma}_{ij}^l, \tilde{\gamma}_{ij}^u \right] \right\} & \text{for positive criteria} \\ \left\{ \left[1 - \tilde{\gamma}_{ij}^u, 1 - \tilde{\gamma}_{ij}^l \right] \right\} & \text{for negative criteria} \end{cases} \quad \forall i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \tag{34}$$

3 Proposed soft computing approach based on IVHF-MCGA method

In this section, the proposed IVHF-MCGA approach is introduced, which is composed of three methods: the COPRAS method is tailored based on the IVHFSs for sorting of the candidate alternatives; the IVHF-entropy method and IVHF-CS method are developed to determine the weights of criteria and DMs, respectively. In this respect, the structure of the proposed IVHF-MCGA approach is depicted in Fig. 1.

Step 1. Determine important criteria which satisfy the potential alternatives.

Step 2. Construct the interval-valued hesitant fuzzy decision matrix (IVHF-decision matrix) from a committee of DMs as follows:

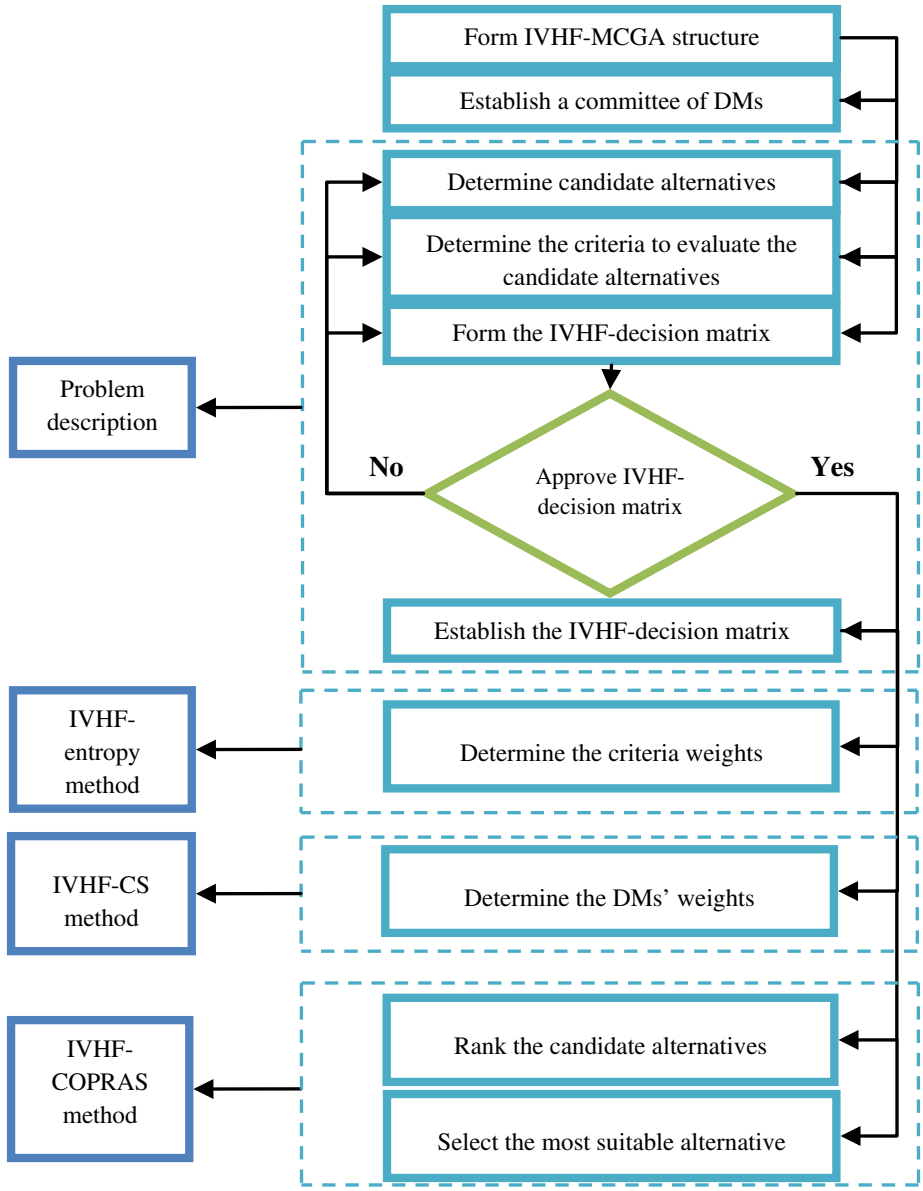

Fig. 1 Structure of the proposed IVHF-MCGA approach

$$M = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \cdots & C_n \end{array} \\ \begin{bmatrix} \{[\mu_{11}^{L1}, \mu_{11}^{U1}], [\mu_{11}^{L2}, \mu_{11}^{U2}], \ldots, [\mu_{11}^{Lk}, \mu_{11}^{Uk}]\} & \cdots & \{[\mu_{1n}^{L1}, \mu_{1n}^{U1}], [\mu_{1n}^{L2}, \mu_{1n}^{U2}], \ldots, [\mu_{1n}^{Lk}, \mu_{1n}^{Uk}]\} \\ \vdots & \ddots & \vdots \\ \{[\mu_{m1}^{L1}, \mu_{m1}^{U1}], [\mu_{m1}^{L2}, \mu_{m1}^{U2}], \ldots, [\mu_{m1}^{Lk}, \mu_{m1}^{Uk}]\} & \cdots & \{[\mu_{mn}^{L1}, \mu_{mn}^{U1}], [\mu_{mn}^{L2}, \mu_{mn}^{U2}], \ldots, [\mu_{mn}^{Lk}, \mu_{mn}^{Uk}]\} \end{bmatrix}_{m \times n} \end{array} \tag{35}$$

Step 3. Estimate criteria weights by the proposed IVHF-entropy method regarding DMs' opinions.

Step 3.1. Construct the aggregated IVHF-decision matrix (M_{Agg}) based on Definition 10 as follows:

$$\left[\mu_{ij}^{L}, \mu_{ij}^{U}\right] = \text{IVHFG}(\tilde{h}_{p1}, \tilde{h}_{p2}, \ldots, \tilde{h}_{pk}) = \left(\bigotimes_{k=1}^{K} (\tilde{h}_{pk})^{\frac{1}{k}}\right)$$
$$= \cup_{\tilde{\gamma}_{p1}\in\tilde{h}_{p1}, \tilde{\gamma}_{p2}\in\tilde{h}_{p2}, \ldots, \tilde{\gamma}_{pk}\in\tilde{h}_{pk}} \left\{\left[\prod_{k=1}^{K} (\tilde{\gamma}_{pk}^{L})^{\frac{1}{k}}, \prod_{k=1}^{K} (\tilde{\gamma}_{pk}^{U})^{\frac{1}{k}}\right]\right\} \tag{36}$$

$$M_{\text{Agg}} = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} [\mu_{11}^{L}, \mu_{11}^{U}] & [\mu_{12}^{L}, \mu_{12}^{U}] & \cdots & [\mu_{1n}^{L}, \mu_{1n}^{U}] \\ \vdots & \vdots & \ddots & \vdots \\ [\mu_{m1}^{L}, \mu_{m1}^{U}] & [\mu_{m2}^{L}, \mu_{m2}^{U}] & \cdots & [\mu_{mn}^{L}, \mu_{mn}^{U}] \end{pmatrix}_{m\times n} \end{array} \tag{37}$$

