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METHODOLOGIES AND APPLICATION

Solving nonlinear fuzzy differential equations by using fuzzy variational iteration method

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Abstract In this paper, the fuzzy variational iteration method is proposed to solve the nonlinear fuzzy differential equation (*NFDE*). The convergence and the maximum absolute truncation error of the proposed method are proved in details. Some examples are investigated to verify convergence results and to illustrate the efficiently of the method.

Keywords Fuzzy differential equations · Fuzzy number · Fuzzy-valued function · h-difference · gh-difference · Generalized differentiability · Fuzzy variational iteration method (FVIM)

1 Introduction

As we know the fuzzy differential equations are one of the important part of the fuzzy analysis theory that play major role in numerical analysis. For example, population models (Guo et al. 2003), the golden mean (Datta 2003), quantum optics and gravity (El Naschie 2005), control chaotic systems (Feng and Chen 2005; Jiang et al. 2005), medicine (Abbod et al. 2001; Barro and Marin 2002). Recently, some mathematicians have studied FDE (Abbasbandy and Allahviranloo 2002; Abbasbandy et al. 2004, 2005; Allahviranloo et al. 2007; Bede 2008; Bede and Gal 2005; Bede et al. 2007; Buckley and Feuring 2000; Buckley et al. 2002; Buckley and Jowers 2006; Chalco-Cano and Romn-Flores 2006; Chalco-Cano et al. 2007; Chen and Ho 1999; Cho and Lan 2007; Congxin and Shiji 1993; Diamond 1999, 2002; Ding et al. 1997; Dubois and Prade 1982; Fard et al. 2009, 2010; Fard 2009a, b; Fard and Kamyad 2011; Fei 2007; Jang et al. 2000; Jowers et al. 2007; Kaleva 1987, 1990, 2006; Lopez 2008; Ma et al. 1999; Mizukoshi et al. 2007; Oberguggenberger and Pittschmann 1999; Papaschinopoulos et al. 2007; Puri and Ralescu 1983; Seikkala 1987; Song et al. 2000; Solaymani Fard and Ghal-Eh 2011). In this work, we present the fuzzy variational iteration method to solve the *NFDE* as follows:

$$L\widetilde{u}(t) + N\widetilde{u}(t) \ominus^{g} \widetilde{g}(t) = \widetilde{0}, \quad t > 0, \tag{1}$$

where the linear operator L is defined as $L = \frac{d^m}{dt^m}$, N is a nonlinear operator and $\widetilde{g}(t)$ is a known fuzzy function.

With fuzzy initial condition:

$$u^{(k)}(0) = \widetilde{c}_k, \quad k = 0, 1, \ldots, m-1, \tag{2}$$

where $\widetilde{c}_k$ are fuzzy constant values.

The structure of this paper is organized as follows: In Sect. 2, some basic notations and definitions in fuzzy calculus are brought. In Sect. 3, are solved Eqs. (1, 2) with FVIM. The existence and uniqueness of the solution and convergence of the proposed method are proved in Sect. 4 respectively. Finally, in Sect. 5, are illustrated the accuracy of method by solving some numerical examples, and a brief conclusion is given in Sect. 6.

2 Basic concepts

Here basic definitions of a fuzzy number are given as follows, Kauffman and Gupta (1991), Zadeh (1965), Zimmermann

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(1991), Dubois and Prade (1980), Allahviramloo (2005), Nguyen (1978).

Definition 2.1 An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:

(i) $\overline{u} : r \to u_r^- \in R$ is a bounded left-continuous non-decreasing function over [0, 1],
(ii) $\underline{u} : r \to u_r^+ \in R$ is a bounded left-continuous non-increasing function over [0, 1],
(iii) $\underline{u} \le \overline{u}, \quad 0 \le r \le 1$.

Definition 2.2 For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$, we use the distance (Hausdorff metric) (Goetschel 1986)

$$D(u(r), v(r)) = \max\left\{ \sup_{r\in[0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup |\overline{u}(r) - \overline{v}(r)| \right\},$$

and it is shown (Puri and Ralescu 1986) that (E^1, D) is a complete metric space and the following properties are well known:

$$D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\, \widetilde{u}, \widetilde{v} \in E^1,$$
$$D(k\widetilde{u}, k\widetilde{v}) = |\, k \,|\, D(\widetilde{u}, \widetilde{v}), \forall\, k \in R, \widetilde{u}, \widetilde{v} \in E^1,$$
$$D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \le D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \forall\, \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E^1.$$

Definition 2.3 A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \ge 0, \beta \ge 0$ with

$$\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \le a \\ R(\frac{x-b}{\beta}) & x \ge a \end{cases} \tag{3}$$

the mean value of $\tilde{A}$, a is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.4 Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,

(1) $\lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$
(2) $-\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$
(3) $\tilde{M} \oplus \tilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$
(4)

$$\tilde{M} \odot \tilde{N} \simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR} & \tilde{M}, \tilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR} & \tilde{M} > 0, \tilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR} & \tilde{M}, \tilde{N} < 0 \end{cases} \tag{4}$$

Definition 2.5 Consider $x, y \in E$. If there exists $z \in E$ such that $x = y + z$ then z is called the H-difference of x and y, and is denoted by $x \ominus y$ (Bede and Gal 2005).

Proposition 1 *If $f : (a, b) \to E$ is a continuous fuzzy-valued function then $g(x) = \int_a^x f(t)\, dt$ is differentiable, with derivative $g'(x) = f(x)$* (Bede and Gal 2005).

