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FOUNDATIONS

General type-2 fuzzy rough sets based on α-plane Representation theory

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Abstract Rough sets theory and fuzzy sets theory are mathematical tools to deal with uncertainty, imprecision in data analysis. Traditional rough set theory is restricted to crisp environments. Since theories of fuzzy sets and rough sets are distinct and complementary on dealing with uncertainty, the concept of fuzzy rough sets has been proposed. Type-2 fuzzy set provides additional degree of freedom, which makes it possible to directly handle highly uncertainties. Some researchers proposed interval type-2 fuzzy rough sets by combining interval type-2 fuzzy sets and rough sets. However, there are no reports about combining general type-2 fuzzy sets and rough sets. In addition, the α-plane representation method of general type-2 fuzzy sets has been extensively studied, and can reduce the computational workload. Motivated by the aforementioned accomplishments, in this paper, from the viewpoint of constructive approach, we first present definitions of upper and lower approximation operators of general type-2 fuzzy sets by using α-plane representation theory and study some basic properties of them. Furthermore, the connections between special general type-2 fuzzy relations and general type-2 fuzzy rough upper and lower approximation operators are also examined. Finally, in axiomatic approach, various classes of general type-2 fuzzy rough approximation operators are characterized by different sets of axioms.

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1 Introduction

Rough sets theory proposed by Pawlak (1982, 1991) is a mathematical tool to deal with uncertainty, imprecision in data analysis. The core of rough set theory and application is a pair of approximation operators derived from approximate space. In general, there are two ways to study approximation operators, including the constructive and axiomatic approaches (Wu et al. 2003). The goal of the constructive method is to construct the lower and upper approximation operators on the basis of binary relation, boolean subalgebra and so on. On the other hand, the basic idea of the axiomatic approach is find to approximate operators which can satisfy certain axioms set and focus on studying the mathematical structure of rough sets, such as algebraic and topologic structures (Zhao et al. 2009). The equivalence relations play an important role in Pawlak's rough set model. However, the equivalence relations are very restrictive requirement, which would limit the application scope of rough sets theory. Hence, the generalization of the Pawlak's rough set model is an important research direction in rough sets theory (Zhou et al. 2009). Thus, many extensions of rough sets, such as rough set model based on general binary relations, variable precision rough set model, covering rough set model, rough fuzzy set model, fuzzy rough set model, intuitionistic fuzzy rough set model and interval type-2 fuzzy rough set model, have been proposed and studied.

Fuzzy set theory (type-1 fuzzy sets) was proposed by Zadeh (1965). Since theories of fuzzy sets and rough sets are distinct and complementary on dealing with uncertainty, the concept of fuzzy rough sets has been proposed. Dubois and Prade (1990) did pioneering work about fuzzy rough sets. They extended crisp concept to fuzzy concept, and proposed rough fuzzy sets. Further, they extended crisp partition to fuzzy partition, and introduced the concept of fuzzy rough sets. Currently, there are two main branches on the researches of fuzzy rough sets: the knowledge representation of fuzzy rough sets and attribute reduction using fuzzy rough sets. About the study of attribute reduction using fuzzy rough sets, readers can refer these literatures (Jensen and Shen 2004; Bhatt and Copal 2005; Jensen and Shen 2005, 2009a,b; Zhao and Tsang 2008; Tsang et al. 2008). The knowledge representation of fuzzy rough sets includes two research ways, i.e., the mentioned constructive and axiomatic approaches. Many valuable results about the knowledge representation of fuzzy rough sets have been extensively studied. In Wu et al. (2003), a general framework for the study of fuzzy rough sets was presented in which both constructive and axiomatic approaches were used. By suing a residual implication, the definitions for generalized fuzzy lower and upper approximation operators were introduced in Mi and Zhang (2004). Based on outer and inner products, the lower and upper approximations of rough sets and fuzzy rough sets were respectively characterized in Liu (2008). The connections with lattice theory and fuzzy topology were also developed in Yeung et al. (2005). In Wu and Zhang (2004), the minimal axiom sets of fuzzy approximation operators were proposed to guarantee the existence of certain types of fuzzy relations producing the same operators. Three classes of fuzzy rough sets were also defined by employing three main classes of implicators in Radzikowska and Kerre (2002).

