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FOUNDATIONS

On pseudo-fractional integral inequalities related to Hermite–Hadamard type

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Abstract General versions of Hermite–Hadamard type inequality for pseudo-fractional integrals of the order $\alpha > 0$ on a semiring $([a, b], \oplus, \odot)$ are studied. These inequalities include both pseudo-integral and fractional integral. The well-known previous results are shown to be special cases of our results. Finally, two open problems for further investigations are given.

Keywords Pseudo-addition · Pseudo-multiplication · Semiring · Pseudo-fractional integrals · Hermite–Hadamard's inequality

1 Introduction

The study on pseudo-analysis, as a generalization of the classical analysis, is an interesting subject for many researchers in different fields, such as functional equations, variational calculus, probability and measure theory, functional analysis, optimization theory, semiring theory, etc. (Pap 1990, 2005, 1993; Pap and Ralević 1998; Pap and Štajner 1999; Pap 2002; Pap and Štrboja 2010). Notice that pseudo-integrals were started to attract mathematicians' attentions in several applications, for example in the area of nonlinear partial differential equations occurring in different applied fields, see Pap (2005) as well as the edited volume (Maslov and Samborskij 1992).

Inequalities play a central and fundamental role in the fields of probability and measure theory, classical analysis, optimization theory, mathematical finance and economics. One of the most well-known inequalities for the class of convex functions is the Hermite–Hadamard inequality. This inequality was first published by Hermite in 1883 in an elementary journal and independently proved in 1893 by Hadamard Hadamard (1983). In the classical analysis, many researchers gave the refinements and generalizations to add a substantial contribution in the literature. For example, in 2010, Farissi (2010) provided a refinement of Hermite–Hadamard inequality. Recently, several papers have treated the extension of Hermite–Hadamard inequality by means of the theory of fractional calculus (Sarikaya et al. 2013). In 2013, in connection with the well-known Riemann–Liouville fractional integral operator (Bardaro; Kilbas et al. 2006; Samko et al. 1993), the Hermite–Hadamard type inequality for fractional integral was considered by Sarikaya et al. in (2013).

In pseudo-analysis, there are some known results concerning with the pseudo-integral inequalities (Agahi 2010; Boccuto et al. 2011; Pap and Štrboja 2010). For example, Chebyshev's inequality for pseudo-integral was provided in Agahi (2010). In 2015, Agahi et al. (2015) generalized the previous results of Agahi (2010) to the case of pseudo-fractional integrals of the order $\alpha > 0$ on a semiring $([a, b], \oplus, \odot)$.

The main motivation of this paper is to obtain a general version of the Hermite–Hadamard inequality for pseudo-fractional integrals of the order $\alpha > 0$ on a semi-

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Table 1 Some different types of Hermite–Hadamard inequality

Types	Name	Conditions
Main result Theorem 3.1	Pseudo-fractional integral inequalities related to Hermite–Hadamard type	$\oplus, \odot, \alpha > 0, \lambda \in [0,1]$
Sarikaya et al. (2013)	Fractional integral inequalities related to Hermite–Hadamard type	$\oplus = +, \odot = \times, \alpha > 0, \lambda = 1.$
Corollary 3.3	Pseudo-integral inequalities related to Hermite–Hadamard type	$\oplus, \odot, \alpha = \lambda = 1.$
El Farissi (2010)	The refinement of Hermite–Hadamard inequality	$\oplus = +, \odot = \times, \alpha = 1,$ $\lambda \in [0,1].$
Niculescu and Persson (2006)	A generalization of the Hermite–Hadamard inequality	$\oplus = +, \odot = \times, \alpha = 1, \lambda = \frac{1}{2}.$
Dragomir et al. (1995)	A new generalized Hermite–Hadamard inequality	$\oplus = +, \odot = \times, \alpha = 1,$ $\lambda = \frac{a}{a+b}, b > a.$
Śandor (1988)	A generalized Hermite–Hadamard inequality	$\oplus = +, \odot = \times, \alpha = 1,$ $\lambda = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}, b > a.$
Hadamard (1983)	The classical Hermite–Hadamard inequality	$\oplus = +, \odot = \times, \lambda = \alpha = 1.$

ring $([a,b], \oplus, \odot)$. This inequality includes both pseudo-integral and fractional integral as special cases, thus generalizing some previous results (see Table 1).

The rest of the paper is organized as follows. Some notions and definitions that are useful in this paper are given in Sect. 2. In Sect. 3.4, we state the main results of this paper. Finally, an Appendix is given and some conclusion are added.

2 Preliminaries

In this section, we recall some well-known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see Agahi 2010; Pap 1993; Pap and Štrboja 2010).

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a,b]$ will be denoted by $\preceq$.

Definition 2.1 A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative, and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x \mid x \in [a,b], \mathbf{0} \preceq x\}$.

Definition 2.2 A binary operation $\odot$ on $[a,b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $\mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b], \oplus, \odot)$ is a *semiring* (see Kuich 1986).

Let X be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X.

Definition 2.3 (Pap and Štajner 1999) A set function $m : \mathcal{A} \to [[a,b]_+$ (or semiclosed interval) is a $\oplus$-measure if there holds:

(1) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);
(2) m is σ-$\oplus$-(decomposable) measure, i.e.,

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $\{A_i\}_{i \in N}$ of pairwise disjoint sets from $\mathcal{A}$. If $\oplus$ is idempotent operation condition (1) can be left out and sets from sequence $\mathcal{A}_i$ do not have to be pairwise disjointed.

