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FOUNDATIONS

Symmetric triangular approximations of fuzzy numbers under a general condition and properties

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Abstract We consider the set $\mathcal{P}$ of real parameters associated to a fuzzy number, in a general form which includes the most important characteristics already introduced for fuzzy numbers. We find the set $\mathcal{P}_s \subset \mathcal{P}$ with the property that for any given fuzzy number there exists at least a symmetric triangular fuzzy number which preserves a fixed parameter $p \in \mathcal{P}$. We compute the symmetric triangular approximation of a fuzzy number which preserves the parameter $p \in \mathcal{P}_s$. The uniqueness is an immediate consequence; therefore, an approximation operator is obtained. The properties of scale and translation invariance, additivity and continuity of this operator are studied. Some applications related with value and expected value, as important parameters, are given too.

1 Introduction

The calculus, existence and uniqueness of trapezoidal or triangular approximations, as well as some properties (additivity, continuity, scale and translation invariance, etc.) included in a list of criteria which should possess the generated trapezoidal or triangular approximation operators (see Grzegorzewski and Mrówka 2005) were studied by many authors (see Abbasbandy et al. 2010; Allahviranloo and Adabitabar Firozja 2007; Ban 2008, 2011; Ban et al. 2011; Ban and Coroianu 2012, 2014; Chanas 2001; Coroianu 2011, 2012; Grzegorzewski and Mrówka 2005, 2007; Grzegorzewski 2008; Li et al. 2012; Yeh 2007, 2008a, b, 2009; Zeng and Li 2007).

In the present paper we consider parameters in the general form

$$p(A) = a l_e(A) + b u_e(A) + c x_e(A) + d y_e(A), \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and $[l_e(A), u_e(A), x_e(A), y_e(A)]$ is the extended trapezoidal approximation of a fuzzy number A. It is worth noting that this form includes the most important characteristics of fuzzy numbers (expected value, ambiguity, value, width, right and left-hand ambiguity, etc.) as well as the linear combinations of them. We obtain the set $\mathcal{P}_s$ of parameters in the general form (1) such that for every fuzzy number A there exists a symmetric triangular fuzzy number X, with the property $p(A) = p(X)$. We compute the nearest symmetric triangular fuzzy number, which, in addition, preserves the parameter $p \in \mathcal{P}_s$ of a given fuzzy number. The average Euclidean distance between fuzzy numbers is considered. The uniqueness of approximation is an immediate consequence. We prove that the obtained approximation operators are continuous and translation invariant. Some properties of scale invariance and additivity are given too. We apply these results when p is the value (that is $a = b = \frac{1}{2}, c = \frac{1}{12}, d = -\frac{1}{12}$ in (1)) and p is the expected value (that is $a = b = \frac{1}{2}, c = d = 0$ in (1)).

2 Preliminaries

We recall some basic notions and notations used in this paper.

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Definition 1 (see Dubois and Prade 1978) A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with the membership function A which is:

(i) normal (i.e. there exists an element x_0 such that $A(x_0) = 1$);
(ii) fuzzy convex (i.e. $A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))$, for every $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$);
(iii) upper semicontinuous on $\mathbb{R}$ (i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that $A(x) - A(x_0) < \varepsilon$, whenever $|x - x_0| < \delta$);
(iv) $\mathrm{cl}\{x \in \mathbb{R} : A(x) > 0\}$ is compact, where $\mathrm{cl}(M)$ denotes the closure of the set M.

The α-cut, $\alpha \in (0,1]$, of a fuzzy number A is a crisp set defined as

$$A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}.$$

The support $suppA$ and the 0-cut A_0 of a fuzzy number A are defined as

$$suppA = \{x \in \mathbb{R} : A(x) > 0\}$$

and

$$A_0 = \mathrm{cl}\{x \in \mathbb{R} : A(x) > 0\}.$$

Every α-cut, $\alpha \in [0,1]$, of a fuzzy number A is a closed interval

$$A_\alpha = [A_L(\alpha), A_U(\alpha)],$$

where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\}$$
$$A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\},$$

for any $\alpha \in (0,1]$ and $[A_L(0), A_U(0)] = A_0$. We denote by $F(\mathbb{R})$ the set of all fuzzy numbers.

Let $A, B \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We consider the sum $A + B$ and the scalar multiplication $\lambda \cdot A$ by (see e.g. Diamond and Kloeden 1994, p. 40)

$$(A+B)_\alpha = A_\alpha + B_\alpha = [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)] \tag{2}$$

and

$$(\lambda \cdot A)_\alpha = \lambda \cdot A_\alpha = [\lambda A_L(\alpha), \lambda A_U(\alpha)] \tag{3}$$

for $\lambda \geq 0$ and

$$(\lambda \cdot A)_\alpha = \lambda \cdot A_\alpha = [\lambda A_U(\alpha), \lambda A_L(\alpha)] \tag{4}$$

for $\lambda < 0$.

Expected value EV, ambiguity Amb, value Val, width w, left-hand ambiguity Amb_L and right-hand ambiguity Amb_R were introduced in Chanas (2001); Delgado et al. (1998), Dubois and Prade (1987), Grzegorzewski (2008), Heilpern (1992) and subsequently accepted as important characteristics associated with a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$, by

$$\mathrm{EV}(A) = \frac{1}{2}\int_0^1 A_L(\alpha)\mathrm{d}\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)\mathrm{d}\alpha \tag{5}$$

$$\mathrm{Amb}(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))\mathrm{d}\alpha \tag{6}$$

$$\mathrm{Val}(A) = \int_0^1 \alpha(A_U(\alpha) + A_L(\alpha))\mathrm{d}\alpha \tag{7}$$

$$w(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha))\mathrm{d}\alpha \tag{8}$$

$$\mathrm{Amb}_L(A) = \int_0^1 \alpha(\mathrm{EV}(A) - A_L(\alpha))\mathrm{d}\alpha \tag{9}$$

$$\mathrm{Amb}_U(A) = \int_0^1 \alpha(A_U(\alpha) - \mathrm{EV}(A))\mathrm{d}\alpha. \tag{10}$$

The average Euclidean distance between fuzzy numbers is an extension of the Euclidean distance. It is defined by Grzegorzewski (1998) as

$$d^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2\mathrm{d}\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2\mathrm{d}\alpha. \tag{11}$$

The most often used fuzzy numbers are the trapezoidal fuzzy numbers denoted by

$$T = (t_1, t_2, t_3, t_4), \tag{12}$$

where $t_1 \leq t_2 \leq t_3 \leq t_4$, and given by

$$T_L(\alpha) = t_1 + (t_2 - t_1)\alpha$$

and

$$T_U(\alpha) = t_4 - (t_4 - t_3)\alpha,$$

for every $\alpha \in [0,1]$. When $t_2 = t_3$ we obtain a triangular fuzzy number. If, in addition, $t_4 - t_3 = t_2 - t_1$ we obtain a symmetric triangular fuzzy number. We denote by $F^T(\mathbb{R})$, $F^t(\mathbb{R})$

and $F^{s}(\mathbb{R})$ the set of all trapezoidal fuzzy numbers, triangular fuzzy numbers and symmetric triangular fuzzy numbers, respectively.