Type-2 fuzzy set was proposed by Zadeh (1975). Type-2 fuzzy set is a fuzzy set whose membership values are type-1 fuzzy sets on [0, 1]. At least four sources of uncertainty in fuzzy logic systems have been summarized in Mendel (2001), which are as follows: (1) uncertainty about the meanings of the words that are used in the rules; (2) uncertainty about the consequent that is used in a rule; (3) uncertainty about the measurements that activate the fuzzy logic systems; and (4) uncertainty about the data that are used to tune the parameters of a fuzzy logic system. All these uncertainties lead to uncertain membership functions. Type-2 fuzzy sets are able to handle the four types of uncertainty because they directly model uncertainties. Due to the computational complexity, general type-2 fuzzy sets are limited in practical applications. Therefore, interval type-2 fuzzy sets, as the special case of general type-2 fuzzy sets, have been considered and studied. Thus, to deal with highly uncertain data, interval type-2 fuzzy rough sets combining the characteristics of rough sets and interval type-2 fuzzy sets have been proposed. A rough approximation of every interval type-2 fuzzy set in the interval type-2 fuzzy information system was proposed in Sun et al. (2008). In Zhang et al. (2009), a general study of (I, T) interval type-2 fuzzy rough sets on two universes of discourse was presented by integrating the rough set theory with the interval type-2 fuzzy set theory, where both constructive and axiomatic approaches were considered. The new definitions of interval type-2 fuzzy rough sets were developed in Wu et al. (2009), and the properties of interval type-2 fuzzy rough approximation operators and a method of attribute reduction within the interval type-2 fuzzy rough set framework were presented.

General type-2 fuzzy sets can handle complex and changing systems, and therefore must be better than the interval type-2 fuzzy sets to deal with uncertainties (John and Coupland 2007). In order to simplify the calculation for the general type-2 fuzzy sets, the α-plane representation of the general type-2 fuzzy sets was proposed in Liu (2008). The α-plane representation of a general type-2 fuzzy set has been proved to be useful for both theoretical and computational studies of and for general type-2 fuzzy logic systems in Mendel et al. (2009).

As is well known, traditional rough sets only could handle the datasets with discrete attributes, and have difficulty in handling real-valued datasets. In fact, real-valued data has certain uncertainty and fuzziness, and the boundary between concepts is not clear. A reasonable approach is to use the method of fuzziness such that the real-valued data can be converted to determined membership degree value. Currently, the type-1 fuzzy rough set model has been proposed to deal with real-valued datasets, and many valuable results have also been obtained. In fact, real-valued data has highly uncertain feature due to the influence of noises. Thus, it may not be very reasonable or be very difficult, that one can convert the real-valued data to determined membership degree value. However, type-2 fuzzy sets can improve the ability of dealing with uncertainties. Therefore, it may be useful to handle real-valued datasets having uncertainty and fuzziness, by combining general type-2 fuzzy sets and rough sets. Furthermore, we find that the α-plane representation method of general type-2 fuzzy sets can reduce the computational workload. Motivated by the aforementioned accomplishments, in this paper, we present a general framework for the study of general type-2 fuzzy rough sets in which both constructive and axiomatic approaches are used. The rest of our work is organized as follows. In Sect. 2, the basic definitions and terminologies on type-2 fuzzy sets will be reviewed briefly. In Sect. 3, we define upper and lower approximation operators of general type-2 fuzzy sets by using α-plane representation theory and study some basic properties of them. In Sect. 4, the connections between special general type-2 fuzzy relations and general type-2 fuzzy rough upper and lower approximation operators are also discussed. In Sect. 5, we show that general type-2 fuzzy rough approximation operators can be characterized by axioms. The last section concludes this paper.

2 Preliminaries

In this section, we recall the basic definitions and terminologies on type-2 fuzzy sets and interval type-2 fuzzy rough sets, which are necessary preliminaries for studying general type-2 fuzzy rough sets.

A type-2 fuzzy set, as an extension of a type-1 fuzzy set, was first proposed by Zadeh. In this part, we give the relative definitions of α-plane representation for type-2 fuzzy sets with some modified notations.