Definition 2.4 (Pap 1993; Pap and Ralević 1998) The first class of pseudo-integrals is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and}$$
$$x \odot y = g^{-1}(g(x)\, g(y)).$$

Then the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the g-integral,

$$\int_{[c,d]}^{\oplus} f \odot \mathrm{d}m = g^{-1}\left(\int_c^d g(f(x))\,\mathrm{d}x\right).$$

Since the generator g is an increasing function, then f is said to be integrable if $\int_{[c,d]}^{\oplus} f \odot \mathrm{d}m < \infty$.

Definition 2.5 (Mesiar and Pap 1999) The second class of pseudo-integrals is when $x \oplus y = \sup(x, y)$ and $x \odot y = g^{-1}(g(x)\, g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f(x) \odot \mathrm{d}m = \sup_{x\in\mathbb{R}} (f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m.

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned} m(A) &= ess \sup_{\mu} (x \mid x \in A) \\ &= \sup\{a \mid \mu(\{x \mid x \in A, x > a\}) > 0\}. \end{aligned}$$

Theorem 2.6 (Mesiar and Pap 1999) *Let m be a sup-measure on $([0, \infty], \mathcal{B}([0, \infty]))$, where $\mathcal{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = ess \sup_{\mu} (\psi(x) \mid x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density function. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\gamma\}$ of $\oplus_\gamma$ -measure on $([0, \infty), \mathcal{B})$, where $\oplus_\gamma$ is generated by g^γ (the function g of the power γ), $\gamma \in (0, \infty)$, such that $\lim_{\gamma\to\infty} m_\gamma = m$.*

Theorem 2.7 (Mesiar and Pap 1999) *Let $([0, \infty], \sup, \odot)$ be a semiring with $\odot$ with a generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a, b]$. Let m be the same as in Theorem 2.6. Then there exists a family $\{m_\gamma\}$ of $\oplus_\gamma$-measures, where $\oplus_\gamma$ is generated by g^γ, $\gamma \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$*

$$\begin{aligned} \int^{\sup} f \odot \mathrm{d}m &= \lim_{\gamma\to\infty} \int^{\oplus_\gamma} f \odot \mathrm{d}m_\gamma \\ &= \lim_{\gamma\to\infty} \left(g^\gamma\right)^{-1} \left(\int g^\gamma (f(x))\, \mathrm{d}x\right). \end{aligned}$$

3 Main results

Now, we state and prove the main results of this paper.

Theorem 3.1 *Let $b > a \geq 0$. Let $f : [a, b] \to [a, b]$ be a measurable convex function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and increasing function. Then the following inequalities*

$$\begin{aligned} &\varphi_{\oplus,\odot}^{(\alpha,\lambda)}(a, b) \\ &\quad\leqslant \Big[\Big(\mathbb{J}_{\oplus,\odot,a+}^{\alpha} f(\lambda b + (1-\lambda) a) \oplus \mathbb{J}_{\oplus,\odot,(\lambda b+(1-\lambda)a)^-}^{\alpha} f(a)\Big) \\ &\quad\oplus \Big(\mathbb{J}_{\oplus,\odot,(\lambda b+(1-\lambda)a)^+}^{\alpha} f(b) \oplus \mathbb{J}_{\oplus,\odot,b^-}^{\alpha} f(\lambda b + (1-\lambda) a)\Big)\Big] \\ &\quad\leqslant \Phi_{\oplus,\odot}^{(\alpha,\lambda)}(a, b), \end{aligned}$$

hold for all $\lambda \in [0, 1]$, where

$$\begin{aligned} &\varphi_{\oplus,\odot}^{(\alpha,\lambda)}(a, b) \\ &:= \left(\left[g^{-1}\left(\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot f\left(\frac{\lambda b + (2-\lambda) a}{2}\right)\right]\right. \\ &\left.\oplus \left[g^{-1}\left(\frac{2(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot f\left(\frac{(1+\lambda) b + (1-\lambda) a}{2}\right)\right]\right), \\ &\Phi_{\oplus,\odot}^{(\alpha,\lambda)}(a, b) \\ &:= \left[\left(g^{-1}\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot (f(a) \oplus f(\lambda b + (1-\lambda) a))\right)\right. \\ &\left.\oplus \left(g^{-1}\left(\frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot (f(\lambda b + (1-\lambda) a) \oplus f(b))\right)\right], \end{aligned}$$

and the symbol $\mathbb{J}_{\oplus,\odot}^{\alpha} f$ denotes pseudo-fractional integral of the order $\alpha > 0$ that is defined by

$$\begin{aligned} \mathbb{J}_{\oplus,\odot,a+}^{\alpha} f(t) = \int_{[a,t]}^{\oplus} &\left(g^{-1}\left((\Gamma(\alpha))^{-1} (t-x)^{\alpha-1}\right) \odot f(x)\right) \\ &\times \odot \mathrm{d}m, \quad t > a, \end{aligned}$$

and

$$\begin{aligned} \mathbb{J}_{\oplus,\odot,b-}^{\alpha} f(t) = \int_{[t,b]}^{\oplus} &\left(g^{-1}\left((\Gamma(\alpha))^{-1} (x-t)^{\alpha-1}\right) \odot f(x)\right) \\ &\times \odot \mathrm{d}m, \quad t < b. \end{aligned}$$

Here, $\Gamma(\alpha)$ is the gamma function.