Sometimes (see Yeh 2008b) another notation for trapezoidal fuzzy numbers is useful, namely

$$T = [l, u, x, y]$$

with $l, u, x, y \in \mathbb{R}$ and

$$x \geq 0 \tag{13}$$

$$y \geq 0 \tag{14}$$

$$2u - 2l \geq x + y. \tag{15}$$

It is immediate that

$$T_{\mathrm{L}}(\alpha) = l + x\left(\alpha - \frac{1}{2}\right) \tag{16}$$

and

$$T_{\mathrm{U}}(\alpha) = u - y\left(\alpha - \frac{1}{2}\right), \tag{17}$$

for every $\alpha \in [0, 1]$. Then

$$l = \frac{t_1 + t_2}{2} \tag{18}$$

$$u = \frac{t_3 + t_4}{2} \tag{19}$$

$$x = t_2 - t_1 \tag{20}$$

$$y = t_4 - t_3 \tag{21}$$

or, equivalently,

$$t_1 = \frac{2l - x}{2} \tag{22}$$

$$t_2 = \frac{2l + x}{2} \tag{23}$$

$$t_3 = \frac{2u - y}{2} \tag{24}$$

$$t_4 = \frac{2u + y}{2}. \tag{25}$$

The distance introduced in (11) between $T = [l, u, x, y]$ and $T' = [l', u', x', y']$ becomes (see Yeh 2008a)

$$d^2(T, T') = (l - l')^2 + (u - u')^2 + \frac{1}{12}(x - x')^2 + \frac{1}{12}(y - y')^2. \tag{26}$$

It is obvious that a trapezoidal fuzzy number $T = [l, u, x, y]$ is triangular if and only if

$$2u - 2l = x + y \tag{27}$$

and symmetric triangular if and only if (27) and

$$x = y \tag{28}$$

are satisfied simultaneously.

The addition and scalar multiplication in $F^{\mathrm{T}}(\mathbb{R})$ become (from (2)–(4))

$$[l, u, x, y] + [l', u', x', y'] = [l + l', u + u', x + x', y + y'] \tag{29}$$

$$\lambda \cdot [l, u, x, y] = [\lambda l, \lambda u, \lambda x, \lambda y] \tag{30}$$

for $\lambda \geq 0$ and

$$\lambda \cdot [l, u, x, y] = [\lambda u, \lambda l, -\lambda y, -\lambda x] \tag{31}$$

for $\lambda < 0$.

The below version of the well-known Karush–Kuhn–Tucker theorem is an important tool in the approximation of fuzzy numbers by trapezoidal or triangular fuzzy numbers (see Ban 2008, 2011; Grzegorzewski and Mrówka 2007).

Theorem 1 (see Rockafeller 1970, pp. 281–283) *Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable functions. Then $\overline{x}$ solves the convex programming problem*

$$\min f(x)$$
$$s.t.\ g_i(x) \leq h_i,\ i \in \{1, \ldots, m\}$$

if and only if there exists ξ_i, $i \in \{1, \ldots, m\}$, such that

(i) $\nabla f(\overline{x}) + \sum_{i=1}^{m} \xi_i \nabla g_i(\overline{x}) = 0$
(ii) $g_i(\overline{x}) - h_i \leq 0$
(iii) $\xi_i \geq 0$
(iv) $\xi_i (h_i - g_i(\overline{x})) = 0.$

3 Extended trapezoidal approximation operator and properties

According with its definition in Yeh (2008a), an extended trapezoidal fuzzy number is an ordered pair of polynomial functions of degree less than or equal to 1. We denote by $F_{\mathrm{e}}^{\mathrm{T}}(\mathbb{R})$ the set of all extended trapezoidal fuzzy numbers. The α-cuts of an extended trapezoidal fuzzy number have the same form as in (16)–(17), but l, u, x, y may fail to satisfy (13)–(15); therefore, $F^{\mathrm{T}}(\mathbb{R}) \subset F_{\mathrm{e}}^{\mathrm{T}}(\mathbb{R})$. The distance between two extended trapezoidal fuzzy numbers is similarly defined as in (11) or (26).

The extended trapezoidal approximation $T_{\mathrm{e}}(A) = [l_{\mathrm{e}}(A), u_{\mathrm{e}}(A), x_{\mathrm{e}}(A), y_{\mathrm{e}}(A)]$ of a fuzzy number A is the extended trapezoidal fuzzy number which minimizes the distance

$d(A,X)$, where $X\in F_e^T(\mathbb{R})$. It is not always a fuzzy number (see Allahviranloo and Adabitabar Firozja 2007) and it is determined by (see Yeh 2008a)

$$l_e(A)=\int_0^1 A_L(\alpha)d\alpha \quad (32)$$

$$u_e(A)=\int_0^1 A_U(\alpha)d\alpha \quad (33)$$

$$x_e(A)=12\int_0^1\left(\alpha-\frac{1}{2}\right)A_L(\alpha)d\alpha \quad (34)$$

$$y_e(A)=-12\int_0^1\left(\alpha-\frac{1}{2}\right)A_U(\alpha)d\alpha. \quad (35)$$

It is known that

$$l_e(A)\le u_e(A), \quad (36)$$

(see Yeh 2008a, Lemma 2.1 or Ban 2008, Lemma 1)

$$x_e(A)\ge 0 \quad (37)$$

$$y_e(A)\ge 0 \quad (38)$$

and (see Ban et al. 2011, Theorem 5)

$$6u_e(A)-6l_e(A)\ge x_e(A)+y_e(A), \quad (39)$$

for every $A\in F(\mathbb{R})$.

In addition, the following distance properties were proved.

Proposition 1 (Yeh 2007, Proposition 4.4.) *Let A be a fuzzy number. Then*

$$d^2(A,B)=d^2(A,T_e(A))+d^2(T_e(A),B),$$

for any trapezoidal fuzzy number B.

Proposition 2 (Yeh 2007, Proposition 4.4.) *For all fuzzy numbers A and B,*

$$d(T_e(A),T_e(B))\le d(A,B).$$

At the end of this section we mention that, from (5)–(10) and (32)–(35) we immediately obtain

$$\mathrm{EV}(A)=\frac{1}{2}l_e(A)+\frac{1}{2}u_e(A) \quad (40)$$

$$\mathrm{Amb}(A)=-\frac{1}{2}l_e(A)+\frac{1}{2}u_e(A)-\frac{1}{12}x_e(A)-\frac{1}{12}y_e(A) \quad (41)$$

$$\mathrm{Val}(A)=\frac{1}{2}l_e(A)+\frac{1}{2}u_e(A)+\frac{1}{12}x_e(A)-\frac{1}{12}y_e(A) \quad (42)$$

$$w(A)=u_e(A)-l_e(A) \quad (43)$$

$$\mathrm{Amb}_L(A)=-\frac{1}{4}l_e(A)+\frac{1}{4}u_e(A)-\frac{1}{12}x_e(A) \quad (44)$$

$$\mathrm{Amb}_U(A)=-\frac{1}{4}l_e(A)+\frac{1}{4}u_e(A)-\frac{1}{12}y_e(A), \quad (45)$$

therefore, all these parameters have the form

$$p(A)=al_e(A)+bu_e(A)+cx_e(A)+dy_e(A),$$

where $a,b,c,d\in\mathbb{R}$.

