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FOUNDATIONS

Involutive right-residuated l-groupoids

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Abstract A common generalization of orthomodular lattices and residuated lattices is provided corresponding to bounded lattices with an involution and sectionally extensive mappings. It turns out that such a generalization can be based on integral right-residuated l-groupoids. This general framework is applied to MV-algebras, orthomodular lattices, Nelson algebras, basic algebras and Heyting algebras.

Keywords Right-residuated l-groupoid · Residuated lattice · Antitone involution · MV-algebra · Basic algebra · Congruence regularity

Mathematics Subject Classification Primary 08B05 · 03G25; Secondary 06B05 · 06D15

1 Introduction

Residuated lattices were introduced in Dilworth and Ward (1939), and they are used in several branches of mathematics, including areas of ideal lattices of rings, lattice-ordered groups, formal languages and multi-valued logic. Right-residuated l-groupoids constitute a natural generalization of residuated lattices [see, e.g. Blyth and Janowitz (1972)], and their applications cover even a wider field. We will show that they provide a useful framework for propositional calculus in constructive logic and certain logics related to quantum mechanics, and some computations in universal algebra.

For instance, let $\mathcal{A} = (A, F)$ be an algebra from a congruence modular variety, and $[\varphi, \theta]$ the commutator of two congruences φ, θ. Denote by 0_A and 1_A the least and the greatest element of the congruence lattice $(\mathrm{Con}\mathcal{A}, \vee, \wedge)$, respectively. In Czelakowski (2008), a binary operation $\rightarrow$ on $\mathrm{Con}\mathcal{A}$ was defined as by the following formula:

$$\alpha \rightarrow \beta := \bigvee \{\theta \in \mathrm{Con}\mathcal{A} \mid [\alpha, \theta] \leq \beta\}.$$

If the identity $[1_A, \theta] = \theta$ holds in $\mathrm{Con}\mathcal{A}$ then, in view of Czelakowski (2008), $(\mathrm{Con}\mathcal{A}, \vee, \wedge, [,], \rightarrow, 0_A, 1_A)$ is an integral commutative right-residuated l-groupoid.

Although we will not study the consequences of the previous example in the theory of residuated structures, we can see that integral commutative right-residuated l-groupoids are not exceptional structures in algebra, and hence we will investigate the connections between these structures and lattices having an antitone involution and so-called sectionally extensive antitone mappings.

In our paper we study some particular classes of right-residuted l-groupoids. We aim to show the relevance of these classes of algebras in several research fields. The paper is structured as follows: In Sect. 2 some general notions and facts concerning right-residuated l-groupoids are presented. In Sect. 3 we prove that there is a one-to-one correspondence between involution lattices with sectionally extensive antitone mappings and involutive right-residuated l-groupoids satisfying a certain identity. The case when these residuated l-groupoids form residuated lattices is characterized. In Sect. 4

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some examples of right-residuated l-groupoids belonging to the mentioned class are provided. For instance, we show that residuated lattices corresponding to Nelson algebras belong to this class. We prove that sectionally pseudocomplemented lattices admitting an antitone involution can be characterized as right-residuated l-groupoids satisfying certain identities. A special attention is paid to those right-residuated l-groupoids which are defined by lattices with sectionally antitone involutions. In Sect. 5 is proved that these algebras are term equivalent to the so-called basic algebras which can be viewed as a common generalization of MV-algebras and orthomodular lattices. The fact that these algebras can be reconstructed from their implication reduct is shown in Sect. 6. Finally, in Sect. 7, some congruence properties of right-residuated l-groupoids are investigated.

2 Preliminaries

Definition 1 By a *right-residuated l-groupoid* is meant an algebra
$\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that

1. $(L, \vee, \wedge)$ is a lattice with least element 0 and greatest element 1,
2. $(L, \odot)$ is a groupoid, and $1 \odot x = x$, for all $x \in L$.
3. $\mathcal{G}$ satisfies the *right-adjointness* property, that is $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, for all $x, y, z \in L$ [see, e.g. Bělohlávek (2002)].

In general, right-adjointness does not imply left-adjointness [see Botur et al. (2010)], except the case when $\mathcal{G}$ is *commutative*, that is, $x \odot y = y \odot x$, for all $x, y \in L$.

For our sake, we modify the concept of an integral residuated structure as follows: The algebra $\mathcal{G}$ will be called *integral* if $1 \odot x = x \odot 1 = x$ holds for all $x \in L$. Clearly, $\mathcal{G}$ is integral whenever it is commutative. Let $\rceil x := x \rightarrow 0$. The algebra $\mathcal{G}$ is called *involutive* whenever the mapping $x \mapsto \rceil x$, $x \in L$ is an *antitone involution* on L, i.e. if $x \leq y$ implies $\rceil y \leq \rceil x$ and

$$\rceil(\rceil x) = x, \qquad (*)$$

for all $x, y \in L$. The identity $(*)$ is called the double negation law. Of course, every involutive algebra $\mathcal{G}$ satisfies the double negation law, but not conversely. However, if $\mathcal{G}$ is a residuated lattice, that is, $\odot$ is associative and commutative, then $\mathcal{G}$ is involutive if and only if it satisfies the double negation law. This is because then $\mathcal{G}$ satisfies the implication

$$x \leq y \text{ implies } y \rightarrow z \leq x \rightarrow z,$$

for any $x, y, z \in L$, thus also $\rceil y = y \rightarrow 0 \leq x \rightarrow 0 = \rceil x$, for all $x, y \in L$, $x \leq y$. Further, we say that $\mathcal{G}$ satisfies *divisibility* if

$$(x \rightarrow y) \odot x = x \wedge y,$$

for every $x, y \in L$. Finally, $\mathcal{G}$ satisfies *condition* (C) if

$$z \leq x \odot y \text{ if and only if } y \rightarrow \rceil x \leq \rceil z,$$

for all $x, y, z \in L$. The basic properties of right-residuated l-groupoids are collected in the following lemma:

Lemma 1 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a right-residuated l-groupoid. Then*

(i) $\rceil 0 = 1$;
(ii) $a \leq b$ *if and only if* $a \rightarrow b = 1$;
(iii) $a \odot 0 = 0 \odot a = 0$, *for all* $a \in L$;
(iv) $y \leq z$ *implies* $y \odot x \leq z \odot x$ *and* $x \rightarrow y \leq x \rightarrow z$, *for all* $x, y, z \in L$;
(v) $x \odot y \leq y$ *and* $y \rightarrow z = y \rightarrow (y \wedge z)$, *for all* $x, y, z \in L$;
(vi) *if $\mathcal{G}$ satisfies the double negation law then* $\rceil 1 = 0$.

Proof 1. Since $1 \odot 0 = 0$, we have $1 \leq 0 \rightarrow 0$, and hence $1 = 0 \rightarrow 0 = \rceil 0$.
2. If $a \leq b$ then $1 \odot a = a \leq b$, thus $1 \leq a \rightarrow b$ giving $a \rightarrow b = 1$. If $a \rightarrow b = 1$, then $(a \rightarrow b) \odot a \leq b$ implies $a = 1 \odot a \leq b$.
3. $a \leq 1 = 0 \rightarrow 0$ yields $a \odot 0 = 0$, and $0 \leq a \rightarrow 0$ gives $0 \odot a = 0$.
4. Assume $y \leq z$. Since for all $a, b \in L$, $a \odot b = a \odot b$ yields
$$a \leq b \rightarrow (a \odot b),$$
we get $y \leq z \leq x \rightarrow (z \odot x)$, whence $y \odot x \leq z \odot x$. Further, $x \rightarrow y \leq x \rightarrow y$ yields $(x \rightarrow y) \odot x \leq y \leq z$, whence we deduce $x \rightarrow y \leq x \rightarrow z$, for all $x, y, z \in L$.
5. Since $x \leq 1 = y \rightarrow y$, we obtain $x \odot y \leq y$, for all $x, y \in L$. Thus $x \odot y \leq z$ if and only if $x \odot y \leq y \wedge z$, whence we get $x \leq y \rightarrow z$ if and only if $x \leq y \rightarrow (y \wedge z)$. This implies $y \rightarrow z = y \rightarrow (y \wedge z)$.
6. The double negation law and (i) imply: $\rceil 1 = \rceil(\rceil 0) = 0$. □

An interrelation between condition (C) and the involutive property is stated in the following:

Proposition 1 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a right-residuated l-groupoid. Then $\mathcal{G}$ satisfies the double negation law and condition (C) if and only if $\mathcal{G}$ is involutive and $x \odot y = \rceil(y \rightarrow \rceil x)$ holds for all $x, y, z \in L$.*

Proof The double negation law yields $\rceil(\rceil x) = (x \to 0) \to 0 = x$. If $x \leq y$ then $x \leq 1 \odot y$, and so by (C) we get $\rceil y = y \to 0 = y \to \rceil 1 \leq \rceil x$. Hence $\mathcal{G}$ is involutive, and (C) implies $x \odot y \geq z$ if and only if $y \to \rceil x \leq \rceil z$ if and only if $\rceil(y \to \rceil x) \geq \rceil(\rceil z) = z$. Then $\rceil(y \to \rceil x) \geq x \odot y$, and $x \odot y \geq \rceil(y \to \rceil x)$, whence $x \odot y = \rceil(y \to \rceil x)$.

Conversely, suppose that $\mathcal{G}$ is involutive, and $x \odot y = \rceil(y \to \rceil x)$ holds. Then clearly, $\mathcal{G}$ satisfies the double negation law, and $\rceil(y \to \rceil x) \geq z$ if and only if $y \to \rceil x \leq \rceil z$. This means that $z \leq x \odot y$ if and only if $y \to \rceil x \leq \rceil z$, i.e. (C) holds. □

Remark 1 Observe that in a right-residuated l-groupoid the operations $\odot$ and $\to$ determine completely each other, in other words, if $\mathcal{G}_1 = (L, \vee, \wedge, \odot, \to, 0, 1)$ and $\mathcal{G}_2 = (L, \vee, \wedge, \otimes, \rightsquigarrow, 0, 1)$ are right-residuated l-groupoids having the same underlying lattice $(L, \vee, \wedge)$, then the operations $\odot$ and $\otimes$ coincide if and only if $\to$ and $\rightsquigarrow$ coincide. The proof is the same as that for residuated lattices and hence it is omitted.

Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be a right-residuated l-groupoid and define a binary operation $\Rightarrow$ on L as follows:

$$x \Rightarrow y := \rceil y \to \rceil x, \quad \text{for all } x, y \in L.$$

Then $\Rightarrow$ will be called the *derived implication of* $\mathcal{G}$.