Proof If f is a convex and g is a convex and increasing function, then the composition $g \circ f$ is also a convex function. By "Appendix" [Part I: Eq. (4.2)], we have

$$\begin{aligned} &\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)} g \circ f\left(\frac{\lambda b + (2-\lambda) a}{2}\right) \\ &\leqslant \left[\begin{array}{l} \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)} ((\lambda b + (1-\lambda) a) - u)^{\alpha-1} g \circ f(u)\, \mathrm{d}u \\ + \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)} (v-a)^{\alpha-1} g \circ f(v)\, \mathrm{d}v \end{array}\right]. \end{aligned}$$

Since function g is an increasing function, then g^{-1} is also an increasing function and we have

$$\begin{aligned} &g^{-1}\left(\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)} g \circ f\left(\frac{\lambda b + (2-\lambda) a}{2}\right)\right) \\ &\leqslant g^{-1}\left(\begin{array}{l} \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)} ((\lambda b + (1-\lambda) a) - u)^{\alpha-1} g \circ f(u)\, \mathrm{d}u \\ + \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)} (v-a)^{\alpha-1} g \circ f(v)\, \mathrm{d}v \end{array}\right). \end{aligned} \tag{3.1}$$

For the right side of Eq. (3.1), we have

$$
\begin{aligned}
& g^{-1}\left(\begin{array}{l}\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1} g\circ f(u)\,du \\ +\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} g\circ f(v)\,dv\end{array}\right) \\
& = g^{-1}\left[\begin{array}{l} g\left(g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1} g\circ f(u)\,du\right)\right) \\ +g\left(g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} g\circ f(v)\,dv\right)\right)\end{array}\right] \\
& = \left[\begin{array}{l} g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1} g\circ f(u)\,du\right) \\ \oplus g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} g\circ f(v)\,dv\right)\end{array}\right] \\
& = \left[\begin{array}{l} g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} g\circ g^{-1}\left(\frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1}\right)\cdot g\circ f(u)\,du\right) \\ \oplus g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} g\circ g^{-1}\left(\frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1}\right)\cdot g\circ f(v)\,dv\right)\end{array}\right] \\
& = \left[\begin{array}{l} g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} g\circ\left(g^{-1}\left(\frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1}\right)\odot f(u)\right)du\right) \\ \oplus g^{-1}\left(\int_a^{\lambda b+(1-\lambda)a} g\circ\left(g^{-1}\left(\frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1}\right)\odot f(v)\right)dv\right)\end{array}\right] \\
& = \left[\begin{array}{l} \left(\int_{[a,\lambda b+(1-\lambda)a]}^{\oplus} g^{-1}\left(\frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1}\right)\odot f(u)\odot dm\right) \\ \oplus\left(\int_{[a,\lambda b+(1-\lambda)a]}^{\oplus} g^{-1}\left(\frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1}\right)\odot f(v)\odot dm\right)\end{array}\right] \\
& = \mathbb{J}^{\alpha}_{\oplus,\odot,a+} f(\lambda b+(1-\lambda)a)\oplus \mathbb{J}^{\alpha}_{\oplus,\odot,(\lambda b+(1-\lambda)a)-} f(a). \qquad (3.2)
\end{aligned}
$$

So, (3.1) and (3.2) imply that

$$
\begin{aligned}
& g^{-1}\left(\frac{2\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\odot f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) \\
& = g^{-1}\left(\frac{2\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)} g\circ f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\right) \\
& \leqslant \mathbb{J}^{\alpha}_{\oplus,\odot,a+} f(\lambda b+(1-\lambda)a)\oplus \mathbb{J}^{\alpha}_{\oplus,\odot,(\lambda b+(1-\lambda)a)-} f(a). \qquad (3.3)
\end{aligned}
$$

Similarly, by "Appendix" (Part II: Eq. (4.3), we have

$$
\begin{aligned}
& \frac{2(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)} g\circ f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \\
& \leqslant \left[\begin{array}{l}\int_{\lambda b+(1-\lambda)a}^{b} \frac{1}{\Gamma(\alpha)}(b-u)^{\alpha-1} g\circ f(u)\,du \\ +\int_{\lambda b+(1-\lambda)a}^{b} \frac{1}{\Gamma(\alpha)}(v-(\lambda b+(1-\lambda)a))^{\alpha-1} g\circ f(v)\,dv\end{array}\right]. \qquad (3.4)
\end{aligned}
$$

Then by (3.4), we can prove that

$$
\begin{aligned}
& g^{-1}\left(\frac{2(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\odot f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \\
& \leqslant \mathbb{J}^{\alpha}_{\oplus,\odot,(\lambda b+(1-\lambda)a)+} f(b)\oplus \mathbb{J}^{\alpha}_{\oplus,\odot,b-} f(\lambda b+(1-\lambda)a). \qquad (3.5)
\end{aligned}
$$