Definition 2.6 (see Bede and Gal 2005) Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede–Gal differentiability), if there exists an element $f'(x_0) \in E$, such that:

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or
(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or
(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or
(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

Definition 2.7 Let $f : (a, b) \to E$. We say f is (i)-differentiable on (a, b) if f is differentiable in the sense (i) of Definition (2.7) and similarly for (ii), (iii) and (iv) differentiability.

Definition 2.8 A triangular fuzzy number is defined as a fuzzy set in E^1, that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \le b \le c$ such that $[u]^r = [u_-^r, u_+^r]$ are the endpoints of r-level sets for all $r \in [0, 1]$, where $u_-^r = a + (b - a)r$ and $u_+^r = c - (c - b)r$. Here, $u_-^0 = a, u_+^0 = c, u_-^1 = u_+^1 = b$, which is denoted by u^1. The set of triangular fuzzy numbers will be denoted by E^1.

Definition 2.9 (see Chalco-Cano and Romn-Flores 2006) The mapping $f : T \to E^n$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:

$$[f(t)]^r = [f_-^r(t), f_+^r(t)], \quad t \in T, \ r \in [0,1].$$

Definition 2.10 (see Chalco-Cano and Romn-Flores 2006) Let $f : T \to E^n$ be Hukuhara differentiable and denote $[f(t)]^r = [f_-^r, f_+^r]$. Then, the boundary function f_-^r and f_+^r are differentiable (or Seikkala differentiable) and

$$[f'(t)]^r = [(f_-^r)'(t), (f_+^r)'(t)], \quad t \in T, \ r \in [0,1].$$

Definition 2.11 (see Chalco-Cano et al. 2011) The generalized Hukuhara difference of two intervals, A and B, (gh-difference) is defined as follows

$$A \ominus^g B = C \Leftrightarrow \begin{cases} (a), & A = B + C \\ or\ (b), & B = A + (-1)C. \end{cases}$$

This difference has many interesting new properties, for example $A \ominus^g A = (0)$. Also, the gh-difference of two intervals $A = [a, b]$ and $B = [c, d]$ always exists and it is equal to

$$A \ominus^{gh} B = [\min\{a-c, b-d\}, \max\{a-c, b-d\}].$$

3 Description of the FVIM

We consider the following nonlinear fuzzy differential equation:

$$L\widetilde{u}(t) + N\widetilde{u}(t) \ominus^g \widetilde{g}(t) = \widetilde{0}, \quad t > 0, \tag{5}$$

where the linear operator L is defined as $L = \frac{d^m}{dt^m}$, N is a nonlinear operator, $\widetilde{g}(t)$ is a known fuzzy function and $\widetilde{0}$ is singleton fuzzy zero with membership function as follows:

$$\mu_{\widetilde{0}}(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

With fuzzy initial condition:

$$u^{(k)}(0) = \widetilde{c}_k, \quad k = 0, 1, \ldots, m-1, \tag{6}$$

where $\widetilde{c}_k$ are fuzzy constant values.

In this case, a correction functional can be constructed as follows:

$$\widetilde{u}_{n+1}(t) = \widetilde{u}_n(t) + \int_a^t \lambda(\tau)\{L(\widetilde{u}_n(\tau)) + N(\widetilde{u}_n(\tau)) \ominus^g \widetilde{g}(\tau)\}\, d\tau, \quad n \geq 0, \tag{7}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $\widetilde{u}_n(\tau)$ is a restricted variations which means $\delta\widetilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $\widetilde{u}_n(t)$, $n \geq 0$ of the solution $\widetilde{u}(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $\widetilde{u}_0$. The zeroth approximations $\widetilde{u}_0$ may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $\widetilde{u}_n(t)$, $n \geq 1$ follow immediately. Consequently, the exact solution may be obtained by

$$\widetilde{u}(t) = \lim_{n\to\infty} \widetilde{u}_n(t). \tag{8}$$

Case (1): $\widetilde{u}^{(i)}(t)$ is (1)-differentiable for any i ($1 < i \leq m$), in this case we have,

$$\widetilde{u}_{k+1}(t) = \widetilde{u}_k(t) + \int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) + N\widetilde{u}_k(\tau) \ominus^g \widetilde{g}(\tau)]\, d\tau. \tag{9}$$

$$\delta\widetilde{u}_{k+1}(t) = \delta\widetilde{u}_k(t) + \delta\int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) + N\widetilde{u}_k(\tau) \ominus^g \widetilde{g}(\tau)]\, d\tau. \tag{10}$$

We apply restricted variations to nonlinear term $N\widetilde{u}$ ($\delta N\widetilde{u} = \widetilde{0}$, so, we can write Eq. (10) as follows:

$$\delta\widetilde{u}_{k+1}(t) = \delta\widetilde{u}_k(t) + \delta\int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) \ominus^g \widetilde{g}(\tau)]\, d\tau. \tag{11}$$

