Soft Comput (2016) 20:3117–3124
DOI 10.1007/s00500-015-1934-8

FOUNDATIONS

A Hadamard-type inequality for fuzzy integrals based on r-convex functions

Sadegh Abbaszadeh[1] · **Madjid Eshaghi**[1]

Published online: 14 November 2015

Abstract In this paper, it is shown that the Hadamard integral inequality for r-convex functions is not satisfied in the fuzzy context. Using the classical Hadamard integral inequality, we give an upper bound for the Sugeno integral of r-convex functions. In addition, we generalize the results related to the Hadamard integral inequality for Sugeno integral from 1-convex functions (ordinary convex functions) to r-convex functions. We present a geometric interpretation and some examples in the framework of the Lebesgue measure to illustrate the results.

Keywords Sugeno integral · The Hadamard inequality · r-convex function · Seminormed Sugeno integral

1 Introduction

The process of combining several numerical values into a single representative one is called aggregation, and the numerical function performing this process is called an aggregation function, see Grabisch et al. (2009). Several methods for combining evidence produced by multiple information sources have been studied by different researchers and some synthesizing functions have been proposed. For example, arithmetic mean, geometric mean and median can be regarded as a basic class, because they are often used and very classic. However, these operators are not able to model an interaction between criteria. For having a representation of interaction phenomena between criteria, fuzzy integrals have been proposed by Sugeno (1974). The fuzzy integral is a nonlinear functional that is defined with respect to a fuzzy measure, which in turn is either a belief or a plausibility measure in the sense of Dempster–Shafer belief theory (Wierzchon 1982). The fuzzy integral combines objective evidence for a hypothesis with the system's expectation of the importance of that evidence to the hypothesis, see Wang and Klir (1992) and Chen et al. (2014).

Two main classes of the fuzzy integrals are Choquet and Sugeno integrals. The properties and applications of the Sugeno integral have been studied by many authors. Ralescu and Adams (1980) studied several equivalent definitions of Sugeno integral. Román-Flores et al. (2007a, b) (Flores-Franulič and Román-Flores 2007; Román-Flores and Chalco-Cano 2006, 2007) studied the level-continuity of Sugeno integral, H-continuity of fuzzy measures and geometric inequalities for fuzzy measures and integrals, respectively. Wang and Klir (1992) had a general overview on fuzzy measurement and fuzzy integration theory.

Recently, many authors have studied the most well-known integral inequalities for Sugeno integral. Agahi et al. (2010a, b, 2011, 2012a, b) and Agahi and Eslami (2011) proved general Minkowski-type inequalities, general extensions of Chebyshev-type inequalities and general Barnes–Godunova–Levin-type inequalities for Sugeno integrals. Caballero and Sadarangani (2009, 2010a, b, c, 2011) proved Hermite–Hadamard-type inequalities, Chebyshev-type inequalities, Cauchy and Fritz Carlson's type inequalities for Sugeno integral. Kaluszka et al. (2014) gave the necessary and sufficient conditions guaranteeing the validity of Chebyshev type inequalities for the general-

Communicated by A. Di Nola.

✉ Sadegh Abbaszadeh
abbaszadeh@semnan.ac.ir

Madjid Eshaghi
meshaghi@semnan.ac.ir

1 Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan 35195-363, Iran

ized Sugeno integral in the case of functions belonging to a much wider class than the comonotone functions. Wu et al. (2010) proved two inequalities for the Sugeno integral on abstract spaces generalizing all previous Chebyshev's inequalities.

Most of the integral inequalities studied in the Sugeno integration context normally consider conditions such as monotonicity or comonotonicity. In this paper, the main purpose is to consider the Sugeno integral for r-convex functions using the Hadamard integral inequality. The classical measure theory is closely connected with probability theory. A probability measure, as any other classical measure, is a set function that assigns 0 to the empty set and a nonnegative number to any other set, and that is additive. However, a probability measure requires, in addition, that 1 be assigned to the universal set in question. Hence, probability theory may be viewed as a part of classical measure theory. It is shown by Klir and Folger (1988) that classical (additive) probability measure can capture only one of several types of uncertainty that can clearly be recognized when the additivity property is abandoned. On the other hand, in many applications, assumptions about the convexity of a probability distribution allow just enough special structure to yield a workable theory. So, we believe that our results will be useful in non-additive probabilities.

The paper is organized as follows: Some necessary preliminaries and summarization of some previous known results are presented in Sect. 2. In Sect. 3, we deal with the upper bound of Sugeno integral for r-convex functions and give some examples. In Sect. 4, a geometric interpretation is presented to illustrate the results. Finally, a conclusion is given in Sect. 5.