Lemma 2 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive right-residuated l-groupoid. Then the operation $\Rightarrow$ for all $x, y, z \in L$ satisfies the following conditions:*

(I0) $(x \vee y) \Rightarrow y = x \Rightarrow y$, $x \Rightarrow x = 1$, $1 \Rightarrow x = x$;
(I1) $(x \Rightarrow y) \wedge y = y$;
(I2) $x \leq y$ implies $y \Rightarrow z \leq x \Rightarrow z$;

Moreover, we have $x \leq y$ if and only if $x \Rightarrow y = 1$.

Proof Since $\mathcal{G}$ is involutive, we have $\rceil 1 = 0$, and hence

$$1 \Rightarrow x = \rceil(\rceil x), \quad \text{for all } x \in L. \tag{1}$$

By definition $x \Rightarrow x = \rceil x \to \rceil x = 1$, and $(x \vee y) \Rightarrow y = \rceil y \to \rceil(x \vee y)$, for all $x, y \in L$. Since $x \mapsto \rceil x$, $x \in L$ is an antitone involution on L, we have $\rceil(x \vee y) = \rceil x \wedge \rceil y$, and hence $(x \vee y) \Rightarrow y = \rceil y \to (\rceil y \wedge \rceil x) = \rceil y \to \rceil x = x \Rightarrow y$, by (v) of Lemma 1. Since $\mathcal{G}$ is involutive, it satisfies the double negation law, and because (1) holds true, (I0) is clear. By Lemma 1(iv) for any $x, y \in L$ we get $y = \rceil(\rceil y) = \rceil y \to 0 \leq \rceil y \to \rceil x = x \Rightarrow y$, which proves (I1).

(I2). Since $\mathcal{G}$ is involutive, we have $x \leq y$ if and only if $\rceil y \leq \rceil x$. By Lemma 1(iv) $\rceil y \leq \rceil x$ implies $\rceil z \to \rceil y \leq \rceil z \to \rceil x$. Hence $x \leq y$ implies $y \Rightarrow z \leq x \Rightarrow z$.

Finally, $x \leq y$ if and only if $\rceil y \leq \rceil x$, and Lemma 1(ii) yields $\rceil y \leq \rceil x$ if and only if $\rceil y \to \rceil x = 1$. However, $\rceil y \to \rceil x = 1$ means that $x \Rightarrow y = 1$. □

3 Lattices with sectionally antitone mappings

An algebraic axiomatization of Łukasiewicz many-valued logic can be provided by means of MV-algebras, and analogously, orthomodular lattices constitute an important algebraic framework for logical computations related to quantum mechanics. As will be shown in Sect. 4, both of these classes of algebras can be recognized as bounded lattices with sectionally antitone involutions. However, not in all the algebraic structures used for the formalization of non-classical logics the corresponding sectional mappings (derived by the logical connective implication) must be involutions. For example, in the case of Heyting algebras or BCK-algebras these mappings are antitone, but not necessarily they are involutions. Hence we introduce formally the concept of a lattice with sectionally antitone mappings which will be used here.

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. For an $a \in L$ the interval $[a, 1] = \{x \in L \mid a \leq x \leq 1\}$ is called a *section*. The algebra $\mathcal{L} = (L, \vee, \wedge, \{^a \mid a \in L\}, 0, 1)$ is called *a lattice with sectionally antitone extensive mappings* if for each $a \in L$ there exists a mapping $x \mapsto x^a$ of $[a, 1]$ into itself, such that

$x \leq y$ implies $x^a \geq y^a$, for all $x, y \in [a, 1]$, and
(i.e. $x \mapsto x^a$ is antitone)
$x^{aa} \geq x$, for all $x \in [a, 1]$.(i.e. $x \mapsto x^a$ is extensive)

In this case $1^a = a$ implies $a^a = 1$. Indeed, $1^{aa} = 1$ yields $a^a = (1^a)^a = 1$.

In particular, if each mapping $x \mapsto x^a$, $x \in [a, 1]$ is an involution, i.e. $x^{aa} = x$, for all $x \in [a, 1]$, then $\mathcal{L}$ is called a *lattice with sectionally antitone involutions* [see, e.g. Chajda et al. (2005)].

Let us note that in our example $(\mathrm{Con}\mathcal{A}, \vee, \wedge, [,], \to, 0_A, 1_A)$ from the introduction, for any $\alpha, \theta \in \mathrm{Con}\mathcal{A}$, with $\alpha \leq \theta$ we can define

$$\theta^\alpha := \theta \to \alpha = \bigvee \{\varphi \in \mathrm{Con}\mathcal{A} \mid [\theta, \varphi] \leq \alpha\}.$$

Since $[\theta, \varphi] \leq \theta \wedge \varphi$ holds in any congruence modular variety, we get $[\theta, \alpha] \leq \alpha$, and hence $\theta^\alpha \geq \alpha$. Since for any $\theta_1, \theta_2, \varphi \in \mathrm{Con}\mathcal{A}$ $\theta_1 \leq \theta_2$ implies $[\theta_1, \varphi] \leq [\theta_2, \varphi]$, we get $\theta_1^\alpha \geq \theta_2^\alpha$ whenever $\alpha \leq \theta_1 \leq \theta_2$. Finally, $[\theta^\alpha, \theta] = [\theta \to \alpha, \theta] \leq \alpha$ implies $\theta^{\alpha\alpha} \geq \theta$. Thus for any $\alpha \in \mathrm{Con}\mathcal{A}$ the mapping $\theta \mapsto \theta^\alpha$, $\theta \in [\alpha, 1_A]$ is a sectionally antitone extensive mapping.

Proposition 2 *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and $\Rightarrow$ a binary operation on L, and define $x^a := x \Rightarrow a$, for any $a, x \in L$, with $x \geq a$. Then the following are equivalent:*

(i) *The binary operation $\Rightarrow$ satisfies (I0), (I1), (I2) and*

$$[(x \Rightarrow y) \Rightarrow y] \wedge (x \vee y) = (x \vee y), \quad \text{for all } x, y \in L. \tag{I3}$$

(ii) *For each $a \in L$ the mapping $x \mapsto x^a$, $x \in [a, 1]$, is an antitone extensive mapping on $[a, 1]$ such that $1^a = a$ and $x \Rightarrow y = (x \vee y)^y$, for all $x, y \in L$.*

Proof (i)$\Rightarrow$(ii). Take $a, x \in L$ arbitrary with $x \geq a$. Then in view of (I1) we get $a \leq x \Rightarrow a = x^a$, and this means that the assignment $x \mapsto x^a$, $x \in [a, 1]$ is a mapping of $[a, 1]$ into itself. Let $a \leq x \leq y$. Then (I2) yields $y^a = y \Rightarrow a \leq x \Rightarrow a = x^a$; hence $x \mapsto x^a$, $x \in [a, 1]$ is antitone. By using (I3), for every $x \in [a, 1]$ we obtain $x^{aa} = (x \Rightarrow a) \Rightarrow a \geq x \vee a = x$, i.e. the mapping $x \mapsto x^a$, $x \in [a, 1]$ is extensive. Finally, (I0) implies $1^a = 1 \Rightarrow a = a$, and $x \Rightarrow y = (x \vee y) \Rightarrow y = (x \vee y)^y$, for all $x, y \in L$.

(ii)$\Rightarrow$(i). Let $\mathcal{L} = (L, \vee, \wedge, \{^a |\ a \in L\}, 0, 1)$ be a lattice with sectionally extensive antitone mappings $x \mapsto x^a$, $x \in [a, 1]$ such that $1^a = a$, for all $a \in L$, and suppose that, for all $x, y \in L$ the operation $\Rightarrow$ satisfies

$$x \Rightarrow y = (x \vee y)^y.$$

Then $(x \vee y) \Rightarrow y = x \Rightarrow y$. Since $1^a = a$ implies $a^a = 1$, we get $x \Rightarrow x = x^x = 1$ and $x \Rightarrow 1 = 1 \Rightarrow 1 = 1$, and also $1 \Rightarrow x = 1^x = x$, for all $x \in L$. Thus (I0) is satisfied. As by definition $x \Rightarrow y = (x \vee y)^y \geq y$, we get $(x \Rightarrow y) \wedge y = y$, for all $x, y \in L$, i.e. (I1) holds. Now assume $x \leq y$. Then $x \vee z \leq y \vee z$, for all $z \in L$, and hence $y \Rightarrow z = (y \vee z)^z \leq (x \vee z)^z = x \Rightarrow z$, for all $x, y, z \in L$ because the map $x \mapsto x^z$, $x \in [z, 1]$ is antitone. Thus (I2) holds for $\Rightarrow$. To prove (I3), let us observe that $(x \Rightarrow y) \Rightarrow y = ((x \vee y) \Rightarrow y) \Rightarrow y = (x \vee y)^{yy}$, for all $x, y \in L$. Since by extensive property $(x \vee y)^{yy} \geq x \vee y$, we obtain $[(x \Rightarrow y) \Rightarrow y] \wedge (x \vee y) = (x \vee y)^{yy} \wedge (x \vee y) = x \vee y$, for all $x, y \in L$. □

The mutual interrelation between involutive right-residuated l-groupoids satisfying condition (I3) and bounded lattices with an antitone involution and sectionally extensive antitone mappings is established in the next theorem. This gives us an alternative approach to involutive right-residuated l-groupoids which is more suitable to algebras used for axiomatization of several non-classical logics.

Theorem 1 (a) *Let $\mathcal{L} = (L, \vee, \wedge, \{^a |\ a \in L\}, \sim, 0, 1)$ be a bounded lattice with an antitone involution $\sim$ and sectionally antitone extensive mappings $x \mapsto x^a$, $x \in [a, 1]$ such that $1^a = a$, for all $a \in L$. If we define*

$$x \to y := (\sim x \vee \sim y)^{\sim x} \tag{2}$$

$$x \odot y := \sim (y \to \sim x) = \sim [(x \vee \sim y)^{\sim y}], \tag{3}$$

for all $x, y \in L$, then $\mathcal{G}(\mathcal{L}) = (L, \vee, \wedge, \odot, \to, 0, 1)$ is an involutive right-residuated l-groupoid such that $\rceil x = \sim x$ holds, and its derived implication $x \Rightarrow y := \rceil y \to \rceil x$ satisfies condition (I3).