We can write,

$$\delta\underline{u}_{k+1} = \delta\underline{u}_k + \int_0^t \lambda\underline{u}_k^{(m)}(\tau)\, d\tau,$$

$$\delta\overline{u}_{k+1} = \delta\overline{u}_k + \int_0^t \lambda\overline{u}_k^{(m)}(\tau)\, d\tau.$$

$$\delta\underline{u}_{k+1} = \delta\overline{u}_{k+1} = 0.$$

$$\int_0^t \lambda\underline{u}_k^{(m)}(\tau)\, d\tau = \lambda\underline{u}_k^{(m-1)} - \int_0^t \lambda'\underline{u}_k^{(m-1)}(\tau)\, d\tau$$
$$= \lambda\underline{u}_k^{(m-1)} - \left(\lambda'\underline{u}_k^{(m-2)} - \int_0^t \lambda''\,\underline{u}_k^{(m-2)}(\tau)\, d\tau\right)$$
$$= \lambda\underline{u}_k^{(m-1)} - \lambda'\underline{u}_k^{(m-2)} + \int_0^t \lambda''\underline{u}_k^{(m-2)}(\tau)\, d\tau.$$

$$\int_0^t \lambda\overline{u}_k^{(m)}(\tau)\, d\tau = \lambda\overline{u}_k^{(m-1)} - \int_0^t \lambda'\overline{u}_k^{(m-1)}(\tau)\, d\tau$$
$$= \lambda\overline{u}_k^{(m-1)} - \left(\lambda'\overline{u}_k^{(m-2)} - \int_0^t \lambda''\overline{u}_k^{(m-2)}(\tau)\, d\tau\right)$$
$$= \lambda\overline{u}_k^{(m-1)} - \lambda'\overline{u}_k^{(m-2)} + \int_0^t \lambda''\overline{u}_k^{(m-2)}(\tau)\, d\tau.$$

Finally, we can write

$$0=\delta\underline{u}_{k+1}=\delta\underline{u}_k+\Big(\lambda\delta\underline{u}_k^{(m-1)}+\lambda'\delta\underline{u}_k^{(m-2)}+\cdots+\lambda^{(m-1)}\delta\underline{u}_k\Big)+\int_0^t\lambda^{(m)}\delta\underline{u}_k(\tau)\,d\tau.$$

$$0=\delta\overline{u}_{k+1}=\delta\overline{u}_k+\Big(\lambda\delta\overline{u}_k^{(m-1)}+\lambda'\delta\overline{u}_k^{(m-2)}+\cdots+\lambda^{(m-1)}\delta\overline{u}_k\Big)+\int_0^t\lambda^{(m)}\delta\overline{u}_k(\tau)\,d\tau.$$

So, we have

$$\begin{cases}1+\lambda^{(m-1)}=0,\\ \lambda^{(m)}=0,\\ \lambda=\lambda'=\cdots=\lambda^{(m-1)}=0.\end{cases}$$

Finally, we obtain λ as follows

$$\lambda=\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1},\quad 0<t<\tau<T. \tag{12}$$

For example if $m=1$ then $\lambda=-1$ and if $m=2$ then $\lambda=\tau-t$.

Therefore, substituting (12) into functional (9), we obtain the following iteration formula,

$$\widetilde{u}_{k+1}(t)=\widetilde{u}_k(t)+\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}(L\widetilde{u}_k(\tau)+N\widetilde{u}_k(\tau)\ominus^g\widetilde{g}(\tau)\right]d\tau. \tag{13}$$

Now, define the operator $A[\widetilde{u}]$ as,

$$A[\widetilde{u}]=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}(L\widetilde{u}_k(\tau)+N\widetilde{u}_k(\tau)\ominus^g\widetilde{g}(\tau)\right]d\tau,$$

and define the components $\widetilde{v}_k$, $k=0,1,2,\ldots$ as,

$$\begin{aligned}&\widetilde{v}_0=\widetilde{u}_o,\\ &\widetilde{v}_1=A[\widetilde{v}_0],\\ &\vdots\\ &\widetilde{v}_{k+1}=A[\widetilde{v}_0+\widetilde{v}_1+\cdots+\widetilde{v}_k].\end{aligned}$$

We have $\widetilde{u}(t)=\lim_{k\to\infty}\widetilde{u}_k(t)=\sum_{k=0}^{\infty}\widetilde{v}_k(t)$, therefore, we can write recursive relations as follows:

$$\widetilde{v}_0(t)=\widetilde{c}_0+\sum_{k=1}^{m}\frac{\widetilde{c}_k t^k}{k!},$$

$$\widetilde{v}_{k+1}(t)=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)+N[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)\ominus^g\widetilde{g}(\tau)\right)\right]d\tau. \tag{14}$$

Case (2): $\widetilde{u}^{(i)}(t)$ is (2)-differentiable for any i $(1<i\le m)$, in this case we have,

$$\delta\underline{u}_{k+1}=\delta\underline{u}_k+\int_0^t\lambda\overline{u}_k^{(m)}(\tau)\,d\tau,$$

$$\delta\overline{u}_{k+1}=\delta\overline{u}_k+\int_0^t\lambda\underline{u}_k^{(m)}(\tau)\,d\tau.$$

$$\delta\underline{u}_{k+1}=\delta\overline{u}_{k+1}=0.$$

$$\begin{aligned}\int_0^t\lambda\underline{u}_k^{(m)}(\tau)\,d\tau&=\lambda\underline{u}_k^{(m-1)}-\int_0^t\lambda'\,\underline{u}_k^{(m-1)}(\tau)\,d\tau\\ &=\lambda\underline{u}_k^{(m-1)}-\left(\lambda'\underline{u}_k^{(m-2)}-\int_0^t\lambda''\,\underline{u}_k^{(m-2)}(\tau)\,d\tau\right)\\ &=\lambda\underline{u}_k^{(m-1)}-\lambda'\underline{u}_k^{(m-2)}+\int_0^t\lambda''\underline{u}_k^{(m-2)}(\tau)\,d\tau.\end{aligned}$$

$$\begin{aligned}\int_0^t\lambda\overline{u}_k^{(m)}(\tau)\,d\tau&=\lambda\overline{u}_k^{(m-1)}-\int_0^t\lambda'\,\overline{u}_k^{(m-1)}(\tau)\,d\tau\\ &=\lambda\overline{u}_k^{(m-1)}-\left(\lambda'\overline{u}_k^{(m-2)}-\int_0^t\lambda''\,\overline{u}_k^{(m-2)}(\tau)\,d\tau\right)\\ &=\lambda\overline{u}_k^{(m-1)}-\lambda'\overline{u}_k^{(m-2)}+\int_0^t\lambda''\overline{u}_k^{(m-2)}(\tau)\,d\tau.\end{aligned}$$