Definition 2.1 (Mendel 2001) A type-2 fuzzy set, denoted A, is characterized by a type-2 membership function $u_A(x, u)$, where $x \in U$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$A = \{((x, u), u_A(x, u)) | \forall x \in U, \forall u \in J_x \subseteq [0, 1]\}$$

in which $0 \leq u_A(x, u) \leq 1$. A can also be expressed as:

$$A = \int_{x \in U} \int_{u \in J_x} u_A(x, u)/(x, u) \quad J_x \subseteq [0, 1]$$

where $\int\int$ denotes union over admissible x and u.

Based on the concept of secondary sets (Mendel 2001), we can re-expressed A as:

$$A = \int_{x \in U} u_A(x)/x = \int_{x \in U} \left[\int_{u \in J_x} f_x(u)/u \right] \Big/ x \quad J_x \subseteq [0, 1]$$

where J_x is the primary membership of x; $f_x(u)$ is a secondary membership grade.

When $f_x(u) = 1, \forall u \in J_x \subseteq [0, 1]$, then the secondary membership functions are interval sets, and, if this is true for $\forall x \in X$, we have the case of interval type-2 fuzzy set. i.e.,

$$A = \int_{x \in U} u_A(x)/x = \int_{x \in U} \left[\int_{u \in J_x} 1/u \right] \Big/ x, \quad J_x \subseteq [0, 1].$$

In the following sections, the class of all general type-2 fuzzy sets of the universe U is denoted as $F_2(U)$, and the class of all interval type-2 fuzzy sets of the universe U is denoted as $IF_2(U)$. In practical applications, in order to simplify calculation, we often choose these type-2 fuzzy sets whose secondary membership functions are normal and convex, for example, Gaussian type-2 fuzzy sets and triangular type-2 fuzzy sets. In this paper, we only study these type-2 fuzzy sets whose secondary membership functions are normal and convex.

Definition 2.2 (Mendel 2001) Let U be a nonempty universe. A type-2 fuzzy set R is defined as a type-2 fuzzy binary relation from U to U

$$R \equiv \{((x, y), u_R((x, y), u)) | (x, y) \in U \times U, u \in J_{(x,y)} \subseteq [0, 1]\}$$

where $0 \leq u_R((x, y), u) \leq 1$. R can also be expressed as

$$\begin{aligned} R &\equiv \int_{(x,y) \in U \times U} \int_{t \in J_{(x,y)}} u_R((x, y), t)/((x, y), t) \\ &\equiv \int_{(x,y) \in U \times U} u_R(x, y)/(x, y) \\ &\equiv \int_{(x,y) \in U \times U} \left[\int_{u \in J_{(x,y)}} f_{(x,y)}(u)/u \right] \Big/ (x, y) \end{aligned}$$

where $f_{(x,y)}(u) = u_R((x, y), u)$.

Definition 2.3 (Mendel 2001) Uncertainty in the primary memberships of a type-2 fuzzy set, A, consists of a bounded region that we call the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e.,

$$FOU(A) = \bigcup_{x \in U} J_x$$

Definition 2.4 (Mendel 2001) An upper membership function and a lower membership function are two type-1 membership functions that are bounds for the $FOU(A)$ of a type-2 fuzzy set A. The upper membership function is associated with the upper bound of $FOU(A)$, and is denoted $u_{\bar{A}}(x)$. The lower membership function is associated with the lower bound of $FOU(A)$, and is denoted $u_{\underline{A}}(x)$.

Definition 2.5 Mendel et al. (2009) An α-plane for general type-2 fuzzy set A, which is denoted by A_α, is defined as follow:

$$\begin{aligned} A_\alpha &= \int_{x \in U} \int_{u \in J_x} \{(x, u) | f_x(u) \geq \alpha\} \\ &= \int_{x \in U} \left[\int_{u \in [S_L^A(x|\alpha), S_U^A(x|\alpha)]} u \right] \Big/ x \qquad \alpha \in [0, 1]. \end{aligned}$$

where $[S_L^A(x|\alpha), S_U^A(x|\alpha)]$ denote an α-cut of the secondary membership function $u_A(x)$.