4 Symmetric triangular approximation under a general condition

Let us denote

$$\mathcal{P}=\{p:F(\mathbb{R})\to\mathbb{R}\,|\,p(A)=al_e(A)+bu_e(A)+cx_e(A)+dy_e(A),\,a,b,c,d\in\mathbb{R}\}.$$

In Ban and Coroianu (2014), Theorem 7, the set

$$\mathcal{P}_T=\{p\in\mathcal{P}\,\big|\,\forall A\in F(\mathbb{R}),\exists X\in F^T(\mathbb{R}) \text{ such that } p(A)=p(X)\}$$

was determined as

$$\mathcal{P}_T=P_1\cup P_2\cup P_3\cup P_4, \quad (46)$$

where

$$P_1=\{p\in\mathcal{P}:a+b\neq 0\} \quad (47)$$

$$P_2=\{p\in\mathcal{P}:a=b=0\} \quad (48)$$

$$P_3=\{p\in\mathcal{P}:a+b=0,a\neq 0 \text{ and } (c/a>1/2 \text{ or } d/a>1/2)\} \quad (49)$$

$$P_4=\{p\in\mathcal{P}:a+b=0,a\neq 0, c/a\le 1/6 \text{ and } d/a\le 1/6\}. \quad (50)$$

Our aim is to find the set

$$\mathcal{P}_s=\left\{p\in\mathcal{P}:\forall A\in F(\mathbb{R}),\exists X\in F^s(\mathbb{R}) \text{ such that } p(A)=p(X)\right\}$$

and, in addition, to find the nearest symmetric triangular approximation of $A\in F(\mathbb{R})$, with respect to d, which preserves $p\in\mathcal{P}_s$. Because any symmetric triangular fuzzy number is a trapezoidal fuzzy number we have $\mathcal{P}_s\subseteq\mathcal{P}_T$. The inclusion is strict as the following example proves.

Example 1 Let us consider

$$p(A) = x_e(A) - y_e(A) = 12\int_0^1 \left(\alpha - \frac{1}{2}\right)(A_L(\alpha) + A_U(\alpha))\,d\alpha.$$

Because $a = b = 0$ we have $p \in P_2 \subset \mathcal{P}_T$. If $A = (0, 1, 1, 1)$, that is $A_L(\alpha) = \alpha$ and $A_U(\alpha) = 1, \alpha \in [0, 1]$, then

$$p(A) = 1.$$

On the other hand,

$$p(X) = (t_2 - t_1) - (t_3 - t_2) = 0,$$

for every symmetric triangular fuzzy number $X = (t_1, t_2, t_3)$, therefore, the equation $p(A) = p(X)$ has no solutions in $F^s(\mathbb{R})$, which means $p \notin \mathcal{P}_s$.

Let us denote

$$\Phi_{a,b,c,d} = \Big\{A \in F(\mathbb{R}) : (a^2+b^2+(c+d)(b-a))\,(l_e(A)-u_e(A)) > \left(2c^2 + 2cd + c(b-a) + \frac{1}{6}(a+b)^2\right) x_e(A) + \left(2d^2 + 2cd + d(b-a) + \frac{1}{6}(a+b)^2\right) y_e(A)\Big\}.$$

The main result of the present paper is the following.

Theorem 2

$$\mathcal{P}_s = Q_1 \cup Q_2 \cup Q_3 \cup Q_4,$$

where

$$Q_1 = \{p \in \mathcal{P} : a + b \neq 0\} \quad (51)$$

$$Q_2 = \{p \in \mathcal{P} : a = b = c = d = 0\} \quad (52)$$

$$Q_3 = \{p \in \mathcal{P} : a = b = 0,\ c^2 + d^2 > 0 \text{ and } cd \geq 0\} \quad (53)$$

$$Q_4 = \{p \in \mathcal{P} : a + b = 0, a \neq 0,\ c/a \leq 1/6 \text{ and } d/a \leq 1/6\}. \quad (54)$$

In addition, for every $p \in \mathcal{P}_s$ there exists a unique symmetric triangular fuzzy number

$$s_p(A) = [l_s(A), u_s(A), x_s(A), y_s(A)] = [l_s, u_s, x_s, y_s],$$

nearest to a given fuzzy number A with respect to metric d, having the extended trapezoidal approximation

$$T_e(A) = [l_e(A), u_e(A), x_e(A), y_e(A)] = [l_e, u_e, x_e, y_e],$$

such that $p(A) = p(s_p(A))$, as follows

(i) *If $p \in Q_1$ and $A \in \Phi_{a,b,c,d}$ then*

$$l_s = \frac{a}{a+b}l_e + \frac{b}{a+b}u_e + \frac{c}{a+b}x_e + \frac{d}{a+b}y_e \quad (55)$$

$$u_s = \frac{a}{a+b}l_e + \frac{b}{a+b}u_e + \frac{c}{a+b}x_e + \frac{d}{a+b}y_e \quad (56)$$

$$x_s = y_s = 0. \quad (57)$$

If $p \in Q_1$ and $A \in F(\mathbb{R}) \setminus \Phi_{a,b,c,d}$ then

$$l_s = \frac{a}{a+b}l_e + \frac{b}{a+b}u_e + \frac{c}{a+b}x_e + \frac{d}{a+b}y_e - \frac{b+c+d}{a+b}x_s, \quad (58)$$

$$u_s = \frac{a}{a+b}l_e + \frac{b}{a+b}u_e + \frac{c}{a+b}x_e + \frac{d}{a+b}y_e + \frac{a-c-d}{a+b}x_s, \quad (59)$$

$$x_s = y_s = -\frac{a^2+b^2+(c+d)(b-a)}{(b+c+d)^2+(a-c-d)^2+\frac{1}{6}(a+b)^2} \times (l_e - u_e) + \frac{2c^2+2cd+c(b-a)+\frac{1}{6}(a+b)^2}{(b+c+d)^2+(a-c-d)^2+\frac{1}{6}(a+b)^2}x_e + \frac{2d^2+2cd+d(b-a)+\frac{1}{6}(a+b)^2}{(b+c+d)^2+(a-c-d)^2+\frac{1}{6}(a+b)^2}y_e. \quad (60)$$

(ii) *If $p \in Q_2$ then*

$$l_s = \frac{7}{8}l_e + \frac{1}{8}u_e - \frac{1}{16}x_e - \frac{1}{16}y_e \quad (61)$$

$$u_s = \frac{1}{8}l_e + \frac{7}{8}u_e + \frac{1}{16}x_e + \frac{1}{16}y_e \quad (62)$$

$$x_s = y_s = -\frac{3}{4}l_e + \frac{3}{4}u_e + \frac{1}{8}x_e + \frac{1}{8}y_e. \quad (63)$$

(iii) *If $p \in Q_3$ then*

$$l_s = \frac{1}{2}l_e + \frac{1}{2}u_e - \frac{c}{2(c+d)}x_e - \frac{d}{2(c+d)}y_e \quad (64)$$

$$u_s = \frac{1}{2}l_e + \frac{1}{2}u_e + \frac{c}{2(c+d)}x_e + \frac{d}{2(c+d)}y_e \quad (65)$$

$$x_s = y_s = \frac{c}{c+d}x_e + \frac{d}{c+d}y_e. \quad (66)$$

(iv) *If* $p \in Q_4$ *then*

$$l_s = \frac{c+d-2a}{2(c+d-a)} l_e + \frac{c+d}{2(c+d-a)} u_e - \frac{c}{2(c+d-a)} x_e - \frac{d}{2(c+d-a)} y_e \quad (67)$$

$$u_s = \frac{c+d}{2(c+d-a)} l_e + \frac{c+d-2a}{2(c+d-a)} u_e + \frac{c}{2(c+d-a)} x_e + \frac{d}{2(c+d-a)} y_e \quad (68)$$

$$x_s = y_s = \frac{a}{c+d-a} l_e - \frac{a}{c+d-a} u_e + \frac{c}{c+d-a} x_e + \frac{d}{c+d-a} y_e. \quad (69)$$