Finally, we can write

$$0=\delta\underline{u}_{k+1}=\delta\underline{u}_k+(\lambda\delta\overline{u}_k^{(m-1)}+\lambda'\delta\overline{u}_k^{(m-2)}+\cdots+\lambda^{(m-1)}\delta\overline{u}_k)+\int_0^t\lambda^{(m)}\delta\overline{u}_k(\tau)\,d\tau.$$

$$0=\delta\overline{u}_{k+1}=\delta\overline{u}_k+(\lambda\delta\underline{u}_k^{(m-1)}+\lambda'\delta\underline{u}_k^{(m-2)}+\cdots+\lambda^{(m-1)}\delta\underline{u}_k)+\int_0^t\lambda^{(m)}\delta\underline{u}_k(\tau)\,d\tau.$$

So, we have

$$\begin{cases} 1+\lambda^{(m-1)}=0, \\ \lambda^{(m)}=0, \\ \lambda=\lambda'=\cdots=\lambda^{(m-1)}=0. \end{cases}$$

Finally, we obtain λ as follows

$$\lambda=\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}, \quad 0<t<\tau<T. \tag{15}$$

Therefore, we can write recursive relations as follows:

$$\widetilde{v}_0(t)=\widetilde{c}_0\ominus(-1)\sum_{k=1}^{m}\frac{\widetilde{c}_k t^k}{k!},$$

$$\widetilde{v}_{k+1}(t)=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)\right.\right.$$
$$\left.\left.+N[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)\ominus^g\widetilde{g}(\tau)\right)\right]d\tau. \tag{16}$$

Case (3): $\widetilde{u}^{(i)}(t)$ is (1)-differentiable for some i, $(1\le i\le m)$ and for another is (2)-differentiable. In this case let:

$$P=\{1\le i\le m \mid \widetilde{u}^{(i)}(t)\ is\ (1)\text{-}differentiable\},$$
$$P'=\{1\le i\le m \mid \widetilde{u}^{(i)}(t)\ is\ (2)\text{-}differentiable\}.$$

λ in this case is similar to the previous cases.

$$\lambda=\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}, \quad 0<t<\tau<T. \tag{17}$$

If $u^{(i)}(t)\in P$ then $\widetilde{s}_i=\frac{\widetilde{c}_i t^i}{i!}$ and if $u^{(i)}(t)\in P'$ then $\widetilde{s}_i=\ominus(-1)\frac{\widetilde{c}_i t^i}{i!}$.

Therefore, we can write recursive relations as follows:

$$\widetilde{v}_0(t)=\widetilde{c}_0+\widetilde{s}_1+\widetilde{s}_2+\cdots+\widetilde{s}_{m-1},$$

$$\widetilde{v}_{k+1}(t)=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)\right.\right.$$
$$\left.\left.+N[\widetilde{v}_0+\cdots+\widetilde{v}_k](\tau)\ominus^g\widetilde{g}(\tau)\right)\right]d\tau. \tag{18}$$

Remark 1 Consider the following system of nonlinear fuzzy differential equations,

$$\begin{cases} \frac{d\widetilde{u}_1^m}{dt^m}+N_1(\widetilde{u}_1,\widetilde{u}_2,\ldots,\widetilde{u}_n)\ominus^g\widetilde{g}_1(t)=\widetilde{0}, \\ \frac{d\widetilde{u}_2^m}{dt^m}+N_2(\widetilde{u}_1,\widetilde{u}_2,\ldots,\widetilde{u}_n)\ominus^g\widetilde{g}_2(t)=\widetilde{0} \\ \vdots \\ \frac{d\widetilde{u}_n^m}{dt^m}+N_n(\widetilde{u}_1,\widetilde{u}_2,\ldots,\widetilde{u}_n)\ominus^g\widetilde{g}_n(t)=\widetilde{0}. \end{cases} \tag{19}$$

where $n,m\in N$, $N_1,N_2,\ldots,N_n$ are nonlinear operators and $\widetilde{g}_1(t),\widetilde{g}_2(t),\ldots,\widetilde{g}_n(t)$ are known fuzzy functions, subject to the initial conditions

$$\begin{cases} \widetilde{u}_1^{(k)}(0)=\widetilde{c}_{1,k}, \\ \widetilde{u}_2^{(k)}(0)=\widetilde{c}_{2,k}, \quad k=0,1,\ldots,m-1, \\ \vdots \\ \widetilde{u}_n^{(k)}(0)=\widetilde{c}_{n,k}. \end{cases} \tag{20}$$

We can write recursive relations as follows:

Case (1):

$$\widetilde{v}_{i,0}=\widetilde{c}_{i,0}+\sum_{k=1}^{m-1}\frac{\widetilde{c}_{i,k}}{k!}t^k,$$

$$\widetilde{v}_{i,k+1}=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_{i,0}+\cdots+\widetilde{v}_{i,k}](\tau)\right.\right.$$
$$+N_i[(\widetilde{v}_{1,0}+\cdots+\widetilde{v}_{1,k}),(\widetilde{v}_{2,0}+\cdots+\widetilde{v}_{2,k}),\ldots,$$
$$\left.\left.(\widetilde{v}_{n,0}+\cdots+\widetilde{v}_{n,k})](\tau)\ominus^g\widetilde{g}_i(\tau)\right)\right]d\tau. \tag{21}$$