Step 3.2. Aggregate the DMs' judgments about the relative importance of criteria ($\tilde{h}_{wk}$) by the following relation:

$$\upsilon_j = \text{IVHFG}(\tilde{h}_{w1}, \tilde{h}_{w2}, \ldots, \tilde{h}_{wk}) = \left(\bigotimes_{k=1}^{K} (\tilde{h}_{wk})^{\frac{1}{k}}\right)$$
$$= \cup_{\tilde{\gamma}_{w1}\in\tilde{h}_{w1}, \tilde{\gamma}_{w2}\in\tilde{h}_{w2}, \ldots, \tilde{\gamma}_{wk}\in\tilde{h}_{wk}} \left\{\left[\prod_{k=1}^{K} (\tilde{\gamma}_{wk}^{L})^{\frac{1}{k}}, \prod_{k=1}^{K} (\tilde{\gamma}_{wk}^{U})^{\frac{1}{k}}\right]\right\} \tag{38}$$

Step 3.3. Specify the $T_{ij} = \left[T_{ij}^{l}, \mathrm{T}_{ij}^{u}\right]$ by the following relations:

$$T_{ij}^{l} = \begin{cases} \dfrac{\mu_{ij}^{l}}{1-\prod_{i=1}^{m}(1-\mu_{ij}^{l})} & \text{if } \mu_{ij}^{l} \le 1 - \prod_{i=1}^{m}(1-\mu_{ij}^{l}) \text{ and } 1 - \prod_{i=1}^{m}(1-\mu_{ij}^{l}) \neq 0 \\ 1 & \text{Otherwise} \end{cases} \tag{39}$$

$$T_{ij}^{u} = \begin{cases} \dfrac{\mu_{ij}^{u}}{1-\prod_{i=1}^{m}(1-\mu_{ij}^{u})} & \text{if } \mu_{ij}^{u} \le 1 - \prod_{i=1}^{m}(1-\mu_{ij}^{u}) \text{ and } 1 - \prod_{i=1}^{m}(1-\mu_{ij}^{u}) \neq 0 \\ 1 & \text{Otherwise} \end{cases} \tag{40}$$

Step 3.4. Specify the interval-valued hesitant fuzzy degree of deviation or interval-valued hesitant fuzzy unreliability of each criterion regarding interval-valued hesitant fuzzy entropy ($E_j = [E_j^{l}, E_j^{u}]$) as follows:

$$d_j = \left[\left(\prod_{i=1}^{m}(1-T_{ij}^{l})^{Ln(T_{ij}^{l})}\right)^{-\frac{1}{Ln(m)}}, \left(\prod_{i=1}^{m}(1-T_{ij}^{u})^{Ln(T_{ij}^{u})}\right)^{-\frac{1}{Ln(m)}}\right] \tag{41}$$

Step 3.5. Determine the final criteria weights (w_j) by according to DMs' opinions.

$$w_j = \frac{\upsilon_j^{l}.d_j^{l}\left(\sum_{j=1}^{n}\upsilon_j^{u}.d_j^{u}\right) + \upsilon_j^{u}.d_j^{u}\left(\sum_{j=1}^{n}\upsilon_j^{l}.d_j^{l}\right)}{2\left(\sum_{j=1}^{n}\upsilon_j^{l}.d_j^{l}\right)\left(\sum_{j=1}^{n}\upsilon_j^{u}.d_j^{u}\right)} \quad \forall j \tag{42}$$

Step 4. Estimate the DMs' weights by the proposed IVHF-CS method.

*Step 4.1.*Establish the weighted normalized IVHF-decision matrix for each DM based on Definition 12.

$$M_k = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} [\mu_{11}^{Lk}, \mu_{11}^{Uk}] & [\mu_{12}^{Lk}, \mu_{12}^{Uk}] & \cdots & [\mu_{1n}^{Lk}, \mu_{1n}^{Uk}] \\ \vdots & \vdots & \ddots & \vdots \\ [\mu_{m1}^{Lk}, \mu_{m1}^{Uk}] & [\mu_{m2}^{Lk}, \mu_{m2}^{Uk}] & \cdots & [\mu_{mn}^{Lk}, \mu_{mn}^{Uk}] \end{pmatrix}_{m\times n} \end{array} \tag{43}$$

Step 4.2. Estimate the interval-valued hesitant fuzzy positive ideal solution (IVHF-PIS) and interval-valued hesitant fuzzy negative ideal solution (IVHF-NIS) based on the following relations:

$$A^{*} = \left(\left[\mu_{ij}^{*L}, \mu_{ij}^{*U}\right]\right)_{m\times n}$$
$$= \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} [\mu_{11}^{L*}, \mu_{11}^{U*}] & [\mu_{12}^{L*}, \mu_{12}^{U*}] & \cdots & [\mu_{1n}^{L*}, \mu_{1n}^{U*}] \\ \vdots & \vdots & \ddots & \vdots \\ [\mu_{m1}^{L*}, \mu_{m1}^{U*}] & [\mu_{m2}^{L*}, \mu_{m2}^{U*}] & \cdots & [\mu_{mn}^{L*}, \mu_{mn}^{U*}] \end{pmatrix}_{m\times n} \end{array} \tag{44}$$

$$A^{-} = \left(\left[\mu_{ij}^{-L}, \mu_{ij}^{-U}\right]\right)_{m\times n} = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \left[\mu_{11}^{L-}, \mu_{11}^{U-}\right] & \left[\mu_{12}^{L-}, \mu_{12}^{U-}\right] & \cdots & \left[\mu_{1n}^{L-}, \mu_{1n}^{U-}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\mu_{m1}^{L-}, \mu_{m1}^{U-}\right] & \left[\mu_{m2}^{L-}, \mu_{m2}^{U-}\right] & \cdots & \left[\mu_{mn}^{L-}, \mu_{mn}^{U-}\right] \end{pmatrix}_{m\times n} \end{array} \tag{45}$$

where the average of group decision matrix is computed as follows:

$$\mu_{ij}^{*L} = \frac{1}{K}\sum_{k=1}^{K}\mu_{ij}^{Lk} \tag{46}$$

$$\mu_{ij}^{*U} = \frac{1}{K}\sum_{k=1}^{K}\mu_{ij}^{Uk} \tag{47}$$

$$\mu_{ij}^{-L} = \min_{k}\left\{\mu_{ij}^{Lk}\right\} \tag{48}$$

$$\mu_{ij}^{-U} = \max_{k}\left\{\mu_{ij}^{Uk}\right\} \tag{49}$$

Step 4.3. Compute the separation measure for the matrix of each DM from the IVHF-PIS and IVHF-NIS by applying the interval-valued hesitant fuzzy Euclidean distance measure. In this regard, the distance measure is developed to the n-dimensional interval-valued hesitant fuzzy Euclidean distance measure as below:

$$\xi_k^{*} = \sqrt{\frac{1}{2l_{x_i}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\lambda=1}^{l_{x_i}}\left(\left|\mu_{ij}^{Lk\sigma(\lambda)}(x_i) - A_{ij}^{*L\sigma(\lambda)}(x_i)\right|^2 + \left|\mu_{ij}^{Uk\sigma(\lambda)}(x_i) - A_{ij}^{*U\sigma(\lambda)}(x_i)\right|^2\right)} \quad \forall k \tag{50}$$

$$\xi_k^{-} = \sqrt{\frac{1}{2l_{x_i}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\lambda=1}^{l_{x_i}}\left(\left|\mu_{ij}^{Lk\sigma(\lambda)}(x_i) - A_{ij}^{-L\sigma(\lambda)}(x_i)\right|^2 + \left|\mu_{ij}^{Uk\sigma(\lambda)}(x_i) - A_{ij}^{-U\sigma(\lambda)}(x_i)\right|^2\right)} \quad \forall k \tag{51}$$

Step 4.4. Determine the weight of each DM (ϖ_k) by considering the following relation:

$$\varpi_k = \frac{\xi_k^{-}}{(\xi_k^{-} + \xi_k^{*})\left(\sum_{k=1}^{K}\frac{\xi_k^{-}}{\xi_k^{-}+\xi_k^{*}}\right)} \forall k \tag{52}$$

Step 5. Calculate sums $p_i^k = \left[p_i^{lk}, p_i^{uk}\right]$ of positive criteria values with respect to the weighted normalized IVHF-decision matrix for each DM in Step 4.1:

$$p_i^k = \left[1 - \prod_{j=1}^{r}(1-\mu_{ij}^{lk}), 1 - \prod_{j=1}^{r}(1-\mu_{ij}^{uk})\right] \quad \forall k, i \tag{53}$$

where r is the number of positive criteria. It is supposed that in the IVHF-decision matrix, first columns are placed by positive criteria and negative criteria are placed after.

