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METHODOLOGIES AND APPLICATION

# On interval-valued hesitant fuzzy rough approximation operators

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**Abstract** Interval-valued hesitant fuzzy set is a generalization of classical interval-valued fuzzy set by returning a family of the interval-valued membership degrees for each object in the universe. By combining interval-valued hesitant fuzzy set and rough set models, the concept of an interval-valued hesitant fuzzy rough set is explored in this paper. Both constructive and axiomatic approaches are considered for this study. In constructive approach, by employing an interval-valued hesitant fuzzy relation, a pair of lower and upper interval-valued hesitant fuzzy rough approximation operators is first defined. The connections between special interval-valued hesitant fuzzy relations and interval-valued hesitant fuzzy rough approximation operators are further established. In axiomatic approach, an operators-oriented characterization of the interval-valued hesitant fuzzy rough set is presented, that is, interval-valued hesitant fuzzy rough approximation operators are defined by axioms, and then, different axiom sets of lower and upper interval-valued hesitant fuzzy set-theoretic operators guarantee the existence of different types of interval-valued hesitant fuzzy relations producing the same operators. Finally, a practical application is provided to illustrate the validity of the interval-valued hesitant fuzzy rough set model.

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## 1 Introduction

The concept of rough sets, proposed by Pawlak (1982, 1991) as a framework for the construction of approximations of concepts, is a formal tool for modeling and processing insufficient and incomplete information. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. There are mainly two methods for the development of this theory (Yao 1998a; Lin 1996), namely the constructive and axiomatic approaches.

In the constructive approach, binary relations on the universe of discourse, partition (or coverings) of the universe of discourse, neighborhood systems, and Boolean algebras are all primitive notions (Pawlak 1991; Yao 1998a, b; Wu and Zhang 2002). The lower and upper approximation operators are constructed by means of these notions. Recently, rough set approximations have been developed into the fuzzy environment, and the results are called rough fuzzy sets (Dubois and Prade 1990; Li and Zhang 2008; Thiele 2001b; Wu et al. 2006) and fuzzy rough sets (Dubois and Prade 1990; Radzikowska and Kerre 2002; Yeung et al. 2005; Wu et al. 2005, 2003; Tiwari and Srivastava 2013). Moreover, many authors also extended rough set theory into interval-valued fuzzy sets and intuitionistic fuzzy (IF) sets (Cornelis et al. 2003; Chakrabarty et al. 1998; Jena and Ghosh 2002; Rizvi et al. 2002; Samanta and Mondal 2001; Zhou and Wu 2008, 2009; Zhang et al. 2012; Zhang 2012a, b, 2013). For example, according to fuzzy rough sets in the sense of Nanda and

Majumda (1992), Jena and Ghosh (2002), Chakrabarty et al. (1998), and Samanta and Mondal (2001) presented the concept of IF rough sets that are not defined by an approximation space. Comparing with the above approaches, Rizvi et al. (2002) proposed the concept of rough IF sets base on a Pawlak approximation space $(U, R)$ in which the lower and upper approximations are not IF sets in the universe of discourse $U$, but IF sets in the family of equivalence classes derived by equivalence relation $R$. To remedy this difficulty, on the basis of an IF triangular norm $\mathcal{T}_L$ and IF implicator $\mathcal{I}_L$, Cornelis et al. (2003) introduced a concept of $(\mathcal{T}_L, \mathcal{I}_L)$ IF rough sets in which the lower and upper approximation operators are both IF sets in the universe. However, they have not investigated the essential properties of the lower and upper approximation operators generated by other relations, such as reflexive relation, symmetric relation, and transitive relation. Therefore, in Zhou and Wu (2008), various relation-based IF rough approximation operators were discussed by Zhou and Wu by using a special type of IF triangular norm min. Meanwhile, on the basis of IF implicator, Zhou and Wu (2009) investigated IF rough approximations on one universe, but they have not studied properties of $(\mathcal{I}, \mathcal{T})$-IVF rough sets on two universes of discourse. Therefore, Zhang et al. (2009) studied $(\mathcal{I}, \mathcal{T})$-IVF rough approximation operators on two universes of discourse by the constructive and axiomatic approaches. Very recently, rough set theory has been developed into hesitant fuzzy environment, and the result is called hesitant fuzzy rough sets (Yang et al. 2014).

On the other hand, the axiomatic approach (Morsi and Yakout 1998; Radzikowska and Kerre 2002; Thiele 2001a, b, c; Wu et al. 2002, 2005; Wu and Zhang 2004; Liu 2013) takes the lower and upper approximation operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators produced by the constructive approach. Under this point of view, lower and upper approximation operators are strongly related to the necessity (box) and possibility (diamond) operator in modal logic, the interior and closure operators in topological space, and the belief and plausibility functions in the Dempster–Shafer theory of evidence (Chuchro 1994, 1993; Kortelainen 1994; Vakarelov 1991; Thiele 2000, 2001a, b, c; Yao 1998a, 1996; Yao and Lin 1996). Many authors explored and developed the axiomatic approach in the study of crisp rough set theory (Comer 1991, 1993; Thiele 2000; Skardowska 1989; Yang and Li 2006; Yao 1996, 1998a, b). The research of the axiomatic approach has also been extended to approximation operators in fuzzy environment (Morsi and Yakout 1998; Radzikowska and Kerre 2002; Thiele 2001a, b; Wu et al. 2003, 2005; Wu and Zhang 2004; Yang 2007; Mi and Zhang 2004). For example, a set of axioms on fuzzy rough sets was investigated by Morsi and Yakout (1998). Thiele (2001a, b) explored axiomatic characterizations of fuzzy rough approximation operators and rough fuzzy approximation operators within modal logic. Furthermore, Wu et al. (2003, 2006), Wu and Zhang (2002, 2004) studied various generalized fuzzy approximation operators that are characterized by different sets of axioms. Recently, the axiomatic approach to approximation operators has been investigated by many authors in IF environment (Zhang et al. 2009, 2012; Zhang 2012a, 2013; Zhou and Wu 2008, 2009) and hesitant fuzzy environment (Yang et al. 2014).

Another important concept used to cope with imperfect and/or imprecise information is hesitant fuzzy (HF) set originated by Torra and Narukawa (2009), Torra (2010). It is an intuitively straightforward extension of Zadeh's fuzzy sets (Zadeh 1965). Torra and Narukawa (2009), Torra (2010) extended fuzzy sets to HF sets, because they found that under a group setting, it is difficult to determine the membership of an element to a set due to doubts between a few different values (Torra 2010). For example, two decision makers discuss the membership degree of $x$ into $A$. One wants to assign 0.4 and the other 0.6, and they cannot persuade with each other; thus, the membership degrees of $x$ into $A$ can be represented by $\{0.4, 0.6\}$. After it was introduced by Torra, HF set theory has attracted more and more scholars' attention (Rodrguez et al. 2012; Xia and Xu 2011; Xu and Xia 2011). Very recently, the study of hybrid models combining HF sets with other mathematical structures is also emerging as an active research topic of HF set theory. By combining HF set with rough set models, Yang et al. (2014) introduced the concept of HF rough sets and proposed an axiomatic approach to the model. Babitha and John (2013) defined a hybrid model called HF soft sets and investigated some of their basic properties. They also presented an algorithm to solve decision-making problems based on HF soft sets.

In many real decision-making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0, 1]. Based on this consideration, Chen et al. (2013a, b) introduced the concept of interval-valued hesitant fuzzy (IVHF) sets, which permits the membership degrees of an element to a given set to have a few different interval values. Since IVHF sets were introduced, IVHF set theory has been applied in dealing with fuzzy decision-making problems Chen et al. (2013b), Wei et al. (2013). Very recently, similarity, distance, and entropy measures for IVHF sets have been investigated by Farhadinia (2013). On the one hand, although Yang et al. (2014) proposed HF rough set theory that can deal with some decision making problems to exactly quantify decision makers' opinions with a crisp number, one of the main characteristics of decision-making events is incomplete and inaccuracy of available data information. So when facing the problem, the decision makers are easy to lose information and cannot supply correct policies by using HF rough set theory. Instead, the basic characteristics of the decision-making problems described by an interval number within [0, 1] can overcome such a situation. So, it is very natural to extend concepts from HF rough set theory to their generalizations in IVHF set theory. On the other hand, up to now, many of researches about IVHF sets are mainly focusing on IVHF set itself. The discussions about fusions of IVHF set theory and other mathematical structures are rarely found in the related literatures. Meanwhile, we know that IVHF set and rough set can both capture particular facets of the imprecision. Considering the above facts, we are mainly focusing on the combination of IVHF set and rough set in this paper. Because the new hybrid model includes both ingredients of IVHF set and rough set, it is more flexible and effective to cope with imperfect and imprecise information than IVHF set and rough set. Therefore, the research about fusions of IVHF set and rough set is important and necessary to us. The purpose of the present paper is to present a new hybrid model called IVHF rough sets by combining IVHF set and rough set. Then, we are mainly devoted to investigating axiomatic approaches to IVHF rough set and its application in medical diagnosis.

In the next section, we review some basic notions related to HF sets and IVHF sets. In Sect. 3, we define rough approximations of IVHF sets with respect to arbitrary IVHF approximation spaces. Then, properties of the IVHF rough approximation operators are examined. Further, the connections between special IVHF relations and IVHF rough approximation operators are established. Section 4 explores axiomatic characterizations of the IVHF relation-based approximation operators in which various classes of IVHF approximation operators are characterized by different sets of axioms. In Sect. 5, a general approach to decision making based on IVHF rough sets over two universes is established under the background of application in medical diagnosis. Section 6 illustrates the principal steps of the proposed decision method by a numerical example. We then conclude the paper with a summary and outlook for further research in Sect. 7.

## 2 Preliminaries

### 2.1 Lattice and hesitant fuzzy sets

First, we recall briefly a special complete lattice on $[0, 1]^2$ with its logical operations originated by Cornelis et al. (2003, 2004). These concepts may be seen as generalizations of the logical connectives in $([0, 1], \leq)$.

**Definition 2.1** (Cornelis et al. 2004) Let $L^I = \{[\mu, \nu] \in [0, 1] \times [0, 1] | \mu \leq \nu\}$ and denote $[\mu_1, \nu_1] \leq_{L^I} [\mu_2, \nu_2] \Leftrightarrow \mu_1 \leq \mu_2$ and $\nu_1 \leq \nu_2, \forall [\mu_1, \nu_1], [\mu_2, \nu_2] \in L^I$. Then, the pair $(L^I, \leq_{L^I})$ is called a complete, bounded lattice.

The operators $\wedge$ and $\vee$ on $(L^I, \leq_{L^I})$ are defined as follows:

$$[\mu_1, \nu_1] \wedge [\mu_2, \nu_2] = [\min\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}],$$
$$[\mu_1, \nu_1] \vee [\mu_2, \nu_2] = [\max\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\}],$$

for $[\mu_1, \nu_1], [\mu_2, \nu_2] \in L^I$.

Obviously, a complete lattice on $L^I$ has the smallest element $0_{L^I} = [0, 0]$ and the greatest element $1_{L^I} = [1, 1]$. The definitions of fuzzy logical operators can be straightforwardly extended to the interval-valued fuzzy case. The strict partial order $<_{L^I}$ is defined by

$$[\mu_1, \nu_1] <_{L^I} [\mu_2, \nu_2] \Leftrightarrow [\mu_1, \nu_1] \leq_{L^I} [\mu_2, \nu_2] \text{ and } [\mu_1, \nu_1] \neq [\mu_2, \nu_2].$$

Next, we review some basic concepts related to HF sets introduced by Torra and Narukawa (2009), Torra (2010):

**Definition 2.2** (Torra and Narukawa 2009; Torra 2010) Let $U$ be a nonempty and finite universe of discourse; a HF set $\tilde{A}$ on $U$ is in terms of a function $h_{\tilde{A}}(x)$ that when applied to $U$ returns a subset of [0, 1], that is,

$$\tilde{A} = \{\langle x, h_{\tilde{A}}(x)\rangle | x \in U\},$$

where $h_{\tilde{A}}(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in U$ to $\tilde{A}$.

For convenience, we call $h_{\tilde{A}}(x)$ a HF element.

*Example 2.3* Let $U = \{x_1, x_2, x_3\}$ be a universe set, $h_{\tilde{A}}(x_1) = \{0.7, 0.4, 0.5\}$, $h_{\tilde{A}}(x_2) = \{0.2, 0.4\}$, and $h_{\tilde{A}}(x_3) = \{0.3, 0.1, 0.7, 0.6\}$, be the HF elements of $x_i (i = 1, 2, 3)$ to a set $\tilde{A}$, respectively. Then, $\tilde{A}$ can be considered as a HF set, that is,

$$\tilde{A} = \{\langle x_1, \{0.7, 0.4, 0.5\}\rangle, \langle x_2, \{0.2, 0.4\}\rangle, \langle x_3, \{0.3, 0.1, 0.7, 0.6\}\rangle\}.$$

### 2.2 Interval-valued hesitant fuzzy sets

#### 2.2.1 *Concept of interval-valued hesitant fuzzy sets*

In the subsection, we review some basic concepts related to IVHF sets introduced by Chen et al. (2013a).

**Definition 2.4** (Chen et al. 2013a) Let $U$ be a nonempty and finite universe of discourse, and Int[0, 1] be the set of all closed subintervals of [0, 1]. An IVHF set $\mathbb{A}$ on $U$ is defined as

$$\mathbb{A} = \{\langle x, h_{\mathbb{A}}(x)\rangle | x \in U\},$$

where $h_{\mathbb{A}}(x) : U \to \text{Int}[0, 1]$ denotes all possible interval-valued membership degrees of the element $x \in U$ to $\mathbb{A}$.

For convenience, we call $h_{\mathbb{A}}(x)$ an IVHF element. The set of all IVHF sets on $U$ is denoted by IVHF($U$).

*Remark 2.5* (1) From Definition 2.4, we can note that an IVHF set $\mathbb{A}$ can be seen as an interval-valued fuzzy set if there is only one element in $h_{\mathbb{A}}(x)$, which indicates that interval-valued fuzzy sets are a special type of IVHF sets.
(2) It should be noted that when the upper and lower limits of the interval values in $h_{\mathbb{A}}(x)$ are identical, IVHF set $\mathbb{A}$ degenerates to HF set $\tilde{A}$, indicating that the latter is a special case of the former.

*Example 2.6* Let $U = \{x_1, x_2\}$ be a universe set, $h_{\mathbb{A}}(x_1) = \{[0.2, 0.3], [0.4, 0.6], [0.5, 0.6]\}$ and $h_{\mathbb{A}}(x_2) = \{[0.3, 0.5], [0.4, 0.7]\}$ be the IVHF elements of $x_i (i = 1, 2)$ to a set $\mathbb{A}$, respectively. Then, $\mathbb{A}$ can be considered as an IVHF set, that is,

$$\mathbb{A} = \{\langle x_1, \{[0.2, 0.3], [0.4, 0.6], [0.5, 0.6]\}\rangle, \langle x_2, \{[0.3, 0.5], [0.4, 0.7]\}\rangle\}.$$

Here, we define several special IVHF sets as follows: $\forall \mathbb{A} \in$ IVHF($U$),

1. $\mathbb{A}$ is referred to as an empty IVHF set if and only if $h_{\mathbb{A}}(x) = \{[0, 0]\}$ for all $x \in U$. In that case, the empty IVHF set is denoted by $\emptyset$;
2. $\mathbb{A}$ is referred to as a full IVHF set if and only if $h_{\mathbb{A}}(x) = \{[1, 1]\}$ for all $x \in U$. In that case, the full IVHF set is denoted by $\mathbb{U}$;
3. $\mathbb{A}$ is referred to as a constant IVHF set if and only if $h_{\mathbb{A}}(x) = \{[a_1^L, a_1^U], [a_2^L, a_2^U], \ldots, [a_m^L, a_m^U]\}$ for all $x \in U$, where $[a_i^L, a_i^U] \in \mathrm{Int}[0, 1], i = 1, \ldots, m$, i.e., $h_{\mathbb{A}}(x) \in 2^{\mathrm{Int}[0,1]}$. In this case, the constant IVHF set is denoted by $[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}]$.