Case (2):

$$\widetilde{v}_{i,0}=\widetilde{c}_{i,0}\ominus(-1)\sum_{k=1}^{m-1}\frac{\widetilde{c}_{i,k}}{k!}t^k,$$

$$\widetilde{v}_{i,k+1}=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_{i,0}+\cdots+\widetilde{v}_{i,k}](\tau)\right.\right.$$
$$+N_i[(\widetilde{v}_{1,0}+\cdots+\widetilde{v}_{1,k}),(\widetilde{v}_{2,0}+\cdots+\widetilde{v}_{2,k}),\ldots,$$
$$\left.\left.(\widetilde{v}_{n,0}+\cdots+\widetilde{v}_{n,k})](\tau)\ominus^g\widetilde{g}_i(\tau)\right)\right]d\tau. \tag{22}$$

Case (3):

If $u_k^{(i)}(t)\in P$ then $\widetilde{s}_{i,k}=\frac{\widetilde{c}_{i,k}t^i}{i!}$ and if $u_k^{(i)}(t)\in P'$ then $\widetilde{s}_{i,k}=\ominus(-1)\frac{\widetilde{c}_{i,k}t^i}{i!}$.

$$\widetilde{v}_{i,0}=\widetilde{c}_{i,0}+s_{1,0}+\cdots+s_{m-1,0},$$

$$\widetilde{v}_{i,k+1}=\int_0^t\left[\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}\left(\frac{d^m}{d\tau^m}[\widetilde{v}_{i,0}+\cdots+\widetilde{v}_{i,k}](\tau)\right.\right.$$
$$+N_i[(\widetilde{v}_{1,0}+\cdots+\widetilde{v}_{1,k}),(\widetilde{v}_{2,0}+\cdots+\widetilde{v}_{2,k}),\ldots,$$
$$\left.\left.(\widetilde{v}_{n,0}+\cdots+\widetilde{v}_{n,k})](\tau)\ominus^g\widetilde{g}_i(\tau)\right)\right]d\tau. \tag{23}$$

4 Existence and convergence analysis

In this section we are going to prove the convergence and the maximum absolute truncation error of the proposed method.

Theorem 4.1 *The series solution $\widetilde{u}(t)=\sum_{k=0}^{\infty}\widetilde{v}_k(t)$ obtained from the relation (14) using FVIM converges to the exact solution of the problems (1, 2) if $\exists 0<\gamma<1$ such that $D(\widetilde{v}_{k+1},\widetilde{0})\le\gamma D(\widetilde{v}_k,\widetilde{0})$.*

Proof Define the sequence $\{\widetilde{s}_n\}_{n=0}^{\infty}$ as,

$$\begin{aligned}
&\widetilde{s}_0 = \widetilde{v}_0,\\
&\widetilde{s}_1 = \widetilde{v}_0 + \widetilde{v}_1,\\
&\vdots\\
&\widetilde{s}_n = \widetilde{v}_0 + \widetilde{v}_1 + \cdots + \widetilde{v}_n,
\end{aligned}$$

and we show that $\{\widetilde{s}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space. According to the property (1) from Hausdorff metric we can write,

$$\begin{aligned}
D(\widetilde{s}_{n+1}, \widetilde{s}_n) = D(\widetilde{v}_{n+1}, \widetilde{0}) \le \gamma D(\widetilde{v}_n, \widetilde{0}) \le \gamma^2 D(\widetilde{v}_{n-1}, \widetilde{0})\\
\le \cdots \le \gamma^{n+1} D(\widetilde{v}_0, \widetilde{0}).
\end{aligned}$$

For every $n, J \in N, n \ge j$, we have,

$$\begin{aligned}
D(\widetilde{s}_n, \widetilde{s}_j) &\le D(\widetilde{s}_n, \widetilde{s}_{n-1}) + D(\widetilde{s}_{n-1}, \widetilde{s}_{n-2})\\
&\quad + \cdots + D(\widetilde{s}_{j+1}, \widetilde{s}_j)\\
&\le \gamma^n D(\widetilde{v}_0, \widetilde{0}) + \gamma^{n-1} D(\widetilde{v}_0, \widetilde{0})\\
&\quad + \cdots + \gamma^{j+1} D(\widetilde{v}_0, \widetilde{0})\\
&= \frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1} D(\widetilde{v}_0, \widetilde{0}),
\end{aligned}$$

and since $0 < \gamma < 1$, we get,

$$\lim_{n,j\to\infty} D(\widetilde{s}_n, \widetilde{s}_j) = 0.$$

Therefore, $\{\widetilde{s}_n\}_{n=0}^{\infty}$ is a Cachy sequence in the Banach space. □

Theorem 4.2 *The maximum absolute truncation error of the series solution $\widetilde{u}(t) = \sum_{k=0}^{\infty} \widetilde{v}_k(t)$ to problems (1, 2) by using FVIM is estimated to be*

$$E_j(t) = D(\widetilde{u}(t), \widetilde{u}_j(t)) \le \frac{1}{1-\gamma}\gamma^{j+1} D(\widetilde{v}_0, \widetilde{0}).$$