Step 6. Computing sums $R_i^k = [R_i^{lk}, R_i^{uk}]$ of criteria values which are negative criteria for each potential alternative with respect to weighted normalized hesitant fuzzy decision matrix for each DM in Step 4.1.

$$R_i^k = \left[1 - \prod_{j=r+1}^{n}(1-\mu_{ij}^{lk}), 1 - \prod_{j=r+1}^{n}(1-\mu_{ij}^{uk})\right] \quad \forall k, i \tag{54}$$

Step 7. Determine the smallest value of $R_i^k = [R_i^{lk}, R_i^{uk}]$ as follows:

$$R_{\min}^k = \left[\min_i(R_i^{lk}), \min_i(R_i^{uk})\right] \quad \forall k \tag{55}$$

Step 8. Calculate the interval-valued hesitant fuzzy relative importance of each alternative ($Q_i^k = \left[Q_i^{lk}, Q_i^{uk}\right]$) regarding each DM as follows:

$$Q_i^k = \left[P_i^{lk} + (1-P_i^{lk})\left(\frac{1-\left(\prod_i^m(1-R_i^{lk})\right)^{R_{\min}^{lk}}}{1-\left(\prod_i^m\left(1-\frac{R_{\min}^{lk}}{R_i^{lk}}\right)\right)^{R_i^{lk}}}\right), P_i^{uk} + (1-P_i^{uk})\left(\frac{1-\left(\prod_i^m(1-R_i^{uk})\right)^{R_{\min}^{uk}}}{1-\left(\prod_i^m\left(1-\frac{R_{\min}^{uk}}{R_i^{uk}}\right)\right)^{R_i^{uk}}}\right)\right] \forall k, i \tag{56}$$

Step 9. Determine each Q_i value by utilizing the IVHFG by considering the DMs' weights as follows:

$$Q_i = \text{IVHFWG}(Q_i^{l1}, Q_i^{l2}, \ldots, Q_i^{lk}) = \left(\bigotimes_{k=1}^{K} (\tilde{Q}_i^k)^{\varpi_k}\right)$$
$$= \cup_{\tilde{Q}_1 \in \tilde{h}_{q1}, \tilde{Q}_2 \in \tilde{h}_{q2}, \ldots, \tilde{Q}_k \in \tilde{h}_{qk}}$$
$$\times \left\{ \left[\prod_{k=1}^{K} (Q_i^{lk})^{\varpi_k^l}, \prod_{k=1}^{K} (Q_i^{uk})^{\varpi_k^u} \right] \right\} \forall i \quad (57)$$

Step 10. Compute the interval-valued hesitant fuzzy utility degree of each possible alternative as below:

$$N_i = \left[\frac{Q_i^l}{\max(Q_i^l)}, \frac{Q_i^u}{\max(Q_i^u)} \right] 100\,\% \quad (58)$$

Step 11. Rank the alternatives by selecting the maximum value of interval-valued hesitant fuzzy utility degree by ordering relation based on Definition 8.

4 Application of the proposed IVHF-MCGA method in solving the MCGDM problems

4.1 Illustrative example 1: robot selection

The practical application from Vahdani et al. (2014) is used to show the suitability and feasibility of the proposed IVHF-MCGA method. Suppose that a manufacturing company needs a robot for implementing the material-handling assessment. In this respect, for further assessment, three robots (R_i, $i = 1, 2, 3$) are selected for the MCGDM problem. Also, six criteria are considered for rating the potential alternatives as follows:

- Man–machine interface (C_1);
- Programming flexibility (C_2);
- Vendor's service contract (C_3);
- Load capacity (C_4);
- Positioning accuracy (C_5); and
- Purchase cost (C_6).

In the robot selection problem, consider four DM ($\text{DM}_k, k = 1, 2, 3, 4$) to evaluate the alternatives with respect to criteria and to choose the best robot candidate. Tables 2 and 3 provide values of hesitant linguistic variables for relative importance of criteria and ratings of the alternatives. Table 4 indicates the IVHF-decision matrix by linguistic variables, and also the relative importance of each criterion as assessed by DMs' opinions is indicated in Table 5.

The interval-valued hesitant fuzzy entropy method is proposed to determine the weight of each criterion. As indicated in Table 6, firstly, we establish an aggregated IVHF-decision matrix and then construct the T_{ij} matrix. In this regard, the interval-valued hesitant fuzzy degree of deviation or the interval-valued hesitant fuzzy unreliability is computed and

Table 2 Linguistic variables for rating the importance of criteria

Linguistic variables	Interval-valued hesitant fuzzy elements
Very high (VH)	[0.90, 0.90]
High (H)	[0.75, 0.80]
Medium high (MH)	[0.60,0.70]
Medium (M)	[0.50, 0.55]
Medium low (ML)	[0.40,0.50]
Low (L)	[0.35, 0.40]
Very low (VL)	[0.10, 0.10]

Table 3 Linguistic variables for rating the potential alternatives

Linguistic variables	Interval-valued hesitant fuzzy elements
Extremely good (EG)	[1.00, 1.00]
Very very good (VVG)	[0.90, 0.90]
Very good (VG)	[0.80, 0.90]
Good (G)	[0.70, 0.80]
Moderately good (MG)	[0.60, 0.70]
Fair (F)	[0.50, 0.60]
Moderately poor (MP)	[0.40, 0.50]
Poor (P)	[0.25, 0.40]
Very poor (VP)	[0.10, 0.25]
Very very poor (VVP)	[0.10, 0.10]

Table 4 Performance rating of the alternatives by linguistic variables for the first illustrative example

Criteria	Alternatives	Decision makers			
		DM_1	DM_2	DM_3	DM_4
C_1	R_1	F	F	G	VG
	R_2	F	G	F	F
	R_3	G	F	VG	G
C_2	R_1	G	P	G	F
	R_2	VG	G	VG	F
	R_3	G	F	VG	G
C_3	R_1	F	F	G	F
	R_2	G	F	VG	G
	R_3	G	G	G	VG
C_4	R_1	G	MG	F	F
	R_2	MG	G	MG	MG
	R_3	F	F	G	G
C_5	R_1	F	P	MP	F
	R_2	MG	F	F	MG
	R_3	G	G	MG	G
C_6	R_1	G	G	G	MG
	R_2	MG	MG	F	F
	R_3	F	F	P	MP

Table 5 Decision makers' judgments about criteria weights for the first illustrative example

Criteria	Decision makers			
	DM_1	DM_2	DM_3	DM_4
C_1	H	VH	VH	H
C_2	VH	H	VH	M
C_3	M	L	M	L
C_4	VH	VH	H	VH
C_5	VH	H	H	H
C_6	M	M	M	L

the DMs' opinions for criteria weights are considered for decreasing the error of determining the criteria weights. Thus, the DMs' judgments are aggregated and the final weights of each criterion are computed by Eq. (42). Regarding the mentioned results given in Table 7, the weight of each DM is also computed by the proposed IVHF-CS method as mentioned in Steps 4.1–4.4. As represented in Table 8, the separation measure is calculated using the n-dimensional interval-valued fuzzy Euclidean distance. In addition, the relative closeness and the final weight of each DM are computed.