Proof First of all, we recall (see Ban and Coroianu 2014) that $p(T) = al + bu + cx + dy$, for any $p \in \mathcal{P}$ and $T = [l, u, x, y] \in F^{\mathrm{T}}(\mathbb{R})$. By Proposition 1 it follows that

$$d^2(A, B) = d^2(A, T_e(A)) + d^2(T_e(A), B),$$

for any $B \in F^{s}(\mathbb{R})$. Since $d^2(A, T_e(A))$ is constant and taking into account (13), (14), (26), (27) and (28) it results that $[l_s, u_s, x_s, y_s]$ is the symmetric triangular approximation of $A \in F(\mathbb{R})$, preserving the parameter p, if and only if l_s, u_s, x_s, y_s minimize the function

$$h(l, u, x, y) = (l - l_e)^2 + (u - u_e)^2 + \tfrac{1}{12}(x - x_e)^2 + \tfrac{1}{12}(y - y_e)^2 \quad (70)$$

and the following conditions are satisfied

$$x_s \geq 0 \quad (71)$$

$$y_s \geq 0 \quad (72)$$

$$x_s + y_s = 2(u_s - l_s) \quad (73)$$

$$x_s = y_s \quad (74)$$

$$al_s + bu_s + cx_s + dy_s = al_e + bu_e + cx_e + dy_e. \quad (75)$$

(i) If $a + b \neq 0$ then from (70)–(75) we obtain

$$l_s = \frac{a}{a+b} l_e + \frac{b}{a+b} u_e + \frac{c}{a+b} x_e + \frac{d}{a+b} y_e - \frac{b+c+d}{a+b} x_s \quad (76)$$

$$u_s = \frac{a}{a+b} l_e + \frac{b}{a+b} u_e + \frac{c}{a+b} x_e + \frac{d}{a+b} y_e + \frac{a-c-d}{a+b} x_s \quad (77)$$

and

$$y_s = x_s, \quad (78)$$

where x_s is a solution of the problem

$$\min \left\{ \left(\frac{b+c+d}{a+b} x + \frac{b}{a+b} l_e - \frac{b}{a+b} u_e - \frac{c}{a+b} x_e - \frac{d}{a+b} y_e \right)^2 + \left(\frac{a-c-d}{a+b} x + \frac{a}{a+b} l_e - \frac{a}{a+b} u_e + \frac{c}{a+b} x_e + \frac{d}{a+b} y_e \right)^2 + \frac{1}{12}(x - x_e)^2 + \frac{1}{12}(x - y_e)^2 \right\}$$

$$x \geq 0.$$

It is easy to see that $x_s = 0$ if $A \in \Phi_{a,b,c,d}$ and x_s is given by (60), contrariwise, that is $A \in F(\mathbb{R}) \setminus \Phi_{a,b,c,d}$. We obtain l_s, u_s and y_s from (76)–(78).

(ii) Relation (75) does not furnish any information. We have $y_s = x_s$, $u_s = l_s + x_s$ and (l_s, x_s) is the minimum point of the function

$$h_1(l, x) = (l - l_e)^2 + (l + x - u_e)^2 + \frac{1}{12}(x - x_e)^2 + \frac{1}{12}(x - y_e)^2$$

under condition $x \geq 0$. To solve this problem we use the Karush–Kuhn–Tucker theorem (see Theorem 1). We obtain (l_s, x_s) is a solution if and only if there exists μ such that the following system is satisfied

$$2(l_s - l_e) + 2(l_s + x_s - u_e) = 0 \quad (79)$$

$$2(l_s + x_s - u_e) + \frac{1}{6}(x_s - x_e) + \frac{1}{6}(x_s - y_e) - \mu = 0 \quad (80)$$

$$x_s \geq 0 \quad (81)$$

$$\mu \geq 0 \quad (82)$$

$$\mu x_s = 0. \quad (83)$$

If $\mu \neq 0$ then $x_s = 0$ and

$$l_s = \frac{1}{2} l_e + \frac{1}{2} u_e.$$

We get (see (36)–(38))

$$\mu = l_e - u_e - \frac{1}{6} x_e - \frac{1}{6} y_e < 0,$$

therefore, we have not a solution in this case. If $\mu = 0$ then

$$l_s = \frac{7}{8} l_e + \frac{1}{8} u_e - \frac{1}{16} x_e - \frac{1}{16} y_e$$

$$x_s = -\frac{3}{4} l_e + \frac{3}{4} u_e + \frac{1}{8} x_e + \frac{1}{8} y_e \geq 0$$

and (61)–(63) are immediate.

(iii) In this case $c+d \neq 0$ and (73)–(75) imply

$$x_s = y_s = \frac{c}{c+d}x_e + \frac{d}{c+d}y_e$$
$$u_s = \frac{c}{c+d}x_e + \frac{d}{c+d}y_e + l_s.$$

We obtain l_s as the minimum point of the function

$$h_2(l) = (l-l_e)^2 + \left(l + \frac{c}{c+d}x_e + \frac{d}{c+d}y_e - u_e\right)^2$$
$$= 2l^2 - 2l\left(l_e + u_e - \frac{c}{c+d}x_e - \frac{d}{c+d}y_e\right)$$
$$+ l_e^2 + \left(\frac{c}{c+d}x_e + \frac{d}{c+d}y_e - u_e\right)^2.$$

We immediately obtain l_s and u_s as in (64) and (65).

(iv) If $a+b=0$, $a \neq 0$, $c/a \leq 1/6$ and $d/a \leq 1/6$ then $\frac{c}{a} + \frac{d}{a} - 1 < 0$ and ((37)–(39) are used here)

$$l_e - u_e + \frac{c}{a}x_e + \frac{d}{a}y_e \leq -\frac{1}{6}x_e - \frac{1}{6}y_e$$
$$+ \frac{c}{a}x_e + \frac{d}{a}y_e \leq 0.$$

From (73)–(75) we obtain

$$x_s = y_s = \frac{al_e - au_e + cx_e + dy_e}{c+d-a}$$
$$= \frac{l_e - u_e + \frac{c}{a}x_e + \frac{d}{a}y_e}{\frac{c}{a} + \frac{d}{a} - 1} \geq 0$$

and

$$u_s = l_s + x_s. \tag{84}$$

We obtain l_s as the minimum point of the function

$$h_3(l) = (l-l_e)^2$$
$$+ \left(l + \frac{a}{c+d-a}l_e - \frac{c+d}{c+d-a}u_e + \frac{c}{c+d-a}x_e + \frac{d}{c+d-a}y_e\right)^2$$
$$= 2l^2 - 2l$$
$$\times \left(\frac{c+d-2a}{c+d-a}l_e + \frac{c+d}{c+d-a}u_e - \frac{c}{c+d-a}x_e - \frac{d}{c+d-a}y_e\right) + l_e^2$$
$$+ \left(\frac{a}{c+d-a}l_e - \frac{c+d}{c+d-a}u_e + \frac{c}{c+d-a}x_e + \frac{d}{c+d-a}y_e\right)^2$$

that is l_s is given by (67). We get (68) from (84).