Definition 2.6 (Mendel et al. 2009) The α-plane representation (theorem) for type-2 fuzzy set A is

$$A = \bigcup_{\alpha \in [0,1]} \alpha / A_\alpha.$$

Theorem 2.1 (Mendel et al. 2009) *Let $(A \cup B)_\alpha$ and $(A \cap B)_\alpha$ be α-plane of $A \cup B$ and $A \cap B$, respectively, we have*

$$A \cup B = \bigcup_{\alpha \in [0,1]} \alpha / (A_\alpha \cup B_\alpha);$$
$$A \cap B = \bigcup_{\alpha \in [0,1]} \alpha / (A_\alpha \cap B_\alpha).$$

where

$$A_\alpha \cup B_\alpha = \int_{x\in U}\left[\int_{u\in[S_L^A(x|\alpha)\vee S_L^B(x|\alpha),\,S_U^A(x|\alpha)\vee S_U^B(x|\alpha)]} u\right]\Big/x;$$

$$A_\alpha \cap B_\alpha = \int_{x\in U}\left[\int_{u\in[S_L^A(x|\alpha)\wedge S_L^B(x|\alpha),\,S_U^A(x|\alpha)\wedge S_U^B(x|\alpha)]u}\right]\Big/x.$$

Obviously, $S_L^{A\cup B}(x|\alpha) = S_L^A(x|\alpha) \vee S_L^B(x|\alpha)$, $S_U^{A\cup B}(x|\alpha) = S_U^A(x|\alpha) \vee S_U^B(x|\alpha)$, $S_L^{A\cap B}(x|\alpha) = S_L^A(x|\alpha) \wedge S_L^B(x|\alpha)$, *and* $S_U^{A\cap B}(x|\alpha) = S_U^A(x|\alpha) \wedge S_U^B(x|\alpha)$ *hold.*

Theorem 2.2 *Let* $(A^c)_\alpha$ *be* α*-plane of* A^c*, we have*

$$A^c = \bigcup_{\alpha\in[0,1]} \alpha/(A^c)_\alpha = \bigcup_{\alpha\in[0,1]} \alpha/(A_\alpha)^c.$$

where $A_\alpha = \int_{x\in U}[\int_{u\in[S_L^A(x|\alpha),S_U^A(x|\alpha)]} u]/x$.

Proof Let $A^c = \int_{x\in U}[\int_{u\in J_x} f_x(u)/1-u]/x$, $J_x \subseteq [0,1]$. From Definition 2.5, it follows that for $(A^c)_\alpha$ and therefore,

$$(A^c)_\alpha = \int_{x\in U}\int_{u\in J_x} \{(x, 1-u)|f_x(u) \ge \alpha\}$$

$$= \int_{x\in U}\left[\int_{1-u\in[S_L^A(x|\alpha),S_U^A(x|\alpha)]} u\right]\Big/x$$

$$= \int_{x\in U}\left[\int_{u\in[1-S_U^A(x|\alpha),1-S_L^A(x|\alpha)]} u\right]\Big/x.$$

In addition, $(A_\alpha)^c = \int_{x\in U}[\int_{u\in[1-S_U^A(x|\alpha),1-S_L^A(x|\alpha)]} u]/x$. Thus, $(A^c)_\alpha = (A_\alpha)^c$. This completes the proof of the theorem.

Obviously, $S_L^{A^c}(x|\alpha) = 1 - S_U^A(x|\alpha)$ and $S_U^{A^c}(x|\alpha) = 1 - S_L^A(x|\alpha)$ hold. □

Definition 2.7 Let $A, B \in F_2(U)$, define $A \subseteq B$ if $S_L^A(x|\alpha) \le S_L^B(x|\alpha)$ and $S_U^A(x|\alpha) \le S_U^B(x|\alpha)$ hold for any $\alpha \in [0,1]$ and $\forall x \in U$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Definition 2.8 The α-plane representation (theorem) for type-2 fuzzy relation R is

$$R = \bigcup_{\alpha\in[0,1]} \alpha/R_\alpha.$$

where $R_\alpha = \int_{(x,y)\in U\times U}\int_{u\in J_x}\{((x,y),u)|f_{(x,y)}(u) \ge \alpha\} = \int_{(x,y)\in U\times U}[\int_{u\in[S_L^R((x,y)|\alpha),S_U^R((x,y)|\alpha)]} u]/(x,y)\quad \alpha \in [0,1]$, and $[S_L^R((x,y)|\alpha), S_U^R((x,y)|\alpha)]$ denote an α-cut of the secondary membership function $u_R(x,y)$.