In the following, the process of the proposed IVHF-MCGA method, the sum of positive criteria values/negative criteria values and the minimum value of negative criteria values are determined by Eqs. (53)–(55). The above-mentioned results are indicated in Tables 9 and 10. In addition, as shown in Table 11, the interval-valued hesitant fuzzy relative importance of each potential alternative and each DM (Q_i^k) is computed by Eq. (56). Hence, the Q_i^k values are aggregated for determining the interval-valued hesitant fuzzy utility degree. Finally, the potential alternatives are ranked by selecting the maximum value of interval-valued hesitant fuzzy utility degrees. In this regard, two types of ordering as component-wise ordering and total ordering are considered to sort the interval-valued hesitant fuzzy utility degree. Thus, the proposed approach shows that the R_3 is the best alternative for the MCGDM problem. The results are compared to the Vahdani et al. (2014) method that indicates the verification and validation of the proposed IVHF-MCGA method. Results are provided in Table 12.

In addition, the aforementioned case study is solved by the decision method of Liu et al. (2014), and the same ranking results are achieved and are shown in Table 12. Therefore, the proposed IVHF-MCGA approach is investigated by com-

Table 6 Aggregated IVHF-decision matrix and T_{ij} matrix for the first illustrative example

	Aggregated IVHF-decision matrix		
	R_1	R_2	R_3
C_1	[0.611690, 0.713524]	[0.543878, 0.644741]	[0.665371, 0.766731]
C_2	[0.497481, 0.626033]	[0.687958, 0.789644]	[0.665371, 0.766731]
C_3	[0.543878, 0.644741]	[0.665371, 0.766731]	[0.723762, 0.823906]
C_4	[0.569241, 0.670073]	[0.623573, 0.723762]	[0.591607, 0.692820]
C_5	[0.397635, 0.518004]	[0.547722, 0.648071]	[0.673536, 0.773734]
C_6	[0.673536, 0.773734]	[0.547722, 0.648071]	[0.397635, 0.518004]
	Constructed T_{ij} matrix		
	R_1	R_2	R_3
C_1	[0.650228, 0.730875]	[0.578141, 0.660420]	[0.707291, 0.785376]
C_2	[0.525032, 0.637736]	[0.726055, 0.804405]	[0.702218, 0.781064]
C_3	[0.567819, 0.654289]	[0.694659, 0.778086]	[0.755621, 0.836108]
C_4	[0.609611, 0.689373]	[0.667795, 0.744608]	[0.633562, 0.712774]
C_5	[0.436453, 0.538678]	[0.601192, 0.673941]	[0.739289, 0.804616]

Table 7 Interval-valued hesitant fuzzy unreliability (d_j), aggregated DMs' opinions (υ_j) and final weights of criteria (w_j) for the first illustrative example

	d_j		υ_j		w_j
d_1	[0.292491, 0.326003]	υ_1	[0.821583, 0.848528]	w_1	0.198899
d_2	[0.300071, 0.339370]	υ_2	[0.742384, 0.772652]	w_2	0.186451
d_3	[0.305751, 0.350415]	υ_3	[0.418331, 0.469041]	w_3	0.111991
d_4	[0.287705, 0.317675]	υ_4	[0.859898, 0.873885]	w_4	0.202201
d_5	[0.292817, 0.313084]	υ_5	[0.784976, 0.823906]	w_5	0.187873
d_6	[0.292817, 0.313084]	υ_6	[0.457340, 0.507912]	w_6	0.112582

Table 8 Computational results for determining the DMs' weights for the first illustrative example

	ξ_k^*	ξ_k^-	ϖ_k
$k=1$	0.059581	0.118151	0.266057
$k=2$	0.091493	0.116004	0.223751
$k=3$	0.066614	0.119305	0.256826
$k=4$	0.063577	0.109687	0.253367

paring the ranking results with two recent fuzzy decision methods in the literature.

4.2 Illustrative example 2: industrial site selection

Another illustrative example from Wang (2014) is provided for industrial site selection problem to further investigate the proposed IVHF-MCGA approach. In this case, four DMs are employed (k=1,2,…,4) and also three candidate alternatives (A_1, A_2, A_3) are considered to select the most suitable site for building a new factory under the following six criteria:

- Climate condition (C_1);
- Regional demand (C_2);
- Expansion possibility (C_3);
- Transportation availability (C_4);
- Labor force (C_5); and
- Investment cost (C_6).

The DMs have expressed their preferences values about the candidate sites versus the selected criteria by linguistic variables and indicated in Table 13. In addition, the DMs'

Table 9 P_i values regarding each DM for the first illustrative example

	k_1	k_2	k_3	k_4
P_1	[0.425061, 0.480156]	[0.321282, 0.396428]	[0.426575, 0.481432]	[0.413758, 0.469734]
P_2	[0.449459, 0.502547]	[0.450458, 0.503638]	[0.444634, 0.498038]	[0.413266, 0.469111]
P_3	[0.461521, 0.513690]	[0.412473, 0.468361]	[0.497767, 0.547176]	[0.491995, 0.541779]

Table 10 R_i values regarding each DM for the first illustrative example

	k_1	k_2	k_3	k_4
R_1	[0.022516, 0.033775]	[0.022516, 0.033775]	[0.022516, 0.033775]	[0.033775, 0.045033]
R_2	[0.033775, 0.045033]	[0.033774, 0.045033]	[0.045033, 0.056291]	[0.045033, 0.056291]
R_3	[0.045033, 0.056291]	[0.045032, 0.056291]	[0.067549, 0.084437]	[0.056291, 0.067549]
R_{min}	[0.022516, 0.033775]	[0.022516, 0.033775]	[0.022516, 0.033775]	[0.033775, 0.045033]

Table 11 Final Q_i values regarding each DM for the first illustrative example

	Q_1	Q_2	Q_3
k_1	[0.426396, 0.482580]	[0.450730, 0.504867]	[0.462770, 0.515957]
k_2	[0.322858, 0.399242]	[0.451733, 0.505952]	[0.4138373, 0.47084]
k_3	[0.428364, 0.484584]	[0.446367, 0.501089]	[0.499334, 0.549928]
k_4	[0.416492, 0.473871]	[0.416002, 0.473254]	[0.494364, 0.545354]
Final Q_i	[0.398759, 0.460898]	[0.440785, 0.495943]	[0.468009, 0.521112]
	N_1	N_2	N_3
N_i	[85.20340, 88.44524]	[94.18298, 95.17026]	[100, 100]

Table 12 Final rankings based on the Q_i values by considering two types of ordering and the comparative analysis for the first illustrative example

	Component-wise ordering	Total ordering	Ranked by the proposed soft computing approach	Ranked by Vahdani et al. (2014) method	Ranked by Liu et al. (2014) method
N_1	$N_2^l \geq N_1^l,\ N_2^u \geq N_1^u$	86.824326	3	3	3
N_2	$N_3^l \geq N_2^l,\ N_3^u \geq N_2^u$	94.676623	2	2	2
N_3	$N_3^l \geq N_2^l \geq N_1^l,\ N_3^u \geq N_2^u \geq N_1^u$	100	1	1	1