2 Preliminaries

In this section, we are going to review some well-known results from the theory of non-additive measures. Let X be a non-empty set and Σ be a σ-algebra of subsets of X.

Definition 1 (Ralescu and Adams 1980) Suppose that $\mu : \Sigma \longrightarrow [0, \infty)$ is a set function. We say that μ is a fuzzy measure if it satisfies

1. $\mu(\emptyset) = 0$.
2. $E, F \in \Sigma$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$.
3. $E_n \in \Sigma\ (n \in \mathbb{N}), E_1 \subset E_2 \subset . . ,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$ (continuity from below).
4. $E_n \in \Sigma\ (n \in \mathbb{N}),\ E_1 \supset E_2 \supset \ldots,\ \mu(E_1) < \infty$, imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$ (continuity from above).

The triple (X, Σ, μ) is called a fuzzy measure space.

Let (X, Σ, μ) be a fuzzy measure space. By $\mathcal{F}_\mu(X)$ we denote the set

$$\mathcal{F}_\mu(X) = \{f : X \longrightarrow [0, \infty) : f \text{ is measurable with respect to } \Sigma\}.$$

For $f \in \mathcal{F}_\mu(X)$ and $\alpha > 0$, we denote by F_α and $F_{\tilde{\alpha}}$ the following sets

$$F_\alpha = \{x \in X : f(x) \geq \alpha\} \text{ and } F_{\tilde{\alpha}} = \{x \in X : f(x) > \alpha\}.$$

Note that if $\alpha \leq \beta$, then $F_\beta \subset F_\alpha$ and $F_{\tilde{\beta}} \subset F_{\tilde{\alpha}}$.

Definition 2 (Durante and Sempi 2005) A function $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a semicopula if, and only if, it satisfies the two following conditions:

(T_1) $T(x, 1) = T(1, x) = x$ for any $x \in [0, 1]$.
(T_2) For any $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, $T(x_1, y_1) \leq T(x_2, y_2)$.

A semicopula T is a t-norm if

(T_3) $T(x, y) = T(y, x)$ for any $x, y \in [0, 1]$.
(T_4) $T\big(T(x, y), z\big) = T\big(x, T(y, z)\big)$ for any $x, y, z \in [0, 1]$.

A function $S : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a t-conorm (Klement et al. 2000), if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 1 The following functions are t-norms:

1. $T_M(x, y) = \min\{x, y\}$.
2. $T_P(x, y) = x \cdot y$.
3. $T_L(x, y) = \max\{x + y - 1, 0\}$.
4. $T_D(x, y) = \begin{cases} 0, & \text{if}(x, y) \in [0, 1)^2; \\ \min(x, y), & \text{otherwise.} \end{cases}$

Notice that if T is a t-norm, as an immediate consequence of (T_1), (T_3) and (T_4), the drastic product T_D is the weakest, and the minimum T_M is the strongest t-norm, i.e., for each t-norm T we have

$$T_D \leq T \leq T_M. \tag{1}$$

Between the four basic t-norms we have these strict inequalities (see Klement et al. 2000, 2004):

$$T_D < T_L < T_P < T_M.$$

Definition 3 (Pap 1995; Sugeno 1974; Wang and Klir 1992) Let (X, Σ, μ) be a fuzzy measure space, $f \in \mathcal{F}_\mu(X)$ and

$A \in \Sigma$, then the Sugeno integral of f on A with respect to the fuzzy measure μ is defined by

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \geq 0} \big(\alpha \wedge \mu(A \cap F_\alpha)\big),$$

where $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum. If $A = X$, then

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)) .$$

The following properties of Sugeno integral are well known and can be found in Pap (1995) and Wang and Klir (1992).

Theorem 1 *Let (X, Σ, μ) be a fuzzy measure space, $A, B \in \Sigma$ and $f, g \in \mathcal{F}_\mu(X)$ then*

(F_1) $⨍_A f \mathrm{d}\mu \leq \mu(A)$.
(F_2) $⨍_A k \mathrm{d}\mu = k \wedge \mu(A)$, *$k$ non-negative constant.*
(F_3) *If $f \leq g$ on A then* $⨍_A f \mathrm{d}\mu \leq ⨍_A g \mathrm{d}\mu$.
(F_4) *If $A \subset B$ then* $⨍_A f \mathrm{d}\mu \leq ⨍_B f \mathrm{d}\mu$.

By using the concept of semicopulas, García and Álvarez (1986) proposed the following family of fuzzy integrals.