Meanwhile, for any $y \in U$, several special IVHF sets: IVHF singleton set $[1, 1]_y$, its complement $[1, 1]_{U-y}$, and IVHF mediocre set $[1, 1]_M$ are, respectively, defined as follows: For $x \in U$

$$h_{[1,1]_y}(x) = \begin{cases} \{[1, 1]\}, & x = y, \\ \{[0, 0]\}, & x \neq y. \end{cases}$$

$$h_{[1,1]_{U-\{y\}}}(x) = \begin{cases} \{[0, 0]\}, & x = y, \\ \{[1, 1]\}, & x \neq y. \end{cases}$$

$$h_{[1,1]_M}(x) = \begin{cases} \{[1, 1]\}, & x \in M, \\ \{[0, 0]\}, & \text{otherwise}. \end{cases}$$

### *2.2.2 Operations of interval-valued hesitant fuzzy sets*

Given two IVHF sets represented by $\mathbb{A}$ and $\mathbb{B}$, Chen et al. (2013a) defined some operations on them as follows:

**Definition 2.7** Let $U$ be a nonempty and finite universe of discourse. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two IVHF sets, namely, $\forall \mathbb{A}, \mathbb{B} \in$ IVHF($U$), then, for all $x \in U$

(1) the complement of $\mathbb{A}$, denoted by $\mathbb{A}^c$, is given by

$$h_{\mathbb{A}^c}(x) = \sim h_{\mathbb{A}}(x) = \{[1 - \gamma^+, 1 - \gamma^-] | \gamma \in h_{\mathbb{A}}(x)\};$$

(2) the union of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cup \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cup\mathbb{B}}(x) = h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x) = \{[\gamma^- \vee \mu^-, \gamma^+ \vee \mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\};$$

(3) the intersection of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cap \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cap\mathbb{B}}(x) = h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x) = \{[\gamma^- \wedge \mu^-, \gamma^+ \wedge \mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\};$$

(4) the ring sum of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxplus \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxplus\mathbb{B}}(x) = h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x) = \{[\gamma^- + \mu^- - \gamma^-\mu^-, \gamma^+ + \mu^+ - \gamma^+\mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\};$$

(5) the ring product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxtimes \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxtimes\mathbb{B}}(x) = h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x) = \{[\gamma^-\mu^-, \gamma^+\mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\}.$$

It should be noticed that in Definition 2.7, operations $\Cup$, $\Cap$, $^c$, $\boxplus$, and $\boxtimes$ are defined on IVHF sets, respectively, while operations $\veebar$, $\barwedge$, $\sim$, $\oplus$, and $\otimes$ are defined on the corresponding IVHF elements, respectively.

Further, Chen et al. (2013a) established some relationships for the above operations on IVHF sets and IVHF elements.

**Theorem 2.8** *Let $U$ be a nonempty and finite universe of discourse. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two IVHF sets; then, we have*

*(1)* $(\mathbb{A} \Cup \mathbb{B})^c = \mathbb{A}^c \Cap \mathbb{B}^c, \sim (h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \barwedge (\sim h_{\mathbb{B}}(x))$,
*(2)* $(\mathbb{A} \Cap \mathbb{B})^c = \mathbb{A}^c \Cup \mathbb{B}^c, \sim (h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \veebar (\sim h_{\mathbb{B}}(x))$,
*(3)* $(\mathbb{A} \boxplus \mathbb{B})^c = \mathbb{A}^c \boxtimes \mathbb{B}^c, \sim (h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \otimes (\sim h_{\mathbb{B}}(x))$,
*(4)* $(\mathbb{A} \boxtimes \mathbb{B})^c = \mathbb{A}^c \boxplus \mathbb{B}^c, \sim (h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x)) = (\sim h_{\mathbb{A}}(x)) \oplus (\sim h_{\mathbb{B}}(x))$.

*Example 2.9* Let $\mathbb{A}$ and $\mathbb{B}$ be two IVHF sets. Suppose that $h_{\mathbb{A}}(x) = \{[0.5, 0.6], [0.3, 0.8], [0.3, 0.6]\}$ and $h_{\mathbb{B}}(x) = \{[0.4, 0.5], [0.4, 0.7]\}$ are two IVHF elements of $x$ to $\mathbb{A}$ and $\mathbb{B}$, respectively. By the operational laws of IVHF elements given in Definition 2.7, we have

$$\begin{aligned}
h_{\mathbb{A}\Cup\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x)\\
&= \{[\gamma^- \vee \mu^-, \gamma^+ \vee \mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\}\\
&= \{[0.5 \vee 0.4, 0.6 \vee 0.5],\\
&\quad [0.5 \vee 0.4, 0.6 \vee 0.7], [0.3 \vee 0.4, 0.8 \vee 0.5],\\
&\quad [0.3 \vee 0.4, 0.8 \vee 0.7], [0.3 \vee 0.4, 0.6 \vee 0.5],\\
&\quad [0.3 \vee 0.4, 0.6 \vee 0.7]\}\\
&= \{[0.5, 0.6], [0.5, 0.7],\\
&\quad [0.4, 0.8], [0.4, 0.8], [0.4, 0.6], [0.4, 0.7]\},\\
h_{\mathbb{A}\Cap\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x)\\
&= \{[\gamma^- \wedge \mu^-, \gamma^+ \wedge \mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\}\\
&= \{[0.5 \wedge 0.4, 0.6 \wedge 0.5],\\
&\quad [0.5 \wedge 0.4, 0.6 \wedge 0.7], [0.3 \wedge 0.4, 0.8 \wedge 0.5],\\
&\quad [0.3 \wedge 0.4, 0.8 \wedge 0.7], [0.3 \wedge 0.4, 0.6 \wedge 0.5],\\
&\quad [0.3 \wedge 0.4, 0.6 \wedge 0.7]\}\\
&= \{[0.4, 0.5], [0.4, 0.6], [0.3, 0.5],\\
&\quad [0.3, 0.7], [0.3, 0.5], [0.3, 0.6]\},
\end{aligned}$$

$$\begin{aligned}
h_{\mathbb{A}\boxplus\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x)\\
&= \{[\gamma^- + \mu^- - \gamma^-\mu^-, \gamma^+ + \mu^+ - \gamma^+\mu^+] | \gamma\\
&\quad \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\}\\
&= \{[0.5 + 0.4 - 0.5 \cdot 0.4, 0.6 + 0.5 - 0.6 \cdot 0.5],\\
&\quad [0.5 + 0.4 - 0.5 \cdot 0.4, 0.6 + 0.7 - 0.6 \cdot 0.7],\\
&\quad [0.3 + 0.4 - 0.3 \cdot 0.4, 0.8 + 0.5 - 0.8 \cdot 0.5],\\
&\quad [0.3 + 0.4 - 0.3 \cdot 0.4, 0.8 + 0.7 - 0.8 \cdot 0.7],\\
&\quad [0.3 + 0.4 - 0.3 \cdot 0.4, 0.6 + 0.5 - 0.6 \cdot 0.5],\\
&\quad [0.3 + 0.4 - 0.3 \cdot 0.4, 0.6 + 0.7 - 0.6 \cdot 0.7]\}\\
&= \{[0.7, 0.8], [0.7, 0.88], [0.58, 0.9],\\
&\quad [0.58, 0.94], [0.58, 0.8], [0.58, 0.88]\},\\
h_{\mathbb{A}\boxtimes\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x)\\
&= \{[\gamma^-\mu^-, \gamma^+\mu^+] | \gamma \in h_{\mathbb{A}}(x), \mu \in h_{\mathbb{B}}(x)\}\\
&= \{[0.5 \cdot 0.4, 0.6 \cdot 0.5],\\
&\quad [0.5 \cdot 0.4, 0.6 \cdot 0.7], [0.3 \cdot 0.4, 0.8 \cdot 0.5],\\
&\quad [0.3 \cdot 0.4, 0.8 \cdot 0.7], [0.3 \cdot 0.4, 0.6 \cdot 0.5],\\
&\quad [0.3 \cdot 0.4, 0.6 \cdot 0.7]\}\\
&= \{[0.2, 0.3], [0.2, 0.42], [0.12, 0.4],\\
&\quad [0.12, 0.56], [0.12, 0.3], [0.12, 0.42]\}.
\end{aligned}$$

It is noted that the number of interval values in different IVHF elements may be different and the interval values are usually out of order. In order to rank the interval values, Xu and Da (2002) gave the definition as follows:

**Definition 2.10** (Xu and Da 2002) Let $a = [a^L, a^U]$, and $b = [b^L, b^U]$; then, the degree of possibility of $a \geq b$ is defined as

$$p(a \geq b) = \max\left\{1 - \max\left(\frac{b^U - a^L}{a^U - a^L + b^U - b^L}, 0\right), 0\right\} \tag{1}$$

Similarly, the degree of possibility of $b \geq a$ is defined as:

$$p(b \geq a) = \max\left\{1 - \max\left(\frac{a^U - b^L}{a^U - a^L + b^U - b^L}, 0\right), 0\right\} \tag{2}$$

Equations (1) and (2) are proposed in order to compare two interval values and to rank all the input arguments. Further details could be found in Xu and Da (2002).

Suppose that $l(h_{\mathbb{A}}(x))$ stands for the number of interval values in the IVHF element $h_{\mathbb{A}}(x)$. To operate correctly, Chen et al. (2013a) gave the following assumptions:

(A1) All the elements in each IVHF element $h_{\mathbb{A}}(x)$ are arranged in increasing order by Eq. (1). Let $h_{\mathbb{A}}^{\sigma(k)}(x)$ stands for the $k$th largest interval numbers in the IVHF element $h_{\mathbb{A}}(x)$. In this case, $h_{\mathbb{A}}^{\sigma(k)}(x)$ is denoted by

$$h_{\mathbb{A}}^{\sigma(k)}(x) = \left[h_{\mathbb{A}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x)\right],$$

where $h_{\mathbb{A}}^{\sigma(k)L}(x) = \inf h_{\mathbb{A}}^{\sigma(k)}(x)$ and $h_{\mathbb{A}}^{\sigma(k)U}(x) = \sup h_{\mathbb{A}}^{\sigma(k)}(x)$, respectively, represent the lower and upper limits of $h_{\mathbb{A}}^{\sigma(k)}(x)$.

(A2) If, for two IVHF elements $h_{\mathbb{A}}(x), h_{\mathbb{B}}(x), l(h_{\mathbb{A}}(x)) \neq l(h_{\mathbb{B}}(x))$, then $l = \max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$. To have a correct comparison, the two IVHF elements $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$ should have the same length $l$. If there are fewer elements in $h_{\mathbb{A}}(x)$ than in $h_{\mathbb{B}}(x)$, an extension of $h_{\mathbb{A}}(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_{\mathbb{B}}(x)$.

From Example 2.9, we can see that the dimension of the derived IVHF element may increase as the addition or multiplicative operations are done, which may increase the complexity of the calculations. To overcome the difficulty, we develop some new methods to decrease the dimension of the derived IVHF element when operating the IVHF elements on the premise of assumptions given by Chen et al. (2013a). The adjusted operational laws are defined as follows:

**Definition 2.11** Let $U$ be a nonempty and finite universe of discourse. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two IVHF sets, namely $\forall \mathbb{A}, \mathbb{B} \in \text{IVHF}(U)$, then, for all $x \in U$

(1) the complement of $\mathbb{A}$, denoted by $\mathbb{A}^c$, is given by

$$h_{\mathbb{A}^c}(x) = \sim h_{\mathbb{A}}(x) = \left\{\left[1 - h_{\mathbb{A}}^{\sigma(k)U}(x), 1 - h_{\mathbb{A}}^{\sigma(k)L}(x)\right] \middle| k = 1, 2, \ldots, l\right\},$$

(2) the union of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cup \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cup\mathbb{B}}(x) = h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) \vee h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x) \vee h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, \ldots, l\right\},$$

(3) the intersection of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \Cap \mathbb{B}$, is given by

$$h_{\mathbb{A}\Cap\mathbb{B}}(x) = h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) \wedge h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x) \wedge h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, \ldots, l\right\},$$

(4) the ring sum of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxplus \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxplus\mathbb{B}}(x) = h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) + h_{\mathbb{B}}^{\sigma(k)L}(x) - h_{\mathbb{A}}^{\sigma(k)L}(x)h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{B}}^{\sigma(k)U}(x) + h_{\mathbb{B}}^{\sigma(k)U}(x) - h_{\mathbb{A}}^{\sigma(k)U}(x)h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, \ldots, l\right\},$$

(5) the ring product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \boxtimes \mathbb{B}$, is given by

$$h_{\mathbb{A}\boxtimes\mathbb{B}}(x) = h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x)h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x)h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, \ldots, l\right\},$$

where $l = \max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$.

It is noted that Theorem 2.8 is still valid for the new operations above.

*Example 2.12* Reconsider Example 2.9. By Eq. (1) and the assumptions given by Chen et al. (2013a), then $h_{\mathbb{A}}(x) = \{[0.3, 0.6], [0.3, 0.8], [0.5, 0.6]\}$ and $h_{\mathbb{B}}(x) = \{[0.4, 0.5], [0.4, 0.7], [0.4, 0.7]\}$. By virtue of Definition 2.11, we have

$$\begin{aligned} h_{\mathbb{A}\Cup\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \veebar h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) \vee h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x) \vee h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, 3\right\} \\ &= \{[0.3 \vee 0.4, 0.6 \vee 0.5], [0.3 \vee 0.4, 0.8 \vee 0.7], [0.5 \vee 0.4, 0.6 \vee 0.7]\} \\ &= \{[0.4, 0.6], [0.4, 0.8], [0.5, 0.7]\}, \end{aligned}$$

$$\begin{aligned} h_{\mathbb{A}\Cap\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \barwedge h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) \wedge h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x) \wedge h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, 3\right\} \\ &= \{[0.3 \wedge 0.4, 0.6 \wedge 0.5], [0.3 \wedge 0.4, 0.8 \wedge 0.7], [0.5 \wedge 0.4, 0.6 \wedge 0.7]\} \\ &= \{[0.3, 0.5], [0.3, 0.7], [0.4, 0.6]\}, \end{aligned}$$

$$\begin{aligned} h_{\mathbb{A}\boxplus\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \oplus h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x) + h_{\mathbb{B}}^{\sigma(k)L}(x) - h_{\mathbb{A}}^{\sigma(k)L}(x)h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{B}}^{\sigma(k)U}(x) + h_{\mathbb{B}}^{\sigma(k)U}(x) - h_{\mathbb{A}}^{\sigma(k)U}(x)h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, 3\right\} \\ &= \{[0.3 + 0.4 - 0.3 \cdot 0.4, 0.6 + 0.5 - 0.6 \cdot 0.5], [0.3 + 0.4 - 0.3 \cdot 0.4, 0.8 + 0.7 - 0.8 \cdot 0.7], [0.5 + 0.4 - 0.5 \cdot 0.4, 0.6 + 0.7 - 0.6 \cdot 0.7]\} \\ &= \{[0.58, 0.8], [0.58, 0.94], [0.7, 0.88]\}, \end{aligned}$$

$$\begin{aligned} h_{\mathbb{A}\boxtimes\mathbb{B}}(x) &= h_{\mathbb{A}}(x) \otimes h_{\mathbb{B}}(x) = \left\{\left[h_{\mathbb{A}}^{\sigma(k)L}(x)h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x)h_{\mathbb{B}}^{\sigma(k)U}(x)\right] \middle| k = 1, 2, 3\right\} \\ &= \{[0.3 \cdot 0.4, 0.6 \cdot 0.5], [0.3 \cdot 0.4, 0.8 \cdot 0.7], [0.5 \cdot 0.4, 0.6 \cdot 0.7]\} \\ &= \{[0.12, 0.3], [0.12, 0.56], [0.2, 0.42]\}. \end{aligned}$$

Comparing with Examples 2.9 and 2.12, we can note that the adjusted operational laws given in Definition 2.11 indeed decrease the dimension of the derived IVHF element when operating the IVHF elements, which brings grievous advantage for the practicing application.

In this study, unless otherwise stated, the comparisons and operations on IVHF elements are carried out by using Definition 2.11 and the assumptions given by Chen et al. (2013a).

In what follows, we will introduce the concept of IVHF subset to compare two IVHF sets.

**Definition 2.13** Let $U$ be a nonempty and finite universe of discourse. For all $\mathbb{A}, \mathbb{B} \in \text{IVHF}(U)$, $\mathbb{A}$ is said to be an IVHF subset of $\mathbb{B}$, if $h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x)$ holds for any $x \in U$ such that

$$h_{\mathbb{A}}(x) \preceq h_{\mathbb{B}}(x) \Leftrightarrow [h_{\mathbb{A}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x)] \leq_{L^I} [h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{B}}^{\sigma(k)U}(x)] \Leftrightarrow h_{\mathbb{A}}^{\sigma(k)L}(x) \leq h_{\mathbb{B}}^{\sigma(k)L}(x), h_{\mathbb{A}}^{\sigma(k)U}(x) \leq h_{\mathbb{B}}^{\sigma(k)U}(x), \quad k = 1, 2, \ldots, l.$$

We denote it by $\mathbb{A} \sqsubseteq \mathbb{B}$.