Table 13 Linguistic evaluations by DMs for the second illustrative example

Criteria	Alternatives	Decision makers			
		DM_1	DM_2	DM_3	DM_4
C_1	A_1	MG	MG	G	VG
	A_2	VG	G	G	G
	A_3	F	MG	F	MG
C_2	A_1	G	VG	VG	MG
	A_2	F	MG	F	F
	A_3	VG	VG	VG	G
C_3	A_1	F	MG	MG	MG
	A_2	VG	G	VG	VG
	A_3	MG	MG	G	MG
C_4	A_1	VG	VG	G	G
	A_2	G	G	G	MG
	A_3	G	G	VG	F
C_5	A_1	MG	G	G	VG
	A_2	MG	MG	VG	G
	A_3	VG	G	G	G
C_6	A_1	VVG	VG	VVG	EG
	A_2	F	B	B	MG
	A_3	VVG	VVG	VG	VVG

Table 14 Decision makers' judgments about criteria weights for the second illustrative example

Criteria	Decision makers			
	DM_1	DM_2	DM_3	DM_4
C_1	H	VH	M	H
C_2	M	H	M	H
C_3	H	M	H	VH
C_4	H	L	M	M
C_5	M	H	H	M
C_6	M	M	H	H

judgments about the weight of each criterion are shown based on the linguistic terms in Table 14.

The weight of each criterion is obtained based on the IVHF-entropy method. In this case, the aggregated IVHF-decision matrix and then the T_{ij} matrix are established. Also, the IVHF-unreliability/degree of deviation is calculated and then the DMs' opinions about the weights of criteria are aggregated. Therefore, the final weights are obtained based on the aforementioned results. The results are reported in Tables 15 and 16. In addition, the weight of each DM is obtained based on the proposed IVHF-CS method and represented in Table 17.

The P_i and R_i values regarding preferences of DMs' judgments are computed and indicated in Tables 18 and 19, respectively. Then, the interval-valued hesitant fuzzy relative importance of each candidate site for each DM is determined based on the sum of positive criteria values/negative criteria values. Hence, IVHF-utility degree for each candidate site is calculated based on the aggregated IVHF-relative importance of each candidate site. The mentioned results are shown in Table 20. Consequently, two types of ordering are applied to rank the candidate sites by decreasing the value of the IVHF-utility degrees. However, as represented in Table 21, the most suitable site for building a new factory is the second candidate site and the worst candidate site is A_3. Also, to validate the proposed IVHF-MCGA approach, the ranking results are compared to the method used by Wang (2014), which shows the same results.

In the previous, solving two industrial applications and comparing the ranking results with some recent decision methods from the literature are presented; then, the proposed IVHF-MCGA approach is validated for industrial selection problems. Hence, it can be concluded that the proposed IVHF-MCGA can be more accurate and reliable regarding the above analysis and indicating the review of the literature as shown in Table 1. Finally, this study has the following advantages versus recent decision methods in the literature: (1) some IVHF-operators are introduced for establishing the soft computing approach; (2) an IVHF-entropy method is proposed to determine the criteria weights; (3) a novel method based on the compromise solution is presented to specify the weight of each DM; (4) the COPRAS method is developed with IVHFSs for evaluation and selection process; and (5) the preferences of DMs' judgments are aggregated in the last step to prevent the loss of data.

4.3 Illustrative example 3: rapid prototyping process selection

The third illustrative example about the rapid prototyping process (RPP) selection is provided by Byun and Lee (2005) and Rao (2013) as a well-known decision-making issue in the manufacturing problems to show the further applicability of the proposed IVHF-MCGA approach. In this case, six alternatives ($RPP_1, RPP_2, \ldots, RPP_6$) as rapid prototyping processes (i.e., SLA3500, SLS2500, FDM8000, LOM1015, Quadra, Z402) are judged by four DMs ($k = 1, 2, \ldots, 4$) under six objective and subjective criteria. The following are the aforementioned criteria:

- Accuracy (C_1);
- Part cost (C_2);
- Roughness (C_3);
- Build time (C_4);
- Strength (C_5); and
- Elongation (C_6).

Table 15 Aggregated IVHF-decision matrix and T_{ij} matrix for the second illustrative example

	Aggregated IVHF-decision matrix		
	A_1	A_2	A_3
C_1	[0.670074, 0.770694]	[0.723762, 0.823907]	[0.547723, 0.648074]
C_2	[0.720041, 0.820669]	[0.523318, 0.623574]	[0.773735, 0.873885]
C_3	[0.573266, 0.673537]	[0.773735, 0.873885]	[0.623574, 0.723762]
C_4	[0.748331, 0.848528]	[0.673537, 0.773735]	[0.665371, 0.766732]
C_5	[0.696401, 0.796857]	[0.670074, 0.770694]	[0.723762, 0.823907]
C_6	[0.894958, 0.921702]	[0.370041, 0.509146]	[0.873885, 0.900000]
	Constructed T_{ij} matrix		
	A_1	A_2	A_3
C_1	[0.698881, 0.781804]	[0.754878, 0.835784]	[0.571270, 0.657416]
C_2	[0.742460, 0.827716]	[0.539611, 0.628928]	[0.797825, 0.881389]
C_3	[0.594887, 0.681285]	[0.802917, 0.883938]	[0.647093, 0.732089]
C_4	[0.769487, 0.855367]	[0.692578, 0.779970]	[0.684182, 0.772911]
C_5	[0.716218, 0.803447]	[0.689142, 0.777068]	[0.744358, 0.830721]
C_6	[0.902489, 0.925258]	[0.373155, 0.511110]	[0.881239, 0.903472]

Table 16 Interval-valued hesitant fuzzy unreliability (d_j), aggregated DMs' opinions (υ_j) and final weights of criteria (w_j) for the second illustrative example

	d_j		υ_j		w_j
d_1	[0.306382, 0.351669]	υ_1	[0.709306, 0.750233]	w_1	0.172878
d_2	[0.322405, 0.380545]	υ_2	[0.612372, 0.663325]	w_2	0.161193
d_3	[0.312127, 0.362346]	υ_3	[0.709306, 0.750233]	w_3	0.177114
d_4	[0.318997, 0.380986]	υ_4	[0.506136, 0.557788]	w_4	0.133795
d_5	[0.318168, 0.379653]	υ_5	[0.612372, 0.663325]	w_5	0.159960
d_6	[0.414095, 0.433177]	υ_6	[0.612372, 0.663325]	w_6	0.195061

Table 17 Computational results for determining the DMs' weights for the second illustrative example

	ξ_k^*	ξ_k^-	ϖ_k
$k = 1$	0.040063	0.074690	0.261652
$k = 2$	0.044050	0.075665	0.254081
$k = 3$	0.045824	0.076055	0.250856
$k = 4$	0.061433	0.085054	0.233411

Table 18 P_i values regarding each DM for the second illustrative example

	k_1	k_2	k_3	k_4
P_1	[0.414956, 0.465517]	[0.446715, 0.495065]	[0.449256, 0.497467]	[0.450199, 0.498353]
P_2	[0.442843, 0.491575]	[0.430107, 0.479688]	[0.451777, 0.499858]	[0.433479, 0.482878]
P_3	[0.437903, 0.486891]	[0.438425, 0.487385]	[0.447218, 0.495545]	[0.411145, 0.462069]