Now, (3.3) and (3.5) imply that

$$
\begin{aligned}
& \varphi^{(\alpha,\lambda)}_{\oplus,\odot}(a,b) \\
& := \left[\left(g^{-1}\left(\frac{2\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\odot f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\right)\right. \\
& \left.\oplus\left(g^{-1}\left(\frac{2(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\odot f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right)\right] \\
& \leqslant \left[\left(\mathbb{J}^{\alpha}_{\oplus,\odot,a+} f(\lambda b+(1-\lambda)a)\oplus \mathbb{J}^{\alpha}_{\oplus,\odot,(\lambda b+(1-\lambda)a)-} f(a)\right)\right. \\
& \left.\oplus\left(\mathbb{J}^{\alpha}_{\oplus,\odot,(\lambda b+(1-\lambda)a)+} f(b)\oplus \mathbb{J}^{\alpha}_{\oplus,\odot,b-} f(\lambda b+(1-\lambda)a)\right)\right].
\end{aligned}
$$

Then the first inequality is proved. For the proof of the second inequality, since $g\circ f$ is convex then by "Appendix" (Part III: Eq. 4.4), we have

$$
\begin{aligned}
& \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(\lambda b+(1-\lambda)a-u)^{\alpha-1} g\circ f(u)\,du \\
& \quad+\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} g\circ f(v)\,dv \\
& \leqslant \frac{\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}[g\circ f(a)+g\circ f(\lambda b+(1-\lambda)a)].
\end{aligned}
$$

Since g is an increasing function, then g^{-1} is also an increasing function and we have

$$g^{-1}\begin{pmatrix} \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(\lambda b+(1-\lambda)a-u)^{\alpha-1} g\circ f(u)\,du \\ +\int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} g\circ f(v)\,dv \end{pmatrix} \leqslant g^{-1}\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}[g\circ f(a)+g\circ f(\lambda b+(1-\lambda)a)]\right). \tag{3.6}$$

Then by (3.6), we have

$$\mathbb{J}^\alpha_{\oplus,\odot,a^+} f(\lambda b+(1-\lambda)a) \oplus \mathbb{J}^\alpha_{\oplus,\odot,\lambda b+(1-\lambda)a^-} f(a) \leqslant g^{-1}\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right)\odot (f(a)\oplus f(\lambda b+(1-\lambda)a)). \tag{3.7}$$

Similarly, by "Appendix" (Part IV: Eq. 4.5), we have

$$\begin{bmatrix} \int_{\lambda b+(1-\lambda)a}^{b} \frac{1}{\Gamma(\alpha)}(b-u)^{\alpha-1} g\circ f(u)\,du \\ +\int_{\lambda b+(1-\lambda)a}^{b} \frac{1}{\Gamma(\alpha)}(v-(\lambda b+(1-\lambda)a))^{\alpha-1} g\circ f(v)\,dv \end{bmatrix} \leqslant \frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}[g\circ f(\lambda b+(1-\lambda)a)+g\circ f(b)]. \tag{3.8}$$

Then by (3.8), we can prove that

$$\mathbb{J}^\alpha_{\oplus,\odot,(\lambda b+(1-\lambda)a)^+} f(b) \oplus \mathbb{J}^\alpha_{\oplus,\odot,b^-} f(\lambda b+(1-\lambda)a) \leqslant g^{-1}\left(\frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot [f(\lambda b+(1-\lambda)a)\oplus f(b)]. \tag{3.9}$$

Now, (3.7) and (3.9) imply that

$$\begin{aligned} &\left[\left(\mathbb{J}^\alpha_{\oplus,\odot,a^+} f(\lambda b+(1-\lambda)a)\oplus \mathbb{J}^\alpha_{\oplus,\odot,\lambda b+(1-\lambda)a^-} f(a)\right)\right.\\ &\left.\oplus\left(\mathbb{J}^\alpha_{\oplus,\odot,(\lambda b+(1-\lambda)a)^+} f(b)\oplus \mathbb{J}^\alpha_{\oplus,\odot,b^-} f(\lambda b+(1-\lambda)a)\right)\right]\\ &\leqslant\left[\left(g^{-1}\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right)\odot(f(a)\oplus f(\lambda b+(1-\lambda)a))\right)\right.\\ &\left.\oplus\left(g^{-1}\left(\frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right)\odot(f(\lambda b+(1-\lambda)a)\oplus f(b))\right)\right]\\ &=: \Phi^{(\alpha,\lambda)}_{\oplus,\odot}(a,b) \end{aligned}$$

and the proof is completed. □

Example 3.2 Let $g(x)=x$. The corresponding pseudo-operations are $x\oplus y = x+y$ and $x\odot y = xy$. Then the following inequalities for fractional integrals:

$$\begin{aligned} &\varphi^{(\alpha,\lambda)}_{(+,\times)}(a,b)\\ &\leqslant\left[\mathbb{J}^\alpha_{(+,\times),a+} f(\lambda b+(1-\lambda)a)+\mathbb{J}^\alpha_{(+,\times),(\lambda b+(1-\lambda)a)^-} f(a)\right.\\ &\left.+\mathbb{J}^\alpha_{(+,\times),(\lambda b+(1-\lambda)a)^+} f(b)+\mathbb{J}^\alpha_{(+,\times),b^-} f(\lambda b+(1-\lambda)a)\right]\\ &\leqslant\Phi^{(\alpha,\lambda)}_{(+,\times)}(a,b), \end{aligned}$$

hold for all $\lambda\in[0,1]$, where

$$\begin{aligned} &\varphi^{(\alpha,\lambda)}_{(+,\times)}(a,b)\\ &:=\left(\left[\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)} f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\right]\right.\\ &\left.+\left[\left(\frac{2(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right]\right), \end{aligned}$$

and

$$\begin{aligned} &\Phi^{(\alpha,\lambda)}_{(+,\times)}(a,b)\\ &:=\left[\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\times[f(a)+f(\lambda b+(1-\lambda)a)]\right)\right.\\ &\left.+\left(\frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\times[f(\lambda b+(1-\lambda)a)+f(b)]\right)\right]. \end{aligned}$$