Obviously, we can note that $\sqsubseteq$ is reflexive, transitive, and antisymmetric on IVHF(U).

*Remark 2.14* If the upper and lower limits of all the interval values in the IVHF elements $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$ are identical, it should be noted that $\mathbb{A}, \mathbb{B}$ degenerate to two HF sets. In such case, Definition 2.13 will degenerate to the form such that

$$\mathbb{A} \sqsubseteq \mathbb{B} \Leftrightarrow h_{\mathbb{A}}^{\sigma(k)}(x) \leq h_{\mathbb{B}}^{\sigma(k)}(x), \quad \forall x \in U,$$

where $h_{\mathbb{A}}^{\sigma(k)}(x)$ and $h_{\mathbb{B}}^{\sigma(k)}(x)$ stand for the $k$th largest values in the HFEs $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$, respectively. In that case, the notation $\sqsubseteq$ is reflexive, transitive, and antisymmetric. That is, $\mathbb{A} \sqsubseteq \mathbb{B}, \mathbb{B} \sqsubseteq \mathbb{A} \Rightarrow \mathbb{A} = \mathbb{B}$.

Yang et al. (2014) proposed the concept of the HF subset. Subsequently, they pointed out that the notation $\sqsubseteq$ is not necessarily antisymmetric. However, when IVHF sets $\mathbb{A}, \mathbb{B}$ degenerate to two HF sets, the notation $\sqsubseteq$ given in Definition 2.13 is antisymmetric. Therefore, the comparison of two HF sets in Definition 2.13 is more reasonable than the one by Definition 4 in Yang et al. (2014).

## 3 Construction of interval-valued hesitant fuzzy rough approximation operators

It is generally known that Pawlak's rough set model is based on the equivalence relation. However, the equivalence relation is a very stringent condition that could limit the application of rough sets in practical problems. Therefore, many authors have generalized the notion of approximation operators by using non-equivalence binary relations. This has lead to various other approximation operators, such as generalized rough set approximation operators in fuzzy environment, intuitionistic fuzzy rough approximation operators induced from an arbitrary intuitionistic fuzzy relation, and so on. HF set is a generalization of the classical fuzzy set by returning a family of the membership degrees for each object in the universe. Yang et al. (2014) introduced the concept of HF rough sets by combining HF set with rough set models. Just like HF sets, HF rough sets can also be applied to multiple attribute decision making. However, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number in HF environment. To overcome the difficulty, in the section, we will extend HF rough sets to the case of IVHF and construct IVHF rough approximation operators.

### 3.1 Interval-valued hesitant fuzzy rough sets

In the subsection, inspired by the concept of the HF relation in Yang et al. (2014), we will further extend the HF relation into IVHF environment and first introduce the concept of an IVHF relation which is used to construct IVHF rough approximation operators.

**Definition 3.1** Suppose that $U$ is a nonempty and finite universe of discourse. An IVHF relation $\mathbb{R}$ on $U$ is an IVHF subset of $U \times U$, namely $\mathbb{R}$ is given by

$$\mathbb{R} = \{\langle (x, y), h_{\mathbb{R}}(x, y)\rangle | (x, y) \in U \times U\},$$

where $h_{\mathbb{R}} : U \times U \to \text{Int}[0, 1]$ is a set of interval values in Int[0, 1], denoting the possible membership degrees of the relationships between $x$ and $y$.

For convenience, we denote by IVHFR$(U \times U)$ the family of all IVHF relations on $U$.

Yang et al. (2014) first introduced several special HF relations and pointed out that a HF relation having special property, such as reflexivity, symmetry, and transitivity, can be characterized by the essential properties of the lower and upper HF rough approximation operators. In what follows, following the line of exploration in Yang et al. (2014), we intend to further extend several special HF relations into IVHF environment and propose the concepts of several special IVHF relations. What we could do are to establish the connection between special IVHF relations and properties of IVHF approximation operators.

**Definition 3.2** Let $\mathbb{R} \in$ IVHFR$(U \times U)$.

(1) $\mathbb{R}$ is serial, if for any $x \in U$, there exists a $y \in U$ such that $h_{\mathbb{R}}(x, y) = \{[1, 1]\}$;
(2) $\mathbb{R}$ is reflexive, if $h_{\mathbb{R}}(x, x) = \{[1, 1]\}$ for all $x \in U$;
(3) $\mathbb{R}$ is symmetric, if for all $(x, y) \in U \times U$, $h_{\mathbb{R}}(x, y) = h_{\mathbb{R}}(y, x)$;
(4) $\mathbb{R}$ is transitive if, $h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{R}}(y, z) \preceq h_{\mathbb{R}}(x, z)$ for all $(x, z) \in U \times U$.

Alternatively, $\mathbb{R}$ is transitive if the following conditions are satisfied:

$$h_{\mathbb{R}}^{\sigma(k)L}(x, y) \wedge h_{\mathbb{R}}^{\sigma(k)L}(y, z) \leq h_{\mathbb{R}}^{\sigma(k)L}(x, z),$$
$$h_{\mathbb{R}}^{\sigma(k)U}(x, y) \wedge h_{\mathbb{R}}^{\sigma(k)U}(y, z) \leq h_{\mathbb{R}}^{\sigma(k)U}(x, z), \quad k = 1, 2, \ldots, l,$$

with $l = \max\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{R}}(y, z)), l(h_{\mathbb{R}}(x, z)).\}$

In modal logic, different systems can be constructed by using various types of binary relations. So various types of IVHF relations can construct different systems in modal logic so that it is possible to construct different rough set models with respect to various modal logic systems. The classic rough set model may be extended by using an arbitrary IVHF relation in the same way modal operators are defined. This is something we are working on for the future.

In the following, IVHF rough approximation operators will be introduced and induced from an IVHF approximation space.

**Definition 3.3** Let $U$ be a nonempty and finite universe of discourse and $\mathbb{R} \in$ IVHFR$(U \times U)$; the pair $(U, \mathbb{R})$ is called an IVHF approximation space. For any $\mathbb{A} \in \text{IVHF}(U)$, the lower and upper approximations of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two IVHF sets and are, respectively, defined as follows:

$$\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\underline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}, \tag{3}$$

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\overline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}, \tag{4}$$

where

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \bar{\wedge}_{y \in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}}(y)\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \veebar_{y \in U}\{h_{\mathbb{R}}(x, y) \bar{\wedge} h_{\mathbb{A}}(y)\}.$$

$\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ are, respectively, called the lower and upper approximations of $\mathbb{A}$ with respect to $(U, \mathbb{R})$. The pair $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ is called the IVHF rough set of $\mathbb{A}$ with respect to $(U, \mathbb{R})$, and $\underline{\mathbb{R}}, \overline{\mathbb{R}} : \text{IVHF}(U) \to \text{IVHF}(U)$ are referred to as lower and upper IVHF rough approximation operators, respectively.

Clearly, the above definition implies equivalences of the following form:

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \left\{\left[\bigwedge_{y \in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)L}(y)\right),\right.\right.$$
$$\left.\left.\bigwedge_{y \in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \vee h_{\mathbb{A}}^{\sigma(k)U}(y)\right)\right] \middle| k = 1, 2, \ldots, l\right\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \left\{\left[\bigvee_{y \in U}\left(h_{\mathbb{R}}^{\sigma(k)L}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)L}(y)\right),\right.\right.$$
$$\left.\left.\bigvee_{y \in U}\left(h_{\mathbb{R}}^{\sigma(k)U}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)U}(y)\right)\right] \middle| k = 1, 2, \ldots, l\right\},$$

where $l = \max\{l(h_{\mathbb{R}}(x, y)), l(h_{\mathbb{A}}(y))\}$.

*Remark 3.4* When the upper and lower limits of all the interval values in the IVHF elements $h_{\mathbb{R}}(x, y)$ and $h_{\mathbb{A}}(y)$ are respectively identical, that is, IVHF elements degenerate to HF elements, the IVHF rough set $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ degenerates to a HF rough set introduced by Yang et al. (2014) on the basis of Definition 2.11 and the assumptions given by Chen et al. (2013a), which indicates that HF rough sets are a special type of IVHF rough sets.

*Example 3.5* We assume that if IVHF elements degenerate to HF elements, the comparisons and operations on HF elements are still carried out by using Definition 2.11 and the assumptions given by Chen et al. (2013a).

Let $(U, \mathbb{R})$ be a HF approximation space, where $U = \{x_1, x_2, x_3\}$. Suppose that there are three judges who are invited to evaluate the possible membership degrees of the relationships between $x_i$ and $x_j$ with a crisp number. In that case, $\mathbb{R}$ is a HF relation defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \{1\} & \{0.4, 0.7\} & \{0.6, 0.8\} \\ \{0.4, 0.7\} & \{1\} & \{0.3, 0.7, 0.8\} \\ \{0.6, 0.8\} & \{0.3, 0.7, 0.8\} & \{1\} \end{pmatrix}\end{array}$$

For example, we cannot present the precise membership degrees of the relationships between $x_2$ and $x_1$, but we have a certain hesitancy in providing two possible crisp numbers 0.4 and 0.7 to depict the possible membership degrees of the relationships between $x_2$ and $x_1$.

Let a HF set

$$\mathbb{A} = \{\langle x_1, \{0.3, 0.5\}\rangle, \langle x_2, \{0.4, 0.6\}\rangle, \langle x_3, \{0.5, 0.6, 0.9\}\rangle\};$$

then by the definition of HF approximation operators introduced by Yang et al. (2014) and the above assumption, we have

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x_1) = \{0.3, 0.5, 0.5\}, \quad h_{\underline{\mathbb{R}}(\mathbb{A})}(x_2) = \{0.3, 0.6, 0.6\},$$
$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x_3) = \{0.3, 0.5, 0.5\};$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x_1) = \{0.5, 0.6, 0.8\}, \quad h_{\overline{\mathbb{R}}(\mathbb{A})}(x_2) = \{0.4, 0.6, 0.8\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x_3) = \{0.5, 0.6, 0.9\}.$$

Hence, we can conclude that

$$\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, \{0.3, 0.5, 0.5\}\rangle, \langle x_2, \{0.3, 0.6, 0.6\}\rangle, \langle x_3, \{0.3, 0.5, 0.5\}\rangle\},$$

and

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, \{0.5, 0.6, 0.8\}\rangle, \langle x_2, \{0.4, 0.6, 0.8\}\rangle, \langle x_3, \{0.5, 0.6, 0.9\}\rangle\}.$$

However, as we mentioned above that due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number. Instead, the basic characteristics of the decision-making problems are described by an interval number within [0, 1]. In that case, $\mathbb{R}$ is an IVHF relation instead of a HF relation and is defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \{[1, 1]\} & \{[0.4, 0.6], [0.7, 0.8]\} & \{[0.5, 0.7], [0.6, 0.9]\} \\ \{[0.4, 0.6], [0.7, 0.8]\} & \{[1, 1]\} & \{[0.3, 0.4], [0.5, 0.8], [0.7, 0.8]\} \\ \{[0.5, 0.7], [0.6, 0.9]\} & \{[0.3, 0.4], [0.5, 0.8], [0.7, 0.8]\} & \{[1, 1]\} \end{pmatrix}\end{array}$$

For example, due to insufficiency in available information, we cannot present the precise membership degrees of the relationships between $x_2$ and $x_1$, but we have a certain hesitancy in providing two possible interval values [0.4,0.6] and [0.7,0.8] to depict the possible membership degrees of the relationships between $x_2$ and $x_1$.

If an IVHF set

$$\mathbb{A} = \{\langle x_1, \{[0.3, 0.5], [0.4, 0.6]\}\rangle, \langle x_2, \{[0.3, 0.4], [0.3, 0.7]\}\rangle, \langle x_3, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle\},$$

then by Definition 3.3, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x_1) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x_1, y) \veebar h_{\mathbb{A}}(y)\} \\ &= (\{[0, 0]\} \veebar \{[0.3, 0.5], [0.4, 0.6]\}) \\ &\quad \barwedge (\{[0.2, 0.3], [0.4, 0.6]\} \veebar \{[0.3, 0.4], [0.3, 0.7]\}) \\ &\quad \barwedge (\{[0.1, 0.4], [0.3, 0.5]\} \veebar \{[0.5, 0.5], \\ &\quad [0.5, 0.7], [0.8, 0.9]\}) \\ &= \{[0.3, 0.5], [0.4, 0.6]\} \barwedge \{[0.3, 0.4], [0.4, 0.7]\} \\ &\quad \barwedge \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\} \\ &= \{[0.3, 0.4], [0.4, 0.6], [0.4, 0.6]\}. \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x_2) &= \{[0.3, 0.4], [0.3, 0.6], [0.3, 0.6]\}, \\ h_{\underline{\mathbb{R}}(\mathbb{A})}(x_3) &= \{[0.3, 0.4], [0.3, 0.6], [0.4, 0.6]\}; \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_1) &= \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}, \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_2) &= \{[0.3, 0.5], [0.5, 0.7], [0.7, 0.8]\}, \\ h_{\overline{\mathbb{R}}(\mathbb{A})}(x_3) &= \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}. \end{aligned}$$

Hence, we can conclude that

$$\begin{aligned} \underline{\mathbb{R}}(\mathbb{A}) = \{&\langle x_1, \{[0.3, 0.4], [0.4, 0.6], [0.4, 0.6]\}\rangle, \\ &\langle x_2, \{[0.3, 0.4], [0.3, 0.6], [0.3, 0.6]\}\rangle, \\ &\langle x_3, \{[0.3, 0.4], [0.3, 0.6], [0.4, 0.6]\}\rangle\}, \end{aligned}$$

and

$$\begin{aligned} \overline{\mathbb{R}}(\mathbb{A}) = \{&\langle x_1, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle, \\ &\langle x_2, \{[0.3, 0.5], [0.5, 0.7], [0.7, 0.8]\}\rangle, \\ &\langle x_3, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle\}. \end{aligned}$$

Comparing with the results of HF approximation operators and IVHF approximation operators, we note that the available information in IVHF rough sets is more comprehensive than HF rough sets, and HF rough approximation operators are indeed a special type of IVHF rough approximation operators.

*Remark 3.6* In Definition 3.3, if there is only one element in the IVHF elements $h_{\mathbb{R}}(x, y)$ and $h_{\mathbb{A}}(y)$, respectively, we can note that the IVHF rough set $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ degenerates to an interval-valued fuzzy rough set (Sun et al. 2008). That is to say, IVHF rough sets in Definition 3.3 are an extension of interval-valued fuzzy rough sets proposed by Sun et al. (2008).

*Example 3.7* Let $(U, \mathbb{R})$ be an interval-valued fuzzy approximation space, where $U = \{x_1, x_2, x_3\}$. Suppose that there is only a expert who is invited to evaluate the possible membership degrees of the relationships between $x_i$ and $x_j$ with an interval number within [0, 1]. In that case, $\mathbb{R}$ is an interval-valued fuzzy relation defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} [1, 1] & [0.7, 0.8] & [0.6, 0.9] \\ [0.7, 0.8] & [1, 1] & [0.3, 0.4] \\ [0.6, 0.9] & [0.3, 0.4] & [1, 1] \end{pmatrix} \end{array}$$

If an interval-valued fuzzy set

$$\mathbb{A} = \{\langle x_1, [0.4, 0.6]\rangle, \langle x_2, [0.3, 0.4]\rangle, \langle x_3, [0.5, 0.7]\rangle\},$$

then by the definition of interval-valued fuzzy approximation operators in Sun et al. (2008), we obtain

$$\begin{aligned} &\underline{\mathbb{R}}(\mathbb{A})(x_1) = [0.3, 0.4], \quad \underline{\mathbb{R}}(\mathbb{A})(x_2) = [0.3, 0.4], \\ &\quad \underline{\mathbb{R}}(\mathbb{A})(x_3) = [0.4, 0.6]; \\ &\overline{\mathbb{R}}(\mathbb{A})(x_1) = [0.5, 0.7], \quad \overline{\mathbb{R}}(\mathbb{A})(x_2) = [0.4, 0.6], \\ &\quad \overline{\mathbb{R}}(\mathbb{A})(x_3) = [0.5, 0.7]; \end{aligned}$$

Hence, we can conclude that

$$\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, [0.3, 0.4]\rangle, \langle x_2, [0.3, 0.4]\rangle, \langle x_3, [0.4, 0.6]\rangle\},$$

and

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, [0.5, 0.7]\rangle, \langle x_2, [0.4, 0.6]\rangle, \langle x_3, [0.5, 0.7]\rangle\}.$$

However, in decision-making problems, multiple factors should be considered and the evaluation of these factors are often carried out by several experts. So it is unreasonable to invite only a expert to supply the policy with an interval number in decision-making events. Thus, we should need several experts participating in making the policy in decision-making events. Only in this way may the decision results be more reasonable and accurate. In that case, $\mathbb{R}$ is an IVHF relation and is given in Example 3.5 above. Meanwhile, $\mathbb{A}$ is an IVHF set defined in Example 3.5.