Table 19 R_i values regarding each DM for the second illustrative example

	k_1	k_2	k_3	k_4
A_1	[0.019506, 0.019506]	[0.019506, 0.039012]	[0.019506, 0.019506]	[0.001951, 0.001951]
A_2	[0.078024, 0.097530]	[0.117036, 0.146295]	[0.117036, 0.146295]	[0.058518, 0.078024]
A_3	[0.019506, 0.019506]	[0.019506, 0.019506]	[0.019506, 0.039012]	[0.019506, 0.019506]
$R_{\min}$	[0.019506, 0.019506]	[0.019506, 0.019506]	[0.019506, 0.019506]	[0.001951, 0.001951]

Table 20 Final Q_i values regarding each DM for the second illustrative example

	Q_1	Q_2	Q_3
k_1	[0.416331, 0.466995]	[0.444152, 0.492982]	[0.439224, 0.488311]
k_2	[0.448489, 0.497205]	[0.432926, 0.481893]	[0.440217, 0.489557]
k_3	[0.451014, 0.499596]	[0.453527, 0.501977]	[0.448983, 0.497682]
k_4	[0.450279, 0.498454]	[0.433579, 0.482982]	[0.411249, 0.462177]
Final Q_i	[0.440870, 0.489993]	[0.441101, 0.490019]	[0.435167, 0.484700]
	N_1	N_2	N_3
N_i	[99.94, 99.99]	[100.00, 100.00]	[98.65, 98.91]

Table 21 Final ranking based on the Q_i values by considering two types of ordering and the comparative analysis for the second illustrative example

	Component-wise ordering	Total ordering (%)	Ranked by the proposed soft computing approach	Ranked by Wang (2014) method
N_1	$N_2^l \geq N_1^l,\ N_2^u \geq N_1^u$	99.97	2	2
N_2	$N_2^l \geq N_1^l \geq N_3^l,\ N_2^u \geq N_1^u \geq N_3^u$	100	1	1
N_3	$N_1^l \geq N_3^l,\ N_1^u \geq N_3^u$	98.78	3	3

Table 22 Performance rating of the alternatives versus objective and subjective criteria

Rapid prototyping process	Objective criteria			
	C_1	C_2	C_3	C_4
RPP_1	[0.05, 0.10]	[0.17, 0.22]	[0.00, 0.05]	[0.56, 0.61]
RPP_2	[0.11, 0.16]	[0.51, 0.56]	[0.42, 0.47]	[0.17, 0.22]
RPP_3	[0.06, 0.11]	[1.00, 1.00]	[0.58, 0.63]	[0.00, 0.05]
RPP_4	[0.18, 0.23]	[0.94, 0.99]	[0.67, 0.72]	[0.00, 0.05]
RPP_5	[0.00, 0.05]	[0.00, 0.05]	[0.58, 0.63]	[0.44, 0.49]
RPP_6	[1.00, 1.00]	[0.69, 0.74]	[1.00, 1.00]	[1.00, 1.00]
Subjective criteria	Decision makers			
	DM_1	DM_2	DM_3	DM_4
C_5				
RPP_1	G	G	G	MG
RPP_2	G	MG	G	G
RPP_3	MG	MG	F	MG
RPP_4	F	F	MP	F
RPP_5	G	G	G	MG
RPP_6	VG	G	VG	VG
C_6				
RPP_1	MG	MG	F	MG
RPP_2	F	MG	MG	MG
RPP_3	G	MG	G	G
RPP_4	F	F	MP	F
RPP_5	MP	F	F	F
RPP_6	MP	MP	MP	P

The performance rating of RPP candidates under the objective and subjective criteria is indicated in Table 22. In addition, the weight of each criterion based on the preferences of the DMs is presented in Table 23.

In Table 24, the aggregated DMs' opinions about the criteria weights and the final weights of criteria based on the proposed IVHF-entropy method are demonstrated. Also, the computational results of proposed IVHF-CS method are

Table 23 Decision makers' judgments about criteria weights for the third illustrative example

Criteria	Decision makers			
	DM_1	DM_2	DM_3	DM_4
C_1	MH	M	MH	M
C_2	M	MH	M	MH
C_3	L	ML	L	ML
C_4	ML	L	ML	L
C_5	VH	VH	H	VH
C_6	VH	VH	VH	H

provided for determining the DMs' weights. Finally, the interval-valued hesitant fuzzy utility degree for each candidate is calculated based on the interval-valued hesitant fuzzy relative importance. Then, the RPP alternatives are ranked by decreasing sorting. Computational results are given in Tables 25 and 26. The ranking results by comparing with Byun and Lee (2005) study indicate somewhat similar outcomes.

5 Comparative analysis

In this section, the comparative analysis is presented for the IVHF-MCGA approach and fuzzy group TOPSIS method. The reason of selecting the fuzzy group TOPSIS is that all studies, considered in illustrative examples for comparing the ranking results, are based on the concept of the TOPSIS method (i.e., closer to positive ideal solution and farther from negative ideal solution). In this case, some comparison parameters are regarded to show the profitability of the proposed approach. In fact, the comparison parameters are selected based on the Junior et al. (2014) study and the characteristics of fuzzy decision-making methods presented in Table 1. Junior et al. (2014) have compared the fuzzy AHP and fuzzy TOPSIS methods based on some appropriate comparison parameters, including adequacy to changes of alternatives and criteria, agility in decision process, time complexity, support to group decision making, number of alternatives and criteria, and modeling of uncertainty.

In this paper, some comparison parameters, including adequacy to changes of alternatives and criteria, agility in decision process, influence of DMs' weights, and impact of first and last aggregations, that can show the efficiency and suitability of the proposed IVHF-MCGA approach are considered to compare with fuzzy group TOPSIS method in the selection problem of the rapid prototyping process. The following are the comparative analysis of comparison parameters in detail.

5.1 Adequacy to changes of alternatives or criteria

One of interesting comparison parameters is adequacy to changes of alternatives or criteria in the decision-making process that could lead to ranking reversal in the ordering of potential alternatives or selected criteria. In this case, the adding or excluding one alternative is evaluated by both proposed IVHF-MCGA approach and fuzzy group TOPSIS method; the same ranking results are obtained. The ranking order of the alternatives in the third illustrative example by adding/excluding the candidates is achieved as $RPP_6 > RPP_5 > RPP_1 > RPP_4 > RPP_2 > RPP_3$. In addition, the aforementioned results are provided for

Table 24 Computational results of aggregated DMs' opinions about criteria weights (υ_j), final weights of criteria (w_j) and DMs' weights for the third illustrative example

Criteria	υ_j	w_j	Decision makers	ξ_k^*	ξ_k^-	ϖ_k
C_1	[0.5477, 0.6205]	0.23907	$k = 1$	0.03873	0.03673	0.23865
C_2	[0.5477, 0.6205]	0.20195	$k = 2$	0.03959	0.03947	0.24476
C_3	[0.3742, 0.4472]	0.10833	$k = 3$	0.04351	0.03847	0.23004
C_4	[0.3742, 0.4472]	0.14012	$k = 4$	0.10264	0.14439	0.28655
C_5	[0.8599, 0.8739]	0.16612				
C_6	[0.8599, 0.8739]	0.14441				