Because $\mathcal{P}_s \subset \mathcal{P}_T$, the proof is complete if for any $(a,b,c,d) \in \mathbb{R}^4$ in the following two cases we find $A \in F(\mathbb{R})$ such that the equation $p(A) = p(X)$ has no solutions in $F^s(\mathbb{R})$ (see (46)–(50) and (51)–(54)):

(a) $a = b = 0$, $cd < 0$

(b) $a+b=0$, $a \neq 0$, ($\frac{c}{a} > \frac{1}{2}$ or $\frac{d}{a} > \frac{1}{2}$).

In the case (a), let us consider $c < 0$ and $d > 0$ (the case $c > 0$ and $d < 0$ is similar). If $c+d < 0$ then we take $A \in F(\mathbb{R})$ such that A_L is constant, that is $x_e(A) = 0$, and $y_e(A) > 0$. From (71), (72), (74) and (75) we obtain

$$0 \geq (c+d)\,x_s(A) = dy_e(A) > 0,$$

therefore, it does not exist as a symmetric triangular fuzzy number

$$[l_s(A), u_s(A), x_s(A), y_s(A)]$$

such that

$$p([l_s(A), u_s(A), x_s(A), y_s(A)]) = p(A).$$

Contrariwise, if $c+d > 0$, then we take $A \in F(\mathbb{R})$ such that A_U is constant, that is $y_e(A) = 0$, and $x_e(A) > 0$. From (71), (72), (74) and (75) we obtain

$$0 \leq (c+d)\,x_s(A) = cx_e(A) < 0,$$

therefore, we have the same conclusion as above.

From (73)–(75), in case (b) we obtain

$$\left(1 - \frac{c}{a} - \frac{d}{a}\right)x_s(A)$$
$$= -l_e(A) + u_e(A) - \frac{c}{a}x_e(A) - \frac{d}{a}y_e(A). \tag{85}$$

In the hypothesis $\frac{c}{a} \geq \frac{1}{2}$ and $\frac{d}{a} \geq \frac{1}{2}$ we do not find $X \in F^s(\mathbb{R})$ such that $p(A) = p(X)$. Indeed, by considering $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ and $A \in F(\mathbb{R})$ given by

$$A_L(\alpha) = x_1 + (x_2 - x_1)\sqrt{\alpha},$$
$$A_U(\alpha) = \left(\frac{1}{3} - \frac{8c}{5a}\right)x_1 + \left(\frac{2}{3} + \frac{8c}{5a}\right)x_2$$

we obtain (see (32)–(35))

$$-l_e(A) + u_e(A) - \frac{c}{a}x_e(A) - \frac{d}{a}y_e(A)$$
$$= \frac{4c}{5a}(x_2 - x_1) > 0,$$

and (85) cannot be satisfied. Now, let us consider $\frac{c}{a} > \frac{1}{2}$ and $\frac{d}{a} < \frac{1}{2}$ (or $\frac{c}{a} < \frac{1}{2}$ and $\frac{d}{a} > \frac{1}{2}$). Let $y_1, y_2 \in \mathbb{R}$, $y_1 < y_2$ and $B, C \in F(\mathbb{R})$ given by $B_L(\alpha) = y_1 + (y_2 - y_1)\alpha$, $B_U(\alpha) = y_2$ and $C_L(\alpha) = y_1$, $C_U(\alpha) = y_2 - \alpha(y_2 - y_1)$, $\alpha \in [0,1]$, respectively. Then (see (32)–(35))

$$l_e(B) = \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$u_e(B) = y_2$$
$$x_e(B) = y_2 - y_1$$
$$y_e(B) = 0$$

and

$$l_e(C) = y_1$$
$$u_e(C) = \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$x_e(C) = 0$$
$$y_e(C) = y_2 - y_1,$$

respectively. Taking into account (85) we obtain

$$\left(\frac{c}{a} - \frac{1}{2}\right)(y_2 - y_1) = \left(\frac{c}{a} + \frac{d}{a} - 1\right)x_s(B) \tag{86}$$

and

$$\left(\frac{d}{a} - \frac{1}{2}\right)(y_2 - y_1) = \left(\frac{c}{a} + \frac{d}{a} - 1\right)x_s(C), \tag{87}$$

respectively, where $x_s(B) \geq 0$ and $x_s(C) \geq 0$ (see (13)). In our hypothesis (86) and (87) cannot be satisfied simultaneously; therefore, either $p(B) = p(X)$ or $p(C) = p(X)$ has no solution in $F^s(\mathbb{R})$. □

Remark 1 Passing to the α-cut representation of a fuzzy number (see (32)–(35)) in Theorem 2 we immediately obtain the nearest symmetric triangular fuzzy number of a given fuzzy number preserving $p \in \mathcal{P}_s$ in terms of its α-cuts.

5 Properties

Throughout in this section we denote by s_p, $p \in \mathcal{P}_s$ the well-defined (according with Theorem 2) symmetric triangular approximation operator $s_p : F(\mathbb{R}) \to F^s(\mathbb{R})$, where $s_p(A)$ is the unique nearest symmetric triangular fuzzy number of A, with respect to the average Euclidean metric d, which preserves p, that is

$$d(A, s_p(A)) = \min_{T \in F^s(\mathbb{R}), p(T) = p(A)} d(A, T).$$

In the sequel we discuss the main properties of s_p, $p \in \mathcal{P}_s$: identity, translation invariance, scale invariance, additivity and continuity, considered as important for a such approximation operator (see Grzegorzewski and Mrówka 2005). According with the definitions of the properties, our aim is to find the sets

$$\mathcal{P}_s^I = \{p \in \mathcal{P}_s : s_p(A) = A, \forall A \in F^s(\mathbb{R})\}$$
$$\mathcal{P}_s^{TI} = \{p \in \mathcal{P}_s : s_p(A + z) = s_p(A) + z, \forall A \in F(\mathbb{R}), \forall z \in \mathbb{R}\},$$
$$\mathcal{P}_s^{SI} = \{p \in \mathcal{P}_s : s_p(\lambda \cdot A) = \lambda \cdot s_p(A), \forall A \in F(\mathbb{R}), \forall \lambda \in \mathbb{R}\},$$
$$\mathcal{P}_s^{AD} = \{p \in \mathcal{P}_s : s_p(A + B) = s_p(A) + s_p(B), \forall A, B \in F(\mathbb{R})\}$$

and

$$\mathcal{P}_s^C = \{p \in \mathcal{P}_s : s_p \text{ is continuous with respect to } d\}.$$

5.1 Identity

If $A \in F^s(\mathbb{R})$ then

$$d(A, s_p(A)) = \min_{T \in F^s(\mathbb{R}), p(T) = p(A)} d(A, T) \leq d(A, A) = 0,$$

that is $s_p(A) = A$, for every $p \in \mathcal{P}_s$. We get $\mathcal{P}_s^I = \mathcal{P}_s$.