The definitions of interval type-2 fuzzy rough sets have been proposed in Sun et al. (2008). In this following, the concept of interval type-2 fuzzy rough sets is briefly recalled with some modified notations.

Definition 2.9 (Sun et al. 2008) Let U be a nonempty universe of discourse and $R \in IF_2(U \times U)$, the pair (U, R) is called an interval type-2 fuzzy approximation space. For any $A \in IF_2(U)$, define lower and upper interval type-2 fuzzy rough approximation operators $R_*, R^* : IF_2(U) \to IF_2(U)$ about (U, R) by

$$R_*A \equiv \int_{x\in U}\int_{u\in D\underline{R}(A)(x)} 1/(x,u)$$

$$R^*A \equiv \int_{x\in U}\int_{u\in D\bar{R}(A)(x)} 1/(x,u)$$

where for any $x \in U$, $D\underline{R}(A)(x) \equiv [u_{\underline{R_*A}}(x), u_{\overline{R_*A}}(x)]$, $D\bar{R}(A)(x) \equiv [u_{\underline{R^*A}}(x), u_{\overline{R^*A}}(x)]$, and

$$u_{\underline{R_*A}}(x) = \bigwedge_{y\in U}[(1-u_{\bar{R}}(x,y)) \vee u_{\underline{A}}(y)]$$

$$u_{\overline{R_*A}}(x) = \bigwedge_{y\in U}[(1-u_{\underline{R}}(x,y)) \vee u_{\bar{A}}(y)]$$

$$u_{\underline{R^*A}}(x) = \bigvee_{y\in U}[u_{\underline{R}}(x,y) \wedge u_{\underline{A}}(y)]$$

$$u_{\overline{R^*A}}(x) = \bigvee_{y\in U}[u_{\bar{R}}(x,y) \wedge u_{\bar{A}}(y)].$$

The pair (R_*A, R^*A) is defined as interval type-2 fuzzy rough set.

3 General type-2 fuzzy rough sets

In this section, we will introduce general type-2 fuzzy rough approximation operators induced from a general type-2 fuzzy approximation space and discuss their properties.

3.1 General type-2 fuzzy rough approximation operators based on general type-2 fuzzy relations

In this part, we introduce the constructive definition of general type-2 rough sets and show that the general type-2 fuzzy rough set model is an extension of the classical rough set models.

Definition 3.1 Let U be a nonempty universe of discourse and $R \in F_2(U\times U)$, the pair (U, R) is called a general type-2 fuzzy approximation space. For any $A \in F_2(U)$, define lower and upper general type-2 fuzzy rough approximation operators $\underline{f}, \overline{f} : F_2(U) \to F_2(U)$ about (U, R) by

$$\underline{f}(A) \equiv \bigcup_{\alpha\in[0,1]} \alpha/(\underline{f}(A))_\alpha,$$

$$\overline{f}(A) \equiv \bigcup_{\alpha\in[0,1]} \alpha/(\overline{f}(A))_\alpha.$$

where $(\underline{f}(A))_\alpha = \int_{x\in U}[\int_{u\in[S_L^{\underline{f}(A)}(x|\alpha), S_U^{\underline{f}(A)}(x|\alpha)]} u]/x$, $(\overline{f}(A))_\alpha = \int_{x\in U}[\int_{u\in[S_L^{\overline{f}(A)}(x|\alpha), S_U^{\overline{f}(A)}(x|\alpha)]} u]/x$, and

$$S_L^{\underline{f}(A)}(x|\alpha) = \bigwedge_{y\in U}\left[\left(1 - S_U^R((x,y)|\alpha)\right) \vee S_L^A(y|\alpha)\right]$$
$$S_U^{\underline{f}(A)}(x|\alpha) = \bigwedge_{y\in U}\left[\left(1 - S_L^R((x,y)|\alpha)\right) \vee S_U^A(y|\alpha)\right]$$
$$S_L^{\overline{f}(A)}(x|\alpha) = \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^A(y|\alpha)\right]$$
$$S_U^{\overline{f}(A)}(x|\alpha) = \bigvee_{y\in U}\left[S_U^R((x,y)|\alpha) \wedge S_U^A(y|\alpha)\right]$$

The pair $(\underline{f}(A), \overline{f}(A))$ is defined as general type-2 fuzzy rough set.