(b) *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive right-residuated l-groupoid having the property that its derived implication $\Rightarrow$ satisfies condition (I3). Let $\sim z := z \to 0$, for all $z \in L$, and define*

$$x^a := x \Rightarrow a = \rceil a \to \rceil x, \tag{4}$$

for all $a, x \in L$ with $x \geq a$. Then $\mathcal{L}(\mathcal{G}) = (L, \vee, \wedge, \{^a |\ a \in L\}, \sim, 0, 1)$ is a bounded lattice with an antitone involution $\sim$ and sectionally antitone extensive mappings $x \mapsto x^a$, $x \in [a, 1]$ such that $1^a = a$.

(c) *The correspondence between bounded lattices with an involution $\sim$ and sectionally antitone extensive mappings satisfying $1^a = a$, and involutive right-residuated l-groupoids satisfying condition (I3) is one-to-one, i.e. $\mathcal{G}(\mathcal{L}(\mathcal{G})) = \mathcal{G}$ and $\mathcal{L}(\mathcal{G}(\mathcal{L})) = \mathcal{L}$.*

Before the proof, let us note that the mappings $x \mapsto \sim x$, $x \in L$ and $x \mapsto x^0$, $x \in L$ need not coincide. The second map need not be an involution contrary to the case $x \mapsto \sim x$, $x \in L$.

Proof (a) By definition we have

$$1 \odot x = \sim [(1 \vee \sim x)^{\sim x}] = \sim (1^{\sim x}) \tag{*}$$
$$= \sim (\sim x) = x, \text{ for all } x \in L.$$

Let $x \odot y \leq z$ for some $x, y, z \in L$. Then $\sim [(x \vee \sim y)^{\sim y}] \leq z$ implies that $\sim z \leq (x \vee \sim y)^{\sim y}$. Since $\sim y \leq (x \vee \sim y)^{\sim y}$, together we obtain

$$\sim z \vee \sim y \leq (x \vee \sim y)^{\sim y}.$$

This implies $x \leq x \vee \sim y \leq (x \vee \sim y)^{\sim y \sim y} \leq (\sim z \vee \sim y)^{\sim y} = y \to z$, according to the definition and to the antitony of the mapping $x \mapsto x^{\sim y}$, $x \in [\sim y, 1]$.

Conversely, $x \leq y \to z$ implies $x \vee \sim y \leq (\sim z \vee \sim y)^{\sim y}$, whence we get $(\sim z \vee \sim y)^{\sim y \sim y} \leq (x \vee \sim y)^{\sim y}$; thus $\sim [(x \vee \sim y)^{\sim y}] \leq \sim [(\sim z \vee \sim y)^{\sim y \sim y}]$. Because the map $x \mapsto x^{\sim y}$, $x \in [\sim y, 1]$ is extensive $(\sim z \vee \sim y)^{\sim y \sim y} \geq \sim z$, whence we deduce $\sim [(\sim z \vee \sim y)^{\sim y \sim y}] \leq \sim (\sim z) = z$. Thus we obtain

$$x \odot y = \sim [(x \vee \sim y)^{\sim y}] \leq z.$$

Since $\mathcal{G}(\mathcal{L})$ satisfies the right-adjointness property and $(*)$, it is a right-residuated l-groupoid. Observe also that $\rceil x := x \to 0 = (\sim x \vee \sim 0)^{\sim x} = 1^{\sim x} = \sim x$. Thus the map

$x \mapsto \rceil x$, $x \in L$ is an antitone involution on L, and we can write

$$x \to y = (\rceil x \vee \rceil y)^{\rceil x}, \quad x \odot y = \rceil(y \to \rceil x) = \rceil(x \vee \rceil y)^{\rceil y},$$

and

$$x \Rightarrow y = \rceil y \to \rceil x = (x \vee y)^y.$$

Hence for any $a \in L$ and $x \in [a, 1]$ we get $x^a = (x \vee a)^a = x \Rightarrow a$. Then $\Rightarrow$ satisfies (I3), according to Proposition 2.

(b) Since $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ is involutive, the map $\sim x := x \to 0 = \rceil x$, $x \in L$ is an antitone involution, and by using Lemma 2 we get that $x \Rightarrow y = \rceil y \to \rceil x$ satisfies (I0), (I1) and (I2). Since (I3) is also satisfied by $\Rightarrow$, by defining $x^a := x \Rightarrow a$, for all $a \in L$ and $x \in [a, 1]$, and using Proposition 2, we obtain that $\mathcal{L}(\mathcal{G}) = (L, \vee, \wedge, \{^a | a \in L\}, 0, 1)$ is a lattice with sectionally antitone extensive mappings $x \mapsto x^a$, $x \in [a, 1]$ satisfying $1^a = a$.

(c) First, we prove that $\mathcal{G}(\mathcal{L}(\mathcal{G})) = \mathcal{G}$.

Indeed, in $\mathcal{L}(\mathcal{G})$ we have $\sim x = x \to 0 = \rceil x$, for all $x \in L$ where $\rceil x := x \to 0$ is defined in $\mathcal{G}$. Then by (a), $\rceil x$ has the same meaning as in $\mathcal{G}(\mathcal{L}(\mathcal{G}))$. In view of (2), for all $x, y \in L$ the operation $\to$ in $\mathcal{G}(\mathcal{L}(\mathcal{G}))$ is defined as

$x \to y := (\sim x \vee \sim y)^{\sim x} = (\rceil x \vee \rceil y)^{\rceil x} = (\rceil x \vee \rceil y) \Rightarrow \rceil x$, where $\Rightarrow$ is the derived implication of $\mathcal{G}$. Since (I0) holds in $\mathcal{G}$, we get $(\rceil x \vee \rceil y) \Rightarrow \rceil x = \rceil y \Rightarrow \rceil x$. Thus we obtain $x \to y = \rceil y \Rightarrow \rceil x$. Since in view of (b), $\rceil y \Rightarrow \rceil x$ also equals to $x \to y$ in $\mathcal{G}$, the operation $\to$ in the right-residuated l-groupoid $\mathcal{G}(\mathcal{L}(\mathcal{G}))$ coincides with the operation $\to$ in $\mathcal{G}$. Therefore, in view of Remark 1, $\odot$ represents the same operation in $\mathcal{G}$ and $\mathcal{G}(\mathcal{L}(\mathcal{G}))$. Because these algebras are defined on the same bounded lattice $(L, \vee, \wedge, 0, 1)$, they coincide, i.e. $\mathcal{G}(\mathcal{L}(\mathcal{G})) = \mathcal{G}$.

To prove $\mathcal{L}(\mathcal{G}(\mathcal{L})) = \mathcal{L}$, first observe that for any $x \in L$, $\sim x$ in $\mathcal{L}(\mathcal{G}(\mathcal{L}))$ is defined as $x \to 0 = \rceil x$ in $\mathcal{G}(\mathcal{L})$, and this is the same as $\sim x$ in $\mathcal{L}$, according to (a). Hence the algebras $\mathcal{L}$ and $\mathcal{L}(\mathcal{G}(\mathcal{L}))$ are defined on the same bounded lattice $(L, \vee, \wedge, \sim, 0, 1)$ with an antitone involution. Therefore, it is enough to prove that the mappings $x \mapsto x^a$, $x \in [a, 1]$ are the same in $\mathcal{L}(\mathcal{G}(\mathcal{L}))$ and $\mathcal{L}$. Observe that x^a in $\mathcal{L}(\mathcal{G}(\mathcal{L}))$ by definition is the same as $\rceil a \to \rceil x$ in the right-residuated l-groupoid $\mathcal{G}(\mathcal{L})$. By the definition of $\mathcal{G}(\mathcal{L})$ in (a) we get

$$\rceil a \to \rceil x = \sim a \to \sim x = (a \vee x)^a = x^a,$$

where x^a is defined in $\mathcal{L}$ for all $a, x \in$ with $x \geq a$. Hence x^a in $\mathcal{L}(\mathcal{G}(\mathcal{L}))$ is the same as x^a in $\mathcal{L}$, and this completes the proof. □

Corollary 1 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive right-residuated l-groupoid. Then the following assertions are equivalent:*

(i) *The derived implication $\Rightarrow$ satisfies identity (I3).*
(ii) *$x \odot y = \rceil(y \to \rceil x)$ holds for all $x, y \in L$.*
(iii) *$\mathcal{G}$ satisfies condition (C).*

Proof Since $\mathcal{G}$ satisfies the double negation law, in view of Proposition 1, (ii) and (iii) are equivalent.

(i)⇒(ii). If (i) holds then $\Rightarrow$ satisfies all the conditions (I0),…,(I3), according to Lemma 2. Now (ii) follows by applying Proposition 2 and Theorem 1.

(ii)⇒(i). Since $\mathcal{G}$ is involutive, in view of Lemma 2, $\Rightarrow$ satisfies (I1). This implies $y \leq (x \Rightarrow y) \Rightarrow y$, for any $x, y \in L$. Observe that in order to prove (I3) it is enough to show that $x \leq (x \Rightarrow y) \Rightarrow y$. We have

$(x \Rightarrow y) \Rightarrow y = \rceil y \to \rceil(x \Rightarrow y) = \rceil y \to \rceil(\rceil y \to \rceil x) = \rceil y \to (x \odot \rceil y)$. Now, $x \odot \rceil y \leq x \odot \rceil y$ gives $x \leq \rceil y \to (x \odot \rceil y) = (x \Rightarrow y) \Rightarrow y$, completing the proof. □

Observe that residuated lattices can be characterized as integral residuated l-groupoids where the operation $\odot$ is associative and commutative. Hence it is important in our case to know under what conditions the above properties hold.

Theorem 2 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive right-residuated l-groupoid satisfying $x \odot y = \rceil(y \to \rceil x)$ for all $x, y \in L$ and $\Rightarrow$ its derived implication. Then the following hold true:*

(i) *$\mathcal{G}$ is integral if and only if $x \Rightarrow 0 = x \to 0$, for all $x \in L$.*
(ii) *$\mathcal{G}$ is commutative if and only if $\Rightarrow$ and $\to$ coincide.*
(iii) *$\odot$ is associative if and only if*

$$(x \odot y) \Rightarrow z = x \Rightarrow (y \Rightarrow z), \quad \textit{for all } x, y \in L. \quad \text{(D)}$$

Proof (i) If $1 \odot x = x \odot 1 = x$ holds for all $x \in L$, then $x \leq 1 \to x$, and $1 \to x = (1 \to x) \odot 1 \leq x$, hence $x = 1 \to x$. Then $x \to 0 = \rceil x = 1 \to \rceil x = \rceil(\rceil x) \Rightarrow \rceil 1 = x \Rightarrow 0$, because $\mathcal{G}$ satisfies the double negation law.