Thus, we have

$$\begin{aligned} \underline{\mathbb{R}}(\mathbb{A}) = \{&\langle x_1, \{[0.3, 0.4], [0.4, 0.6], [0.4, 0.6]\}\rangle, \\ &\langle x_2, \{[0.3, 0.4], [0.3, 0.6], [0.3, 0.6]\}\rangle, \\ &\langle x_3, \{[0.3, 0.4], [0.3, 0.6], [0.4, 0.6]\}\rangle\}, \end{aligned}$$

and

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle, \langle x_2, \{[0.3, 0.5], [0.5, 0.7], [0.7, 0.8]\}\rangle, \langle x_3, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle\}.$$

From the above discussions, it should be noted that IVHF rough sets in Definition 3.3 contain more information than interval-valued fuzzy rough sets and are indeed an extension of interval-valued fuzzy rough sets proposed by Sun et al. (2008).

*Remark 3.8* If there is only one interval value in the IVHF elements $h_{\mathbb{R}}(x, y)$ and $h_{\mathbb{A}}(y)$ whose the upper and lower limits are identical, the IVHF rough set $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ in Definition 3.3 degenerates to a classical fuzzy rough set presented by Wu and Zhang (2004). That is, fuzzy rough sets presented by Wu and Zhang (2004) are a special case of IVHF rough sets defined by us.

*Example 3.9* Let $(U, \mathbb{R})$ be a fuzzy approximation space, where $U = \{x_1, x_2, x_3\}$. Suppose that there is only a expert who is invited to evaluate the possible membership degrees of the relationships between $x_i$ and $x_j$ with a crisp number. In that case, $\mathbb{R}$ is a fuzzy relation defined by the matrix as follows:

$$\mathbb{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left( \begin{array}{ccc} 1 & 0.4 & 0.8 \\ 0.4 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{array} \right) \end{array}$$

If a fuzzy set

$$\mathbb{A} = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.9}{x_3},$$

then by the definition of fuzzy approximation operators in Wu and Zhang (2004), we obtain

$$\underline{\mathbb{R}}(\mathbb{A})(x_1) = 0.5, \quad \underline{\mathbb{R}}(\mathbb{A})(x_2) = 0.4, \quad \underline{\mathbb{R}}(\mathbb{A})(x_3) = 0.4;$$
$$\overline{\mathbb{R}}(\mathbb{A})(x_1) = 0.8, \quad \overline{\mathbb{R}}(\mathbb{A})(x_2) = 0.7, \quad \overline{\mathbb{R}}(\mathbb{A})(x_3) = 0.9;$$

Hence, we can conclude that

$$\underline{\mathbb{R}}(\mathbb{A}) = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}, \quad \overline{\mathbb{R}}(\mathbb{A}) = \frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.9}{x_3}.$$

Now, we reconsider this example. On the one hand, it is unreasonable to invite only a expert to make the policy in decision-making events. The number of experts is added to make the decision result more objective and comprehensive. On the other hand, due to the shortage of the experts' experience and available information, it may be difficult for experts to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0, 1]. Considering these facts, it is necessary for us to extend a fuzzy relation to an IVHF relation. In that case, $\mathbb{R}$ is an IVHF relation and is defined in Example 3.5 above. Meanwhile, $\mathbb{A}$ is an IVHF set defined in Example 3.5.

By Example 3.5, we have

$$\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, \{[0.3, 0.4], [0.4, 0.6], [0.4, 0.6]\}\rangle, \langle x_2, \{[0.3, 0.4], [0.3, 0.6], [0.3, 0.6]\}\rangle, \langle x_3, \{[0.3, 0.4], [0.3, 0.6], [0.4, 0.6]\}\rangle\},$$

and

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x_1, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle, \langle x_2, \{[0.3, 0.5], [0.5, 0.7], [0.7, 0.8]\}\rangle, \langle x_3, \{[0.5, 0.5], [0.5, 0.7], [0.8, 0.9]\}\rangle\}.$$

Comparing with the results of two type approximation operators, we can see that the available information in IVHF rough sets is more comprehensive and objective than fuzzy rough sets, and fuzzy rough sets presented by Wu and Zhang (2004) are indeed a special case of IVHF rough sets in Definition 3.3.

**Theorem 3.10** *Let $(U, \mathbb{R})$ be an IVHF approximation space. Then, the lower and upper IVHF rough approximation operators induced from $(U, \mathbb{R})$ satisfy the following properties: $\forall \mathbb{A}, \mathbb{B} \in$ IVHF$(U)$,*

(IVHFL1) $\underline{\mathbb{R}}(\mathbb{A}^c) = (\overline{\mathbb{R}}(\mathbb{A}))^c$, (IVHFU1) $\overline{\mathbb{R}}(\mathbb{A}^c) = (\underline{\mathbb{R}}(\mathbb{A}))^c$,

(IVHFL2) $\mathbb{A} \sqsubseteq \mathbb{B} \Rightarrow \underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}}(\mathbb{B})$, (IVHFU2) $\mathbb{A} \sqsubseteq \mathbb{B} \Rightarrow \overline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{B})$,

(IVHFL3) $\underline{\mathbb{R}}(\mathbb{A} \Cap \mathbb{B}) = \underline{\mathbb{R}}(\mathbb{A}) \Cap \underline{\mathbb{R}}(\mathbb{B})$, (IVHFU3) $\overline{\mathbb{R}}(\mathbb{A} \Cup \mathbb{B}) = \overline{\mathbb{R}}(\mathbb{A}) \Cup \overline{\mathbb{R}}(\mathbb{B})$,

(IVHFL4) $\underline{\mathbb{R}}(\mathbb{A} \Cup \mathbb{B}) \sqsupseteq \underline{\mathbb{R}}(\mathbb{A}) \Cup \underline{\mathbb{R}}(\mathbb{B})$, (IVHFU4) $\overline{\mathbb{R}}(\mathbb{A} \Cap \mathbb{B}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A}) \Cap \overline{\mathbb{R}}(\mathbb{B})$,

(IVHFL5) $\underline{\mathbb{R}}\left(\mathbb{A} \Cup \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \underline{\mathbb{R}}(\mathbb{A}) \Cup \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$,

(IVHFU5) $\overline{\mathbb{R}}\left(\mathbb{A} \Cap \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \overline{\mathbb{R}}(\mathbb{A}) \Cap \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$,

(IVHFL6) $\underline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$, (IVHFU6) $\overline{\mathbb{R}}(\emptyset) = \emptyset$.

*Proof* Since IVHF rough approximation operators $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ are dual to each other, we only investigate the case of $\underline{R}$.

(IVHFL1) By Eq. (3) and Theorem 2.8, we obtain

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A}^c)}(x) &= \overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\, h_{\mathbb{A}^c}(y)\} \\ &= \overline{\wedge}_{y\in U}\{(\sim h_{\mathbb{R}}(x,y))\,\underline{\vee}\,(\sim h_{\mathbb{A}}(y))\} \\ &= \overline{\wedge}_{y\in U}\{\sim(h_{\mathbb{R}}(x,y)\,\overline{\wedge}\, h_{\mathbb{A}}(y))\} \\ &= \sim(\underline{\vee}_{y\in U}\{(h_{\mathbb{R}}(x,y)\,\overline{\wedge}\, h_{\mathbb{A}}(y)\}) = h_{(\overline{\mathbb{R}}(\mathbb{A}))^c}(x), \end{aligned}$$

which implies that

$\underline{\mathbb{R}}(\mathbb{A}^c) = (\overline{\mathbb{R}}(\mathbb{A}))^c$.

(IVHFL2) Since $\mathbb{A} \sqsubseteq \mathbb{B}$, then by Definition 2.13, we have $h_{\mathbb{A}}^{\sigma(k)L}(y) \le h_{\mathbb{B}}^{\sigma(k)L}(y)$, $h_{\mathbb{A}}^{\sigma(k)U}(y) \le h_{\mathbb{B}}^{\sigma(k)U}(y)$ for all $y \in U$. So it follows that

$$\bigwedge_{y\in U}\left(h_{\mathbb{A}}^{\sigma(k)L}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)L}(x,y)\right) \le \bigwedge_{y\in U}\left(h_{\mathbb{B}}^{\sigma(k)L}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)L}(x,y)\right),$$

and

$$\bigwedge_{y\in U}\left(h_{\mathbb{A}}^{\sigma(k)U}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)U}(x,y)\right) \le \bigwedge_{y\in U}\left(h_{\mathbb{B}}^{\sigma(k)U}(y) \vee h_{\mathbb{R}^c}^{\sigma(k)U}(x,y)\right).$$

Hence, for each $x \in U$, $h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \preceq h_{\underline{\mathbb{R}}(\mathbb{B})}(x)$, which means that $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}}(\mathbb{B})$.

(IVHFL3) $\forall x \in U$, by Eq. (3), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A}\Cap\mathbb{B})}(x) &= \overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\, h_{\mathbb{A}\Cap\mathbb{B}}(y)\} \\ &= \overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,(h_{\mathbb{A}}(y)\,\overline{\wedge}\, h_{\mathbb{B}}(y))\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x,y) \vee \left(h_{\mathbb{A}}^{\sigma(k)L}(y) \wedge h_{\mathbb{B}}^{\sigma(k)L}(y)\right)\right), \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x,y) \vee \left(h_{\mathbb{A}}^{\sigma(k)U}(y) \wedge h_{\mathbb{B}}^{\sigma(k)U}(y)\right)\right)\right] \middle| k = 1,2,\ldots,l\right\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x,y) \vee h_{\mathbb{A}}^{\sigma(k)L}(y)\right) \wedge \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x,y) \vee h_{\mathbb{B}}^{\sigma(k)L}(y)\right), \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x,y) \vee h_{\mathbb{A}}^{\sigma(k)U}(y)\right) \wedge \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x,y) \vee h_{\mathbb{B}}^{\sigma(k)U}(y)\right)\right] \middle| k = 1,2,\ldots,l\right\} \\ &= h_{\underline{\mathbb{R}}(\mathbb{A})}(x)\,\overline{\wedge}\, h_{\underline{\mathbb{R}}(\mathbb{B})}(x) = h_{\underline{\mathbb{R}}(\mathbb{A})\Cap\underline{\mathbb{R}}(\mathbb{B})}(x), \end{aligned}$$

where $l = \max\{l(h_{\mathbb{R}^c}(x,y)), l(h_{\mathbb{A}}(y)), l(h_{\mathbb{B}}(y))\}$.

Hence, it follows that (IVHFL3) holds.

(IVHFL4) It can be directly followed from (IVHFL2).

(IVHFL5) $\forall x \in U$, by Eq. (3), we have

$$\begin{aligned} & h_{\underline{\mathbb{R}}\left(\mathbb{A}\Cup\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right)}(x) \\ &= \overline{\wedge}_{y\in U}\left\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\, h_{\left(\mathbb{A}\Cup\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right)}(y)\right\} \\ &= \overline{\wedge}_{y\in U}\left\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\left(h_{\mathbb{A}}(y)\,\underline{\vee}\left\{\left[a^L_{1,\ldots,m}, a^U_{1,\ldots,m}\right]\right\}\right)\right\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x,y) \vee \left(h_{\mathbb{A}}^{\sigma(k)L}(y) \vee a^{\sigma(k)L}_{1,\ldots,m}\right)\right), \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x,y) \vee \left(h_{\mathbb{A}}^{\sigma(k)U}(y) \vee a^{\sigma(k)U}_{1,\ldots,m}\right)\right)\right] \middle| k = 1,2,\ldots,l\right\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)L}(x,y) \vee h_{\mathbb{A}}^{\sigma(k)L}(y)\right) \vee a^{\sigma(k)L}_{1,\ldots,m}, \bigwedge_{y\in U}\left(h_{\mathbb{R}^c}^{\sigma(k)U}(x,y) \vee h_{\mathbb{A}}^{\sigma(k)U}(y)\right) \vee a^{\sigma(k)U}_{1,\ldots,m}\right] \middle| k = 1,2,\ldots,l\right\} \\ &= h_{\underline{\mathbb{R}}(\mathbb{A})}(x)\,\underline{\vee}\, h_{\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]}(x) = h_{\underline{\mathbb{R}}(\mathbb{A})\Cup\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]}(x), \end{aligned}$$

where $l = \max\{l(h_{\mathbb{R}^c}(x,y)), l(h_{\mathbb{A}}(y)), l([\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}])\}$.

Hence, we conclude that (IVHFL5) holds.

(IVHFL6) It can be directly obtained from Eq. (3). □

Properties (IVHFL1) and (IVHFU1) show that IVHF rough approximation operators $\overline{\mathbb{R}}$ and $\underline{\mathbb{R}}$ are dual to each other. Properties with the same number may be also considered as dual properties. Properties (IVHFL6) and (IVHFU6) can be induced, respectively, from (IVHFL5) and (IVHFU5) when we set $[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}] = \{[1,1]\}$.

**Theorem 3.11** *Let $\mathbb{R} \in$ IVHFR$(U \times U)$. If $\mathbb{R}$ is reflexive, the following properties hold:*

*(1)* $\underline{\mathbb{R}}(\emptyset) = \overline{\mathbb{R}}(\emptyset) = \emptyset$,
*(2)* $\underline{\mathbb{R}}(\mathbb{U}) = \overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$,
*(3)* $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \mathbb{A} \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$, $\forall \mathbb{A} \in$ IVHF$(U)$.

*Proof* (1) For all $x \in U$, $h_{\emptyset}(x) = \{[0,0]\}$. Then by Eq. (3), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\emptyset)}(x) &= \overline{\wedge}_{y\in U}\{h_{\mathbb{R}^c}(x,y)\,\underline{\vee}\,\{[0,0]\}\} = \overline{\wedge}_{y\in U} h_{\mathbb{R}^c}(x,y) \\ &= h_{\mathbb{R}^c}(x,x)\,\overline{\wedge}\,(\overline{\wedge}_{y\neq x} h_{\mathbb{R}^c}(x,y)) = \{[0,0]\}. \end{aligned}$$

Thus, it follows that $\underline{\mathbb{R}}(\emptyset) = \emptyset$. By (IVHFU6), we can obtain $\overline{\mathbb{R}}(\emptyset) = \emptyset$.

(2) For all $x \in U$, $h_{\mathbb{U}}(x) = \{[1,1]\}$. Then by Eq. (4), we obtain

$$h_{\overline{\mathbb{R}}(\mathbb{U})}(x) = \veebar_{y\in U}\{h_{\mathbb{R}}(x,y) \barwedge \{[1,1]\}\} = \veebar_{y\in U} h_{\mathbb{R}}(x,y) \\ = h_{\mathbb{R}}(x,x) \veebar (\veebar_{y\neq x} h_{\mathbb{R}}(x,y)) = \{[1,1]\},$$

which implies that $\overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$. By (IVHFL6), we can obtain $\underline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$.

(3) $\forall x \in U$, by Eq. (3), then we obtain

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x,y) \veebar h_{\mathbb{A}}(y)\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right),\right.\right. \\ &\quad \left.\left.\bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right] \middle| k=1,2,\ldots,l\right\} \\ &= \left\{\left[\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,x) \vee h^{\sigma(k)L}_{\mathbb{A}}(x)\right)\right.\right. \\ &\quad \wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right)\right), \\ &\quad \left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,x) \vee h^{\sigma(k)U}_{\mathbb{A}}(x)\right) \\ &\quad \left.\left.\wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right)\right] \middle| k=1,2,\ldots,l\right\} \\ &= \left\{\left[h^{\sigma(k)L}_{\mathbb{A}}(x) \wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right)\right), h^{\sigma(k)U}_{\mathbb{A}}(x)\right.\right. \\ &\quad \left.\left.\wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right)\right] \middle| k=1,2,\ldots,l\right\} \\ &\preceq \left\{\left[h^{\sigma(k)L}_{\mathbb{A}}(x), h^{\sigma(k)U}_{\mathbb{A}}(x)\right] \middle| k=1,2,\ldots,l\right\} = h_{\mathbb{A}}(x). \end{aligned}$$

From the above discussions, we can conclude that $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \mathbb{A}$ holds. Similarly, we can prove that $\mathbb{A} \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$ holds. □

**Theorem 3.12** *Let $U$ be a nonempty and finite universe of discourse. Suppose that $\mathbb{R}_1, \mathbb{R}_2 \in$ IVHFR$(U \times U)$ are two IVHF relations. If $\mathbb{R}_1 \sqsubseteq \mathbb{R}_2$, then the following holds:*

*(1)* $\underline{\mathbb{R}_1}(\mathbb{A}) \sqsupseteq \underline{\mathbb{R}_2}(\mathbb{A}), \quad \forall \mathbb{A} \in \text{IVHF}(U)$,
*(2)* $\overline{\mathbb{R}_1}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}_2}(\mathbb{A}), \quad \forall \mathbb{A} \in \text{IVHF}(U)$.