Definition 4 Let T be a semicopula. Then the seminormed Sugeno integral of a function $f \in \mathcal{F}_\mu(X)$ over $A \in \Sigma$ with respect to T and the fuzzy measure μ is defined by

$$\int_{T,A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T\,(\alpha, \mu(A \cap F_\alpha)) .$$

It should be noted that the seminormed Sugeno integrals were independently introduced not only as (N) fuzzy integral due to Zhao (1981), but also as the weakest universal integrals (on [0, 1]) with respect to a given semicopula, see Klement et al. (2010).

Notice that the Sugeno integral of $f \in \mathcal{F}_\mu(X)$ over $A \in \Sigma$ is the seminormed Sugeno integral of f over $A \in \Sigma$ with respect to the semicopula T_M.

In virtue of (1),

$$\begin{aligned} \int_{T,A} f \mathrm{d}\mu &= \bigvee_{\alpha \in [0,1]} T\big(\alpha, \mu(A \cap F_\alpha)\big) \\ &\leq \bigvee_{\alpha \in [0,1]} \big(\alpha \wedge \mu(A \cap F_\alpha)\big) \\ &= ⨍_A f \mathrm{d}\mu. \end{aligned} \tag{2}$$

Proposition 1 (García and Álvarez 1986) Let (X, Σ, μ) be a fuzzy measure space and T be a semicopula. Then

1. For any $A \in \Sigma$ and $f, g \in \mathcal{F}_\mu(X)$ with $f \leq g$, we have
$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,A} g \mathrm{d}\mu.$$
2. For $A, B \in \Sigma$ with $A \subset B$ and any $f \in \mathcal{F}_\mu(X)$,
$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,B} f \mathrm{d}\mu.$$

The power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined by

$$M_r(x, y; \lambda) = \begin{cases} \big(\lambda f(x)^r + (1-\lambda) f(y)^r\big)^{\frac{1}{r}}, & r \neq 0; \\ f(x)^\lambda f(y)^{1-\lambda}, & r = 0. \end{cases}$$

A positive function f is r-convex on a real interval $[a, b]$ if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1-\lambda)y) \leq M_r(x, y; \lambda).$$

We have that 1-convex functions are ordinary convex functions. Moreover, 0-convex functions are simply log-convex functions. It will be convenient to invoke the generalized logarithmic mean $L_r(x, y)$ of order r of two positive numbers x, y, which is given by

$$L_r(x, y) = \begin{cases} \frac{r}{r+1} \cdot \frac{x^{r+1} - y^{r+1}}{x^r - y^r}, & r \neq 0, -1, \quad x \neq y, \\ \frac{x-y}{\ln(x) - \ln(y)}, & r = 0, \quad x \neq y, \\ xy \cdot \frac{\ln(x) - \ln(y)}{x-y}, & r = -1, \quad x \neq y, \\ x, & x = y. \end{cases}$$

The following Hadamard inequality provides an upper bound for the mean value of an r-convex function $f : [a, b] \longrightarrow \mathbb{R}$, see (Gill et al. 1997):

$$\frac{1}{b-a} \int_a^b f(x) \mathrm{d}x \leq L_r\,(f(a), f(b)) . \tag{3}$$

3 The main results

To simplify the calculation of the Sugeno integral, for a given $f \in \mathcal{F}^\mu(X)$ and $A \in \Sigma$, we write

$$\Gamma = \big\{\alpha \mid \alpha \geq 0, \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \text{ for any } \beta > \alpha\big\} .$$

It is easy to see that

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} \big(\alpha \wedge \mu(A \cap F_\alpha)\big).$$

Notice that if X is the set of real numbers ($X = \mathbb{R}$), Σ is the Borel field and μ is the Lebesgue measure, then (X, Σ, μ) is a fuzzy measure space; but it should be noted that the Sugeno integral is not an extension of the Lebesgue integral.

The following example shows that the Hadamard integral inequality for r-convex functions is not valid in the fuzzy context.

Example 2 Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Take the non-negative and $\frac{1}{2}$-convex function $f(x) = 2^{x-2}$ on $[0, 1]$. We have

$$\begin{aligned} -\!\!\!\!\!\!\int_0^1 2^{x-2}\mathrm{d}\mu &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([0,1]\cap\left\{2^{x-2}\geq\alpha\right\}\right)\right) \\ &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([0,1]\cap\left\{x\geq\frac{\ln(\alpha)}{\ln(2)}+2\right\}\right)\right) \\ &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \left(-1-\frac{\ln(\alpha)}{\ln(2)}\right)\right). \end{aligned}$$

In this expression, $-1 - \frac{\ln(\alpha)}{\ln(2)}$ is a decreasing continuous function of α when $\alpha \geq 0$. Hence, the supremum will be attained at the point which is one of the solutions of the equation

$$\alpha = -1 - \frac{\ln(\alpha)}{\ln(2)},$$

that is, at $\alpha \approx 0.383$. So, we have

$$-\!\!\!\!\!\!\int_0^1 2^{x-2}\mathrm{d}\mu \approx 0.383.$$

On the other hand, $L_{\frac{1}{2}}\left(f(0), f(1)\right) \approx 0.368$. This proves that the Hadamard integral inequality (3) for r-convex functions is not satisfied in the fuzzy context.