5.2 Translation invariance

Because

$$l_e(A + z) = l_e(A) + z$$
$$u_e(A + z) = u_e(A) + z$$
$$x_e(A + z) = x_e(A)$$
$$y_e(A + z) = y_e(A),$$

for every $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$, and, in addition (see (29)),

$$[l, u, x, y] + z = [l + z, u + z, x, y]$$

for $[l, u, x, y] \in F^T(\mathbb{R})$ and $z \in \mathbb{R}$, the translation invariance can be easily obtained by a direct proof taking into account Theorem 2. We have

Proposition 3 *$s_p(A + z) = s_p(A) + z$, for every $p \in \mathcal{P}_s$, $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$, that is $\mathcal{P}_s^{TI} = \mathcal{P}_s$.*

5.3 Scale invariance

Properties of scale invariance can be obtained by direct proofs taking into account Theorem 2, (30), (31) and the following properties:

$$l_e(\lambda \cdot A) = \lambda l_e(A)$$
$$u_e(\lambda \cdot A) = \lambda u_e(A)$$

$$x_e(\lambda \cdot A) = \lambda x_e(A)$$
$$y_e(\lambda \cdot A) = \lambda y_e(A),$$

for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}, \lambda \geq 0$,

$$l_e(\lambda \cdot A) = \lambda u_e(A)$$
$$u_e(\lambda \cdot A) = \lambda l_e(A)$$
$$x_e(\lambda \cdot A) = -\lambda y_e(A)$$
$$y_e(\lambda \cdot A) = -\lambda x_e(A),$$

for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}, \lambda < 0$. The following result is obvious.

Proposition 4 *$s_p(\lambda \cdot A) = \lambda \cdot s_p(A)$, for every $p \in \mathcal{P}_s$, $A \in F(\mathbb{R})$ and $\lambda \geq 0$.*

An immediate conclusion is that the scale invariance of an operator s_p, $p \in \mathcal{P}_s$, is equivalent with $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$. This result helps us to give the following characterization.

Proposition 5 *$\mathcal{P}_s^{SI} = S_1 \cup S_2 \cup S_3 \cup S_4$, where*

$$S_1 = \{p \in Q_1 : a = b \text{ and } c + d = 0\},$$
$$S_2 = Q_2,$$
$$S_i = \{p \in Q_i : c = d\}, i \in \{3, 4\}.$$

Proof Let $p \in Q_1$. If $a = b$ and $c + d = 0$ then the second case in Theorem 2, (i) is applicable for every $A \in F(\mathbb{R})$ and from (58)–(60) we obtain

$$x_s(A) = y_s(A) = -\frac{3}{4}l_e(A) + \frac{3}{4}u_e(A) + \frac{1}{4}x_e(A) + \frac{1}{4}y_e(A)$$
$$l_s(A) = \frac{1}{2}l_e(A) + \frac{1}{2}u_e(A) + \frac{c}{2a}x_e(A) - \frac{c}{2a}y_e(A) - \frac{1}{2}x_s(A)$$
$$u_s(A) = \frac{1}{2}l_e(A) + \frac{1}{2}u_e(A) + \frac{c}{2a}x_e(A) - \frac{c}{2a}y_e(A) + \frac{1}{2}x_s(A).$$

The equalities $l_s(-A) = -u_s(A)$ and $x_s(-A) = y_s(A)$ are immediate, therefore, $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$. Now, let us assume that $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$. Our aim is to prove $a = b$ and $c + d = 0$. If $a^2 + b^2 + (c + d)(b - a) < 0$ then the first case in Theorem 2, (i) is applicable to compute the symmetric triangular approximation of $A = [0, 1, 0, 0]$ and $-A = [-1, 0, 0, 0]$. We get (from (55)–(57))

$$l_s(A) = u_s(A) = \frac{b}{a+b},$$
$$x_s(A) = y_s(A) = 0$$

and

$$l_s(-A) = u_s(-A) = -\frac{a}{a+b},$$
$$x_s(-A) = y_s(-A) = 0.$$

Because $s_p(-A) = -s_p(A)$ implies $l_s(-A) = -u_s(A)$ we obtain $a = b$. The second case in Theorem 2, (i) is applicable to compute the symmetric triangular approximation of $B = [-1, 0, 0, 0]$ and $-B = [0, 1, 0, 0]$. We get (from (58)–(60))

$$l_s(-B) = \frac{1}{2} - \frac{a(a+c+d)}{(a+c+d)^2 + (a-c-d)^2 + \frac{2}{3}a^2},$$
$$-u_s(A) = \frac{1}{2} - \frac{a(a-c-d)}{(a+c+d)^2 + (a-c-d)^2 + \frac{2}{3}a^2}.$$

Because $s_p(-B) = -s_p(B)$ implies $l_s(-B) = -u_s(B)$ we obtain $c + d = 0$.

If $p \in Q_2 = S_2$ then from (61)–(63) we obtain

$$l_s(-A) = -u_s(A) = -\frac{1}{8}l_e(A) - \frac{7}{8}u_e(A) - \frac{1}{16}x_e(A) - \frac{1}{16}y_e(A)$$

and

$$x_s(-A) = y_s(A) = -\frac{3}{4}l_e(A) + \frac{3}{4}u_e(A) + \frac{1}{8}x_e(A) + \frac{1}{8}y_e(A),$$

for every $A \in F(\mathbb{R})$, therefore, $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$.

Let $p \in Q_3$. If $c = d$ then from (64)–(66) we obtain

$$l_s(-A) = -u_s(A) = -\frac{1}{2}l_e(A) - \frac{1}{2}u_e(A) - \frac{1}{4}x_e(A) - \frac{1}{4}y_e(A)$$

and

$$x_s(-A) = y_s(A) = \frac{1}{2}x_e(A) + \frac{1}{2}y_e(A),$$

for every $A \in F(\mathbb{R})$, therefore, $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$. In the hypothesis $s_p(-A) = -s_p(A)$, for every $A \in F(\mathbb{R})$, we consider $A = [0, \frac{1}{2}, 0, 1]$. Then $-A = [-\frac{1}{2}, 0, 1, 0]$ and from $-l_s(A) = u_s(-A)$ we obtain

$$-\frac{1}{4}+\frac{d}{2(c+d)}=-\frac{1}{4}+\frac{c}{2(c+d)},$$

that is $c=d$.

Let $p\in Q_4$. If $c=d$ then from (67)–(69) we obtain

$$l_s(-A)=-u_s(A)=-\frac{c}{2c-a}l_e(A)-\frac{c-a}{2c-a}u_e(A)-\frac{c}{2(2c-a)}x_e(A)-\frac{c}{2(2c-a)}y_e(A)$$

and

$$x_s(-A)=y_s(A)=\frac{a}{2c-a}l_e(A)-\frac{a}{2c-a}u_e(A)+\frac{c}{2c-a}x_e(A)+\frac{c}{2c-a}y_e(A),$$

for every $A\in F(\mathbb{R})$, therefore, $s_p(-A)=-s_p(A)$, for every $A\in F(\mathbb{R})$. As above, we consider $A=[0,\frac{1}{2},0,1]\in F(\mathbb{R})$. In the hypothesis $s_p(-A)=-s_p(A)$ we obtain

$$-\frac{c+d-2a}{4(c+d-a)}-\frac{c}{2(c+d-a)}=l_s(-A)=-u_s(A)=-\frac{c+d-2a}{4(c+d-a)}-\frac{d}{2(c+d-a)}$$

which implies $c=d$. □

5.4 Additivity

The additivity of s_p is not generally valid for $p\in Q_1$, as the below example proves. It is enough to find $A,B\in F(\mathbb{R})$ such that the first case in Theorem 2, (i) is applicable to A and $A+B$ and the second case in Theorem 2, (i) is applicable to B. If $x_s(B)\neq 0$ then

$$x_s(A+B)=0\neq x_s(B)=x_s(A)+x_s(B),$$

which implies $s_p(A+B)\neq s_p(A)+s_p(B)$ (see (29)).