Conversely, suppose that $x \Rightarrow 0 = x \to 0$, for all $x \in L$. Then $x \odot 1 = \rceil(1 \to \rceil x) = \rceil(\rceil(\rceil x) \Rightarrow \rceil 1) = \rceil(x \Rightarrow 0) = \rceil(x \to 0) = \rceil(\rceil x) = x$.

(ii) By our assumption, $x \odot \rceil y = \rceil(\rceil y \to \rceil x) = \rceil(x \Rightarrow y)$. Hence, $x \Rightarrow y = \rceil(x \odot \rceil y)$, for all $x, y \in L$. If $\odot$ is commutative, then $x \Rightarrow y = \rceil(x \odot \rceil y) = \rceil(\rceil y \odot x) = \rceil(\rceil(x \to \rceil(\rceil y)) = x \to y$, for all $x, y \in L$.

Conversely, $x \Rightarrow y = x \to y$ implies $x \Rightarrow \rceil y = x \to \rceil y$. This means that $\rceil(\rceil y) \to \rceil x = x \to \rceil y$, i.e. $y \to \rceil x = x \to \rceil y$. Then for all $x, y \in L$ we have $x \odot y = \rceil(y \to \rceil x) = \rceil(x \to \rceil y) = y \odot x$; hence $\mathcal{G}$ is commutative.

(iii) We have $(x \odot y) \odot z = \rceil(z \to \rceil(x \odot y)) = \rceil(z \to (y \to \rceil x))$, for all $x, y, z \in L$. Observe that $(x \odot y) \Rightarrow \rceil z = \rceil(\rceil z) \to \rceil(x \odot y) = z \to (y \to \rceil x)$. Hence $(x \odot y) \odot$

$z = \rceil((x \odot y) \Rightarrow \rceil z)$. Similarly, we get $x \odot (y \odot z) = \rceil((y \odot z) \rightarrow \rceil x) = \rceil(\rceil(z \rightarrow \rceil y) \rightarrow \rceil x) = \rceil(x \Rightarrow (z \rightarrow \rceil y)) = \rceil(x \Rightarrow (y \Rightarrow \rceil z))$.

First, suppose that $\odot$ is associative. Then $(x \odot y) \odot z = x \odot (y \odot z)$ implies

$$(x \odot y) \Rightarrow \rceil z = x \Rightarrow (y \Rightarrow \rceil z),$$

and $(x \odot y) \Rightarrow z = (x \odot y) \Rightarrow \rceil(\rceil z) = x \Rightarrow (y \Rightarrow \rceil(\rceil z)) = x \Rightarrow (y \Rightarrow z)$, for all $x, y, z \in L$, which is (D).

Conversely, suppose that (D) holds. Then $\rceil((x \odot y) \Rightarrow \rceil z) = \rceil(x \Rightarrow (y \Rightarrow \rceil z))$ is also satisfied, for all $x, y, z \in L$. In view of the above formulas, this means that $(x \odot y) \odot z = x \odot (y \odot z)$, for all $x, y, z \in L$. Thus $\odot$ is associative. □

Corollary 2 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be an involutive right-residuated l-groupoid such that $\Rightarrow$ satisfies condition (I3). Then $\mathcal{G}$ is an integral commutative residuated lattice if and only if $\odot$ is associative.*

Proof Since the only if part is clear, and $\mathcal{G}$ is integral whenever it is commutative, we have to show only that $\odot$ is commutative, whenever it is associative.

Suppose that $\odot$ is associative. Since we have $x \odot y = \rceil(y \rightarrow \rceil x)$ by Corollary 1, Theorem 2 yields $(x \odot y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$. Then Lemma 2 implies

$$x \odot y \leq z \Leftrightarrow (x \odot y) \Rightarrow z = 1$$
$$\Leftrightarrow x \Rightarrow (y \Rightarrow z) = 1 \Leftrightarrow x \leq y \Rightarrow z.$$

Thus we get $x \leq y \rightarrow z$ if and only if $x \odot y \leq z$ if and only if $x \leq y \Rightarrow z$, and this implies $y \rightarrow z \leq y \Rightarrow z$ and $y \Rightarrow z \leq y \rightarrow z$. Hence $y \rightarrow z = y \Rightarrow z$, for all $y, z \in L$, and now by using Theorem 2(ii) we obtain that $\odot$ is commutative. □

It is known that any integral commutative residuated lattice $\mathcal{L}$ satisfying the double negation is involutive [see, e.g. Kondo (2011)]. Moreover, $x \odot y = \rceil(y \rightarrow \rceil x)$ holds in $\mathcal{L}$, according to [2; Theorem 2.40]. Hence, by Theorem 3(ii) $\Rightarrow$ and $\rightarrow$ coincide in $\mathcal{L}$, and in view of Corollary 1 and Theorem 1(b) we obtain the following:

Corollary 3 *Let $\mathcal{L}=(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a (commutative, integral) residuated lattice satisfying the double negation law. Then $\Rightarrow$ and $\rightarrow$ coincide, and for each $a \in L$, $x^a := x \rightarrow a$, $x \in [a, 1]$ is an antitone extensive mapping.*

4 Examples and applications

4.1 Sectionally pseudocomplemented lattices with an added involution

In this section we show how useful can be lattices with an antitone involution and sectionally extensive mappings. This will be shown by examples of algebras used frequently in mathematics as well as in applications.

A bounded lattice L is called *pseudocomplemented* if for any $x \in L$ there exists an element $x^* \in L$ such that

$$y \wedge x = 0 \quad \text{if and only if } y \leq x^*.$$

It is evident that $x^{**} \geq x$, and $x \leq y$ implies $y^* \leq x^*$, for any $x, y \in L$. If for any $a \in L$ the section $[a, 1]$ is a pseudocomplemented lattice, then L is called *sectionally pseudocomplemented*.

It is worth mentioning that sectionally pseudocomplemented lattices capture the relativity of the pseudocomplement slightly better than the so-called relatively pseudocomplemented lattices. Namely in a relatively pseudocomplemented lattice L, the relative pseudocomplement $x \rightarrow y$ of an element $x \in L$ with respect to $y \in L$ need not belong to the interval $[y, 1]$: however, it is known that any relatively pseudocomplemented bounded lattice is also sectionally pseudocomplemented [see Chajda (2003)]. Moreover, as it is shown in Chajda (2003), sectionally pseudocomplemented lattices enable us to extend the concept of relative pseudocomplementation also for nondistributive lattices. For instance, in Chajda and Radeleczki (2003) is proved that any algebraic $\wedge$-semidistributive lattice is sectionally pseudocomplemented; in particular, finite sublattices of free lattices are sectionally pseudocomplemented lattices which are not distributive, in general.

Let L be a bounded sectionally pseudocomplemented lattice. For any $a \in L$ denote by x^a the pseudocomplement of an element $x \in [a, 1]$ in the sublattice $([a, 1], \leq)$, and define $x \Rightarrow y := (x \vee y)^y$, for all $x, y \in L$. Observe that $x \mapsto x^a$, $x \in [a, 1]$ is an antitone extensive mapping of $[a, 1]$ into itself for each $a \in L$.

Indeed, $x^a \in [a, 1]$ by definition, and for any $a \leq x \leq y$ we have $y^a \leq x^a$, and $x^{aa} \geq x$. Then by Proposition 2, $\Rightarrow$ satisfies the conditions (I0),…, (I3).

Now let $\sim$ be an antitone involution on L. If we define

$$x \rightarrow y := (\sim x \vee \sim y)^{\sim x} \quad \text{and}$$
$$x \odot y :=\sim [(x \vee \sim y)^{\sim y}] =\sim (x \Rightarrow \sim y),$$

for all $x, y \in L$, then by Theorem 1(a) we obtain an involutive right-residuated l-groupoid $\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ such that $\rceil x = x \rightarrow 0 =\sim x$, for all $x \in L$, and its derived implication coincides with $\Rightarrow$.

A well-known example for a sectionally pseudocomplemented lattice admitting an antitone involution is the five-element nondistributive lattice N_5. In view of Chajda (2003) and Chajda and Radeleczki (2003) sectionally pseudocomplemented bounded lattices are characterized by the following identities:

(P1) $x \Rightarrow x = 1$, $1 \Rightarrow x = x$, for all $x \in L$;
(P2) $(x \vee y) \Rightarrow y = x \Rightarrow y$, $y \wedge (x \Rightarrow y) = y$, for all $x, y \in L$;
(P3) $[(x \Rightarrow y) \Rightarrow y] \wedge (x \vee y) = (x \vee y)$, for all $x, y \in L$;
(P4) $([(x \vee z) \wedge (y \vee z)] \Rightarrow z) \wedge ([(x \vee z) \wedge (y \Rightarrow z)] \Rightarrow z) = x \wedge z$, for all $x, y, z \in L$.

Let us observe that the conjunction of (P1), (P2) and (P3) is equivalent to the conjunction of (I0), (I1), (I2) and (I3). By the above characterization $\Rightarrow$ in $\mathcal{G}$ also satisfies (P4). Moreover, using this characterization and Theorem 1, we deduce the following:

Proposition 3 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be an involutive right-residuated l-groupoid. Then its derived implication $\Rightarrow$ satisfies condition (P3) and (P4) if and only if $(L, \vee, \wedge)$ is a sectionally pseudocomplemented lattice with an antitone involution such that for any $x, y \in L$ with $x \geq y$, $x \Rightarrow y$ is equal to the pseudocomplement of x in $[y, 1]$.*

We note that $\mathcal{G}$ is neither integral nor associative, in general. Clearly, if $\odot$ is associative, then $\mathcal{G}$ is integral by Corollary 2. If $\mathcal{G}$ is integral, then we have $x^* = x \Rightarrow 0 = x \rightarrow 0 = \sim x$, according to Theorem 2. It is known that the map $x \mapsto x^*$, $x \in L$ is an involution on L if and only if $(L, \vee, \wedge)$ is a Boolean lattice. Hence $\mathcal{G}$ is integral if and only if $(L, \vee, \wedge)$ is a Boolean lattice.