Notice that

- If $\lambda=1$, then we have

$$\begin{aligned} \varphi^{(\alpha,1)}_{(+,\times)}(a,b) &= \frac{2(b-a)^\alpha}{\Gamma(\alpha+1)} f\left(\frac{b+a}{2}\right)\\ &\leqslant\left[\mathbb{J}^\alpha_{(+,\times),a^+} f(b)+\mathbb{J}^\alpha_{(+,\times),b^-} f(a)\right]\\ &\leqslant\frac{2(b-a)^\alpha}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}\\ &=\Phi^{(\alpha,1)}_{(+,\times)}(a,b), \end{aligned}$$

  which is Hermite–Hadamard type inequality for fractional integrals Sarikaya et al. (2013).
- If $\alpha=1$, then we have

$$\varphi^{(1,\lambda)}_{+,\times}(a,b)\leqslant 2\int_a^b f(x)\,dx\leqslant\Phi^{(1,\lambda)}_{+,\times}(a,b),$$

  for all $\lambda\in[0,1]$, where

$$\varphi^{(1,\lambda)}_{+,\times}(a,b) := 2(b-a)\left[\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right],$$

  and

$$\Phi^{(1,\lambda)}_{+,\times}(a,b) = (b-a)[f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)],$$

  which is a refinement of Hermite–Hadamard inequality (El Farissi 2010).
- If $\alpha=\lambda=1$, we have the classical Hermite–Hadamard inequality:

$$f\left(\frac{b+a}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(b)+f(a)}{2}.$$

As special cases of Theorem 3.1, we have the following corollaries.

Corollary 3.3 *Let $b > a \geq 0$. Let $f : [a, b] \to [a, b]$ be a convex function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and increasing function. Then the following inequalities*

$$\begin{aligned}
&\varphi^{(1,\lambda)}_{\oplus,\odot}(a,b)\\
&\leqslant \Big[\Big(\mathbb{J}^1_{\oplus,\odot,a+} f(\lambda b + (1-\lambda)a) \oplus \mathbb{J}^1_{\oplus,\odot,(\lambda b+(1-\lambda)a)^-} f(a)\Big)\\
&\quad \oplus \Big(\mathbb{J}^1_{\oplus,\odot,(\lambda b+(1-\lambda)a)^+} f(b) \oplus \mathbb{J}^1_{\oplus,\odot,b^-} f(\lambda b + (1-\lambda)a)\Big)\Big]\\
&\leqslant \Phi^{(1,\lambda)}_{\oplus,\odot}(a,b),
\end{aligned}$$

hold for all $\lambda \in [0, 1]$, where

$$\begin{aligned}
&\varphi^{(1,\lambda)}_{\oplus,\odot}(a,b)\\
&:= \left(\left[g^{-1}(2\lambda(b-a)) \odot f\left(\frac{\lambda b + (2-\lambda)a}{2}\right)\right]\right.\\
&\quad \left.\oplus \left[g^{-1}(2(1-\lambda)(b-a)) \odot f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)\right]\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi^{(1,\lambda)}_{\oplus,\odot}(a,b)\\
&:= \left[\left(g^{-1}(\lambda(b-a)) \odot (f(a) \oplus f(\lambda b + (1-\lambda)a))\right)\right.\\
&\quad \left.\oplus \left(g^{-1}((1-\lambda)(b-a)) \odot (f(\lambda b + (1-\lambda)a) \oplus f(b))\right)\right],
\end{aligned}$$

and the symbol $\mathbb{J}^1_{\oplus,\odot} f$ denotes pseudo-integrals that are defined by

$$\mathbb{J}^1_{\oplus,\odot,a+} f(t) = \int_{[a,t]}^{\oplus} f \odot \mathrm{d}m, \quad t > a \tag{3.10}$$

and

$$\mathbb{J}^1_{\oplus,\odot,b-} f(t) = \int_{[t,b]}^{\oplus} f \odot \mathrm{d}m, \quad t < b. \tag{3.11}$$

Corollary 3.4 *Let $b > a \geq 0$. Let $f : [a, b] \to [a, b]$ be a measurable convex function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and increasing function. Then the following inequalities*

$$\varphi^{(1,1)}_{\oplus,\odot}(a,b) \leqslant \mathbb{J}^1_{\oplus,\odot,a+} f(b) \oplus \mathbb{J}^1_{\oplus,\odot,b^-} f(a) \leqslant \Phi^{(1,1)}_{\oplus,\odot}(a,b),$$

hold where

$$\varphi^{(1,1)}_{\oplus,\odot}(a,b) := \left[g^{-1}(2(b-a)) \odot f\left(\frac{b+a}{2}\right)\right],$$

$$\Phi^{(1,1)}_{\oplus,\odot}(a,b) := \left(g^{-1}((b-a)) \odot (f(a) \oplus f(b))\right)$$

and the symbol $\mathbb{J}^1_{\oplus,\odot,a+} f$ and $\mathbb{J}^1_{\oplus,\odot,b^-} f$ are defined in (3.10) and (3.11), respectively.