*Proof* (1) Since $\mathbb{R}_1 \sqsubseteq \mathbb{R}_2$, then for any $(x,y) \in U \times U$, we have $h^{\sigma(k)L}_{\mathbb{R}_1^c}(x,y) \geq h^{\sigma(k)L}_{\mathbb{R}_2^c}(x,y), h^{\sigma(k)U}_{\mathbb{R}_1^c}(x,y) \geq h^{\sigma(k)U}_{\mathbb{R}_2^c}(x,y)$.

On the other hand, by Eq. (3), for all $x \in U$, we obtain

$$\begin{aligned} h_{\underline{\mathbb{R}_1}(\mathbb{A})}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}_1^c}(x,y) \veebar h_{\mathbb{A}}(y)\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}_1^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right),\right.\right. \\ &\quad \left.\left.\bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}_1^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right] \middle| k=1,2,\ldots,l\right\} \\ &\succeq \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}_2^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right),\right.\right. \\ &\quad \left.\left.\bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}_2^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right] \middle| k=1,2,\ldots,l\right\} \\ &= h_{\underline{\mathbb{R}_2}(\mathbb{A})}(x). \end{aligned}$$

From the above discussions, it follows that $\underline{\mathbb{R}_1}(\mathbb{A}) \sqsupseteq \underline{\mathbb{R}_2}(\mathbb{A})$. Similarly, by Eq. (4), we can easily prove that $\overline{\mathbb{R}_1}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}_2}(\mathbb{A})$ holds.

### 3.2 Connections between special interval-valued hesitant fuzzy relations and approximation operators

In rough set theory, many authors started from the properties of binary relations, for instance, reflexivity, symmetry, and transitivity, to investigate the essential properties of the lower and upper approximation operations generated by such relations and achieved a lot. For example, Yao (1998c), Yao and Lin (1996) studied generalized rough sets and established the connections between a binary relation and generalized approximation operators in which various classes of algebraic rough set model can be derived by the properties satisfied by a binary relation, such as serial, reflexive, symmetric, transitive, and Euclidean. Wu et al. (2003, 2005, 2006), Wu and Zhang (2004) investigated the connections between fuzzy relations and fuzzy rough approximation operators in fuzzy environment. Subsequently, the connections between special intuitionistic fuzzy relations and intuitionistic fuzzy rough approximation operators are further established in Zhou and Wu (2008, 2009) where Zhou et al. pointed out that an intuitionistic fuzzy relation having special property, such as reflexivity, symmetry, and transitivity, can be characterized by the essential properties of the lower and upper intuitionistic fuzzy rough approximation operators. More recently, generalizations of rough set have developed to HF environment. By combining HF set with rough set models, Yang et al. (2014) first proposed the concept of the HF rough sets in which the relationships between HF rough approximations and several special HF relations are further discussed.

In this subsection, along the lines of the above-mentioned rough set models, we will show that some special IVHF relations could be characterized by IVHF rough approximation operators.

**Theorem 3.13** *Let $\mathbb{R} \in$ IVHFR$(U \times U)$; then $\forall x \in U, (x,y) \in U \times U, M \subseteq U$,*

*(1)* $h_{\underline{\mathbb{R}}([1,1]_M)}(x) = \barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y)$,
*(2)* $h_{\overline{\mathbb{R}}([1,1]_M)}(x) = \veebar_{y\in M} h_{\mathbb{R}}(x, y)$,
*(3)* $h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(x) = h_{\mathbb{R}^c}(x, y)$,
*(4)* $h_{\overline{\mathbb{R}}([1,1]_y)}(x) = h_{\mathbb{R}}(x, y)$.

*Proof* (1) For all $x \in U$, using Eq. (3), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}([1,1]_M)}(x) &= \barwedge_{y\in U}\{h_{\mathbb{R}^c}(x, y) \veebar h_{[1,1]_M}(y)\} \\ &= \{[1, 1]\} \barwedge (\barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y)) \\ &= \barwedge_{y\notin M} h_{\mathbb{R}^c}(x, y) \end{aligned}$$

(2) It follows immediately from (1) and the duality.
(3) For all $x \in U$, by virtue of Eq. (3), we obtain

$$\begin{aligned} h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(x) &= \barwedge_{z\in U}\{h_{\mathbb{R}^c}(x, z) \veebar h_{[1,1]_{U-\{y\}}}(z)\} \\ &= h_{\mathbb{R}^c}(x, y) \barwedge \{[1, 1]\} = h_{\mathbb{R}^c}(x, y). \end{aligned}$$

(4) It follows immediately from (3) and the duality. □

Theorem 3.13 above shows that an IVHF relation can be represented by the IVHF rough approximation operators.

Now we discuss the relationships between the properties of several special IVHF relations and the properties of IVHF rough approximation operators. The following Theorems 3.14 and 3.15 show that an IVHF relation having special property, such as serializability, reflexivity, symmetry, and transitivity, can be characterized by the essential properties of the lower and upper IVHF rough approximation operators.

**Theorem 3.14** *Let* $\mathbb{R} \in \mathrm{IVHFR}(U \times U)$. *Suppose that* $\underline{\mathbb{R}}$ *and* $\overline{\mathbb{R}}$ *are the lower and upper IVHF rough approximation operators given in Definition 3.3; then,* $\mathbb{R}$ *is serial iff one of the following properties holds:*

(IVHFL0) $\underline{\mathbb{R}}(\emptyset) = \emptyset$,

(IVHFU0) $\overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$,

(IVHFLU0) $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$, $\quad \forall \mathbb{A} \in \mathrm{IVHF}(U)$,

(IVHFL0)′ $\underline{\mathbb{R}}\left(\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$,

(IVHFU0)′ $\overline{\mathbb{R}}\left(\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$.

*Proof* First, we can deduce from the dual properties of $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ that (IVHFL0) and (IVHFU0) are equivalent. Similarly, (IVHFL0)′ and (IVHFU0)′ are also equivalent.

Second, we are to prove that $\mathbb{R}$ is serial $\Longleftrightarrow$ (IVHFU0).

Assume that $\mathbb{R}$ is serial. For any $x \in U$, there exists a $z \in U$ such that $h_{\mathbb{R}}(x, z) = \{[1, 1]\}$. By virtue of Eq. (4), we have

$$\begin{aligned} h_{\overline{\mathbb{R}}(\mathbb{U})}(x) &= \left\{\left[\bigvee_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}}(x, y) \wedge h^{\sigma(k)L}_{\mathbb{U}}(y)\right),\right.\right. \\ &\qquad \left.\left.\bigvee_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}}(x, y) \wedge h^{\sigma(k)U}_{\mathbb{U}}(y)\right)\right] \middle| k = 1, 2, \ldots, l\right\} \\ &= \left\{\left[\bigvee_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}}(x, y) \wedge 1\right), \bigvee_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}}(x, y) \wedge 1\right)\right]\right. \\ &\qquad \left.| k = 1, 2, \ldots, l\right\} \\ &= \left\{\left[h^{\sigma(k)L}_{\mathbb{R}}(x, z) \vee \left(\bigvee_{y\neq z} h^{\sigma(k)L}_{\mathbb{R}}(x, y)\right),\right.\right. \\ &\qquad \left.h^{\sigma(k)U}_{\mathbb{R}}(x, z) \vee \left(\bigvee_{y\neq z} h^{\sigma(k)U}_{\mathbb{R}}(x, y)\right)\right] \\ &\qquad \left.| k = 1, 2, \ldots, l\right\} = \{[1, 1]\}, \end{aligned}$$

which implied that $\overline{\mathbb{R}}(\mathbb{U}) = \mathbb{U}$.

Assume that (IVHFU0) holds, then $\forall x \in U$, $h_{\overline{\mathbb{R}}(\mathbb{U})}(x) = \{[1, 1]\}$. If $\mathbb{R}$ is not serial, then $\exists x \in U, \forall y \in U$ such that $h_{\mathbb{R}}(x, y) \neq \{[1, 1]\}$. Since $h_{\mathbb{U}}(y) = \{[1, 1]\}$ for all $y \in U$, then $h_{\mathbb{R}}(x, y) \barwedge h_{\mathbb{U}}(y) = h_{\mathbb{R}}(x, y) \neq \{[1, 1]\}$.

From the above discussions, it follows that $h_{\overline{\mathbb{R}}(\mathbb{U})}(x) \neq \{[1, 1]\}$, which contradicts the assumption.

Third, we are to prove that $\mathbb{R}$ is serial $\Longleftrightarrow$ (IVHFLU0). If $\mathbb{R}$ is serial, then $\forall x \in U$, there exists a $z \in U$ such that $h_{\mathbb{R}}(x, z) = \{[1, 1]\}$, which implies that $h_{\mathbb{R}^c}(x, z) = \{[0, 0]\}$. By Eq. (3), we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\mathbb{A})}(x) &= \barwedge_{y\in U}\ \{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}}(y)\} \\ &= \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x, y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right),\right.\right. \\ &\qquad \left.\left.\bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x, y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right] \middle| k = 1, 2, \ldots, l\right\} \\ &= \left\{\left[\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x, z) \vee h^{\sigma(k)L}_{\mathbb{A}}(z)\right) \wedge \left(\bigwedge_{y\neq z}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x, y)\right.\right.\right.\right. \\ &\qquad \left.\left.\vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right)\right), \left(h^{\sigma(k)U}_{\mathbb{R}^c}(x, z) \vee h^{\sigma(k)U}_{\mathbb{A}}(z)\right) \\ &\qquad \left.\wedge \left(\bigwedge_{y\neq z}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x, y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right)\right] \\ &\qquad \left.| k = 1, 2, \ldots, l\right\} \\ &= \left\{\left[h^{\sigma(k)L}_{\mathbb{A}}(z) \wedge \left(\bigwedge_{y\neq z}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x, y) \vee h^{\sigma(k)L}_{\mathbb{A}}(y)\right)\right),\right.\right. \\ &\qquad \left.h^{\sigma(k)U}_{\mathbb{A}}(z) \wedge \left(\bigwedge_{y\neq z}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x, y) \vee h^{\sigma(k)U}_{\mathbb{A}}(y)\right)\right)\right] \end{aligned}$$

$$|\, k = 1, 2, \ldots, l\}$$
$$\preceq \left\{ \left[ h_{\mathbb{A}}^{\sigma(k)L}(z), h_{\mathbb{A}}^{\sigma(k)U}(z) \right] \middle| k = 1, 2, \ldots, l \right\} = h_{\mathbb{A}}(z).$$

Moreover, by Eq. (4), we have

$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \underline{\vee}_{y\in U}\{h_{\mathbb{R}}(x, y) \,\overline{\wedge}\, h_{\mathbb{A}}(y)\}$$
$$= \left\{ \left[ \bigvee_{y\in U} \left(h_{\mathbb{R}}^{\sigma(k)L}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)L}(y)\right), \bigvee_{y\in U} \left(h_{\mathbb{R}}^{\sigma(k)U}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)U}(y)\right) \right] \middle| k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ \left(h_{\mathbb{R}}^{\sigma(k)L}(x, z) \wedge h_{\mathbb{A}}^{\sigma(k)L}(z)\right) \vee \left(\bigvee_{y\neq z} (h_{\mathbb{R}}^{\sigma(k)L}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)L}(y))\right), \left(h_{\mathbb{R}}^{\sigma(k)U}(x, z) \wedge h_{\mathbb{A}}^{\sigma(k)U}(z)\right) \vee \left(\bigvee_{y\neq z} \left(h_{\mathbb{R}}^{\sigma(k)U}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)U}(y)\right)\right) \right] \middle| k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ h_{\mathbb{A}}^{\sigma(k)L}(z) \vee \left(\bigvee_{y\neq z} \left(h_{\mathbb{R}}^{\sigma(k)L}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)L}(y)\right)\right), h_{\mathbb{A}}^{\sigma(k)U}(z) \vee \left(\bigvee_{y\neq z} \left(h_{\mathbb{R}}^{\sigma(k)U}(x, y) \wedge h_{\mathbb{A}}^{\sigma(k)U}(y)\right)\right) \right] |k = 1, 2, \ldots, l \right\}$$
$$\succeq \left\{ \left[ h_{\mathbb{A}}^{\sigma(k)L}(z), h_{\mathbb{A}}^{\sigma(k)U}(z) \right] \middle| k = 1, 2, \ldots, l \right\} = h_{\mathbb{A}}(z).$$

From the above discussions, we can conclude that $h_{\underline{\mathbb{R}}(\mathbb{A})}(x) \preceq h_{\overline{\mathbb{R}}(\mathbb{A})}(x)$, which means that $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A})$.

Conversely, supposing that (IVHFLU0) holds, then for any $x \in U$, we have $h_{\underline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)L}(x) \leq h_{\overline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)L}(x)$ and $h_{\underline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)U}(x) \leq h_{\overline{\mathbb{R}}(\mathbb{A})}^{\sigma(k)U}(x)$, from which it follows that $h_{\underline{\mathbb{R}}(\emptyset)}^{\sigma(k)L}(x) \leq h_{\overline{\mathbb{R}}(\emptyset)}^{\sigma(k)L}(x)$ and $h_{\underline{\mathbb{R}}(\emptyset)}^{\sigma(k)U}(x) \leq h_{\overline{\mathbb{R}}(\emptyset)}^{\sigma(k)U}(x)$.

On the other hand, by Eqs. (3) and (4), we have

$$h_{\underline{\mathbb{R}}(\emptyset)}(x) = \overline{\wedge}_{y\in U} h_{\mathbb{R}^c}(x, y)$$
$$= \left\{ \left[ \bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)L}(x, y), \bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \right] \middle| k = 1, 2, \ldots, l \right\}$$

and $h_{\overline{\mathbb{R}}(\emptyset)}(x) = \{[0, 0]\}$. Hence, for any $x \in U$, there exists a $y \in U$ such that $h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) = 0$, and $h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) = 0$, which implies that $h_{\mathbb{R}}(x, y) = \{[1, 1]\}$. So $\mathbb{R}$ is serial.

At last, we are to prove that $\mathbb{R}$ is serial $\Longleftrightarrow$ (IVHFL0)$'$. Assume that $\mathbb{R}$ is serial. For any $x \in U$, there exists a $z \in U$ such that $h_{\mathbb{R}}(x, z) = \{[1, 1]\}$. By virtue of Eq. (3), we have

$$h_{\underline{\mathbb{R}}\left(\left[\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}\right]\right)}(x) = \overline{\wedge}_{y\in U} \left\{h_{\mathbb{R}^c}(x, y) \,\underline{\vee}\, h_{\left[\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}\right]}(y)\right\}$$
$$= \left\{ \left[ \left(h_{\mathbb{R}^c}^{\sigma(k)L}(x, z) \vee a_{1,\ldots,m}^{\sigma(k)L}\right) \wedge \left(\bigwedge_{y\neq z} \left(h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)L}\right)\right), \left(h_{\mathbb{R}^c}^{\sigma(k)U}(x, z) \vee a_{1,\ldots,m}^{\sigma(k)U}\right) \wedge \left(\bigwedge_{y\neq z} \left(h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)U}\right)\right) \right] |k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ a_{1,\ldots,m}^{\sigma(k)L} \wedge \left(\bigwedge_{y\neq z} h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)L}\right), a_{1,\ldots,m}^{\sigma(k)U} \wedge \left(\bigwedge_{y\neq z} h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)U}\right) \right] \middle| k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ a_{1,\ldots,m}^{\sigma(k)L}, a_{1,\ldots,m}^{\sigma(k)U} \right] \middle| k = 1, 2, \ldots, l \right\} = h_{\left[\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}\right]}(x).$$

Hence, we conclude that $\underline{\mathbb{R}}([\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}]) = [\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}]$ holds.