Example 3.1 Let $U = \{x_1, x_2, x_3\}$. $A \in F_2(U)$ and $R \in F_2(U \times U)$ are given as follows:

$$u_A(x_1) = \text{trimf}(0.18, 0.36, 0.54);$$
$$u_A(x_2) = \text{trapmf}(0.38, 0.52, 0.74, 0.86);$$
$$u_A(x_3) = \text{trimf}(0.39, 0.61, 0.72);$$
$$u_R(x_1,x_1) = \frac{1}{1};\quad u_R(x_2,x_2) = \frac{1}{1};\quad u_R(x_3,x_3) = \frac{1}{1};$$
$$u_R(x_1,x_2) = u_R(x_2,x_1) = \text{trapmf}(0.2, 0.3, 0.37, 0.45);$$
$$u_R(x_1,x_3) = u_R(x_3,x_1) = \text{trimf}(0.1, 0.23, 0.39);$$
$$u_R(x_2,x_3) = u_R(x_3,x_2) = \text{trimf}(0.8, 0.95, 1).$$

Where trapmf$(\cdot,\cdot,\cdot,\cdot)$ denotes trapezoid function, the first parameter and four parameter of () denote bottom left and right endpoint, respectively, and the second parameter and third parameter of () denote top left and right endpoint, respectively. Furthermore, trimf$(\cdot,\cdot,\cdot)$ denotes triangular function, the first parameter and third parameter of () denote bottom left and right endpoint, respectively, and the second parameter of () denotes apex.

In the following, we should decide on how many α-planes will be used, where $\alpha \in [0, 1]$. Call that number $\Delta + 1$. Its choice will depend on the accuracy that is required. Regardless of $\Delta + 1$, $\alpha = 0$ and $\alpha = 1$ must always be used. If $\Delta+1 = 101$, then $\alpha = 0, 0.01, 0.02, 0.03, 0.04, \ldots, 0.99, 1$. When $\alpha = 0.5$, we have

$$S_L^A(x_1|0.5) = 0.27,\quad S_L^A(x_2|0.5) = 0.45,\quad S_L^A(x_3|0.5) = 0.5,$$
$$S_U^A(x_1|0.5) = 0.44,\quad S_U^A(x_2|0.5) = 0.79,\quad S_U^A(x_3|0.5) = 0.66,$$

$$S_L^R((x,y)|0.5) = \begin{bmatrix} 1 & 0.25 & 0.17 \\ 0.25 & 1 & 0.88 \\ 0.17 & 0.88 & 1 \end{bmatrix},$$
$$S_U^R((x,y)|0.5) = \begin{bmatrix} 1 & 0.40 & 0.31 \\ 0.40 & 1 & 0.97 \\ 0.31 & 0.97 & 1 \end{bmatrix}.$$

According to Definition 3.1, we can obtain:

$$S_L^{\underline{f}(A)}(x_1|0.5) = 0.27,\quad S_L^{\underline{f}(A)}(x_2|0.5) = 0.45,\quad S_L^{\underline{f}(A)}(x_3|0.5) = 0.45,$$
$$S_U^{\underline{f}(A)}(x_1|0.5) = 0.44,\quad S_U^{\underline{f}(A)}(x_2|0.5) = 0.66,\quad S_U^{\underline{f}(A)}(x_3|0.5) = 0.66,$$
$$S_L^{\overline{f}(A)}(x_1|0.5) = 0.27,\quad S_L^{\overline{f}(A)}(x_2|0.5) = 0.50,\quad S_L^{\overline{f}(A)}(x_3|0.5) = 0.50,$$
$$S_U^{\overline{f}(A)}(x_1|0.5) = 0.44,\quad S_U^{\overline{f}(A)}(x_2|0.5) = 0.79,\quad S_U^{\overline{f}(A)}(x_3|0.5) = 0.79.$$

Similarly, we can compute the other approximation results for $\alpha \neq 0.5$.

In the following, we will establish the relationships between the general type-2 fuzzy rough set with the other classical rough set models. It is easy to prove that the general type-2 fuzzy rough set model is an extension of the classical rough set models.