4.2 Residuated lattices corresponding to Nelson algebras

Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice with an antitone involution $\sim$. If for all $x, y \in L$ the inequality

$$x \wedge \sim x \leq y \vee \sim y$$

holds; then $\mathcal{K} = (L, \vee, \wedge, \sim 0, 1)$ is called a *Kleene algebra*. If for $a, b \in L$ there exists a greatest element $x \in L$ such that $a \wedge x \leq b$, then this x is called *the relative pseudocomplement of a with respect to b*, and it is denoted by $a \rhd b$. A *quasi-Nelson* algebra is a Kleene algebra $\mathcal{K}$ such that $a \rhd (\sim a \vee b)$ exists for all a,$b \in L$. [see, e.g. Cignoli (1986)]. $a \rhd (\sim a \vee b)$ is denoted simply by $a \rightarrow b$. A *Nelson algebra* is an algebra $\mathcal{N} = (A, \vee, \wedge, \rightarrow, \sim 0, 1)$ of type (2,2,2,1,0,0), such that $(A, \vee, \wedge, \sim 0, 1)$ is a quasi-Nelson algebra with $\rightarrow$, and $\rightarrow$ satisfies

$$(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z), \quad \text{for all } x, y, z \in A, \quad \text{(N)},$$

i.e. the so-called *Nelson-identity*.

Nelson algebras are the algebraic counterparts of the *constructive logic with strong negation* [see Järvinen et al. (2013); Jarvinen and Radeleczki (2014)]. Spinks and Veroff (2010) proved that to any Nelson algebra $\mathcal{N} = (A, \vee, \wedge, \rightarrow, \sim 0, 1)$ corresponds an integral commutative residuated lattice $\mathcal{L}(\mathcal{N}) = (A, \vee, \wedge, *, \Rightarrow, 0, 1)$. For any $x, y \in A$ the operations $\Rightarrow$ and $*$ are defined as follows:

$$x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x),$$
$$x * y := \sim (x \rightarrow \sim y) \vee \sim (y \rightarrow \sim x)$$

In view of Spinks and Veroff (2010) we have $\urcorner x := x \Rightarrow 0 = \sim x$, for all $x \in A$, which is an antitone involution. Thus $\urcorner(\urcorner x) = x$, and applying Theorem 2.40 in Bělohlávek (2002), we obtain

$$x * y = \urcorner(y \Rightarrow \urcorner x),$$

for all $x, y \in A$, and hence $\Rightarrow$ and the derived implication of $\mathcal{L}(\mathcal{N})$ coincide. Clearly, the residuated lattice $\mathcal{L}(\mathcal{N})$ satisfies the condition (C) and (I3) (see, e.g. Corollary 1). Let $x^a := x \Rightarrow a$, for all $x, y \in A$. Then for each $a \in L$ the assignment $x \mapsto x^a$, $x \in [a, 1]$ is an antitone extensive mapping, according to Corollary 3. An other important property of $\mathcal{L}(\mathcal{N})$ is 3-*potency* [see Spinks and Veroff (2010)], which means that it satisfies the identity:

$$x \Rightarrow (x \Rightarrow (x \Rightarrow y)) = x \Rightarrow (x \Rightarrow y), \text{ for all } x, y \in A.$$

Nelson algebras are also fundamental structures in Rough set theory [see Pagliani and Chakraborty (2008) or Jarvinen and Radeleczki (2014)]. During the last decade new approaches have been developed that combine tools of Fuzzy set theory with that one of Rough set theory, like the investigations of intuitionistic fuzzy sets, and fuzzy rough sets [see, e.g. Cornelis et al. (2007)]. Our expectation is that the algebraic structures behind these constructions can be reduced to involutive right-residuated l-groupoids.

4.3 Bounded lattices with sectionally antitone involutions

In this paragraph we are going to show that bounded lattices with sectionally antitone involutions are common structures equivalent to involutive right-residuated l-grupoids having the property that their induced implication $\Rightarrow$ satisfies a condition which will be denoted by (I3*). This will be applied in the next Sect. 5.

Let $\mathcal{L} = (L, \vee, \wedge, \{^a | \ a \in L\}, 0, 1)$ be a lattice with sectionally antitone mappings $x \mapsto x^a$, $x \in [a, 1]$ and define the operation $x \Rightarrow y := (x \vee y)^y$, for all $x, y \in L$.

Remark 2 Since $(x \vee y)^y \geq y$, we have $(x \Rightarrow y) \Rightarrow y = (x \vee y)^{yy}$. Hence the identity $(x \Rightarrow y) \Rightarrow y = x \vee y$, $x, y \in L$ holds if and only if $(x \vee y)^{yy} = x \vee y$, for all $x, y \in L$. Of course, this is equivalent to the condition that $x^{aa} = x$, for all $a \in L$ and $x \in [a, 1]$. Therefore, operation $\Rightarrow$ satisfies the identity

$$(x \Rightarrow y) \Rightarrow y = x \vee y, \quad \text{for all } x, y \in L \tag{I3*}$$

if and only if $\mathcal{L}$ is a lattice with sectionally antitone involutions. In that case, define $\sim x := x^0$, for all $x \in L$. Then $x \mapsto \sim x$, $x \in L$ is an antitone involution on the lattice L; moreover, $x \Rightarrow 0 = x^0 = \sim x$, for all $x \in L$.

Since (I3*) implies condition (I3), we can apply Theorem 1 to get

Theorem 3 (a) *Let $\mathcal{L} = (L, \vee, \wedge, \{^a|\ a \in L\}, 0, 1)$ be a bounded lattice with sectionally antitone involutions $x \mapsto x^a$, $x \in [a, 1]$. If we define $\sim x := x^0$, $x \to y := (\sim x \vee \sim y)^{\sim x}$ and $x \odot y := \sim (y \to \sim x) = \sim [(x \vee \sim y)^{\sim y}]$, for all $x, y \in L$, then $\mathcal{G}(\mathcal{L}) = (L, \vee, \wedge, \odot, \to, 0, 1)$ is an involutive integral right-residuated l-groupoid with $\rceil x = \sim x$, and its derived implication $\Rightarrow$ satisfies (I3*).*

(b) *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive integral right-residuated l-groupoid such that its derived implication $\Rightarrow$ satisfies condition (I3*), and define $x^a := x \Rightarrow a$, for all $a, x \in L$ with $x \geq a$. Then $\mathcal{L}(\mathcal{G}) = (L, \vee, \wedge, \{^a|\ a \in L\}, 0, 1)$ is bounded lattice with sectionally antitone involutions $x \mapsto x^a$, $x \in [a, 1]$, and $x^0 = x \to 0$.*

(c) *The correspondence between bounded lattices with sectionally antitone involutions and involutive integral right-residuated l-groupoids satisfying (I3*) is one-to-one, i.e. $\mathcal{G}(\mathcal{L}(\mathcal{G})) = \mathcal{G}$ and $\mathcal{L}(\mathcal{G}(\mathcal{L})) = \mathcal{L}$.*

Proof (a) We have to show only that $\mathcal{G}(\mathcal{L}) = (L, \vee, \wedge, \odot, \to, 0, 1)$ is integral. Since $x \Rightarrow 0 = x^0 = \sim x$ and $x \to 0 = 1^{\sim x} = \sim x$ for all $x \in L$ by definition, we get $x \Rightarrow 0 = x \to 0$. Hence $\mathcal{G}(\mathcal{L})$ is integral, according to Theorem 2(i).

(b) In view of Theorem 1(b), now it suffices to prove $x^0 = x \to 0$. Since $\mathcal{G}$ is integral, using the definition of $\Rightarrow$ and Theorem 2(i) we obtain $x^0 = x \Rightarrow 0 = x \to 0$, for all $x \in L$. (c) is clear. □

Proposition 4 *Let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be a right-residuated l-groupoid. Then the following assertions are equivalent:*

(i) *$(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$, for all $x, y \in L$, and $\mathcal{G}$ is involutive.*
(ii) *$\Rightarrow$ satisfies (I3*), and $\mathcal{G}$ is involutive.*
(iii) *$\mathcal{G}$ satisfies the double negation law, divisibility, and condition (C).*

Proof (i)⇒(ii). Let $x, y, z \in L$ arbitrary. Since $\mathcal{G}$ is involutive, by Lemma 2 we have $(x \vee y) \Rightarrow y = x \Rightarrow y$, $1 \Rightarrow x = x$, and $y \leq z$ implies $y \Rightarrow z = 1$. Now, using (i) we deduce (I3*). Indeed, $(x \Rightarrow y) \Rightarrow y = ((x \vee y) \Rightarrow y) \Rightarrow y = (y \Rightarrow (x \vee y)) \Rightarrow (x \vee y) = 1 \Rightarrow (x \vee y) = x \vee y$, for all $x, y \in L$.

(ii)⇒(iii). Since $\mathcal{G}$ is involutive, it satisfies the double negation law. Because (I3*) implies (I3), by Corollary 1 we deduce that $\mathcal{G}$ satisfies (C) and for any $x, y \in L$ we have $x \odot y = \rceil(y \to \rceil x)$. By using this formula and (I3*) we obtain

$$(x \to y) \odot x = \rceil(x \to \rceil(x \to y)) = \rceil(x \to \rceil(\rceil y \Rightarrow \rceil x)) =$$
$$= \rceil((\rceil y \Rightarrow \rceil x) \Rightarrow \rceil x) = \rceil(\rceil y \vee \rceil x) = x \wedge y, \text{ for all } x, y \in L,$$

which proves divisibility.

(iii)⇒(i). Since $\mathcal{G}$ satisfies (C) and the double negation law, in view of Proposition 1 it is involutive and satisfies $x \odot y = \rceil(y \to \rceil x)$, for all $x, y \in L$. Hence repeating the previous proof we get $(x \to y) \odot x = \rceil((\rceil y \Rightarrow \rceil x) \Rightarrow \rceil x)$. Now, substituting x by $\rceil x$ and y by $\rceil y$, for any $x, y \in L$ we get

$$\rceil((y \Rightarrow x) \Rightarrow x) = (\rceil x \to \rceil y) \odot (\rceil x),$$

and then interchanging x and y we obtain

$$\rceil((x \Rightarrow y) \Rightarrow y) = (\rceil y \to \rceil x) \odot (\rceil y).$$

Since $(\rceil x \to \rceil y) \odot (\rceil x) = \rceil x \wedge \rceil y = (\rceil y \to \rceil x) \odot (\rceil y)$ by divisibility, we deduce $(y \Rightarrow x) \Rightarrow x = (x \Rightarrow y) \Rightarrow y$, for all $x, y \in L$. □

We note that the identity from Proposition 4(i) is called *Łukasiewicz identity*. Hence we can introduce the following concept:

Definition 2 If an integral involutive right-residuated l-groupoid $\mathcal{G}$ satisfies Łukasiewicz identity, then we say that $\mathcal{G}$ has *Łukasiewicz type.*

If $\mathcal{G}$ has Łukasiewicz type, then in view of the proof of (ii)⇒(iii) from Proposition 4, $\mathcal{G}$ also satisfies $x \odot y = \rceil(y \to \rceil x)$, for all $x, y \in L$ and (I3).