Conversely, if we assume that (IVHFL0)$'$ holds, then for any $x \in U$, we have

$$h_{\underline{\mathbb{R}}\left(\left[\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}\right]\right)}(x) = \overline{\wedge}_{y\in U} \left\{h_{\mathbb{R}^c}(x, y) \,\underline{\vee}\, h_{\left[\widehat{a_{1,\ldots,m}^L, a_{1,\ldots,m}^U}\right]}(y)\right\}$$
$$= \left\{ \left[ \bigwedge_{y\in U} \left(h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)L}\right), \bigwedge_{y\in U} \left(h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \vee a_{1,\ldots,m}^{\sigma(k)U}\right) \right] \middle| k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ \left(\bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)L}(x, y)\right) \vee a_{1,\ldots,m}^{\sigma(k)L}, \left(\bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)U}(x, y)\right) \vee a_{1,\ldots,m}^{\sigma(k)U} \right] \middle| k = 1, 2, \ldots, l \right\}$$
$$= \left\{ \left[ a_{1,\ldots,m}^{\sigma(k)L}, a_{1,\ldots,m}^{\sigma(k)U} \right] \middle| k = 1, 2, \ldots, l \right\},$$

which implies that $\bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)L}(x, y) \leq a_{1,\ldots,m}^{\sigma(k)L}$ and $\bigwedge_{y\in U} h_{\mathbb{R}^c}^{\sigma(k)U}(x, y) \leq a_{1,\ldots,m}^{\sigma(k)U}$ for any $1 \leq k \leq l$. If we set $a_{1,\ldots,m}^{\sigma(k)L} = a_{1,\ldots,m}^{\sigma(k)U} = 0$, then $\forall x \in U$, there exists a $y \in U$ such that $h_{\mathbb{R}}(x, y) = \{[1, 1]\}$. Hence, $\mathbb{R}$ is serial. □

**Theorem 3.15** *Let $(U, R)$ be an IVHF approximation space. $\underline{R}$ and $\overline{R}$ are the IVHF approximation operators induced from $(U, \mathbb{R})$, then*

$$(1)\ \mathbb{R} \text{ is reflexive} \Longleftrightarrow \text{(IVHFLR) } \underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \mathbb{A}, \forall \mathbb{A} \in \text{IVHF}(U), \Longleftrightarrow \text{(IVFHUR) } \mathbb{A} \sqsubseteq \overline{\mathbb{R}}(\mathbb{A}), \forall \mathbb{A} \in \text{IVHF}(U).$$

$$(2)\ \mathbb{R} \textit{ is symmetric} \Longleftrightarrow \text{(IVHFLS) } h_{\underline{\mathbb{R}}([1,1]_{U-\{x\}})}(y) = h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(x), \Longleftrightarrow \textit{(IVHFUS) } h_{\overline{\mathbb{R}}([1,1]_x)}(y) = h_{\overline{\mathbb{R}}([1,1]_y)}(x).$$

$$(3)\ \mathbb{R} \text{ is transitive} \Longleftrightarrow \text{(IVHFLT) } \underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A})), \forall \mathbb{A} \in \text{IVHF}(U), \Longleftrightarrow \text{(IVHFUT) } \overline{\mathbb{R}}(\overline{\mathbb{R}}(\mathbb{A})) \sqsubseteq \overline{\mathbb{R}}(\mathbb{A}), \forall \mathbb{A} \in \text{IVHF}(U).$$

*Proof* (1) By the dual properties of IVHF rough approximation operators, it is only to prove that

$\mathbb{R}$ is reflexive $\Longleftrightarrow$ (IVHFLR) $\underline{\mathbb{R}}(\mathbb{A}) \sqsubseteq \mathbb{A}$.

Assume that $\mathbb{R}$ is a reflexive IVHF relation; then by Theorem 3.11, (IVHFLR) holds.

Conversely, if (IVHFLR) holds, then for any $x \in U$, we obtain $h^{\sigma(k)L}_{\underline{\mathbb{R}}(\mathbb{A})}(x) \leq h^{\sigma(k)L}_{\mathbb{A}}(x)$ and $h^{\sigma(k)U}_{\underline{\mathbb{R}}(\mathbb{A})}(x) \leq h^{\sigma(k)U}_{\mathbb{A}}(x)$. So $h^{\sigma(k)L}_{\underline{\mathbb{R}}([1,1]_{U-\{x\}})}(x) \leq h^{\sigma(k)L}_{[1,1]_{U-\{x\}}}(x) = 0$ and $h^{\sigma(k)U}_{\underline{\mathbb{R}}([1,1]_{U-\{x\}})}(x) \leq h^{\sigma(k)U}_{[1,1]_{U-\{x\}}}(x) = 0$, from which we conclude that $h_{\underline{\mathbb{R}}([1,1]_{U-\{x\}})}(x) = \{[0,0]\}$.

On the other hand, by Eq. (3), then

$$\begin{aligned}
h_{\underline{\mathbb{R}}([1,1]_{U-\{x\}})}(x) &= \overline{\wedge}_{y\in U}\left\{h_{\mathbb{R}^c}(x,y) \veebar h_{[1,1]_{U-\{x\}}}(y)\right\}\\
&= \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{[1,1]_{U-\{x\}}}(y)\right),\right.\right.\\
&\quad \left.\bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{[1,1]_{U-\{x\}}}(y)\right)\right]\\
&\quad \left.|k = 1,2,\ldots,l\right\}\\
&= \left\{\left[\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,x) \vee h^{\sigma(k)L}_{[1,1]_{U-\{x\}}}(x)\right)\right.\right.\\
&\quad \wedge\left(\bigwedge_{y\neq x}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{[1,1]_{U-\{x\}}}(y)\right)\right),\\
&\quad \left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,x) \vee h^{\sigma(k)U}_{[1,1]_{U-\{x\}}}(x)\right)\\
&\quad \left.\wedge\left(\bigwedge_{y\neq x}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{[1,1]_{U-\{x\}}}(y)\right)\right)\right]\\
&\quad \left.|k = 1,2,\ldots,l\right\}\\
&= \left\{\left[h^{\sigma(k)L}_{\mathbb{R}^c}(x,x) \wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee 1\right)\right),\right.\right.\\
&\quad \left.h^{\sigma(k)U}_{\mathbb{R}^c}(x,x) \wedge \left(\bigwedge_{y\neq x}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee 1\right)\right)\right]\\
&\quad \left.|k = 1,2,\ldots,l\right\}\\
&= \{[h^{\sigma(k)L}_{\mathbb{R}^c}(x,x), h^{\sigma(k)U}_{\mathbb{R}^c}(x,x)]\\
&\quad |k = 1,2,\ldots,l\}\\
&= h_{\mathbb{R}^c}(x,x) = \{[0,0]\}.
\end{aligned}$$

So $h_{\mathbb{R}}(x,x) = \{[1,1]\}$, which implies that $\mathbb{R}$ is reflexive.

(2) It follows immediately from Theorem 3.13.

(3) (IVHFLT) and (IVHFUT) are equivalent because of the duality of IVHF rough approximation operators. We are only to prove that the transitivity of $\mathbb{R}$ is equivalent to (IVHFLT).

Let us assume that $\mathbb{R}$ is transitive. So we have $h^{\sigma(k)L}_{\mathbb{R}}(x,y) \wedge h^{\sigma(k)L}_{\mathbb{R}}(y,z) \leq h^{\sigma(k)L}_{\mathbb{R}}(x,z)$, and $h^{\sigma(k)U}_{\mathbb{R}}(x,y) \wedge h^{\sigma(k)U}_{\mathbb{R}}(y,z) \leq h^{\sigma(k)U}_{\mathbb{R}}(x,z)$. Moreover, by Eq. (3), we have

$$\begin{aligned}
h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A}))}(x) &= \overline{\wedge}_{y\in U}\left\{h_{\mathbb{R}^c}(x,y) \veebar h_{\underline{\mathbb{R}}(\mathbb{A})}(y)\right\}\\
&= \left\{\left[\bigwedge_{y\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee \left(\bigwedge_{z\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(y,z)\right.\right.\right.\right.\right.\\
&\quad \left.\left.\left.\vee h^{\sigma(k)L}_{\mathbb{A}}(z)\right)\right)\right), \bigwedge_{y\in U}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y)\right.\\
&\quad \left.\left.\left.\vee\left(\bigwedge_{z\in U}\left(h^{\sigma(k)U}_{\mathbb{R}^c}(y,z) \vee h^{\sigma(k)U}_{\mathbb{A}}(z)\right)\right)\right)\right]\right| k = 1,2,\ldots,l\Bigg\}\\
&= \left\{\left[\bigwedge_{y\in U}\bigwedge_{z\in U}\left(h^{\sigma(k)L}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)L}_{\mathbb{R}^c}(y,z) \vee h^{\sigma(k)L}_{\mathbb{A}}(z)\right),\right.\right.\\
&\quad \bigwedge_{y\in U}\bigwedge_{z\in U}(h^{\sigma(k)U}_{\mathbb{R}^c}(x,y) \vee h^{\sigma(k)U}_{\mathbb{R}^c}(y,z)\\
&\quad \left.\left.\vee h^{\sigma(k)U}_{\mathbb{A}}(z)\right)\right]\Big| k = 1,2,\ldots,l\Big\}\\
&= \left\{\left[\bigwedge_{z\in U}\bigwedge_{y\in U}\left(\left(1 - h^{\sigma(k)L}_{\mathbb{R}}(x,y)\right)\right.\right.\right.\\
&\quad \left.\vee\left(1 - h^{\sigma(k)L}_{\mathbb{R}}(y,z)\right)\right) \vee h^{\sigma(k)L}_{\mathbb{A}}(z),\\
&\quad \bigwedge_{z\in U}\bigwedge_{y\in U}\left(\left(1 - h^{\sigma(k)U}_{\mathbb{R}}(x,y)\right) \vee \left(1 - h^{\sigma(k)U}_{\mathbb{R}}(y,z)\right)\right)\\
&\quad \left.\left.\vee h^{\sigma(k)U}_{\mathbb{A}}(z)\right)\right]\Big| k = 1,2,\ldots,l\Big\}\\
&= \left\{\left[\bigwedge_{z\in U}\left(\bigwedge_{y\in U}\left(1 - \left(h^{\sigma(k)L}_{\mathbb{R}}(x,y) \wedge h^{\sigma(k)L}_{\mathbb{R}}(y,z)\right)\right)\right)\right.\right.\\
&\quad \left.\vee h^{\sigma(k)L}_{\mathbb{A}}(z)\right), \bigwedge_{z\in U}\left(\bigwedge_{y\in U}\left(1 - \left(h^{\sigma(k)U}_{\mathbb{R}}(x,y)\right.\right.\right.\\
&\quad \left.\left.\left.\left.\wedge h^{\sigma(k)U}_{\mathbb{R}}(y,z)\right)\right)\right) \quad \vee h^{\sigma(k)U}_{\mathbb{A}}(z)\right)\right]\Big| k = 1,2,\ldots,l\Big\}
\end{aligned}$$

$$\succeq \left\{ \left[ \bigwedge_{z \in U} \left( h_{\mathbb{R}^c}^{\sigma(k)L}(x,z) \vee h_{\mathbb{A}}^{\sigma(k)L}(z) \right), \bigwedge_{z \in U} \left( h_{\mathbb{R}^c}^{\sigma(k)U}(x,z) \vee h_{\mathbb{A}}^{\sigma(k)U}(z) \right) \right] \middle| k = 1, 2, \ldots, l \right\} = h_{\underline{\mathbb{R}}(\mathbb{A})}(x).$$

Hence, it follows that (IVHFLT) holds.

Conversely, assume that $h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A}))}(x) \succeq h_{\underline{\mathbb{R}}(\mathbb{A})}(x)$ for all $\mathbb{A} \in \text{IVHF}(U)$. Then for any $x \in U$, $h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}([1,1]_{U-\{y\}}))}(x) \succeq h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(x)$. On the other hand, by virtue of Eq. (3) and Theorem 3.13, we have

$$\begin{aligned} h_{\underline{\mathbb{R}}(\underline{\mathbb{R}}([1,1]_{U-\{y\}}))}(x) &= \bar{\wedge}_{z \in U} \{h_{\mathbb{R}^c}(x,z) \veebar h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(z)\} \\ &= \bar{\wedge}_{z \in U} \{h_{\mathbb{R}^c}(x,z) \veebar h_{\mathbb{R}^c}(z,y)\}, \end{aligned}$$

and $h_{\underline{\mathbb{R}}([1,1]_{U-\{y\}})}(x) = h_{\mathbb{R}^c}(x,y)$, from which we can conclude that for any $1 \le k \le l$, $\bigwedge_{z \in U}(h_{\mathbb{R}^c}^{\sigma(k)L}(x,z) \vee h_{\mathbb{R}^c}^{\sigma(k)L}(z,y)) \ge h_{\mathbb{R}^c}^{\sigma(k)L}(x,y)$ and $\bigwedge_{z \in U}(h_{\mathbb{R}^c}^{\sigma(k)U}(x,z) \vee h_{\mathbb{R}^c}^{\sigma(k)U}(z,y)) \ge h_{\mathbb{R}^c}^{\sigma(k)U}(x,y)$. That is, for $\forall z \in U$, $h_{\mathbb{R}^c}^{\sigma(k)L}(x,z) \vee h_{\mathbb{R}^c}^{\sigma(k)L}(z,y) \ge h_{\mathbb{R}^c}^{\sigma(k)L}(x,y)$ and $h_{\mathbb{R}^c}^{\sigma(k)U}(x,z) \vee h_{\mathbb{R}^c}^{\sigma(k)U}(z,y) \ge h_{\mathbb{R}^c}^{\sigma(k)U}(x,y)$. So it follows that $h_{\mathbb{R}}^{\sigma(k)L}(x,z) \wedge h_{\mathbb{R}}^{\sigma(k)L}(z,y) \le h_{\mathbb{R}}^{\sigma(k)L}(x,y)$ and $h_{\mathbb{R}}^{\sigma(k)U}(x,z) \wedge h_{\mathbb{R}}^{\sigma(k)U}(z,y) \le h_{\mathbb{R}}^{\sigma(k)U}(x,y)$, $1 \le k \le l$ with $l = \max\{l(h_{\mathbb{R}}(x,z)), l(h_{\mathbb{R}}(z,y)), l(h_{\mathbb{R}}(x,y))\}$. By the definition of transitivity, we know that $\mathbb{R}$ is transitive. □

Combining (1) and (3) in Theorem 3.15, we can easily obtain the conclusion as follows.

**Corollary 3.16** *Let $\mathbb{R} \in \text{IVHFR}(U \times U)$. If $\mathbb{R}$ is reflexive and transitive, and $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ are the lower and upper IVHF rough approximation operators defined in Definition 3.3, then*

(IVHFLRT) $\underline{\mathbb{R}}(\mathbb{A}) = \underline{\mathbb{R}}(\underline{\mathbb{R}}(\mathbb{A})), \forall \mathbb{A} \in \text{IVHF}(U)$,

(IVHFURT) $\overline{\mathbb{R}}(\overline{\mathbb{R}}(\mathbb{A})) = \overline{\mathbb{R}}(\mathbb{A}), \forall \mathbb{A} \in \text{IVHF}(U)$.

## 4 Axiomatic characterization of interval-valued hesitant fuzzy rough sets

In an axiomatic approach, rough sets are axiomatized by abstract operators. In recent years, many scholars attach great importance to the research on axiomatic approach and achieved a lot. The most important axiomatic studies for crisp rough sets were made by Yao (1998a, c) where various classes of rough set algebras are characterized by different sets of axioms. Morsi and Yakout (1998) studied a set of axioms on fuzzy rough set, but their studies were restricted to fuzzy $T$-rough sets. Subsequently, the studies of axiomatic research on various generalized approximation operators in fuzzy environment were made by Wu et al. (2003, 2005, 2006), Wu and Zhang (2004) in which various classes of fuzzy approximation operators are characterized by different sets of axioms. Along the lines of fuzzy rough sets, Zhou and Wu (2008, 2009) explored and developed the axiomatic approach in the study of intuitionistic fuzzy rough approximation operators. More recently, rough set approximations were introduced into HF sets. Yang et al. (2014) introduced the concept of the HF rough sets and investigated axiomatic characterizations of HF rough approximation operators.

In the section, we will extend the axiomatic approach to HF rough approximation operators in IVHF environment. The results may be viewed as the generalization counterparts of Yang et al. (2014). For the case of IVHF rough sets, the primitive notion is a system $(\text{IVHF}(U), \Cup, \Cap, ^c, L, H)$, where $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ are operators from $\text{IVHF}(U)$ to $\text{IVHF}(U)$.