Proof We have,

$$\begin{aligned}
D(\widetilde{s}_n, \widetilde{s}_j) &\le D(\widetilde{s}_n, \widetilde{s}_{n-1}) + D(\widetilde{s}_{n-1}, \widetilde{s}_{n-2})\\
&\quad + \cdots + D(\widetilde{s}_{j+1}, \widetilde{s}_j)\\
&\le \gamma^n D(\widetilde{v}_0, \widetilde{0}) + \gamma^{n-1} D(\widetilde{v}_0, \widetilde{0})\\
&\quad + \cdots + \gamma^{j+1} D(\widetilde{v}_0, \widetilde{0})\\
&= \frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1} D(\widetilde{v}_0, \widetilde{0}),
\end{aligned}$$

for $n \ge j$, then $\lim_{n\to infty} \widetilde{s}_n = \widetilde{u}(t)$. So,

$$D\left(\widetilde{u}(t), \sum_{k=0}^{j} \widetilde{v}_k\right) \le \frac{1-\gamma^{n-1}}{1-\gamma}\gamma^{j+1} D(\widetilde{v}_0, \widetilde{0}).$$

Also, since $0 < \gamma < 1$ we have $(1-\gamma^{n-j}) < 1$. Therefore the above inequality becomes,

$$D\left(\widetilde{u}(t), \sum_{k=0}^{j} \widetilde{v}_k\right) \le \frac{1}{1-\gamma}\gamma^{j+1} D(\widetilde{v}_0, \widetilde{0}).$$

□

Table 1 Numerical results for Example 5.1

				$\underline{v}$	$n = 17$						
t	$r = 0$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$	$r = 0.9$	$r = 1$
0.1	0.2234561	0.2478652	0.2788435	0.2908766	0.32472841	0.34012978	0.3701324	0.3907581	0.4078695	0.4276405	0.4401785
0.2	0.2644012	0.2831405	0.3098662	0.3245372	0.3506574	0.3714885	0.3988621	0.4278453	0.4456621	0.4622614	0.4768903
0.3	0.3267895	0.3577893	0.3714485	0.3965231	0.4258109	0.4507024	0.4802788	0.5066983	0.5317995	0.5478907	0.5612685
0.4	0.3855608	0.4013674	0.4317439	0.4601235	0.4988026	0.5123558	0.5463189	0.5647256	0.5780956	0.5867548	0.5968496
0.5	0.4233897	0.4489975	0.4603174	0.4812765	0.5194122	0.5378436	0.5522465	0.5850871	0.6033762	0.6240423	0.6503073
0.6	0.4654138	0.4823178	0.5043655	0.5234614	0.5518709	0.5766204	0.6064382	0.6189728	0.6257848	0.6465792	0.6578698

Table 2 Numerical results for Example 5.1

				$\overline{v}$	$n = 17$						
t	$r = 1$	$r = 0.9$	$r = 0.8$	$r = 0.7$	$r = 0.6$	$r = 0.5$	$r = 0.4$	$r = 0.3$	$r = 0.2$	$r = 0.1$	$r = 0$
0.1	0.2537862	0.2744567	0.3012783	0.32653784	0.3589705	0.3743105	0.4065482	0.41995715	0.4297863	0.4413554	0.4568249
0.2	0.3077845	0.3246347	0.3588703	0.3703512	0.3944358	0.4230412	0.4456285	0.4703058	0.4870423	0.4968359	0.5156483
0.3	0.3566259	0.3702483	0.3933185	0.4277358	0.4446204	0.4689021	0.4945289	0.5216503	0.5419703	0.5601326	0.5852781
0.4	0.4105523	0.4416578	0.4765427	0.4955831	0.5120753	0.5322089	0.5567842	0.5718057	0.5867504	0.5987616	0.6134679
0.5	0.4689571	0.4877642	0.5167792	0.5478905	0.5622451	0.5942759	0.6134265	0.6289607	0.6407164	0.6589652	0.6689652
0.6	0.4954126	0.5133528	0.5346704	0.5505179	0.5766809	0.6014057	0.6217685	0.6345732	0.6503415	0.6704285	0.6877493

Table 3 Numerical results for Example 5.1

				$\underline{v}$	$n=22$						
t	$r=0$	$r=0.1$	$r=0.2$	$r=0.3$	$r=0.4$	$r=0.5$	$r=0.6$	$r=0.7$	$r=0.8$	$r=0.9$	$r=1$
0.1	0.2576802	0.2630675	0.2801534	0.2968415	0.3065441	0.3166302	0.3256742	0.3356478	0.3488609	0.3566208	0.3627803
0.2	0.2744236	0.2977564	0.3056803	0.3234821	0.3376589	0.3456407	0.3519607	0.3698574	0.3748305	0.3814707	0.3976845
0.3	0.3011547	0.3188907	0.3299755	0.3356702	0.3512436	0.3616945	0.3865307	0.4066739	0.4156743	0.4335786	0.4568321
0.4	0.3288605	0.3426739	0.3603272	0.379184	0.3925476	0.4128557	0.4374528	0.4528652	0.4748372	0.4918635	0.5138469
0.5	0.3508726	0.3668952	0.3804435	0.4175209	0.4355739	0.4539781	0.4719505	0.5039883	0.5235886	0.5419728	0.5638795
0.6	0.3755608	0.3914537	0.4211317	0.4435589	0.4616542	0.4817325	0.5223467	0.5417602	0.5624839	0.5866359	0.6044789