In the sequel, we will establish an upper bound on Sugeno integral of r-convex functions ($r \neq 0$). Some specific examples will be given to illustrate the results.

Theorem 2 *Let $([a, b], \Sigma, \mu)$ be a fuzzy measure space. Let $r > 0$ and $f : [a, b] \longrightarrow [0, \infty)$ be an r-convex function with $f(a) \neq f(b)$. If $f(b) > f(a)$, then*

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq \bigvee_{\alpha\in\Gamma}\left(\alpha\wedge\mu\left(\left[(b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a, b\right]\right)\right),$$

where $\Gamma = \left[f(a), f(b)\right)$. If $f(a) > f(b)$, then

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq \bigvee_{\alpha\in\Gamma}\left(\alpha\wedge\mu\left(\left[a, (b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a\right]\right)\right),$$

where $\Gamma = \left[f(b), f(a)\right)$.

Proof As f is r-convex, for $x \in [a, b]$ we have

$$\begin{aligned} f(x) &= f\left(\left(1-\frac{x-a}{b-a}\right)a + \frac{x-a}{b-a}b\right) \\ &\leq \left(\left(1-\frac{x-a}{b-a}\right)f(a)^r + \frac{x-a}{b-a}f(b)^r\right)^{\frac{1}{r}} \\ &= g(x). \end{aligned}$$

By (F_3) of Theorem 1 and Definition 3, we get

$$\begin{aligned} -\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu &\leq -\!\!\!\!\!\!\int_a^b \left(\left(1-\frac{x-a}{b-a}\right)f(a)^r + \frac{x-a}{b-a}f(b)^r\right)^{\frac{1}{r}}\mathrm{d}\mu \\ &= -\!\!\!\!\!\!\int_a^b g\mathrm{d}\mu = \bigvee_{\alpha\geq 0}\left(\alpha\wedge G(\alpha)\right), \end{aligned} \tag{4}$$

where G is the distribution function associated with g given by

$$G(\alpha) = \mu\left([a, b]\cap\{g\geq\alpha\}\right).$$

If $f(b) > f(a)$, then

$$\begin{aligned} G(\alpha) &= \mu\left([a,b]\cap\left\{\left(\left(1-\frac{x-a}{b-a}\right)f(a)^r + \frac{x-a}{b-a}f(b)^r\right)^{\frac{1}{r}}\geq\alpha\right\}\right) \\ &= \mu\left([a,b]\cap\left\{x\geq(b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a\right\}\right) \\ &= \mu\left(\left[(b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a, b\right]\right). \end{aligned} \tag{5}$$

Thus, $\Gamma = \left[f(a), f(b)\right)$ and we only need to consider $\alpha \in \left[f(a), f(b)\right)$.

If $f(b) < f(a)$, then

$$\begin{aligned} G(\alpha) &= \mu\left([a,b]\cap\left\{\left(\left(1-\frac{x-a}{b-a}\right)f(a)^r + \frac{x-a}{b-a}f(b)^r\right)^{\frac{1}{r}}\geq\alpha\right\}\right) \\ &= \mu\left([a,b]\cap\left\{x\leq(b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a\right\}\right) \\ &= \mu\left(\left[a, (b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r}+a\right]\right). \end{aligned} \tag{6}$$

Thus, $\Gamma = \left[f(b), f(a)\right)$ and we only need to consider $\alpha \in \left[f(b), f(a)\right)$.

Finally, the assertions of this theorem are true in view of (4), (5) and (6). □

Remark 1 In the case $f(a) = f(b)$ in Theorem 2, we have $g(x) = f(a)$ and using (F_2) and (F_3) of Theorem 1, we get

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq -\!\!\!\!\!\!\int_a^b g\mathrm{d}\mu = -\!\!\!\!\!\!\int_a^b f(a)\mathrm{d}\mu = f(a)\wedge\mu([a,b]).$$

Remark 2 In Theorem 2, if we suppose $r < 0$, then for $f(b) > f(a)$,

$$\fint_a^b f\,\mathrm{d}\mu \le \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left(\left[a, (b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r} + a\right]\right)\right),$$

where $\Gamma = \big(f(a), f(b)\big]$. If $f(a) > f(b)$, then

$$\fint_a^b f\,\mathrm{d}\mu \le \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left(\left[(b-a)\frac{\alpha^r - f(a)^r}{f(b)^r - f(a)^r} + a, b\right]\right)\right),$$

where $\Gamma = \big(f(b), f(a)\big]$.