**Definition 4.1** Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be two operators. $L$ and $H$ are referred to as dual operators if the following axioms are satisfied: $\forall \mathbb{A} \in \text{IVHF}(U)$,

(AL1) $L(\mathbb{A}) = (H(\mathbb{A}^c))^c$,

(AU1) $H(\mathbb{A}) = (L(\mathbb{A}^c))^c$.

**Theorem 4.2** *Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be two dual operators. Then, there exists an IVHF relation $\mathbb{R}$ on $U$ such that $L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$, and $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in \text{IVHF}(U)$ iff $L$ satisfies axioms (AL2) and (AL3), or equivalently, $H$ satisfies axioms (AU2) and (AU3): $\forall \mathbb{A}, \mathbb{B} \in \text{IVHF}(U)$, $\widehat{[a_{1,\ldots,m}^L, a_{1,\ldots,m}^U]} \in 2^{\text{Int}[0,1]}$,*

(AL2) $L\left(\mathbb{A} \Cup \widehat{\left[a_{1,\ldots,m}^L, a_{1,\ldots,m}^U\right]}\right) = L(\mathbb{A}) \Cup \widehat{\left[a_{1,\ldots,m}^L, a_{1,\ldots,m}^U\right]}$,

(AL3) $L(\mathbb{A} \Cap \mathbb{B}) = L(\mathbb{A}) \Cap L(\mathbb{B})$,

(AU2) $H\left(\mathbb{A} \Cap \widehat{\left[a_{1,\ldots,m}^L, a_{1,\ldots,m}^U\right]}\right) = H(\mathbb{A}) \Cap \widehat{\left[a_{1,\ldots,m}^L, a_{1,\ldots,m}^U\right]}$,

(AU3) $H(\mathbb{A} \Cup \mathbb{B}) = H(\mathbb{A}) \Cup H(\mathbb{B})$.

*Proof* "$\Longrightarrow$" follows immediately from Theorem 3.10.

"$\Longleftarrow$" Assume that the operator $H$ satisfies axioms (AU2) and (AU3). Then, we can define an IVHF relation $\mathbb{R} = \{\langle (x,y), h_{\mathbb{R}}(x,y)\rangle | (x,y) \in U \times U\}$ on $U$ by $H$ as follows:

$$h_{\mathbb{R}}(x,y) = h_{H([1,1]_y)}(x), \ (x,y) \in U \times U.$$

Moreover, we can prove that for any $\mathbb{A} \in \text{IVHF}(U)$,

$$\mathbb{A} = \Cup_{y \in U}\left([1,1]_y \Cap \widehat{h_{\mathbb{A}}(y)}\right).$$

In fact, for all $x \in U$, then

$$\begin{aligned} h_{\Cup_{y \in U}\left([1,1]_y \Cap \widehat{h_{\mathbb{A}}(y)}\right)}(x) &= \underline{\vee}_{y \in U} h_{\left([1,1]_y \Cap \widehat{h_{\mathbb{A}}(y)}\right)}(x) \\ &= \underline{\vee}_{y \in U}\left(h_{[1,1]_y}(x) \bar{\wedge} h_{\widehat{h_{\mathbb{A}}(y)}}(x)\right) \end{aligned}$$

$$= (\{[1,1]\} \barwedge h_{\mathbb{A}}(x)) \veebar \{[0,0]\}$$
$$= h_{\mathbb{A}}(x),$$

which implies that $\mathbb{A} = \Cup_{y\in U}([1,1]_y \Cap \widehat{h_{\mathbb{A}}(y)})$.

For any $x \in U$, by Eq. (4), (AU2), and (AU3), we have

$$\begin{aligned} h_{\overline{\mathbb{R}}(\mathbb{A})}(x) &= \veebar_{y\in U} \{h_{\mathbb{R}}(x,y) \barwedge h_{\mathbb{A}}(y)\} \\ &= \veebar_{y\in U} \left\{h_{H([1,1]_y)}(x) \barwedge h_{\widehat{h_{\mathbb{A}}(y)}}(x)\right\} \\ &= \veebar_{y\in U} \left\{h_{H([1,1]_y)\Cap\widehat{h_{\mathbb{A}}(y)}}(x)\right\} \\ &= \veebar_{y\in U} \left\{h_{H\left([1,1]_y\Cap\widehat{h_{\mathbb{A}}(y)}\right)}(x)\right\} \\ &= h_{\Cup_{y\in U} H\left([1,1]_y\Cap\widehat{h_{\mathbb{A}}(y)}\right)}(x) \\ &= h_{H\left(\Cup_{y\in U}\left([1,1]_y\Cap\widehat{h_{\mathbb{A}}(y)}\right)\right)}(x) \\ &= h_{H(A)}(x). \end{aligned}$$

Thus, $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$.

$L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$ follows immediately from the conclusion $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ and the dual axioms (AL1) and (AU1). □

From Theorem 4.2, we can note that axioms (AL1), (AU1), (AL2), and (AL3), or equivalently, axioms (AL1), (AU1), (AU2), and (AU3) are considered as basic axioms to characterize IVHF rough approximation operators. So we have the following definition of IVHF rough set algebra.

**Definition 4.3** Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be two dual operators. If $L$ satisfies axioms (AL2) and (AL3), or equivalently, $H$ satisfies axioms (AU2) and (AU3), then the system $(\text{IVHF}(U), \Cup, \Cap, ^c, L, H)$ is referred to as an IVHF rough set algebra; $L$ and $H$ are called the lower and upper IVHF approximation operators, respectively.

The following Theorems show that IVHF approximation operators generated by special IVHF relations can be characterized by axioms.

**Theorem 4.4** *Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be a pair of dual operators, i.e., $L$ satisfies axioms* (AL1), (AL2), *and* (AL3), *and $H$ satisfies axioms* (AU1), (AU2), *and* (AU3). *Then, there exists a serial IVHF relation $\mathbb{R}$ on $U$ such that $L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$, and $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in \text{IVHF}(U)$ iff $L$ satisfies axioms* (ALS), *or equivalently, $H$ satisfies axioms* (AUS)*:*

(ALS) $L\left(\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$,

(AUS) $H\left(\left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]\right) = \left[\widehat{a^L_{1,\ldots,m}, a^U_{1,\ldots,m}}\right]$.

*Proof* "$\Longrightarrow$" follows immediately from Theorem 3.14, and "$\Longleftarrow$" follows immediately from Theorems 4.2 and 3.14. □

*Remark 4.5* By Theorem 3.14, it can be easily seen that axioms (ALS) and (AUS) can be replaced by one of the following axioms:

(ALS)′ $L(\emptyset) = \emptyset$,

(AUS)′ $H(\mathbb{U}) = \mathbb{U}$,

(ALUS) $L(\mathbb{A}) \sqsubseteq H(\mathbb{A})$.

**Theorem 4.6** *Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be a pair of dual operators, i.e., $L$ satisfies axioms* (AL1), (AL2), *and* (AL3), *and $H$ satisfies axioms* (AU1), (AU2), *and* (AU3). *Then, there exists a reflexive IVHF relation $\mathbb{R}$ on $U$ such that $L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$, and $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in \text{IVHF}(U)$ iff $L$ satisfies axioms* (ALR), *or equivalently, $H$ satisfies axioms* (AUR)*:*

(ALR) $L(\mathbb{A}) \sqsubseteq \mathbb{A}, \quad \forall \mathbb{A} \in \text{IVHF}(U)$,

(AUR) $\mathbb{A} \sqsubseteq H(\mathbb{A}), \quad \forall \mathbb{A} \in \text{IVHF}(U)$.

*Proof* "$\Longrightarrow$" follows immediately from Theorem 3.15, and "$\Longleftarrow$" follows immediately from Theorems 4.2 and 3.15. □

**Theorem 4.7** *Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be a pair of dual operators, i.e., $L$ satisfies axioms* (AL1), (AL2), *and* (AL3), *and $H$ satisfies axioms* (AU1), (AU2), *and* (AU3). *Then, there exists a symmetric IVHF relation $\mathbb{R}$ on $U$ such that $L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$, and $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in \text{IVHF}(U)$ iff $L$ satisfies axioms* (ALSY), *or equivalently, $H$ satisfies axioms* (AUSY)*:*

(ALSY) $h_{L([1,1]_{U-\{x\}})}(y) = h_{L([1,1]_{U-\{y\}})}(x)$, $\forall (x,y) \in U \times U$,

(AUSY) $h_{H([1,1]_x)}(y) = h_{H([1,1]_y)}(x), \quad \forall (x,y) \in U \times U$.

*Proof* "$\Longrightarrow$" follows immediately from Theorem 3.15, and "$\Longleftarrow$" follows immediately from Theorems 4.2 and 3.15. □

**Theorem 4.8** *Let $L, H : \text{IVHF}(U) \longrightarrow \text{IVHF}(U)$ be a pair of dual operators, i.e., $L$ satisfies axioms* (AL1), (AL2), *and* (AL3), *and $H$ satisfies axioms* (AU1), (AU2), *and* (AU3). *Then, there exists a transitive IVHF relation $\mathbb{R}$ on $U$ such that $L(\mathbb{A}) = \underline{\mathbb{R}}(\mathbb{A})$, and $H(\mathbb{A}) = \overline{\mathbb{R}}(\mathbb{A})$ for all $\mathbb{A} \in \text{IVHF}(U)$ iff $L$ satisfies axioms* (ALT), *or equivalently, $H$ satisfies axioms* (AUT)*:*

(ALT) $L(\mathbb{A}) \sqsubseteq L(L(\mathbb{A})), \quad \forall \mathbb{A} \in \text{IVHF}(U)$,

(AUT) $H(H(\mathbb{A})) \sqsubseteq H(\mathbb{A}), \quad \forall \mathbb{A} \in \text{IVHF}(U)$.

*Proof* "$\Longrightarrow$" follows immediately from Theorem 3.15, and "$\Longleftarrow$" follows immediately from Theorems 4.2 and 3.15. □

## 5 Application of the IVHF rough set model in medical diagnosis

Rough set theory was developed by Pawlak (1982, 1991) as a mathematical approach to handle imprecision, vagueness, and uncertainty in data analysis. This theory has been successfully applied in solving a variety of problems, especially in the areas of multi-criteria decision making and group decision making. After it was introduced by Pawlak, rough set theory has attracted more and more scholars' attention. Up to now, many of researches about rough sets are mainly focusing on the same universe. But in reality, the possible two or more different universes and their interrelationship may invalidate rough set theory on the one universe, which makes the study of two universes or multi-universes become a necessity. For instance, in clinic, a patient maybe shows many symptoms at the same time. Meanwhile, a concrete disease could also include many basic symptoms in clinic. In that case, it is very difficult for a doctor to determine whether a patient is suffering from a certain disease or not. Then, an effectively method to describe this problem is to use two different universes in which the one is the set of all patients and the other one is the set of all possible symptoms in clinic.

In this section, we extend the IVHF rough set on one same universe in Sect. 3 and give the concept of IVHF rough set on two universes. Then, an approach to the decision making based on the IVHF rough set is presented in order to illustrate the validity of IVHF rough sets on two universes.

Firstly, we generalize the IVHF relation on one same universe in Sect. 3 and give an IVHF relation on two universes.

**Definition 5.1** Let $U, V$ be two nonempty and finite universes. An IVHF subset $\mathbb{R}$ of the universe $U \times V$ is called an IVHF relation from $U$ to $V$, namely $\mathbb{R}$ is given by

$$\mathbb{R} = \{\langle (x, y), h_{\mathbb{R}}(x, y)\rangle | (x, y) \in U \times V\},$$

where $h_{\mathbb{R}} : U \times V \to \mathrm{Int}[0, 1]$ is a set of interval values in $\mathrm{Int}[0, 1]$.

It is noted that if $U = V$, then $\mathbb{R}$ degenerates to an IVHF relation on $U$ given in Definition 3.1.

In generally, for any $x \in U$, $y \in V$, $h_{\mathbb{R}}(x, y)$ denotes the possible interval membership degrees of the relationships between $x$ and $y$. We denote by $\mathrm{IVHFR}(U \times V)$ the family of all IVHF relations on $U \times V$.

Based on the IVHF relation on two universes, we extend the IVHF rough sets in Definition 3.3 and construct lower and upper IVHF approximation operators induced from a generalized IVHF approximation space over two universes.

**Definition 5.2** Let $U$ and $V$ be two nonempty and finite universes and $\mathbb{R} \in \mathrm{IVHFR}(U \times V)$; the pair $(U, V, \mathbb{R})$ is called a generalized IVHF approximation space. For any $\mathbb{A} \in \mathrm{IVHF}(V)$, the lower and upper approximations of $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two IVHF sets of $U$ and are, respectively, defined as follows:

$$\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\underline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}, \tag{5}$$

$$\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\overline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}, \tag{6}$$

where

$$h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \bar{\wedge}_{y \in V}\{h_{\mathbb{R}^c}(x, y) \veebar h_{\mathbb{A}}(y)\},$$
$$h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \veebar_{y \in V}\{h_{\mathbb{R}}(x, y) \bar{\wedge} h_{\mathbb{A}}(y)\}.$$

$\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ are, respectively, called the lower and upper approximations of $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$. The pair $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ is called the IVHF rough set of $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$, and $\underline{\mathbb{R}}, \overline{\mathbb{R}} : \mathrm{IVHF}(V) \to \mathrm{IVHF}(U)$ are referred to as lower and upper IVHF rough approximation operators, respectively.

To facilitate and to compare the magnitude of different IVHFEs, Xu and Da (2002) gave the properties of interval numbers.

**Definition 5.3** (Xu and Da 2002) Let $a = [a^L, a^U]$, and $b = [b^L, b^U]$ be two interval numbers, and $\lambda \geq 0$, then

(1) $a = b \Leftrightarrow a^L = b^L$ and $a^U = b^U$;
(2) $a + b = [a^L + b^L, a^U + b^U]$;
(3) $\lambda a = [\lambda a^L, \lambda a^U]$, especially, $\lambda a = 0$, if $\lambda = 0$.

In Chen et al. (2013a), on the basis of Definition 5.3, Chen et al introduced the score function of IVHF elements as follows:

**Definition 5.4** (Chen et al. 2013a) For an IVHF element $h_{\mathbb{A}}(x)$,

$$s(h_{\mathbb{A}}(x)) = \frac{\sum_{\gamma \in h_{\mathbb{A}}(x)} \gamma}{l(h_{\mathbb{A}}(x))}$$

is called the score function of $h_{\mathbb{A}}(x)$, where $l(h_{\mathbb{A}}(x))$ is the number of the elements in $h_{\mathbb{A}}(x)$, and $s(h_{\mathbb{A}}(x))$ is an interval value belonging to $[0, 1]$. For two IVHFEs $h_{\mathbb{A}}(x)$ and $h_{\mathbb{B}}(x)$, if $s(h_{\mathbb{A}}(x)) \geq s(h_{\mathbb{B}}(x))$, then $h_{\mathbb{A}}(x) \geq h_{\mathbb{B}}(x)$.

Note that we can compare two score functions using Eq. 1. Moreover, by Definition 5.4, we can judge the magnitude of two IVHFEs.

In what follows, we will apply IVHF rough set model on two universes to medical diagnosis problems. Suppose that the universe $U = \{x_1, x_2, \ldots, x_m\}$ denotes a symptom set, and the universe $V = \{y_1, y_2, \ldots, y_n\}$ denotes a disease set. Let $\mathbb{R} \in \mathrm{IVHFR}(U \times V)$ be an IVHF relation from $U$ to $V$. For any $(x_i, y_j) \in U \times V$, $h_{\mathbb{R}}(x_i, y_j)$ represents interval membership degree of the relationships between the symptom $x_i (x_i \in U)$ and the disease $y_j (y_j \in V)$, which

**Table 1** Symptoms characteristic for the considered diagnoses

| $\mathbb{R}$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | {[0.3, 0.4], [0.5, 0.6]} | {[0.2, 0.3], [0.4, 0.5]} | {[0.5, 0.8], [0.6, 0.9]} | {[0.6, 0.7], [0.7, 0.8]} |
| $x_2$ | {[0.4, 0.6], [0.5, 0.7]} | {[0.7, 0.9], [0.8, 0.9]} | {[0.6, 0.6], [0.7, 0.8]} | {[0.4, 0.5], [0.6, 0.6]} |
| $x_3$ | {[0.4, 0.5], [0.4, 0.6]} | {[0.3, 0.5], [0.4, 0.6]} | {[0.4, 0.5], [0.5, 0.7]} | {[0.3, 0.6], [0.5, 0.7]} |
| $x_4$ | {[0.4, 0.5], [0.5, 0.5]} | {[0.7, 0.8], [0.9, 0.9]} | {[0.3, 0.5], [0.2, 0.4]} | {[0.2, 0.3], [0.1, 0.3]} |
| $x_5$ | {[0.8, 0.9], [0.7, 0.9]} | {[0.4, 0.5], [0.3, 0.6]} | {[0.5, 0.5], [0.5, 0.6]} | {[0.5, 0.6], [0.3, 0.5]} |

are evaluated by several doctors in advance. In clinical practice, a patient can see different doctors and may get different diagnoses. To decrease the risk of misdiagnosis, we should carefully consider all the doctors' comments. In that case, for any a patient set $\mathbb{A}$ who has some symptoms in universe $U$, patient set $\mathbb{A}$ is an IVHF set on symptom set $U$. That is, $\mathbb{A} = \{\langle x_i, h_{\mathbb{A}}(x_i)\rangle | x_i \in U\}$, where $h_{\mathbb{A}}(x_i)$ is a set of some different interval values in [0, 1], representing the possible membership degrees to the symptom $x_i \in U$ of $\mathbb{A}$. Now, the problem is that a decision maker needs to make a reasonable decision about how to judge what kind of the disease $y_j$ patient $\mathbb{A}$ is suffering from.