Table 4 Numerical results for Example 5.1

				$\overline{v}$	$n=22$						
t	$r=1$	$r=0.9$	$r=0.8$	$r=0.7$	$r=0.6$	$r=0.5$	$r=0.4$	$r=0.3$	$r=0.2$	$r=0.1$	$r=0$
0.1	0.2688704	0.2744826	0.2936057	0.3026854	0.3144968	0.3240879	0.3384259	0.3426897	0.3588609	0.3638975	0.3755309
0.2	0.2876403	0.3067501	0.3258307	0.3345816	0.3465793	0.3546308	0.3688904	0.3720415	0.3856312	0.3968503	0.4011746
0.3	0.3102385	0.3354203	0.3439858	0.3519406	0.3688406	0.3765402	0.3961428	0.4183264	0.4363552	0.4428619	0.4529873
0.4	0.3389692	0.3566804	0.3710503	0.3955196	0.4119815	0.4329808	0.4512773	0.4759825	0.4988126	0.5127805	0.5350612
0.5	0.3650514	0.3730972	0.4067421	0.4368794	0.4537902	0.4765431	0.4955406	0.5212758	0.5468952	0.5665724	0.5877309
0.6	0.3903215	0.4229805	0.4317655	0.4612408	0.4911605	0.5230768	0.553081	0.5766308	0.5945503	0.6133258	0.6343258

Table 5 Numerical results for Example 5.1

r	$(\underline{v}, n=20, t=0.6)$	$(\overline{v}, n=20, t=0.6)$
0.0	0.3725667	0.6823409
0.1	0.3855794	0.6944202
0.2	0.3933572	0.6870843
0.3	0.4119982	0.6637559
0.4	0.4380251	0.6523187
0.5	0.4498431	0.6373561
0.6	0.4668436	0.6123817
0.7	0.4766382	0.6088303
0.8	0.5060981	0.5817429
0.9	0.5222504	0.568605
1.0	0.5388901	0.547409

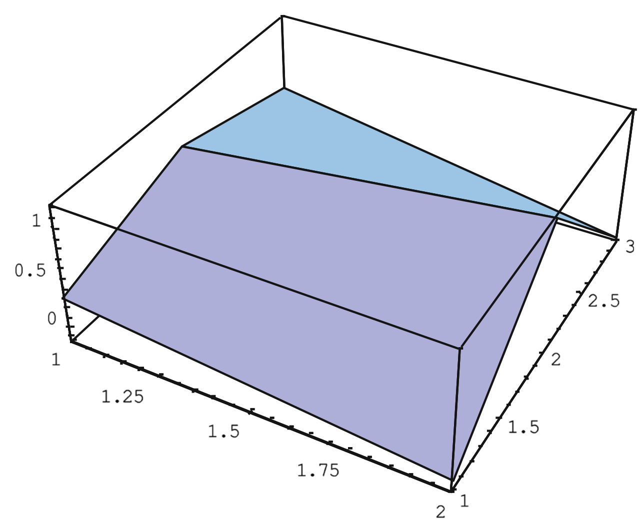

Fig. 1 The results of Example 5.1 (Case (3)) for $(\underline{v}(0.1, r), \overline{v}(0.1, r))$

5 Numerical examples

In this section, we solve NFDE by using the FVIM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm:

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (14) or (16) or (18).

Step 3. If $D(\widetilde{v}_{n+1}, \widetilde{v}_n) < \varepsilon$ then go to step 4, else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $\sum_{i=0}^{\infty} \widetilde{v}_i(t)$ as the approximate of the exact solution.

Example 5.1 Consider the FDE as follows:

$$\widetilde{u}''(t) + \widetilde{u}(t) = \widetilde{0}, \quad 0 < t \leq 0.6. \tag{24}$$

With initial conditions:

$$\widetilde{u}(0) = (0, 0, 0),$$
$$\widetilde{u}'(0) = (0.02, 0.03, 0.04). \tag{25}$$
$$\epsilon = 10^{-4}.$$

Case (1): Tables 1 and 2 show that, the approximation solution of the FDE is convergent with 17 iterations by using the FVIM when $\widetilde{u}''$ and $\widetilde{u}'$ are (1)-differentiable.

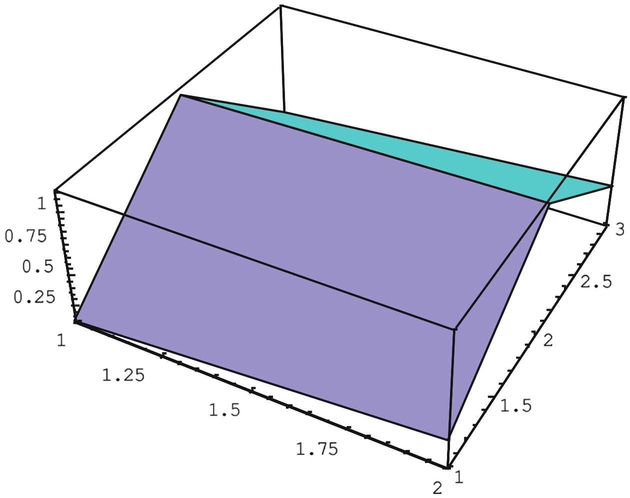

Fig. 2 The results of Example 5.2 (Case (3)) for $(\underline{v}(0.1, r), \overline{v}(0.1, r))$

Table 6 Numerical results for Example 5.2

r	$(\underline{v}, n = 29, t = 0.6)$	$(\overline{v}, n = 29, t = 0.6)$
0.0	0.2056403	0.5425742
0.1	0.2128721	0.5255016
0.2	0.2245319	0.5172813
0.3	0.2433864	0.4907523
0.4	0.2566112	0.4857613
0.5	0.2739551	0.4673809
0.6	0.2822814	0.4581443
0.7	0.3037861	0.4355014
0.8	0.3167574	0.4244813
0.9	0.3308615	0.4077926
1.0	0.3468768	0.3867426

Case (2): Tables 3 and 4 show that, the approximation solution of the FDE is convergent with 22 iterations by using the FVIM when $\widetilde{u}''$ and $\widetilde{u}'$ are (2)-differentiable.