Example 2 Let $p\in Q_1$, $p(A)=u_e(A)+x_e(A)-y_e(A)$. According with Theorem 2, (i) we obtain $x_s(A)=0$ if

$$12l_e(A)-12u_e(A)\geq 13x_e(A)-11y_e(A) \tag{88}$$

and $x_s(A)=-\frac{6}{7}l_e(A)+\frac{6}{7}u_e(A)+x_e(A)-\frac{5}{7}y_e(A)$ if

$$12l_e(A)-12u_e(A)<13x_e(A)-11y_e(A). \tag{89}$$

Let us consider $A\in F(\mathbb{R})$ given by $A_L(\alpha)=1+\alpha$, $A_U(\alpha)=9-7\alpha$, $\alpha\in[0,1]$ and $B=(0,1,2)\in F^s(\mathbb{R})$. Then $s_p(B)=B$, therefore, $x_s(B)=1$. Because A and $A+B$ satisfy (88) we have $x_s(A)=x_s(A+B)=0$.

Because

$$l_e(A+B)=l_e(A)+l_e(B)$$
$$u_e(A+B)=u_e(A)+u_e(B)$$
$$x_e(A+B)=x_e(A)+x_e(B)$$
$$y_e(A+B)=y_e(A)+y_e(B),$$

for every $A,B\in F(\mathbb{R})$, and taking into account (29) some results of additivity can be formulated.

Proposition 6 *If $p(A)=al_e(A)+bu_e(A)+cx_e(A)+dy_e(A)\in Q_1$ such that $c+d=0$ and $a=b$ then $s_p(A+B)=s_p(A)+s_p(B)$, for every $A,B\in F(\mathbb{R})$.*

Proof It is immediate because the second case in Theorem 2, (i) is applicable for every $A\in F(\mathbb{R})$. □

Proposition 7 *$s_p(A+B)=s_p(A)+s_p(B)$, for every $p\in Q_2\cup Q_3\cup Q_4$ and $A,B\in F(\mathbb{R})$.*

5.5 Continuity

In Ban and Coroianu (2014) we proved that any operator $T_p:F(\mathbb{R})\rightarrow F^T(\mathbb{R})$ such that $T_p(A)$ is the nearest trapezoidal fuzzy number of $A\in F(\mathbb{R})$ with the property $p(A)=p(T_p(A))$, where $p\in\mathcal{P}_T$ (see (46)–(50)), is continuous. In the case of symmetric triangular approximation operators s_p we have an even stronger result, namely Lipschitz-continuity. It is worth noting that the property of the approximation operators (in the sense of the present paper) to be Lipschitz was studied in Coroianu (2011, 2012) too.

Theorem 3 *The symmetric triangular approximation operator $s_p:F(\mathbb{R})\rightarrow F^s(\mathbb{R})$, with $p\in\mathcal{P}_s$, is Lipschitz-continuous with respect to the average Euclidean metric d (see (11)).*

Proof By the proof of Theorem 2 we observe that the algorithm to compute $s_p(A)$ when A goes over $F(\mathbb{R})$ is unique. Actually, there exist the linear functions $f_i:\mathbb{R}^4\rightarrow\mathbb{R}$ which does not depend on any fuzzy number A such that

$$s_p(A)=[l_s(A),u_s(A),x_s(A),y_s(A)]=[f_1(T_e(A)),\ f_2(T_e(A)),\ f_3(T_e(A)),\ f_4(T_e(A))]$$

for every $A\in F(\mathbb{R})$. Here we uniquely identify

$$T_e(A)=[l_e(A),u_e(A),x_e(A),y_e(A)]$$

with

$$\overrightarrow{T}_e(A)=(l_e(A),u_e(A),x_e(A),y_e(A))\in\mathbb{R}^4.$$

Now, let us consider the function $\overline{s}_p:\mathbb{R}^4\rightarrow\mathbb{R}^4$,

$$\overline{s}_p(\overrightarrow{u})=(f_1(\overrightarrow{u}),\ f_2(\overrightarrow{u}),\ f_3(\overrightarrow{u}),\ f_4(\overrightarrow{u})),\ \overrightarrow{u}\in\mathbb{R}^4.$$

Since the functions f_i, $i \in \{1,2,3,4\}$ are all linear it easily results that $\overline{s}_{\mathrm{p}}$ is linear too. Then let us consider the metric D defined on $\mathbb{R}^4$ by

$$D(\overrightarrow{u}, \overrightarrow{v}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \frac{1}{12}(x_3 - y_3)^2 + \frac{1}{12}(x_4 - y_4)^2,$$

where $\overrightarrow{u} = (x_1, x_2, x_3, x_4)$ and $\overrightarrow{v} = (y_1, y_2, y_3, y_4)$. It is well known that linear operators between finite dimensional spaces are of Lipschitzian type. Therefore, there exists a constant k which depends only on D and the functions f_i, $i \in \{1,2,3,4\}$, such that

$$D(\overline{s}_{\mathrm{p}}(\overrightarrow{u}), \overline{s}_{\mathrm{p}}(\overrightarrow{v})) \leq kD(\overrightarrow{u}, \overrightarrow{v})$$

for every $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^4$. In particular, we have

$$D(\overline{s}_{\mathrm{p}}(\overrightarrow{T}_{\mathrm{e}}(A)), \overline{s}_{\mathrm{p}}(\overrightarrow{T}_{\mathrm{e}}(B))) \leq kD((\overrightarrow{T}_{\mathrm{e}}(A)), (\overrightarrow{T}_{\mathrm{e}}(B))),$$

for every fuzzy numbers A and B. Since by the construction of $\overline{s}_{\mathrm{p}}$ we have $s_{\mathrm{p}}(T_{\mathrm{e}}(A)) = \overline{s_{\mathrm{p}}}(\overrightarrow{T_{\mathrm{e}}}(A))$ for any fuzzy number A and since by the definition of D we have $D((\overrightarrow{T}_{\mathrm{e}}(A)), (\overrightarrow{T}_{\mathrm{e}}(B))) = d((T_{\mathrm{e}}(A)), (T_{\mathrm{e}}(B)))$ for any fuzzy numbers A and B, we get

$$d(s_{\mathrm{p}}(T_{\mathrm{e}}(A)), s_{\mathrm{p}}(T_{\mathrm{e}}(B))) \leq kd((T_{\mathrm{e}}(A)), (T_{\mathrm{e}}(B)))$$

for all $A, B \in F(\mathbb{R})$. This easily implies that

$$d(s_{\mathrm{p}}(A), s_{\mathrm{p}}(B)) \leq kd((T_{\mathrm{e}}(A)), (T_{\mathrm{e}}(B)))$$

for all $A, B \in F(\mathbb{R})$, which by Proposition 2 implies that

$$d(s_{\mathrm{p}}(A), s_{\mathrm{p}}(B)) \leq kd(A, B)$$

for all $A, B \in F(\mathbb{R})$. The proof is complete. □

6 Applications

The trapezoidal, symmetric trapezoidal, triangular, symmetric triangular approximations of fuzzy numbers, without conditions or preserving the ambiguity, value, ambiguity and value or expected interval were given in Ban (2008); Ban et al. (2011); Ban and Coroianu (2012, 2014); Yeh (2008a, b). We apply the results in the previous sections to complete this list with the calculus of the symmetric triangular approximation preserving value (see (7), (42)) and the symmetric triangular approximation preserving the expected value (see (5), (40)) and to study their properties.