Table 25 Final Q_i values regarding each DM for the third illustrative example

	k_1	k_2	k_3	k_4	Final Q_i	N_i
Q_1	[0.2265, 0.2944]	[0.2237, 0.2999]	[0.2321, 0.2887]	[0.2110, 0.3869]	[0.2225, 0.3184]	[62.93, 74.64]
Q_2	[0.2178, 0.2863]	[0.2149, 0.2919]	[0.2234, 0.2806]	[0.2021, 0.3799]	[0.2137, 0.3105]	[60.44, 72.79]
Q_3	[0.2142, 0.2830]	[0.2113, 0.2886]	[0.2199, 0.2772]	[0.1985, 0.3770]	[0.2101, 0.3073]	[59.43, 72.03]
Q_4	[0.2218, 0.2900]	[0.2189, 0.2955]	[0.2274, 0.2842]	[0.2062, 0.3831]	[0.2178, 0.3141]	[61.58, 73.62]
Q_5	[0.2631, 0.3280]	[0.2604, 0.3332]	[0.2685, 0.3226]	[0.2484, 0.4161]	[0.2594, 0.3511]	[73.35, 82.32]
Q_6	[0.3569, 0.4056]	[0.3545, 0.4102]	[0.3615, 0.4008]	[0.3440, 0.4835]	[0.3536, 0.4266]	[100, 100]

Table 26 Final ranking based on the Q_i values by considering two types of ordering and the comparative analysis for the third illustrative example

	Total ordering (%)	Ranked by the proposed soft computing approach	Ranked by Byun and Lee (2005) method
N_1	68.78	3	4
N_2	66.61	5	5
N_3	65.73	6	6
N_4	67.60	4	2
N_5	77.83	2	3
N_6	100	1	1

adding/excluding the criteria; it means that the importance order (i.e., $C_1 > C_2 > C_5 > C_6 > C_4 > C_3$) has no changes.

5.2 Agility in decision process

This factor assesses the amount of required preference judgments that are assigned by the DMs or experts in both proposed IVHF-MCGA approach and fuzzy TOPSIS method under group decision procedures. Let m the number of alternatives, n the number of criteria and k the number of the DMs. As presented in Eq. (59), the fuzzy group TOPSIS method requires mnk judgments to establish the group decision matrix and also nk judgments for specifying the criteria weights.

$$\mathrm{Ag}_{n,m,k}^{\mathrm{TOPSIS}} = mnk + nk = nk(m+1) \tag{59}$$

In the case of the proposed approach, the IVHF-entropy method is presented to determine the criteria weights. Although the experts' judgments about the relative importance of each criterion can be considered in the procedure of proposed IVHF-entropy to provide an accurate solution, it is optional in the proposed decision-making process. In fact, the IVHF-entropy method can compute the weight of each criterion without the experts' opinions about the criteria weights. Thus, this can be illustrated in Eq. (60).

$$\mathrm{Ag}_{n,m,k}^{\mathrm{IVHF-MCGA}} = mnk \tag{60}$$

Therefore, as indicated in Eqs. (59) and (60), the number of judgments for the proposed IVHF-MCGA method is nk values less than the fuzzy group TOPSIS method. For example, in the third illustrative example, the fuzzy group TOPSIS has required 168 judgments while the proposed IVHF-MCGA has required 144 judgments.

5.3 Influence of decision makers' weights

DMs' expertise might be different in the industry based on different backgrounds; thus, considering the weight of each DM by regarding their judgments could help us to achieve a precise solution. In this respect, the effect of applying the DMs' weights is explained based on the third illustrative example. In the application case, the weight of each DM is considered equal; then the ranking order results are:$\mathrm{RPP}_6 > \mathrm{RPP}_5 > \mathrm{RPP}_4 > \mathrm{RPP}_1 > \mathrm{RPP}_2 > \mathrm{RPP}_3$, which is slightly different from the ranking order results of fuzzy group decision method (i.e., $\mathrm{RPP}_6 > \mathrm{RPP}_4 > \mathrm{RPP}_5 > \mathrm{RPP}_1 > \mathrm{RPP}_2 > \mathrm{RPP}_3$). The comparative analysis based on this comparison parameter shows that the final ranking order of six candidates is affected from the DMs' weights. It is obvious that considering the weight of each DM could lead to an accurate result of the selection problem.

5.4 Impact of first and last aggregations

In the group decision-making process, two approaches as first aggregation and last aggregation are considered to integrate the judgments of the DMs or experts: in the first aggregation approach, the preference judgments of the DMs for the rating of candidates versus selected criteria are aggregated at the first and then the desired decision-making technique is implemented. In the last aggregation approach, ratings of alternatives are provided by the DMs and considered in desired decision-making technique directly. In other words, the opinions of the DMs are aggregated in the last step of desired group decision-making technique and it is capable of preventing the loss of data. In the third illustrative example, the ranking order results based on the proposed IVHF-MCGA approach are: $\mathrm{RPP}_6 > \mathrm{RPP}_5 > \mathrm{RPP}_1 > \mathrm{RPP}_4 > \mathrm{RPP}_2 > \mathrm{RPP}_3$ and also based on the fuzzy group TOPSIS method are: $\mathrm{RPP}_6 > \mathrm{RPP}_4 > \mathrm{RPP}_5 > \mathrm{RPP}_1 > \mathrm{RPP}_2 > \mathrm{RPP}_3$. Then, this study attempts to understand the main cause in the ranking order of RPP_1, RPP_4, and RPP_5. In this respect, in the first step of the proposed IVHF-MCGA, all judgments of the DMs are aggregated and the aggregated group decision-making matrix is considered for input parameters of the proposed approach. Then, it indicates that the ranking order results are: $\mathrm{RPP}_6 > \mathrm{RPP}_4 > \mathrm{RPP}_1 > \mathrm{RPP}_5 > \mathrm{RPP}_2 > \mathrm{RPP}_3$, and it can be concluded that the ranking order from the first aggregation of the proposed IVHF-MCGA is partially different from the ranking order of the fuzzy group TOPSIS.

The comparative analysis of the proposed soft computing approach and the fuzzy group TOPSIS method based on four comparison parameters indicates some interesting outcomes that show the efficiency and suitability of the proposed approach. The obtained results based on these comparison parameters are investigated for the RPP selection problem as

Table 27 Summarized comparative analysis of the proposed soft computing approach and fuzzy group decision method

Comparison parameters	Results
Adequacy to changes of alternatives or criteria	The ranking order of both group decision methods by adding or excluding the alternatives or criteria is not changed
Agility in decision process	Proposed soft computing approach performs better than fuzzy group decision TOPSIS method and requires less judgments in the group decision-making process
Influence of decision makers' weights	Considering the DMs' weights in the procedure of the proposed approach could lead to a precise solution
Impact of first and last aggregations	Considering the last aggregation approach could prevent the loss of data and lead to an accurate solution

the third illustrative example, and they might be different for other contexts in the manufacturing industry. In summary, the outcomes and findings are reported in Table 27.

6 Conclusions and suggestions

For many manufacturing companies, the complex decision problems like the industrial selections are serious issues to increase the productivity. The multi-criteria group decision-making (MCGDM) approach could be considered in an interval-valued hesitant fuzzy (IVHF) environment to solve the industrial selection problems. This study proposed a novel group decision-making method based on the interval-valued hesitant fuzzy complex proportional assessment, called IVHF-MCGA, for the selection of the most suitable industrial candidates. The preferences of potential alternatives and the criteria weights that are judged by decision makers (DMs) are expressed by linguistic variables and then are transformed into IVHFEs. In addition, the weight of each criterion is computed by developing a new IVHF-entropy method according to the DMs' judgments. Also, the relative importance of each DM is calculated by introducing a new IVHF-compromise solution method. Moreover, for the prevention of the data loss, the DMs' opinions are aggregated at the end of the proposed soft computing approach along with some IVHF-operators introduced in this study. The proposed approach under uncertainty is efficient and easy to understand with the IVHF-setting. This approach improved the group decision-making process and developed the concept and theory of compromise solution and utility degree, based on interval-valued hesitant fuzzy positive and negative ideal solutions and, therefore, it could be considered as new soft computing approach for solving the evaluation and selection problems to handle uncertainty. In addition, the validity and applicability of the proposed IVHF-MCGA approach are tested based on solving three different industrial selection problems. In this respect, the obtained ranking order results are compared with some fuzzy decision methods from the literature, which are developed from the concept of TOPSIS method. Computational results show the same ranking of industrial alternatives. Also, the fuzzy group TOPSIS method has been considered to establish a comparative analysis based on four selected comparison parameters to indicate the efficiency and suitability of the proposed soft computing approach. Although the presented approach has been applied to industrial selection problems, it could be used in other fields of management and engineering for making a suitable decision, especially for manufacturing decision problems.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Abdel-Malek L, Resare LJ (2000) Algorithm based decision support system for the concerted selection of equipment in machining/assembly cells. Int J Prod Res 38:323–339