Case (3): Table 5 shows that, the approximation solution of the FDE is convergent with 20 iterations by using the FVIM when $\widetilde{u}'$ is (1)-differentiable and $\widetilde{u}''$ is (2)-differentiable (Figs. 1, 2).

Example 5.2 Consider the NFDE as follows:

$$\widetilde{u}''(t) + \widetilde{u}^3(t) \ominus^g \widetilde{g}(t) = \widetilde{0}. \tag{26}$$

where,

$$\widetilde{g}(t) = (t^2, t^2 + 1, t^2 + 2).$$

With initial conditions:

$$\widetilde{u}(0) = (0.01, 0.03, 0.05),$$
$$\widetilde{u}'(0) = (0.02, 0.04, 0.06). \tag{27}$$
$$\epsilon = 10^{-5}.$$

Case (1): Table 6 shows that, the approximation solution of the NFDE is convergent with 29 iterations by using the FVIM when $\widetilde{u}''$ and $\widetilde{u}'$ are (1)-differentiable.

Case (2): Tables 7 and 8 show that, the approximation solution of the NFDE is convergent with 27 iterations by using the FVIM when $\widetilde{u}''$ and $\widetilde{u}'$ are (2)-differentiable.

Case (3): Table 9 shows that, the approximation solution of the NFDE is convergent with 32 iterations by using the FVIM when $\widetilde{u}'$ is (1)-differentiable and $\widetilde{u}''$ is (2)-differentiable.

6 Conclusion

The VIM gives several successive approximations through using the iteration of the correction functional without any transformation and hence the procedure is direct and straightforward. The VIM proved to be easy to use and provides an efficient method for handling nonlinear problems. In this work, we presented the fuzzy variational iteration method. This method has been successfully employed to obtain the approximate solution of the *NFDE* under generalized H-differentiability. We can use this method to solve another nonlinear fuzzy problems, for example fuzzy partial differential equations, fuzzy integral equations and fuzzy integro-differential equations.

Table 7 Numerical results for Example 5.2

				$\underline{v}$	$n = 27$						
t	$r = 0$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$	$r = 0.9$	$r = 1$
0.1	0.12675303	0.1419205	0.1768107	0.1923908	0.2273445	0.2431654	0.2735124	0.2908576	0.3067743	0.3225307	0.3404553
0.2	0.1634607	0.1851885	0.2074428	0.2237103	0.2508914	0.2704913	0.2965125	0.3284316	0.3455966	0.3602684	0.3732215
0.3	0.2284611	0.2563843	0.2725725	0.2935418	0.32481325	0.3528307	0.3803398	0.4036742	0.4327672	0.4465312	0.5632964
0.4	0.2845217	0.3016433	0.3325318	0.3604272	0.3978314	0.4143606	0.4452286	0.4643655	0.4768988	0.4863509	0.4983521
0.5	0.3223709	0.3484355	0.3606467	0.3832869	0.4184315	0.4357732	0.45524648	0.4840237	0.5023413	0.5261913	0.5533044
0.6	0.3656858	0.3803725	0.4072954	0.4223308	0.4528635	0.4786009	0.5056639	0.5184711	0.5264205	0.5468537	0.6541281

Table 8 Numerical results for Example 5.2

				$\overline{v}$	$n=27$						
t	$r=1$	$r=0.9$	$r=0.8$	$r=0.7$	$r=0.6$	$r=0.5$	$r=0.4$	$r=0.3$	$r=0.2$	$r=0.1$	$r=0$
0.1	0.1527514	0.1735663	0.20323845	0.2268712	0.2569635	0.2752807	0.3073903	0.3184522	0.3258415	0.3423603	0.3567114
0.2	0.2057675	0.22373902	0.2557883	0.2715573	0.2934066	0.3229726	0.3442639	0.3724563	0.3881954	0.3965069	0.4137315
0.3	0.2546935	0.2714643	0.2909238	0.3259324	0.3455216	0.3609311	0.3965551	0.4254729	0.4428775	0.4632447	0.4853688
0.4	0.3111646	0.3420738	0.3763449	0.3928127	0.4133812	0.4343069	0.4569529	0.4766186	0.4885963	0.4988791	0.5134679
0.5	0.3677495	0.3883728	0.4154259	0.4473535	0.4628365	0.4942759	0.5144074	0.5280771	0.5414224	0.5568705	0.5658531
0.6	0.3966117	0.4137469	0.4307355	0.4521564	0.4775443	0.5023742	0.5238519	0.5367273	0.5515629	0.5708526	0.5872834

Table 9 Numerical results for Example 5.2

r	$(\underline{v}, n=32, t=0.6)$	$(\overline{v}, n=32, t=0.6)$
0.0	0.2734517	0.6017654
0.1	0.2835413	0.5964302
0.2	0.2968295	0.5862542
0.3	0.3122348	0.5625338
0.4	0.3376642	0.5577267
0.5	0.3458445	0.5354218
0.6	0.3667309	0.5157625
0.7	0.3746558	0.5046268
0.8	0.4063229	0.4835326
0.9	0.42264319	0.4666518
1.0	0.4384637	0.4489275

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