Corollary 1 *Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Let $r > 0$ and $f : [a, b] \longrightarrow [0, \infty)$ be an r-convex function with $f(a) \neq f(b)$. Then for $f(b) > f(a)$,*

$$\fint_a^b f\,\mathrm{d}\mu \le \beta,$$

where β is one of the solutions of the equation

$$(b-a)\alpha^r + (f(b)^r - f(a)^r)\alpha = (b-a)f(b)^r$$

belonging to $\big[f(a), f(b)\big)$. If $f(b) < f(a)$, then

$$\fint_a^b f\,\mathrm{d}\mu \le \beta,$$

where β is one of the solutions of the equation

$$(b-a)\alpha^r + (f(a)^r - f(b)^r)\alpha = (b-a)f(a)^r$$

belonging to $\big[f(b), f(a)\big)$.

Proof According to Theorem 2, we have

$$\fint_a^b f\,\mathrm{d}\mu \le \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\left(\alpha \wedge (b-a)\frac{f(b)^r-\alpha^r}{f(b)^r-f(a)^r}\right), & f(b) > f(a),\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha \wedge (b-a)\frac{f(a)^r-\alpha^r}{f(a)^r-f(b)^r}\right), & f(b) < f(a).\end{cases}$$

In the case $f(b) > f(a)$, $(b-a)\frac{f(b)^r-\alpha^r}{f(b)^r-f(a)^r}$ is a decreasing continuous function of α when $\alpha \in \big[f(a), f(b)\big)$. Hence, the supremum will be attained at the point which is one of the solutions of the equation

$$\alpha = (b-a)\frac{f(b)^r - \alpha^r}{f(b)^r - f(a)^r},$$

i.e.,

$$(b-a)\alpha^r + (f(b)^r - f(a)^r)\alpha = (b-a)f(b)^r.$$

In the case $f(b) < f(a)$, $(b-a)\frac{f(a)^r-\alpha^r}{f(a)^r-f(b)^r}$ is a decreasing continuous function of α when $\alpha \in \big[f(b), f(a)\big)$. Hence, the supremum will be attained at the point which is one of the solutions of the equation

$$\alpha = (b-a)\frac{f(a)^r - \alpha^r}{f(a)^r - f(b)^r},$$

i.e.,

$$(b-a)\alpha^r + (f(a)^r - f(b)^r)\alpha = (b-a)f(a)^r.$$

□

Remark 3 In Corollary 1, if we suppose $r < 0$, then for $f(b) > f(a)$,

$$\fint_a^b f\,\mathrm{d}\mu \le \beta,$$

where β is one of the solutions of the equation

$$(b-a)\alpha^r + (f(a)^r - f(b)^r)\alpha = (b-a)f(a)^r$$

belonging to $\big(f(a), f(b)\big]$. If $f(b) < f(a)$, then

$$\fint_a^b f\,\mathrm{d}\mu \le \beta,$$

where β is one of the solutions of the equation

$$(b-a)\alpha^r + (f(b)^r - f(a)^r)\alpha = (b-a)f(b)^r$$

belonging to $\big(f(b), f(a)\big]$.

Example 3 Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Consider the non-negative and $\frac{1}{2}$-convex function $f(x) = e^{x^2-1}$ on $[0, 1]$. As $f(0) = 1/e$, $f(1) = 1$, using Corollary 1 we can get the following estimate:

$$\fint_0^1 e^{x^2-1}\mathrm{d}\mu \le \bigvee_{\alpha\in[1/e,1)}\left(\alpha \wedge \frac{1-\sqrt{\alpha}}{1-1/\sqrt{e}}\right).$$

In this expression, $\frac{1-\sqrt{\alpha}}{1-1/\sqrt{e}}$ is a decreasing continuous function of α when $\alpha \in \big[1/e, 1\big)$. We put

$$\beta = \bigvee_{\alpha\in[1/e,1)}\left(\alpha \wedge \frac{1-\sqrt{\alpha}}{1-1/\sqrt{e}}\right).$$