6.1 Symmetric triangular approximations of fuzzy numbers preserving the value

We have (see (42)) $a = b = \frac{1}{2}$, $c = \frac{1}{12}$, $d = -\frac{1}{12}$, therefore, $p = \mathrm{Val} \in \mathcal{P}_{\mathrm{s}}$ and the case (i) in Theorem 2 is applicable. Because $\Phi_{a,b,c,d} = \varnothing$ we apply (58)–(60) to calculate the nearest symmetric triangular approximation of a fuzzy number preserving its value. We obtain

Theorem 4 *Let $A \in F(\mathbb{R})$ and*

$$s_{\mathrm{Val}}(A) = [l_{\mathrm{s}}(A), u_{\mathrm{s}}(A), x_{\mathrm{s}}(A), y_{\mathrm{s}}(A)] = [l_{\mathrm{s}}, u_{\mathrm{s}}, x_{\mathrm{s}}, y_{\mathrm{s}}]$$

the nearest symmetric triangular fuzzy number of A preserving its value. Then

$$l_{\mathrm{s}} = \frac{7}{8}l_{\mathrm{e}} + \frac{1}{8}u_{\mathrm{e}} - \frac{1}{24}x_{\mathrm{e}} - \frac{5}{24}y_{\mathrm{e}}$$
$$u_{\mathrm{s}} = \frac{1}{8}l_{\mathrm{e}} + \frac{7}{8}u_{\mathrm{e}} + \frac{5}{24}x_{\mathrm{e}} + \frac{1}{24}y_{\mathrm{e}}$$
$$x_{\mathrm{s}} = y_{\mathrm{s}} = -\frac{3}{4}l_{\mathrm{e}} + \frac{3}{4}u_{\mathrm{e}} + \frac{1}{4}x_{\mathrm{e}} + \frac{1}{4}y_{\mathrm{e}}.$$

Passing to the α-cut representation (see (22)–(25), (32)–(35)) we get:

Theorem 5 *Let $A \in F(\mathbb{R})$ and*

$$s_{\mathrm{Val}}(A) = (s_1(A), s_2(A), s_3(A)) = (s_1, s_2, s_3)$$

the nearest symmetric triangular fuzzy number of A preserving its value. Then

$$s_1 = \int_0^1 \left(\frac{9}{4} - 2\alpha\right) A_{\mathrm{L}}(\alpha)\,\mathrm{d}\alpha + \int_0^1 \left(4\alpha - \frac{9}{4}\right) A_{\mathrm{U}}(\alpha)\,\mathrm{d}\alpha$$
$$s_2 = \int_0^1 \alpha A_{\mathrm{L}}(\alpha)\,\mathrm{d}\alpha + \int_0^1 \alpha A_{\mathrm{U}}(\alpha)\,\mathrm{d}\alpha$$
$$s_3 = \int_0^1 \left(4\alpha - \frac{9}{4}\right) A_{\mathrm{L}}(\alpha)\,\mathrm{d}\alpha + \int_0^1 \left(\frac{9}{4} - 2\alpha\right) A_{\mathrm{U}}(\alpha)\,\mathrm{d}\alpha.$$

We obtain the continuity and the translation invariance of s_{Val} from Theorem 3 and Proposition 3. Moreover, s_{Val} is additive according to Proposition 6 and scalar invariant from Proposition 5.

6.2 Symmetric triangular approximations of fuzzy numbers preserving the expected value

Because (see (40)) $a = b = \frac{1}{2}$ and $c = d = 0$ we apply Theorem 2, (i) to calculate the nearest symmetric triangular approximation of a fuzzy number preserving its expected value. In addition, $\Phi_{a,b,c,d} = \varnothing$. From (58)–(60) we obtain

Theorem 6 *Let* $A \in F(\mathbb{R})$ *and*

$$s_{\mathrm{EV}}(A) = [l_s(A), u_s(A), x_s(A), y_s(A)] = [l_s, u_s, x_s, y_s]$$

the nearest symmetric triangular fuzzy number of A preserving its expected value. Then

$$l_s = \frac{7}{8}l_e + \frac{1}{8}u_e - \frac{1}{8}x_e - \frac{1}{8}y_e$$
$$u_s = \frac{1}{8}l_e + \frac{7}{8}u_e + \frac{1}{8}x_e + \frac{1}{8}y_e$$
$$x_s = y_s = -\frac{3}{4}l_e + \frac{3}{4}u_e + \frac{1}{4}x_e + \frac{1}{4}y_e.$$

Passing to the α-cut representation (see (22)–(25), (32)–(35)) we get:

Theorem 7 *Let* $A \in F(\mathbb{R})$ *and*

$$s_{\mathrm{EV}}(A) = (s_1(A), s_2(A), s_3(A)) = (s_1, s_2, s_3)$$

the nearest symmetric triangular fuzzy number of A preserving its expected value. Then

$$s_1 = \int_0^1 \left(\frac{11}{4} - 3\alpha\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(3\alpha - \frac{7}{4}\right) A_U(\alpha)\, d\alpha$$
$$s_2 = \frac{1}{2}\int_0^1 A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)\, d\alpha$$
$$s_3 = \int_0^1 \left(3\alpha - \frac{7}{4}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(\frac{11}{4} - 3\alpha\right) A_U(\alpha)\, d\alpha.$$

The approximation operator s_{EV} is translation and scalar invariant, additive and continuous, according with Propositions 3, 5, 6 and Theorem 3, respectively.

7 Conclusion

The set

$$\mathcal{P}_s = \{p \in \mathcal{P} \,|\, \forall A \in F(\mathbb{R}), \exists X \in F^s(\mathbb{R}) \text{ such that } p(A) = p(X)\},$$

where

$$\mathcal{P} = \{p : F(\mathbb{R}) \to \mathbb{R} \,|\, p(A) = a l_e(A) + b u_e(A) + c x_e(A) + d y_e(A), a, b, c, d \in \mathbb{R}\}$$

is determined. The nearest (with respect to the average Euclidean distance) symmetric triangular fuzzy number $s_p(A)$ of $A \in F(\mathbb{R})$, which preserves the parameter p, $p \in \mathcal{P}_s$ is calculated. Properties of the approximation operators s_p, $p \in \mathcal{P}_s$ are studied. The results are applied to the case of value and expected value as parameters. The characterization of the set

$$\{p \in \mathcal{P}_s : s_p(A + B) = s_p(A) + s_p(B), \forall A, B \in F(\mathbb{R})\},$$

where s_p is the symmetric triangular approximation operator introduced in the main result of the paper (Theorem 2), is still an open problem.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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