Now we consider the second case, when $\oplus = \sup$ and $\odot = g^{-1}(g(x)g(y))$.

Theorem 3.5 *Let $b > a \geq 0$. Let $f : [a, b] \to [a, b]$ be a continuous convex function and the pseudo-multiplication $\odot$ is represented by a convex and increasing multiplicative generator g and m be the same as in Theorem 2.6. Then the following inequalities*

$$\begin{aligned}
&\varphi^{(\alpha,\lambda)}_{\sup,\odot}(a,b)\\
&\leqslant \sup\Big[\sup\Big(\mathbb{J}^\alpha_{\sup,\odot,a+} f(\lambda b + (1-\lambda)a), \mathbb{J}^\alpha_{\sup,\odot,(\lambda b+(1-\lambda)a)^-} f(a)\Big),\\
&\quad \sup\Big(\mathbb{J}^\alpha_{\sup,\odot,(\lambda b+(1-\lambda)a)^+} f(b), \mathbb{J}^\alpha_{\sup,\odot,b^-} f(\lambda b + (1-\lambda)a)\Big)\Big]\\
&\leqslant \Phi^{(\alpha,\lambda)}_{\sup,\odot}(a,b),
\end{aligned}$$

hold for all $\lambda \in [0, 1]$ and $\alpha > 0$, where

$$\begin{aligned}
&\varphi^{(\alpha,\lambda)}_{\sup,\odot}(a,b)\\
&:= \sup\left(\left[g^{-1}\left(\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot f\left(\frac{\lambda b + (2-\lambda)a}{2}\right)\right],\right.\\
&\quad \left.\left[g^{-1}\left(\frac{2(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)\right]\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi^{(\alpha,\lambda)}_{\sup,\odot}(a,b)\\
&:= \sup\left(\left[g^{-1}\left(\frac{\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot \sup(f(a), f(\lambda b + (1-\lambda)a))\right],\right.\\
&\quad \left.\left[g^{-1}\left(\frac{(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)}\right) \odot \sup(f(\lambda b + (1-\lambda)a), f(b))\right]\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{J}^\alpha_{\sup,\odot,a+} f(t)\\
&= \int_{[a,t]}^{\sup} \left(g^{-1}\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right) \odot f(x)\right)\\
&\odot \mathrm{d}m, \quad t > a,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{J}^\alpha_{\sup,\odot,b-} f(t)\\
&= \int_{[t,b]}^{\sup} \left(g^{-1}\left((\Gamma(\alpha))^{-1}(x-t)^{\alpha-1}\right) \odot f(x)\right)\\
&\odot \mathrm{d}m, \quad t < b.
\end{aligned}$$

Proof Since $f : [a, b] \to [a, b]$ is a continuous convex function, then proof is obtained immediately from Theorem 3.1, Theorem 2.7 and Proposition 1 in Mesiar and Pap (1999). □

Example 3.6 Let $g^\gamma(x) = e^{\gamma x}$ and $\psi(x)$ be from Theorem 2.6. Then

$$x \odot_\gamma y = x + y$$

and

$$\lim_{\gamma\to\infty}\left(\frac{1}{\gamma}\ln\left(e^{\gamma x}+e^{\gamma y}\right)\right)=\max(x,y).$$

Then the following inequalities

$$\begin{aligned}&\varphi_{\sup,+}^{(\alpha,\lambda)}(a,b)\\&\leqslant\sup\Big[\sup\left(\mathbb{J}^{\alpha}_{\sup,+,a+}f(\lambda b+(1-\lambda)a),\mathbb{J}^{\alpha}_{\sup,+,(\lambda b+(1-\lambda)a)-}f(a)\right),\\&\sup\left(\mathbb{J}^{\alpha}_{\sup,+,(\lambda b+(1-\lambda)a)+}f(b),\mathbb{J}^{\alpha}_{\sup,+,b-}f(\lambda b+(1-\lambda)a)\right)\Big]\\&\leqslant\Phi_{\sup,+}^{(\alpha,\lambda)}(a,b),\end{aligned}$$