So, β is one of the solutions of the equation

$$\sqrt{\alpha} + \left(1 - 1/\sqrt{e}\right)\alpha = 1$$

belonging to $\big[1/e, 1\big)$, that is, $\beta \approx 0.590$. Consequently, we have

$$⨍_0^1 e^{x^2-1} \mathrm{d}\mu \leq 0.590.$$

Example 4 Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Consider the non-negative and $\frac{1}{2}$-convex function $f(x) = (3/2)^x - x$ on $[0, 2]$. As $f(0) = 1$, $f(2) = 1/4$, by Corollary 1 we may approximate the upper bound of the Sugeno integral of f on $[0, 2]$ by

$$⨍_0^2 \big((3/2)^x - x\big)\mathrm{d}\mu \leq \bigvee_{\alpha\in[1/4,1)} \left(\alpha \wedge (2-0)\frac{1-\sqrt{\alpha}}{1-1/2}\right)$$
$$= \bigvee_{\alpha\in[1/4,1)} \big(\alpha \wedge 4(1-\sqrt{\alpha})\big).$$

In this expression, $4(1-\sqrt{\alpha})$ is a decreasing continuous function of α when $\alpha \in \big[1/4, 1\big)$. We put

$$\beta = \bigvee_{\alpha\in[1/4,1)} \big(\alpha \wedge 4(1-\sqrt{\alpha})\big).$$

So, β is one of the solutions of the equation

$$2\sqrt{\alpha} + (1-1/2)\alpha = 2$$

belonging to $\big[1/4, 1\big)$, that is, $\beta \approx 0.686$. Therefore,

$$⨍_0^2 \big((3/2)^x - x\big)\mathrm{d}\mu \leq 0.686.$$

It should be noted that the exact solution of $⨍_0^2 ((3/2)^x - x)$ $\mathrm{d}\mu$ cannot be easily calculated. But surely the exact solution is less than or equal to 0.686.

In the following proposition, we deal with the seminormed fuzzy integral on $[0, 1]$ based on the semicopula T_P. It should be noted that T_P-based integrals are standardly known as Shilkret integral, see Pap (1995).

Proposition 2 *Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Let $r > 0$ and $f : [0, 1] \longrightarrow [0, 1]$ be an r-convex function with $f(0) \neq f(1)$. Then*

$$\int_{T_P,[0,1]} f\mathrm{d}\mu \leq \begin{cases} \frac{r}{f(1)^r - f(0)^r}\left(\frac{f(1)}{(r+1)^{\frac{1}{r}}}\right)^{r+1}, & f(1) > f(0),\\ \frac{r}{f(0)^r - f(1)^r}\left(\frac{f(0)}{(r+1)^{\frac{1}{r}}}\right)^{r+1}, & f(1) < f(0).\end{cases}$$

Proof For an r-convex function $f : [0, 1] \longrightarrow [0, 1]$ with $f(0) \neq f(1)$, according to Proposition 1 and Corollary 1 with semicopula T_P, we have

$$\int_{T_P,[0,1]} f\mathrm{d}\mu$$
$$\leq \begin{cases} \bigvee_{\alpha\in(0,1]}\left(\alpha \cdot \frac{f(1)^r-\alpha^r}{\left(f(1)^r - f(0)^r\right)}\right), & f(1) > f(0)\\ \bigvee_{\alpha\in(0,1]}\left(\alpha \cdot \frac{f(0)^r-\alpha^r}{\left(f(0)^r - f(1)^r\right)}\right), & f(1) < f(0)\end{cases}$$
$$= \begin{cases} \frac{r}{f(1)^r - f(0)^r}\left(\frac{f(1)}{(r+1)^{\frac{1}{r}}}\right)^{r+1}, & f(1) > f(0),\\ \frac{r}{f(0)^r - f(1)^r}\left(\frac{f(0)}{(r+1)^{\frac{1}{r}}}\right)^{r+1}, & f(1) < f(0).\end{cases}$$

□

Example 5 Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$ and consider the non-negative and $\frac{1}{2}$-convex function $f(x) = 3^{-x}$ on $[0, 1]$. As $f(0) = 1$ and $f(1) = 1/3$, using Proposition 2, we can get the following estimate:

$$\int_{T_P,[0,1]} 3^{-x}\mathrm{d}\mu \leq \frac{1/2}{1-\sqrt{1/3}}\left(\frac{1}{(3/2)^2}\right)^{3/2} \approx 0.351.$$

4 Geometric interpretation

Let Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. If $f : A \subseteq \mathbb{R} \longrightarrow [0, \infty)$ is a continuous function, then the geometric significance of $⨍_A f\mathrm{d}\mu$ is the edge's length of the largest square between the curve of $f(x)$ and the x-axis.