5 Łukasiewicz type right-residuated l-groupoids and basic algebras

Basic algebras were introduced in Chajda (2011) and Chajda et al. (2009) as a common generalization of MV-algebras and othomodular lattices. The details of this generalization will be mentioned later. It is worth noticing that MV-algebras form an algebraic counterpart of Łukasiewicz many-valued logic, and othomodular lattices represent an algebraic framework for certain logical computations motivated by foundational issues of quantum theory.

Definition 3 By a *basic algebra* is meant an algebra $\mathcal{A} = (A, \oplus, \urcorner, 0)$ of type (2, 1, 0) satisfying the following axioms:

(BA1) $x \oplus 0 = x$, for all $x \in A$
(BA2) $\urcorner\urcorner x = x$, for all $x \in A$
(BA3) $\urcorner(\urcorner x \oplus y) \oplus y = \urcorner(\urcorner y \oplus x) \oplus x$, for all $x, y \in A$
(BA4) $\urcorner(\urcorner(\urcorner(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, for all $x, y, z \in A$, where $1 := \urcorner 0$.

Recall from Chajda (2011), Chajda et al. (2005) and Chajda et al. (2009) that every basic algebra is a bounded lattice where $x \vee y = \urcorner(\urcorner x \oplus y) \oplus y$, $x \wedge y = \urcorner(\urcorner x \vee \urcorner y)$, for all $x, y \in A$ and the induced order $\leq$ is given by

$x \leq y$ if and only if $\urcorner x \oplus y = 1$.

Of course, $0 \leq x \leq 1$, for all $x \in A$. In every basic algebra $\mathcal{A} = (A, \oplus, \urcorner, 0)$ for all $x, y \in L$ we define the term operations $\odot$, $\to$ and $\Rightarrow$ as follows:

$$x \odot y = \urcorner(\urcorner x \oplus \urcorner y),\ x \to y = y \oplus \urcorner x \quad \text{and} \quad x \Rightarrow y = \urcorner x \oplus y.$$

One can observe that $x \Rightarrow 0 = \urcorner x$, and $x \Rightarrow y = \urcorner y \to \urcorner x$, for all $x, y \in L$. The following theorem was established in Chajda et al. (2009):

Theorem 4 (i) *Let $\mathcal{L} = (L, \vee, \wedge, \{^a \mid a \in L\}, 0, 1)$ be a bounded lattice with sectionally antitone involutions. If we define*

$$x \oplus y := (x^0 \vee y)^y \text{ and } \urcorner x := x^0, \quad \text{for all } x, y \in L,$$

then $\mathcal{A}(\mathcal{L}) = (L, \oplus, \urcorner, 0)$ is a basic algebra. We have $x \vee y = \urcorner(\urcorner x \oplus y) \oplus y$, $x \wedge y = \urcorner(\urcorner x \vee \urcorner y)$, for all $x, y \in L$ and $x^a = \urcorner x \oplus a$, for $x \in [a, 1]$.

(ii) *Let $\mathcal{A} = (A, \oplus, \urcorner, 0)$ be a basic algebra and set*

$$x \vee y := \urcorner(\urcorner x \oplus y) \oplus y,\ x \wedge y := \urcorner(\urcorner x \vee \urcorner y), \text{ for all } x, y \in A.$$

Define $x^a := \urcorner x \oplus a$, for all $a, x \in A$ with $a \leq x$, and $1 := \urcorner 0$. Then $\mathcal{L}(\mathcal{A}) = (A, \vee, \wedge, \{^a \mid a \in A\}, 0, 1)$ is a bounded lattice with sectionally antitone involutions $x \mapsto x^a$, $x \in [a, 1]$, where the lattice order is given by $x \leq y$ iff $\urcorner x \oplus y = 1$, and we have $\urcorner x = x^0$, $x \oplus y := (x^0 \vee y)^y$.

(iii) *The correspondence between bounded lattices with sectionally antitone involutions and basic algebras thus established is one-to-one, i.e. $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.*

Now, let $\mathcal{A} = (A, \oplus, \urcorner, 0)$ be a basic algebra and $(A, \vee, \wedge, 0, 1)$ the bounded lattice determined by $\mathcal{A}$, according to Theorem 4(ii). Then $1 := \urcorner 0$, and in view of Theorem 4(ii) this is a lattice with sectionally antitone involutions $x \mapsto x^a$, $x \in [a, 1]$, where $x^a := \urcorner x \oplus a$, for all $a, x \in A$. In particular, $x^0 = \urcorner x$, $x \in A$ determines an involution on the whole lattice. Further, define

$$x \to y = (\urcorner x \vee \urcorner y)^{\urcorner x} \quad \text{and}$$
$$x \odot y = \urcorner[(x \vee \urcorner y)^{\urcorner y}], \quad \text{for all } x, y \in A.$$

Then applying Theorem 3(a) with $\sim x = x^0 = \urcorner x$ we obtain that $\mathcal{G}(\mathcal{A}) = (A, \vee, \wedge, \odot, \to, 0, 1)$ is an involutive integral right-residuated l-groupoid such that $\Rightarrow$ satisfies condition (I3*). By Proposition 4, the identity

$$(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x, \quad \text{for all } x, y \in A$$

holds; thus $\mathcal{G}(\mathcal{A})$ is of a Łukasiewicz type. By Theorem 4(ii), then we obtain $x \oplus y := (x^0 \vee y)^y = (\urcorner x \vee y)^y$. Thus we get $\urcorner(\urcorner x \oplus \urcorner y) = \urcorner[(x \vee \urcorner y)^{\urcorner y}] = x \odot y$, $x \to y = (\urcorner x \vee \urcorner y)^{\urcorner x} = y \oplus \urcorner x$ and $x \Rightarrow y = \urcorner y \to \urcorner x = \urcorner x \oplus y$, for all $x, y \in A$.

Conversely, let $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be an involutive right-residuated l-groupoid of Łukasiewicz type and $\Rightarrow$ its derived implication. Then $(x \vee y) \Rightarrow x = y \Rightarrow x$, and $x \odot y = \urcorner(y \to \urcorner x)$, for all $x, y \in L$, in view of Lemma 2 and Remark 3. For any $a, x \in L$ with $x \geq a$ define $x^a := x \Rightarrow a$. Since $\mathcal{G}$ is integral, and by Proposition 4 $\Rightarrow$ satisfies (I3*), we can apply Theorem 3(b) and we get that $\mathcal{L}(\mathcal{G}) = (L, \vee, \wedge, \{^a \mid a \in L\}, 0, 1)$ is a bounded lattice with sectionally antitone involutions $x \mapsto x^a$, $x \in [a, 1]$, such that $x^0 = x \to 0$. Now, if we define

$$\urcorner x := x \to 0 \quad \text{and}$$
$$x \oplus y := (x^0 \vee y)^y = (\urcorner x \vee y)^y, \quad \text{for all } x, y \in L,$$

by Theorem 4(i) we obtain a basic algebra $\mathcal{A}(\mathcal{G}) = (L, \oplus, \urcorner, 0)$, where $x \vee y = \urcorner(\urcorner x \oplus y) \oplus y$, $x \wedge y = \urcorner(\urcorner x \vee \urcorner y)$ and $x^a = \urcorner x \oplus a$, for $x \in [a, 1]$. We get also

$$\urcorner(\urcorner x \odot \urcorner y) = \urcorner(\urcorner(\urcorner y \to \urcorner(\urcorner x))) = \urcorner y \to \urcorner(\urcorner x) = \urcorner x \Rightarrow y =$$
$$= (\urcorner x \vee y) \Rightarrow y = (\urcorner x \vee y)^y = x \oplus y, \quad \text{and}$$
$$y \oplus \urcorner x = (\urcorner y \vee \urcorner x)^{\urcorner x} = (\urcorner y \vee \urcorner x) \Rightarrow \urcorner x = \urcorner y \Rightarrow \urcorner x = x \to y,$$

for all $x, y \in L$. Now, using the above computations we can formulate the following:

Theorem 5 (a) *Let $\mathcal{A} = (A, \oplus, \urcorner, 0)$ be a basic algebra. For all $x, y \in A$ define*

$$x \odot y := \urcorner(\urcorner x \oplus \urcorner y), \quad \text{and} \quad x \to y := y \oplus \urcorner x.$$

Set $x \vee y := \urcorner(\urcorner x \oplus y) \oplus y$, $x \wedge y := \urcorner(\urcorner x \vee \urcorner y)$, *and* $1 := \urcorner 0$. *Then* $\mathcal{G}(\mathcal{A}) = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a right-residuated l-groupoid of Łukasiewicz type.*

(b) *Let* $\mathcal{G} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *be a right-residuated l-groupoid of Łukasiewicz type. Define* $\urcorner x := x \rightarrow 0$ *and* $x \oplus y := \urcorner(\urcorner x \odot \urcorner y)$, *for all* $x, y \in A$. *Then* $\mathcal{A}(\mathcal{G}) = (A, \oplus, \urcorner, 0)$ *is a basic algebra.*

(c) *The correspondence between basic algebras and right-residuated l-groupoids of Łukasiewicz type thus established is one-to-one, i.e.* $\mathcal{A}(\mathcal{G}(\mathcal{A})) = \mathcal{A}$ *and* $\mathcal{G}(\mathcal{A}(\mathcal{G})) = \mathcal{G}$.