hold for all $\lambda\in[0,1]$ and $\alpha>0$, where

$$\begin{aligned}&\varphi_{\sup,\odot_\gamma}^{(\alpha,\lambda)}(a,b)=\varphi_{\sup,+}^{(\alpha,\lambda)}(a,b)\\&:=\sup\left(\left[\frac{1}{\gamma}\ln\left(\frac{2\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)+f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\right],\right.\\&\left.\left[\frac{1}{\gamma}\ln\left(\frac{2(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)+f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right]\right),\\&\Phi_{\sup,\odot_\gamma}^{(\alpha,\lambda)}(a,b)=\Phi_{\sup,+}^{(\alpha,\lambda)}(a,b)\\&:=\sup\left(\left[\frac{1}{\gamma}\ln\left(\frac{\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)+\sup(f(a),f(\lambda b+(1-\lambda)a))\right],\right.\\&\left.\left[\frac{1}{\gamma}\ln\left(\frac{(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)+\sup(f(\lambda b+(1-\lambda)a),f(b))\right]\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{J}^{\alpha}_{\sup,\odot_\gamma,a+}f(t)\\&=\sup\left(\frac{1}{\gamma}\ln\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right)+f(x)+\psi(x)\right),\\&\quad t>a,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{J}^{\alpha}_{\sup,\odot_\gamma,b-}f(t)\\&=\sup\left(\frac{1}{\gamma}\ln\left((\Gamma(\alpha))^{-1}(x-t)^{\alpha-1}\right)+f(x)+\psi(x)\right),\\&\quad t<b.\end{aligned}$$

4 Conclusions

We have investigated the Hermite–Hadamard inequality for pseudo-fractional integrals of the order $\alpha>0$ on a semiring $([a,b],\oplus,\odot)$. This inequality includes both pseudo-integral and fractional integral as special cases. For the common addition and common multiplication $(+,\cdot)$, the well-known previous results (El Farissi 2010; Sarikaya et al. 2013) are shown to be special cases of our results. As we have seen, for $\alpha=1$, this inequality is related to Hermite–Hadamard type for pseudo-integral. Notice that the third important case $\oplus=\sup$ and $\odot=\min$ has been studied in Caballero and Sadarangani (2009) and Li et al. (2014), where the pseudo-integral in such a case yield the Sugeno integral (1974) when the considered measure is maxitive.

Lastly, we propose the following open problems for future work.

Open problem 1: Recently, a generalization of convex functions was introduced by Varošanec (2007) which is called the h-convex functions. Notice that the class of h-convex functions generalizes the class of convex functions, s-Breckner convex functions (1978), Godunova–Levin functions (1985), P-functions (Dragomir et al. 1995). Let f be a measurable h-convex function in Theorem 3.1. Under what conditions does the Hermite–Hadamard inequality hold for pseudo-fractional integrals of the order $\alpha>0$ on a semiring $([a,b],\oplus,\odot)$?

Open problem 2: Does the pseudo-fractional Hermite–Hadamard inequality in Theorem 3.1 hold with weaker conditions than convexity?

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Since f is a convex function on $[a,b]$, we have for $x,y\in[a,b]$

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2}. \tag{4.1}$$

Let $t\in(0,1)$.

Part I: For $x=ta+(1-t)(\lambda b+(1-\lambda)a)$, $y=(1-t)a+t(\lambda b+(1-\lambda)a)$, Eq. (4.1) implies that

$$\begin{aligned}2f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)&\leqslant f(ta+(1-t)(\lambda b+(1-\lambda)a))\\&\quad+f((1-t)a+t(\lambda b+(1-\lambda)a)).\end{aligned}$$

For $\lambda\neq0$, multiplying both sides by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{2}{\alpha}f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\\&\leqslant\int_0^1 t^{\alpha-1}f(ta+(1-t)(\lambda b+(1-\lambda)a))\,\mathrm{d}t\\&\quad+\int_0^1 t^{\alpha-1}f((1-t)a+t(\lambda b+(1-\lambda)a))\,\mathrm{d}t\\&=\int_{\lambda b+(1-\lambda)a}^{a}\left(\frac{(\lambda b+(1-\lambda)a)-u}{\lambda(b-a)}\right)^{\alpha-1}f(u)\frac{\mathrm{d}u}{\lambda(a-b)}\\&\quad+\int_a^{\lambda b+(1-\lambda)a}\left(\frac{v-a}{\lambda(b-a)}\right)^{\alpha-1}f(v)\frac{\mathrm{d}v}{\lambda(b-a)}\end{aligned}$$

$$= \int_a^{\lambda b+(1-\lambda)a} \left(\frac{(\lambda b+(1-\lambda)a)-u}{\lambda(b-a)}\right)^{\alpha-1} f(u)\frac{du}{\lambda(b-a)} + \int_a^{\lambda b+(1-\lambda)a}\left(\frac{v-a}{\lambda(b-a)}\right)^{\alpha-1} f(v)\frac{dv}{\lambda(b-a)}.$$

So,

$$\frac{2\lambda^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)} f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) \leqslant \int_a^{\lambda b+(1-\lambda)a} \frac{1}{\Gamma(\alpha)}((\lambda b+(1-\lambda)a)-u)^{\alpha-1} f(u)\,du + \int_a^{\lambda b+(1-\lambda)a}\frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} f(v)\,dv. \tag{4.2}$$

Part II: For $x = t(\lambda b+(1-\lambda)a)+(1-t)b$, $y = (1-t)(\lambda b+(1-\lambda)a)+tb$ and Eq. (4.1), we have

$$2f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \leqslant f(t(\lambda b+(1-\lambda)a)+(1-t)b) + f((1-t)(\lambda b+(1-\lambda)a)+tb)$$