In Example 3, for the real $\frac{1}{2}$-convex function $f(x) = e^{x^2-1}$ on $[0, 1]$, there exists the real function

$$g(x) = \left((1-x)\sqrt{1/e} + x\right)^2$$

such that

$$⨍_0^1 e^{x^2-1}\mathrm{d}\mu \leq ⨍_0^1 \left((1-x)\sqrt{1/e} + x\right)^2 \mathrm{d}\mu. \tag{7}$$

Geometric interpretation of (7) is shown in Fig. 1. The lengths of the lines 1 and 2 are the solutions of the integrals in left and right hand sides of (7), respectively. We have a similar geometric interpretation for Example 4 (Fig. 2)

In Example 5, for the real $\frac{1}{2}$-convex function $f(x) = 3^{-x}$ on $[0, 1]$, we have

$$\int_{T_P,[0,1]} 3^{-x}\mathrm{d}\mu \leq 0.351.$$

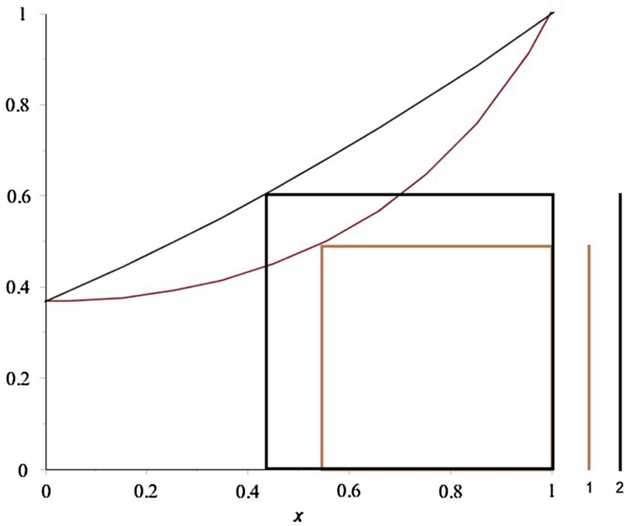

Fig. 1 Geometric interpretation of Example 3

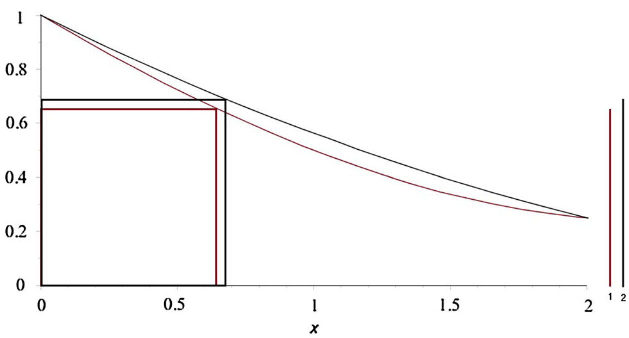

Fig. 2 Geometric interpretation of Example 4

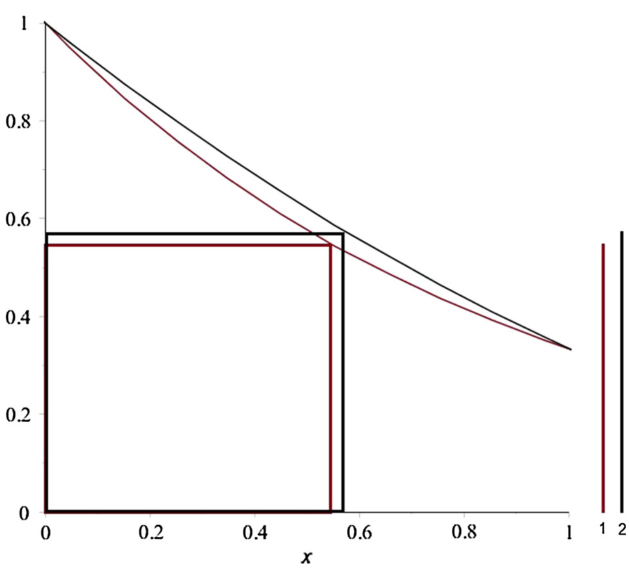

Fig. 3 Geometric interpretation of Example 5

The length of the line 1 in Fig. 3 is the solution of $⨍_0^1 3^{-x} \mathrm{d}\mu$ belonging to the interval (0.5, 0.6). This shows that [cf. the inequality (2)]

$$\int_{T_P,[0,1]} 3^{-x} \mathrm{d}\mu \leq 0.351 \leq ⨍_0^1 3^{-x} \mathrm{d}\mu.$$