Proof Since (a) and (b) follow from the previous computations, we have to check (c) only. If $\mathcal{A} = (A, \oplus, \urcorner, 0)$ is a basic algebra, then in $\mathcal{G}(\mathcal{A})$ we have $x \odot y = \urcorner(\urcorner x \oplus \urcorner y)$, for all $x, y \in A$, and $1 = \urcorner 0$. Then $x = 1 \odot x = \urcorner(x \rightarrow \urcorner 1) = \urcorner(x \rightarrow \urcorner\urcorner 0) = \urcorner(x \rightarrow 0)$. Thus we get $\urcorner x = \urcorner\urcorner(x \rightarrow 0) = x \rightarrow 0$, by using (BA2). This means that $\urcorner$ is the same operation in $\mathcal{A}$ and $\mathcal{A}(\mathcal{G}(\mathcal{A}))$. Since in $\mathcal{G}(\mathcal{A})$ we have also $\urcorner(\urcorner x \odot \urcorner y) = \urcorner\urcorner(\urcorner\urcorner x \oplus \urcorner\urcorner y) = x \oplus y$, in view of the definition in Theorem 5(b) the operations $\oplus$ in $\mathcal{A}$ and $\mathcal{A}(\mathcal{G}(\mathcal{A}))$ coincide. Hence $\mathcal{A}$ and $\mathcal{A}(\mathcal{G}(\mathcal{A})$ are the same algebras. The fact that $\mathcal{G}(\mathcal{A}(\mathcal{G})) = \mathcal{G}$ can be proved similarly. □

The following Corollary is immediate:

Corollary 4 *Any right-residuated l-groupoid of Łukasiewicz type is term equivalent to a basic algebra. Right-residuated l-groupoids of Łukasiewicz type form a variety.*

Remark 4 Let $\mathcal{A} = (A, \oplus, \urcorner, 0)$ be a basic algebra, and $x \odot y = \urcorner(\urcorner x \oplus \urcorner y)$, for all $x, y \in A$. Let us observe that $\odot$ is associative if and only if $\oplus$ is associative, and $\odot$ is commutative if and only if $\oplus$ is commutative. Indeed,

$(x \odot y) \odot z = \urcorner[\urcorner(x \odot y) \oplus \urcorner z] = \urcorner[(\urcorner x \oplus \urcorner y) \oplus \urcorner z]$, and $x \odot (y \odot z) = \urcorner[\urcorner x \oplus \urcorner(y \odot z)] = \urcorner[\urcorner x \oplus (\urcorner y \oplus \urcorner z)]$. Hence $(x \odot y) \odot z = x \odot (y \odot z)$ if and only if $(\urcorner x \oplus \urcorner y) \oplus \urcorner z = \urcorner x \oplus (\urcorner y \oplus \urcorner z)$, and this is equivalent to $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

The proof of the second statement is straightforward.

Examples

1. *MV-algebras* form an important particular case of basic algebras. They can be defined as associative basic algebras (see, e.g. Chajda (2011)). Since to any basic algebra corresponds a right-residuated l-groupoid of Łukasiewicz type, in view of Remark 4 and Corollary 2, this means that to any MV-algebra corresponds an integral commutative residuated lattice of Łukasiewicz type. We note also that these lattices are always distributive.

2. *Orthomodular lattices* are usually defined as bounded orthocomplemented lattices $\mathcal{L} = (L, \vee, \wedge, \sim, 0, 1)$ satisfying the *orthomodular law*

$$x \leq y \text{ implies } x \vee (\sim x \wedge y) = y. \quad \text{(OML)}$$

Here $\sim$ denotes the *orthocomplementation operation* on L, i.e. $\sim$ is an antitone involution such that $x \wedge \sim x = 0$, for all $x \in L$.

Define $x^a := \sim x \vee a$, for all $x, y \in L$. It is known (see Chajda and Radeleczki (2014) or Botur et al. (2010)) that for each $a \in L$ the mapping $x \mapsto x^a$, $x \in [a, 1]$ is an antitone involution on the section $[a, 1]$, moreover $1^a = a$. Hence, in view of Theorem 4 (and Proposition 4), by defining for all $x, y \in L$ the operations

$$\begin{aligned} x \rightarrow y &:= (\sim x \vee \sim y)^{\sim x} \\ &= \sim(\sim x \vee \sim y) \vee \sim x = (x \wedge y) \vee \sim x \quad \text{and} \\ x \odot y &:= \sim[(x \vee \sim y)^{\sim y}] \\ &= \sim[\sim(x \vee \sim y) \vee \sim y] = (x \vee \sim y) \wedge y, \end{aligned}$$

we obtain a right-residuated l-groupoid $\mathcal{G}(\mathcal{L}) = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of Łukasiewicz type, where $\urcorner x = \sim x$. It is easy to check that $\odot$ is not commutative in general. Therefore, in view of Corollary 2, $\odot$ cannot be even associative.

In Chajda (2011) was shown that by defining $x \oplus y := (x \wedge \sim y) \vee y$ for all $x, y \in L$, we obtain a basic algebra $\mathcal{A} = (L, \oplus, \urcorner, 0)$. It was also proved that basic algebras arising from orthomodular lattices form a subvariety characterized by the identity

$$y = y \oplus (x \wedge y), \quad \text{for all } x, y \in L. \quad \text{(OMI)}$$

which implies also $x \oplus x = x$, for all $x \in L$. Observe that $\mathcal{G}(\mathcal{L})$ is just the right-residuated l-groupoid corresponding to the basic algebra $\mathcal{A}$, according to Theorem 5. Now, an easy computation shows that (OMI) is equivalent to $\urcorner y \rightarrow (x \wedge y) = y$, for all $x, y \in L$. Using the derived implication $\Rightarrow$ of $\mathcal{G}(\mathcal{L})$, this can be reformulated as

$$y = (\urcorner x \vee \urcorner y) \Rightarrow y, \text{ for all } x, y \in L. \quad \text{(OMI*)}$$

Hence residuated l-groupoids corresponding to orthomodular lattices are exactly the right-residuated l-groupoids of Łukasiewicz type satisfying (OMI*).

6 Implication reducts of basic algebras

Since the logical connective implication is the most productive one, because it enables to set up some derivation rules as, e.g. Modus Ponens, we are focused now in a description of implication reducts.

Let $\mathcal{A} = (A, \oplus, \urcorner, 0)$ be a basic algebra. For every $x, y \in A$ define

$$x \Rightarrow y := \urcorner x \oplus y,$$

the so-called *implication* in $\mathcal{A}$, and $1 := 0 \Rightarrow 0$. One can easily check that $\Rightarrow$ satisfies the following identities (see Chajda and Kühr (2013)):

(I0*) $x \Rightarrow x = 1, x \Rightarrow 1 = 1, 1 \Rightarrow x = x$, for all $x \in A$;
(I1*) $y \Rightarrow (x \Rightarrow y) = 1$, for all $x, y \in A$;
(Ł) $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$, for all $x, y \in A$;
(I4) $((x \Rightarrow y) \Rightarrow y) \Rightarrow z) \Rightarrow (x \Rightarrow z) = 1$, for all $x, y \in A$.

Now, consider the right-residuated l-groupoid $\mathcal{G}(\mathcal{A}) = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ which corresponds to the basic algebra $\mathcal{A}$ by Theorem 5(a). Since $x \rightarrow y = y \oplus \urcorner x$, it is easy to see that $\Rightarrow$ coincides with the so-called derived implication in $\mathcal{G}(\mathcal{A})$. Since $\mathcal{G}(\mathcal{A})$ is of Łukasiewicz type, in view of Lemma 2 and Proposition 4, for all $x, y \in A$ the following assertions also hold true:

$$x \leq y \Leftrightarrow x \Rightarrow y = 1;\ (x \Rightarrow y) \Rightarrow y = (x \vee y);$$
$$(x \vee y) \Rightarrow y = x \Rightarrow y.$$

Hence the partial order $\leq$ is also determined by $\Rightarrow$. The fact that 0 is the least element in $(A, \vee, \wedge)$ can be expressed by the law:

(I5) $0 \Rightarrow x = 1, \quad$ for all $x \in A$.

Observe that the previous identities can be inferred from (I0*), (I1*), (Ł), (I4) and (I5) only, even more, we have the following:

Proposition 5 *Let $(A; \Rightarrow, 1)$ be an algebra of type $(2, 0)$ satisfying the identities:*

(i) $x \Rightarrow x = 1, x \Rightarrow 1 = 1, 1 \Rightarrow x = x$, *for all* $x \in A$ *;*
(ii) $y \Rightarrow (x \Rightarrow y) = 1$, *for all* $x, y \in A$*;*
(iii) $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$, *for all* $x, y \in A$*;*
(iv) $((x \Rightarrow y) \Rightarrow y) \Rightarrow z) \Rightarrow (x \Rightarrow z) = 1$ *for all* $x, y, z \in A$.

Define a binary relation $\leq$ on A as follows:

$$x \leq y \text{ if and only if } x \Rightarrow y = 1.$$

Then $\leq$ is a partial order on A, and $(A, \leq)$ is a join-semilattice with greatest element 1, where

$$x \vee y = (x \Rightarrow y) \Rightarrow y, \text{ for all } x, y \in A.$$

Moreover, $x \leq y$ implies $y \Rightarrow z \leq x \Rightarrow z$ and $\Rightarrow$ satisfies

$$((x \Rightarrow y) \Rightarrow y) \Rightarrow y = x \Rightarrow y \text{ for all } x, y \in A.$$

Proof By (i) the defined relation $\leq$ is reflexive and $x \leq 1$, for all $x \in A$. Assume $x \leq y$ and $y \leq x$. Then $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$. By (i) and (iii) we conclude $y = 1 \Rightarrow y = (x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x = 1 \Rightarrow x = x$.

Let $x \leq y$ and $y \leq z$. Then $x \Rightarrow y = 1$ and $y \Rightarrow z = 1$, and by (iv) we get

$$\begin{aligned} 1 = (((x \Rightarrow y) \Rightarrow y) \Rightarrow z) \Rightarrow (x \Rightarrow z) &= ((1 \Rightarrow y) \Rightarrow z) \\ \Rightarrow (x \Rightarrow z) &= (y \Rightarrow z) \Rightarrow (x \Rightarrow z) \\ &= 1 \Rightarrow (x \Rightarrow z) = x \Rightarrow z; \end{aligned}$$

thus $x \leq z$. Hence $\leq$ is a partial order on A with the greatest element 1.

By (ii) we get $y \leq x \Rightarrow y$; thus also $y \leq (x \Rightarrow y) \Rightarrow y$ and $x \leq (y \Rightarrow x) \Rightarrow x = (x \Rightarrow y) \Rightarrow y$, i.e. $(x \Rightarrow y) \Rightarrow y$ is a common upper bound for x and y.

Next we prove that $a \leq b$ implies $b \Rightarrow c \leq a \Rightarrow c$. Indeed, $a \leq b$ yields $a \Rightarrow b = 1$, and hence $(b \Rightarrow c) \Rightarrow (a \Rightarrow c) = ((1 \Rightarrow b) \Rightarrow c) \Rightarrow (a \Rightarrow c) = (((a \Rightarrow b) \Rightarrow b) \Rightarrow c) \Rightarrow (a \Rightarrow c) = 1$, by (iv). Hence $b \Rightarrow c \leq a \Rightarrow c$.