For $\lambda \neq 1$, multiplying both sides by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we have

$$\begin{aligned}\frac{2}{\alpha} f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) &\leqslant \int_0^1 t^{\alpha-1} f(t(\lambda b+(1-\lambda)a)+(1-t)b)\,dt + \int_0^1 t^{\alpha-1} f((1-t)(\lambda b+(1-\lambda)a)+tb)\,dt\\ &= \int_b^{\lambda b+(1-\lambda)a}\left(\frac{b-u}{(b-a)(1-\lambda)}\right)^{\alpha-1} f(u)\frac{du}{(a-b)(1-\lambda)} + \int_{\lambda b+(1-\lambda)a}^b \left(\frac{v-(\lambda b+(1-\lambda)a)}{(b-a)(1-\lambda)}\right)^{\alpha-1} f(v)\frac{dv}{(b-a)(1-\lambda)}\\ &= \frac{\Gamma(\alpha)}{(1-\lambda)^\alpha(b-a)^\alpha}\int_{\lambda b+(1-\lambda)a}^b \frac{1}{\Gamma(\alpha)}(b-u)^{\alpha-1} f(u)\,du + \frac{\Gamma(\alpha)}{(1-\lambda)^\alpha(b-a)^\alpha}\int_{\lambda b+(1-\lambda)a}^b \times \frac{1}{\Gamma(\alpha)}(v-(\lambda b+(1-\lambda)a))^{\alpha-1} f(v)\,dv.\end{aligned}$$

Then

$$\frac{2(1-\lambda)^\alpha (b-a)^\alpha}{\Gamma(\alpha+1)} f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \leqslant \left[\begin{array}{l}\int_{\lambda b+(1-\lambda)a}^b \frac{1}{\Gamma(\alpha)}(b-u)^{\alpha-1} f(u)\,du\\ +\int_{\lambda b+(1-\lambda)a}^b \frac{1}{\Gamma(\alpha)}(v-(\lambda b+(1-\lambda)a))^{\alpha-1} f(v)\,dv\end{array}\right]. \tag{4.3}$$

Part III: Since f is a convex, we have

$$f(ta+(1-t)(\lambda b+(1-\lambda)a)) \leqslant tf(a)+(1-t)f(\lambda b+(1-\lambda)a)$$

and

$$f((1-t)a+t(\lambda b+(1-\lambda)a)) \leqslant (1-t)f(a)+tf(\lambda b+(1-\lambda)a).$$

By adding these inequalities, we have

$$\begin{aligned}&f(ta+(1-t)(\lambda b+(1-\lambda)a)) + f((1-t)a+t(\lambda b+(1-\lambda)a))\\ &\leqslant tf(a)+(1-t)f(\lambda b+(1-\lambda)a) + (1-t)f(a)+tf(\lambda b+(1-\lambda)a).\end{aligned}$$

Then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1} f(ta+(1-t)(\lambda b+(1-\lambda)a))\,dt + \int_0^1 t^{\alpha-1} f((1-t)a+t(\lambda b+(1-\lambda)a))\,dt\\ &\leqslant [f(a)+f(\lambda b+(1-\lambda)a)]\int_0^1 t^{\alpha-1}dt.\end{aligned}$$

So,

$$\int_a^{\lambda b+(1-\lambda)a}\frac{1}{\Gamma(\alpha)}(\lambda b+(1-\lambda)a-u)^{\alpha-1} f(u)\,du + \int_a^{\lambda b+(1-\lambda)a}\frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} f(v)\,dv \leqslant \frac{\lambda^\alpha(b-a)^\alpha}{\Gamma(\alpha+1)}(f(a)+f(\lambda b+(1-\lambda)a)). \tag{4.4}$$

Part IV: The convexity of f implies that

$$f(t(\lambda b+(1-\lambda)a)+(1-t)b) \leqslant tf(\lambda b+(1-\lambda)a)+(1-t)f(b)$$

and

$$f((1-t)(\lambda b+(1-\lambda)a)+tb) \leqslant (1-t)f(\lambda b+(1-\lambda)a)+tf(b).$$

By adding these inequalities, we have

$$\begin{aligned}&f(t(\lambda b+(1-\lambda)a)+(1-t)b) + f((1-t)(\lambda b+(1-\lambda)a)+tb)\\ &\leqslant tf(\lambda b+(1-\lambda)a)+(1-t)f(b) + (1-t)f(\lambda b+(1-\lambda)a)+tf(b).\end{aligned}$$

Then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\int_0^1 t^{\alpha-1} f\left(t\left(\lambda b+(1-\lambda) a\right)\right.$$
$$\left.+\ (1-t)\, b\right) \mathrm{d}t+\int_0^1 t^{\alpha-1} f\left((1-t)\left(\lambda b+(1-\lambda) a\right)+t b\right) \mathrm{d}t$$
$$\leqslant \left[f\left(\lambda b+(1-\lambda) a\right)+f(b)\right] \int_0^1 t^{\alpha-1} \mathrm{d}t.$$

Therefore,

$$\begin{bmatrix} \int_{\lambda b+(1-\lambda) a}^{b} \frac{1}{\Gamma(\alpha)}(b-u)^{\alpha-1} f(u)\, \mathrm{d}u \\ +\int_{\lambda b+(1-\lambda) a}^{b} \frac{1}{\Gamma(\alpha)}(v-(\lambda b+(1-\lambda) a))^{\alpha-1} f(v)\, \mathrm{d}v \end{bmatrix}$$
$$\leqslant \frac{(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left(f\left(\lambda b+(1-\lambda) a\right)+f(b)\right). \tag{4.5}$$

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