In the following, we present an approach to the decision making for this kind of problem by using the IVHF rough set theory over two universes with three steps.

First, according to Definition 5.2, we calculate the lower and upper approximations $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ of IVHF set $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$.

Second, from Definition 2.11, we can obtain

$$\begin{aligned}\underline{\mathbb{R}}(\mathbb{A}) \boxplus \overline{\mathbb{R}}(\mathbb{A}) &= \{\langle y_j, h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_j)\rangle : y_j \in V\}\\ &= \{\langle y_j, h_{\underline{\mathbb{R}}(\mathbb{A})}(y_j) \oplus h_{\overline{\mathbb{R}}(\mathbb{A})}(y_j)\rangle : y_j \in V\}.\end{aligned}$$

Furthermore, on the basis of Definition 5.4, the score functions of IVHF elements $h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_j)$ are obtained by us. Denote

$$\lambda_j = s(h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_j)) = s(h_{\underline{\mathbb{R}}(\mathbb{A})}(y_j) \oplus h_{\overline{\mathbb{R}}(\mathbb{A})}(y_j)).$$

Finally, the optimal decision is to select $y_l$ if $\lambda_l = \max_j \lambda_j, j = 1, 2, \ldots, |V|$. In other words, if $\lambda_l = \max_j \lambda_j, j = 1, 2, \ldots, |V|$, we conclude that patient $\mathbb{A}$ is suffering from the disease $y_l$. Note that if $l$ has more than one value, then all the $y_l$ may be chosen, which implies that patient $\mathbb{A}$ is suffering from various diseases.

Therefore, we have established an approach to uncertainty decision making based on the IVHF rough set theory over two universes. In the next section, the application of this method will be shown by using a medical diagnosis decision-making problem.

## 6 A numerical example

In this section, we will apply the decision approach proposed in Sect. 5 to a medical diagnosis problem.

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be five symptoms in clinic, where $x_i$ stand for "temperature," "headache," "stomach pain," "cough," and "chest-pain," respectively, and the universe $V = \{y_1, y_2, y_3, y_4\}$ be four diseases, where $y_j$ stand for "viral fever," " malaria," "typhoid," and "stomach problem," respectively. Let $\mathbb{R}$ be an IVHF relation from $U$ to $V$. And $\mathbb{R}$ is a medical knowledge statistic data of the relationship of the symptom $x_i (x_i \in U)$ and the disease $y_j (y_j \in V)$. The statistic data are given in Table 1.

In clinical practice, a patient can see different doctors and may get different diagnoses. In this example, we suppose that $\mathbb{A}$ represents a patient that can see two different doctors. To decrease the risk of misdiagnosis, we should carefully consider all the doctors' comments. So the symptoms of patient $\mathbb{A}$ are described by an IVHF set on the universe $U$. Let

$$\begin{aligned}\mathbb{A} = \{&\langle x_1, \{[0.4, 0.5], [0.6, 0.9]\}\rangle, \langle x_2, \{[0.1, 0.2], [0.5, 0.6]\}\rangle,\\ &\langle x_3, \{[0.3, 0.5], [0.7, 0.9]\}\rangle, \langle x_4, \{[0.2, 0.3], [0.4, 0.6]\}\rangle,\\ &\langle x_5, \{[0.4, 0.6], [0.5, 0.7]\}\rangle\}.\end{aligned}$$

For example, for $h_{\mathbb{A}}(x_3) = \{[0.3, 0.5], [0.7, 0.9]\}$, doctors cannot present the precise membership degree of how pain the stomach of patient $\mathbb{A}$ is, but they have a certain hesitancy in providing the membership degree of how pain the stomach of patient $\mathbb{A}$ is. Because of different clinic experiences and knowledge backgrounds, doctors may get different diagnosis for the same patient. Thus, one doctor provides possible interval value [0.3,0.5] to depict the membership degree of how pain the stomach of patient $\mathbb{A}$ is, and the other doctor provides possible interval value [0.7,0.9] to depict the membership degree of how pain the stomach of patient $\mathbb{A}$ is.

In what follows, we give the decision-making process by using the three steps given in Sect. 5 in detail.

First, by Definition 5.2, we calculate the lower and upper approximations $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ of patient $\mathbb{A}$ as follows

$$\begin{aligned}\underline{\mathbb{R}}(\mathbb{A}) = \{&\langle y_1, \{[0.3, 0.5], [0.5, 0.6]\}\rangle,\\ &\langle y_2, \{[0.1, 0.2], [0.4, 0.6]\}\rangle,\\ &\langle y_3, \{[0.2, 0.3], [0.5, 0.6]\}\rangle,\\ &\langle y_4, \{[0.3, 0.4], [0.5, 0.6]\}\rangle\},\\ \overline{\mathbb{R}}(\mathbb{A}) = \{&\langle y_1, \{[0.4, 0.6], [0.5, 0.7]\}\rangle,\\ &\langle y_2, \{[0.4, 0.5], [0.5, 0.6]\}\rangle,\end{aligned}$$

$$\langle y_3, \{[0.4, 0.5], [0.6, 0.9]\}\rangle,$$
$$\langle y_4, \{[0.4, 0.6], [0.6, 0.8]\}\rangle\}.$$

Then, we have

$$\begin{aligned}\underline{\mathbb{R}}(\mathbb{A}) \boxplus \overline{\mathbb{R}}(\mathbb{A}) = \{&\langle y_1, \{[0.58, 0.80], [0.75, 0.88]\}\rangle,\\ &\langle y_2, \{[0.46, 0.60], [0.70, 0.84]\}\rangle,\\ &\langle y_3, \{[0.52, 0.65], [0.80, 0.96]\}\rangle,\\ &\langle y_4, \{[0.58, 0.76], [0.80, 0.92]\}\rangle\},\end{aligned}$$

By virtue of Definition 5.4, we obtain the score functions of IVHF elements $h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_j)$ as follows:

$$h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_1) = [0.665, 0.84],$$
$$h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_2) = [0.58, 0.72],$$
$$h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_3) = [0.66, 0.805],$$
$$h_{\underline{\mathbb{R}}(\mathbb{A})\boxplus\overline{\mathbb{R}}(\mathbb{A})}(y_4) = [0.69, 0.84].$$

So according to Eq. 1, it is clear that the maximum score function is $\lambda_4 = [0.69, 0.84]$. Hence, the optimal decision is to select $y_4$. That is, we can conclude that patient $\mathbb{A}$ is suffering from the disease stomach problem ($y_4$).

## 7 Conclusion

In this paper, we develop a general framework for the study of IVHF rough approximation operators which includes both constructive and axiomatic approaches. In our constructive method, IVHF rough approximation operators are defined in terms of IVHF relations. Properties of upper and lower IVHF rough approximation operators are also investigated. By the axiomatic approach, upper and lower IVHF approximation operators are defined by abstract axioms. We prove that axiom sets characterizing IVHF approximation operators guarantee the existence of certain types of IVHF relations which produce the same operators. Finally, the IVHF rough approximation operators are extended to the case of two universes. By using IVHF rough set theory over two universes, we develop a general framework for dealing with uncertainty decision making. Further, we use a medical diagnosis decision-making problem to demonstrate the principal steps of the decision methodology.

In the future, we can further use the proposed rough set model to address the applications to knowledge discovery and reduction. Moreover, it is important and interesting to further investigate relationships between IVHF rough set and other mathematical structures, such as lattice structures and topological structures.

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## References

Babitha KV, John SJ (2013) Hesitant fuzzy soft sets. J New Results Sci 3:98–107
Chakrabarty K, Gedeon T, Koczy L (1998) Intuitionistic fuzzy rough set. In: Proceedings of fourth joint conference on information sciences (JCIS), Durham, NC, pp 211–214
Chen N, Xu ZS, Xia MM (2013a) Interval-valued hesitant preference relations and their applications to group decision making. Knowl-Based Syst 37:528–540
Chen N, Xu ZS, Xia MM (2013b) Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Appl Math Model 37:2197–2211
Chuchro M (1993) A certain conception of rough sets in topological boolean algebras. Bull Sect Logic 22(1):9–12
Chuchro M (1994) On rough sets in topological Boolean algebras. In: Ziarko W (ed) Rough sets, fuzzy sets and knowledge discovery. Springer, Berlin, pp 157–160
Comer S (1991) An algebraic approach to the approximation of information. Fundam Inform 14:492–502
Comer S (1993) On connections between information systems, rough sets, and algebraic logic. In: Rauszer C (ed), Algebraic methods in logic and computer science, vol 28. Banach Center Publisher, Polish Academy of Sciences, pp 117–127
Cornelis C, Cock MD, Kerre EE (2003) Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge. Expert Syst Appl 20:260–270
Cornelis C, Deschrijver G, Kerre EE (2004) Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: constructive, classification, application. Int J Approx Reason 35:55–95
Dubois D, Prade H (1990) Rough fuzzy sets and fuzzy rough sets. Int J Gen Syst 17:191–209
Farhadinia B (2013) Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Inf Sci 240:129–144
Jena SP, Ghosh SK (2002) Intuitionistic fuzzy rough sets. Notes Intuit Fuzzy Sets 8:1–18
Kortelainen J (1994) On relationship between modified sets, topological space and rough sets. Fuzzy Sets Syst 61:91–95
Li TJ, Zhang WX (2008) Rough fuzzy approximations on two universes of discourse. Inf Sci 178:892–906
Lin TY (1996) A rough logic formalism for fuzzy controllers: a hard and soft computing view. Int J Approx Reason 15:359–414
Liu GL (2013) Using one axiom to characterize rough set and fuzzy rough set approximations. Inf Sci 223:285–296
Mi JS, Zhang WX (2004) An axiomatic characterization of a fuzzy generalized of rough sets. Inf Sci 160:235–249
Morsi NN, Yakout MM (1998) Axiomatics for fuzzy rough sets. Fuzzy Sets Syst 100:327–342
Nanda S, Majumda S (1992) Fuzzy rough sets. Fuzzy Sets Syst 45:157–160
Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11:145–172
Pawlak Z (1991) Rough sets—theoretical aspects to reasoning about data. Kluwer, Boston
Radzikowska AM, Kerre EE (2002) A comparative study of fuzzy rough sets. Fuzzy Sets Syst 126:137–155
Rizvi S, Naqvi HJ, Nadeem D (2002) Rough intuitionistic fuzzy set. In: Proceedings of the sixth joint conference on information sciences (JCIS), Durham, NC, pp 101–104
Rodrguez RM, Martnez L, Herrera F (2012) Hesitant fuzzy linguistic term sets for decision making. IEEE Trans Fuzzy Syst 1:109–119

Samanta SK, Mondal TK (2001) Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets. J Fuzzy Math 9:561–582
Skardowska UW (1989) On a generalization of approximation space. Bull Pol Acad Sci Math 37:51–61
Sun BZ, Gong ZT, Chen DG (2008) Fuzzy rough set theory for the interval-valued fuzzy information systems. Inf Sci 178:2794–2815
Thiele H (2000) On axiomatic characterisation of crisp approximation operators. Inf Sci 129:221–226
Thiele H (2001a) On axiomatic characterisation of fuzzy approximation operators: I. The fuzzy rough set based case. In: RSCTC 2000, lecture notes in computer science, vol 205, Springer, Berlin, pp 239–247
Thiele H (2001b) On axiomatic characterisation of fuzzy approximation operators: II. The rough fuzzy set based case. In: Proceedings of the 31st IEEE international symposium on multiple-valued logic, pp 330–335
Thiele H (2001c) On axiomatic characterisation of fuzzy approximation operators: III. The fuzzy diamond and fuzzy box case. In: Proceedings of the 10th IEEE international conference on fuzzy systems, vol 2, pp 1148–1151
Tiwari SP, Srivastava AK (2013) Fuzzy rough sets, fuzzy preorders and fuzzy topologies. Fuzzy Sets Syst 210:63–68
Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25:529–539
Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: The 18th IEEE international conference on fuzzy systems, Jeju Island, Korea, pp 1378–1382
Vakarelov D (1991) A modal logic for similarity relations in Pawlak knowledge representation systems. Fundam Inform 15:61C79.
Wei GW, Zhao XF, Lin R (2013) Some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making. Knowl-Based Syst 46:43–53
Wu WZ, Zhang WX (2002) Neighborhood operator systems and approximations. Inf Sci 144:201–217
Wu WZ, Zhang WX (2004) Constructive and axiomatic approaches of fuzzy approximation operators. Inf Sci 159:233–254
Wu WZ, Leung Y, Zhang WX (2002) Connections between rough-set theory and Dempster–Shafer theory of evidence. Int J Gen Syst 31:405–430
Wu WZ, Mi JS, Zhang WX (2003) Generalized fuzzy rough sets. Inf Sci 151:263–282
Wu WZ, Leung Y, Mi JS (2005) On characterizations of $(\mathcal{I}, \mathcal{T})$-fuzzy rough approximation operators. Fuzzy Sets Syst 154:76–102
Wu WZ, Leung Y, Zhang WX (2006) On generalized rough fuzzy approximation operators. Transactions on Rough sets V, Lecture notes in computer science, vol 4100, pp 263–284
Xia MM, Xu ZS (2011) Hesitant fuzzy information aggregation in decision making. Int J Approx Reason 52:395–407
Xu ZS, Da QL (2002) The uncertain OWA operator. Int J Intell Syst 17:569–575
Xu ZS, Xia MM (2011) Distance and similarity measures for hesitant fuzzy sets. Inf Sci 181:2128–2138
Yang XP (2007) Minimization of axiom sets on fuzzy approximation operators. Inf Sci 177:3840–3854
Yang XP, Li TJ (2006) The minimization of axiom sets characterizing generalized approximation operators. Inf Sci 176:887–899
Yang XB, Song XN, Qi YS, Yang JY (2014) Constructive and axiomatic approaches to hesitant fuzzy rough set. Soft Comput 18:1067–1077
Yao YY (1996) Two views of the theory of rough sets on finite universes. Int J Approx Reason 15:291–317
Yao YY (1998a) Constructive and algebraic methods of the theory of rough sets. Inf Sci 109:21–47
Yao YY (1998b) Relational interpretations of neighborhood operators and rough set approximation operators. Inf Sci 111:239–259
Yao YY (1998c) Generalized rough set model. In: Polkowski L, Skowron A (eds) Rough sets in knowledge discovery 1. Methodology and applications. Physica-Verlag, Heidelberg, pp 286–318
Yao YY, Lin TY (1996) Generalization of rough sets using modal logic. Intell Autom Soft Comput Int J 2:103–120
Yeung DS, Chen DG, Tsang ECC, Lee JWT, Wang XZ (2005) On the generalization of fuzzy rough sets. IEEE Trans Fuzzy Syst 13:343–361
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
Zhang HY, Zhang WX, Wu WZ (2009) On characterization of generalized interval-valued fuzzy rough sets on two universes of discourse. Int J Approx Reason 51:56–70
Zhang XH, Zhou B, Li P (2012) A general frame for intuitionistic fuzzy rough sets. Inf Sci 216:34–49
Zhang ZM (2012a) Generalized intuitionistic fuzzy rough sets based on intuitionistic fuzzy coverings. Inf Sci 198:186–206
Zhang ZM (2012b) On interval type-2 rough fuzzy sets. Knowl-Based Syst 35:1–13
Zhang ZM (2013) On characterization of generalized interval type-2 fuzzy rough sets. Inf Sci 219:124–150
Zhou L, Wu WZ (2008) On generalized intuitionistic fuzzy approximation operators. Inf Sci 178:2448–2465
Zhou L, Wu WZ (2009) On characterization of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators. Inf Sci 179:883–898