Now, if $x, y \leq z$, then $x \Rightarrow y \geq z \Rightarrow y$ and we get also

$$(x \Rightarrow y) \Rightarrow y \leq (z \Rightarrow y) \Rightarrow y = (y \Rightarrow z) \Rightarrow z = 1 \Rightarrow z = z,$$

proving that $(x \Rightarrow y) \Rightarrow y$ is the least common upper bound of x, y, i.e.

$(x \Rightarrow y) \Rightarrow y = x \vee y$, for all $x, y \in A$. Thus $(A, \leq)$ is a join-semilattice with 1.

Finally, using (iii), (ii) and (i), for any $x, y, z \in A$ we infer

$$\begin{aligned} &((x \Rightarrow y) \Rightarrow y) \Rightarrow y = (y \Rightarrow (x \Rightarrow y)) \\ &\Rightarrow (x \Rightarrow y) = 1 \Rightarrow (x \Rightarrow y) = x \Rightarrow y. \end{aligned}$$

□

In what follows, we will consider the algebra $\mathcal{A}_0 = (A, \Rightarrow, 0)$ of type (2,0) which is called an *implication reduct* of the basic algebra $\mathcal{A}$. We are going to show that the basic algebra $(A, \oplus, \urcorner, 0)$ can be reconstructed from this implication reduct; moreover, the following is true:

Theorem 6 *Let $\mathcal{A}_0 = (A, \Rightarrow, 0)$ be an algebra of type $(2, 0)$, $1 := 0 \Rightarrow 0$, such that $\Rightarrow$ satisfies the identities (i),(ii),(iii),(iv) and (I5). Then by defining*

$$\urcorner x := x \Rightarrow 0 \text{ and } x \oplus y := \urcorner x \Rightarrow y, \ \text{for all } x, y \in A \quad (\times)$$

we obtain a basic algebra $\mathcal{B}(\mathcal{A}_0) = (A, \oplus, \urcorner, 0)$ such that the implication in $\mathcal{B}(\mathcal{A}_0)$ coincides with $\Rightarrow$.

Proof In view of Proposition 5, the definition

$$x \leq y \text{ if and only if } x \Rightarrow y = 1,$$

yields a join-semilattice with greatest element 1 on the set A , where $x \vee y = (x \Rightarrow y) \Rightarrow y$, for all $x, y \in A$. In view of (I5), 0 is the least element of $(A, \leq)$. By using Proposition 5, we obtain also $\rceil(\rceil x) = (x \Rightarrow 0) \Rightarrow 0 = x \vee 0 = x$, for all $x \in A$, and we get that for any $x, y \in A$,

$$x \leq y \text{ implies } \rceil y = y \Rightarrow 0 \leq x \Rightarrow 0 = \rceil x.$$

This means that the mapping $x \mapsto \rceil x, x \in A$ is an antitone involution on $(A, \leq)$, and hence $(A, \leq)$ is a lattice where $x \wedge y = \rceil(\rceil x \vee \rceil y)$, for all $x, y \in A$. Since (i),(ii),(iii),(iv) and (I5) together imply the laws (I0),(I1) and (I2) and $(x \Rightarrow y) \Rightarrow y = x \vee y$, by defining $x^a := x \Rightarrow a$ for all $a, x \in A$, in view of Remark 2, we deduce that the mappings $x \mapsto x^a$, $x \in [a, 1]$ are antitone involutions on each section $[a, 1]$ of the bounded lattice $(A, \vee, \wedge)$. In view of Chajda et al. (2009) (see Theorem 4), for the operations

$$x \oplus y := (x^0 \vee y)^y \quad \text{and} \quad \rceil x := x^0$$

we obtain a basic algebra $(A, \oplus, \rceil, 0)$ on the set A. Since $x^0 = x \Rightarrow 0$, $\rceil$ satisfies $(\times)$, and $x \oplus y = (\rceil x \vee y)^y = (\rceil x \vee y) \Rightarrow y = \rceil x \Rightarrow y$, because (i), (ii), (iii), (iv) and (I5) imply also $(x \vee y) \Rightarrow y = x \Rightarrow y$, for all $x, y \in A$, as we pointed out previously. Finally, the implication in $(A, \oplus, \rceil, 0)$ is given by the term $\rceil x \oplus y$, and $x \oplus y = \rceil x \Rightarrow y$ clearly implies $\rceil x \oplus y = x \Rightarrow y$, for all $x, y \in A$. □

We note that Theorem 6 has also a direct proof which does not use Theorem 4. Observe also that the conditions (i), (ii), (iii) and (iv) are in fact the conditions (I0*), (I1*), (Ł) and (I4).

7 Congruence properties

When varieties of algebras are studied, we are usually interested in their congruence properties to reveal their structure.

An algebra $\mathcal{A} = (A, F)$ is said to be *congruence distributive* whenever its congruence lattice Con$\mathcal{A}$ is distributive. $\mathcal{A}$ is called *congruence permutable*, if $\varphi \circ \theta = \theta \circ \varphi$ holds for all $\theta, \varphi \in$ Con$\mathcal{A}$. A variety $\mathcal{V}$ of algebras is *arithmetical* if every algebra $\mathcal{A} \in \mathcal{V}$ of it is both congruence distributive and congruence permutable. An algebra $\mathcal{A} = (A, F)$ is said to be *congruence regular* if every congruence θ of $\mathcal{A}$ is determined by an arbitrary congruence class $\theta[a]$ (for $a \in A$) of it. Let c be a constant of the algebra $\mathcal{A}$. $\mathcal{A}$ is *c-regular* if $\theta[c] = \varphi[c]$ implies $\theta = \varphi$, for every $\theta, \varphi \in$ Con$A$, and $\mathcal{A}$ is called *c-locally regular* if for each $\theta, \varphi \in$ Con$A$ and any $a \in A$ we have that $\theta[a] = \varphi[a]$ implies $\theta[c] = \varphi[c]$. It is known that an algebra $\mathcal{A}$ is congruence regular if and only if it is c-regular and c-locally regular simultaneously [see Chajda (1998)]. It was proved by Csákány (1970), that a variety $\mathcal{V}$ of algebras are congruence c-regular if and only if there exist binary terms $b_1, \ldots, b_n$ such that $\mathcal{V}$ satisfies the following condition:

$$[\, b_1(x, y) = c, \ldots, b_n(x, y) = c \,] \quad \text{if and only if } x = y.$$

It has been proved in Chajda (1998) that $\mathcal{V}$ is c-locally regular if and only if there exist binary terms $p_1, \ldots, p_m$ such that $\mathcal{V}$ satisfies the following condition:

$$[\, p_1(x, y) = x, \ldots, p_m(x, y) = x \,] \quad \text{if and only if } y = c.$$

It is known that any right-residuated l-groupoid $\mathcal{G}$ is congruence 1-regular with the term $b(x, y) = (x \to y) \wedge (y \to x)$ which satisfies $b(x, y) = 1$ if and only if $x = y$. Clearly, $\mathcal{G}$ is also congruence distributive, because its reduct to the signature $\{\vee, \wedge\}$ is a lattice. It is also known that basic algebras form an arithmetical and congruence regular variety (see, e.g. Chajda (2011)). Since, in view of Theorem 3, right-residuated l-groupoids of Łukasiewicz type are term equivalent to basic algebras, it follows that they also form an arithmetical and congruence regular variety. Our last result which is based on some ideas of Bělohlávek (2003) shows that some congruence properties of residuated lattices remain valid in the case of right-residuated l-groupoids also, although in their case the operation $\odot$ is neither associative nor integral, in general.

Proposition 6 *Any right-residuated l-groupoid* $\mathcal{G} = (L, \vee, \wedge, \odot, \to, 0, 1)$ *is congruence permutable and* 1*-regular, and the following hold:*

(a) *If* $\mathcal{G}$ *satisfies the double negation law, then it is* 0*-regular.*
(b) *If* $\mathcal{G}$ *satisfies divisibility and the double negation law, then it is congruence regular.*

Proof It is well known that an algebra $\mathcal{A} = (A, F)$ is congruence permutable whenever it has a Mal'cev term, i.e. a term $p(x, y, z)$ satisfying

$p(x, y, y) = x$ and $p(x, x, y) = y$, for all $x, y \in A$. We can choose the term $p(x, y, z) = [((y \to z) \wedge (z \to y)) \odot x] \vee [((x \to y) \wedge (y \to x)) \odot z]$, from Bělohlávek (2003). Then $p(x, y, y) = x \vee [((x \to y) \wedge (y \to x)) \odot y]$. Since by Lemma 1(iv) we have $((x \to y) \wedge (y \to x)) \odot y \leq (y \to x) \odot y \leq x$, we obtain $p(x, y, y) = x$, for all $x, y \in L$. Similarly we prove $p(x, x, y) = y$, for all $x, y \in L$.

(a) Let us consider the term $t(x, y) = ((x \to y) \wedge (y \to x)) \to 0$. Clearly, $t(x, x) = 1 \to 0 = \rceil 1 = 0$, according to Lemma 1(vi). Conversely, if $t(x, y) = 0$, then $(x \to y) \wedge (y \to x) = (((x \to y) \wedge (y \to x)) \to 0) \to 0 = 0 \to 0 = 1$, by the double negation law. Thus we get $x \to y = 1$ and $y \to x = 1$, whence $x \leq y$ and $y \leq x$, and this implies $x = y$, proving that $\mathcal{G}$ is 0-regular.

(b) Now, in view of (a) and Chajda (1998), it suffices to prove that $\mathcal{G}$ is locally 0-regular. Let $p_1(x, y) = (x \to y) \to 0$, and $p_2(x, y) = x \vee y$. Then obviously $p_2(x, 0) = x$, and $p_1(x, 0) = (x \to 0) \to 0 = x$, for all $x \in L$. Conversely, $p_2(x, y) = x$ implies $y \leq x$ and $p_1(x, y) = x$ yields $(x \to y) \to 0 = x$, whence by double negation we get $x \to y = x \to 0$. Therefore, using divisibility we obtain $y = x \wedge y = (x \to y) \odot x = (x \to 0) \odot x = x \wedge 0 = 0$. This proves that $\mathcal{G}$ is locally 0-regular. □

Corollary 6 *Let $\mathcal{V}$ be a variety consisting of right-residuated l-groupoids satisfying the double negation law and divisibility. Then $\mathcal{V}$ is arithmetical and congruence regular.*

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