Agrawal VP, Kohli V, Gupta S (1991) Computer aided robot selection: the 'multiple attribute decision making' approach. Int J Prod Res 29:1629–1644

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Atanassov KT (1989) More on intuitionistic fuzzy sets. Fuzzy Sets Syst 33:37–45

Atanassov KT (2000) Two theorems for intuitionistic fuzzy sets. Fuzzy Sets Syst 110:267–269

Ayağ Z (2010) A combined fuzzy AHP-simulation approach to CAD software selection. Int J Gen Syst 39:731–756

Bhangale PP, Agrawal VP, Saha SK (2004) Attribute based specification, comparison and selection of a robot. Mech Mach Theory 39:1345–1366

Bustince H (1994) Conjuntos Intuicionistas e Intervalo valorados difusos: propiedades y construcci on. Thesis, Universidad P_ublica de Navarra, Relaciones Intuicionistas Fuzzy

Byun HS, Lee KH (2005) A decision support system for the selection of a rapid prototyping process using the modified TOPSIS method. Int J Adv Manuf Technol 26:1338–1347

Chatterjee P, Athawale VM, Chakraborty S (2010) Selection of industrial robots using compromise ranking and outranking methods. Robot Comput Integr Manuf 26:483–489

Chen N, Zeshui X, Xia M (2013) Interval-valued hesitant preference relations and their applications to group decision making. Knowl Based Syst 37:528–540

Chu T-C, Lin Y-C (2003) A fuzzy TOPSIS method for robot selection. Int J Adv Manuf Technol 21:284–290

Devi K (2011) Extension of VIKOR method in intuitionistic fuzzy environment for robot selection. Expert Syst Appl 38:14163–14168

Farhadinia B (2013) Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Inf Sci 240:129–144

Goh C-H (1997) Analytic hierarchy process for robot selection. J Manuf Syst 16:381–386

Gorzałczany MB (1987) A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst 21:1–17

İç YT, Yurdakul M, Dengiz B (2013) Development of a decision support system for robot selection. Robot Comput Integr Manuf 29:142–157

Junior FR, Lima LO, Carpinetti LCR (2014) A comparison between Fuzzy AHP and Fuzzy TOPSIS methods to supplier selection. Appl Soft Comput 21:194–209

Karsak EE (2005) Choquet integral-based decision making approach for robot selection. In: Khosla R, Howlett RJ, Jain LC (eds) Choquet integral-based decision making approach for robot selection. Knowledge-based intelligent information and engineering systems. Springer, pp 635–641

Li L-G, Peng D-H (2014) Interval-valued hesitant fuzzy Hamacher synergetic weighted aggregation operators and their application to shale gas areas selection. Math Probl Eng 2014:1–25

Li G-D, Yamaguchi D, Nagai M (2007) A grey-based decision-making approach to the supplier selection problem. Math Comput Model 46(3–4):573–581

Liao H, Xu Z (2014) Subtraction and division operations over hesitant fuzzy sets. J Intell Fuzzy Syst 27:65–72

Liu H-C, Ren M-L, Jing W, Lin Q-L (2014) An interval 2-tuple linguistic MCDM method for robot evaluation and selection. Int J Prod Res 52:2867–2880

Mousavi SM, Jolai F, Tavakkoli-Moghaddam R (2013a) A fuzzy stochastic multi-attribute group decision-making approach for selection problems. Group Decis Negot 22(2): 207–233

Mousavi SM, Vahdani B, Tavakkoli-Moghaddam R, Ebrahimnejad S, Amiri M (2013b) A multi-stage decision-making process for multiple attributes analysis under an interval-valued fuzzy environment. Int J Adv Manuf Technol 64:1263–1273

Mousavi SM, Mirdamadi S, Siadat A, Dantan J, Tavakkoli-Moghaddam R (2015) An intuitionistic fuzzy grey model for selection problems with an application to the inspection planning in manufacturing firms. Eng Appl Artif Intell 39:157–167

Ölçer Aİ, Odabaşi AY (2005) A new fuzzy multiple attributive group decision making methodology and its application to propulsion/manoeuvring system selection problem. Eur J Oper Res 166:93–114

Rao RV (2013) Multiple attribute decision making in the manufacturing environment. In: Pham DT (ed) Decision making in manufacturing environment using graph theory and fuzzy multiple attribute decision making methods. Springer

Rao RV, Patel BK, Parnichkun M (2011) Industrial robot selection using a novel decision making method considering objective and subjective preferences. Robot Auton Syst 59:367–375

Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25:529–539

Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: On hesitant fuzzy sets and decision. IEEE international conference on Fuzzy Systems, pp 1378–1382

Turksen IB (1986) Interval valued fuzzy sets based on normal forms. Fuzzy Sets Syst 20:191–210

Türkşen İB, Bilgiç T (1996) Interval valued strict preference with Zadeh triples. Fuzzy Sets Syst 78:183–195

Vahdani B, Hadipour H, Sadaghiani JS, Amiri M (2010) Extension of VIKOR method based on interval-valued fuzzy sets. Int J Adv Manuf Technol 47:1231–1239

Vahdani B, Tavakkoli-Moghaddam R, Mousavi SM, Ghodratnama A (2013) Soft computing based on new interval-valued fuzzy modified multi-criteria decision-making method. Appl Soft Comput 13:165–172

Vahdani B, Mousavi SM, Tavakkoli-Moghaddam R, Ghodratnama A, Mohammadi M (2014) Robot selection by a multiple criteria complex proportional assessment method under an interval-valued fuzzy environment. Int J Adv Manuf Technol 73(5–8):687–697

Vahdani B, Hadipour H (2011) Extension of the ELECTRE method based on interval-valued fuzzy sets. Soft Comput 15:569–579

Wang Y-J (2014) A fuzzy multi-criteria decision-making model by associating technique for order preference by similarity to ideal solution with relative preference relation. Inf Sci 268:169–184

Wang Y-J (2015) A fuzzy multi-criteria decision-making model based on simple additive weighting method and relative preference relation. Appl Soft Comput 30:412–420

Xu Z, Zhang X (2013) Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information. Knowl Based Syst 52:53–64

Yao J-S, Ming-Miin Y (2004) Decision making based on statistical data, signed distance and compositional rule of inference. Int J Uncertain Fuzziness Knowl Based Syst 12:161–190

Yue Z (2011) An extended TOPSIS for determining weights of decision makers with interval numbers. Knowl Based Syst 24:146–153

Zhang Z, Wang C, Tian D, Li K (2014) Induced generalized hesitant fuzzy operators and their application to multiple attribute group decision making. Comput Ind Eng 67:116–138

Zhang X, Zeshui X (2014) Interval programming method for hesitant fuzzy multi-attribute group decision making with incomplete preference over alternatives. Comput Ind Eng 75:217–229

Zhao X, Lin R, Wei G (2014) Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making. Expert Syst Appl 41:1086–1094