5 Conclusion

The Hadamard integral inequality is the first fundamental result for r-convex functions defined on an interval of real numbers with a natural geometrical interpretation and a loose number of applications for particular inequalities. In this paper, we established the Hadamard integral inequality for the Sugeno integral based on r-convex functions which is a useful tool to approximate unsolvable integrals of this kind. In addition, there are numerous applications of Sugeno integral, and thus the study of Hadamard and similar inequalities for Sugeno integral is an important and interesting topic for the further research.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Agahi H, Mesiar R, Ouyang Y (2010a) Chebyshev type inequalities for pseudo-integrals. Nonlinear Anal 72:2737–2743
Agahi H, Mesiar R, Ouyang Y (2010b) General Minkowski type inequalities for Sugeno integrals. Fuzzy Sets Syst 161:708–715
Agahi H, Román-Flores H, Flores-Franulič A (2011) General Barnes–Godunova–Levin type inequalities for Sugeno integral. Inf Sci 181:1072–1079
Agahi H, Mesiar R, Ouyang Y, Pap E, Strboja M (2012a) General Chebyshev type inequalities for universal integral. Inf Sci 187:171–178
Agahi H, Mohammadpour A, Vaezpour SM (2012b) A generalization of the Chebyshev type inequalities for Sugeno integrals. Soft Comput 16:659–666
Agahi H, Eslami E (2011) A general inequality of Chebyshev type for semi(co)normed fuzzy integrals. Soft Comput 15:771–780
Caballero J, Sadarangani K (2009) Hermite–Hadamard inequality for fuzzy integrals. Appl Math Comput 215:2134–2138
Caballero J, Sadarangani K (2010c) Fritz Carlson's inequality for fuzzy integrals. Comput Math Appl 59:2763–2767
Caballero J, Sadarangani K (2010a) A Cauchy–Schwarz type inequality for fuzzy integrals. Nonlinear Anal 73:3329–3335
Caballero J, Sadarangani K (2010b) Chebyshev inequality for Sugeno integrals. Fuzzy Sets Syst 161:1480–1487
Caballero J, Sadarangani K (2011) Sandor's inequality for Sugeno integrals. Appl Math Comput 218:1617–1622
Chen S, Hu Y, Mahadevan S, Deng Y (2014) A visibility graph averaging aggregation operator. Physica A: Stat Mech Appl 403:1–12
Durante F, Sempi C (2005) Semicopulæ. Kybernetika 41:315–328
Flores-Franulič A, Román-Flores H (2007) A Chebyshev type inequality for fuzzy integrals. Appl Math Comput 190:1178–1184
García FS, Álvarez PG (1986) Two families of fuzzy integrals. Fuzzy Sets Syst 18:67–81
Gill PM, Pearce CEM, Pečarić J (1997) Hadamard's inequality for r-convex functions. J Math Anal Appl 215:461–470

Grabisch M, Marichal JL, Mesiar R, Pap E (2009) Aggregation functions. Cambridge Univ. Press, Cambridge

Kaluszka M, Okolewski A, Boczek M (2014) On Chebyshev type inequalities for generalized Sugeno integrals. Fuzzy Sets Syst 244:51–62

Klement EP, Mesiar R, Pap E (2000) Triangular norms. Trends in logic, Studia Logica Library, vol 8. Kluwer Academic Publishers, Dordrecht

Klement EP, Mesiar R, Pap E (2004) Triangular norms. Position paper I: basic analytical and algebraic properties. Fuzzy Sets Syst 143:5–26

Klement EP, Mesiar R, Pap E (2010) A universal integral as common frame for Choquet and Sugeno integral. Fuzzy Syst IEEE Trans 18:178–187

Klir GJ, Folger TA (1988) Prentice-Hall, Englewood Cliffs, New Jersey

Pap E (1995) Null-additive set functions. Kluwer, Dordrecht

Ralescu D, Adams G (1980) The fuzzy integral. J Math Anal Appl 75:562–570

Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2007a) A Jensen type inequality for fuzzy integrals. Inf Sci 177:3192–3201

Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2007b) The fuzzy integral for monotone functions. Appl Math Comput 185:492–498

Román-Flores H, Chalco-Cano Y (2006) H-continuity of fuzzy measures and set defuzzifincation. Fuzzy Sets Syst 157:230–242

Román-Flores H, Chalco-Cano Y (2007) Sugeno integral and geometric inequalities. Int J Uncertain Fuzz Knowl-Based Syst 15:1–11

Sugeno M (1974) Theory of fuzzy integrals and its applications. Ph.d. dissertation, Tokyo Institute of Technology

Wang Z, Klir G (1992) Fuzzy measure theory. Plenum Press, New York

Wierzchon ST (1982) On fuzzy measure and fuzzy integral. Fuzzy information and decision processes. North-Holland, New York, pp 79–86

Wu L, Sun J, Ye X, Zhu L (2010) Hölder type inequality for Sugeno integral. Fuzzy Sets Syst 161:2337–2347

Zhao R (1981) Fuzzy integral. J Math Res Expo 2:55–72