Theorem 3.1 *If A is the interval type-2 fuzzy set on U and R is the interval type-2 fuzzy relation on U, then general type-2 fuzzy rough set model degenerates to interval type-2 fuzzy rough set model defined in 2.9.*

Proof Since A is the interval type-2 fuzzy set on U, we have

$$A = \int_{x\in U} u_A(x)/x = \int_{x\in U}\left[\int_{u\in J_x} 1/u\right]\Big/ x,\quad J_x \subseteq [0,1].$$

Thus, we obtain $S_L^A(x|\alpha) = S_L^A(x|1)$ and $S_U^A(x|\alpha) = S_U^A(x|1)$ for any $x \in U$ and $\alpha \in [0, 1]$.

Similarly, for any $x, y \in U$ and $\alpha \in [0, 1]$, we have $S_L^R((x,y)|\alpha) = S_L^R((x,y)|1)$ and $S_U^R((x,y)|\alpha) = S_U^R((x,y)|1)$.

If we set $S_L^A(x|1) = \underline{A}(x)$, $S_U^A(x|1) = \overline{A}(x)$, $S_L^R((x,y)|1) = \underline{R}(x,y)$, and $S_U^R((x,y)|1) = \bar{R}(x,y)$, then

$$S_L^{\underline{f}(A)}(x|\alpha) = S_L^{\underline{f}(A)}(x|1) = \bigwedge_{y\in U}[(1 - \overline{R}(x,y)) \vee \underline{A}(y)],$$
$$S_U^{\underline{f}(A)}(x|\alpha) = S_U^{\underline{f}(A)}(x|1) = \bigwedge_{y\in U}[(1 - \underline{R}(x,y)) \vee \overline{A}(y)],$$
$$S_L^{\overline{f}(A)}(x|\alpha) = S_L^{\overline{f}(A)}(x|1) = \bigvee_{y\in U}[\underline{R}(x,y) \wedge \underline{A}(y)],$$

and $S_U^{\overline{f}(A)}(x|\alpha) = S_U^{\overline{f}(A)}(x|1) = \vee_{y\in U}[\overline{R}(x,y) \wedge \overline{A}(y)]$ hold for any $x \in U$ and $\alpha \in [0, 1]$.

That is, for any $\alpha \in [0, 1]$, we have $(\underline{f}(A))_\alpha = (\underline{f}(A))_1 = \int_{x\in U}[\int_{u\in[S_L^{\underline{f}(A)}(x|1), S_U^{\underline{f}(A)}(x|1)]} u]/x$ and $(\overline{f}(A))_\alpha = (\overline{f}(A))_1 = \int_{x\in U}[\int_{u\in[S_L^{\overline{f}(A)}(x|1), S_U^{\overline{f}(A)}(x|1)]} u]/x$.

Hence, $\underline{f}(A) = (\underline{f}(A))_1$ and $\overline{f}(A) = (\overline{f}(A))_1$.

This completes the proof of the theorem. □

Theorem 3.2 *If A is the type-1 fuzzy set on U and R is the type-1 fuzzy relation on U, then general type-2 fuzzy rough set model degenerates to type-1 fuzzy rough set model.*

Proof Since A is the type-1 fuzzy set on U, we have

$$A = \int\limits_{x \in U} u_A(x)/x = \int\limits_{x \in U} \left[\int\limits_{u \in J_x} 1/u \right] \Big/ x, \quad J_x \in [0, 1].$$

That is, the primary membership J_x of x only can take a sole value.

Thus, we obtain $S_L^A(x|\alpha) = S_U^A(x|\alpha) = S_L^A(x|1)$ for any $x \in U$ and $\alpha \in [0, 1]$.

Similarly, we have $S_L^R((x, y)|\alpha) = S_U^R((x, y)|\alpha) = S_L^R((x, y)|1)$ for any $x, y \in U$ and $\alpha \in [0, 1]$.

If we set $S_L^A(x|1) = A(x)$ and $S_L^R((x, y)|1) = R(x, y)$, then $S_L^{\underline{f}(A)}(x|\alpha) = S_U^{\underline{f}(A)}(x|\alpha) = S_L^{\underline{f}(A)}(x|1) = \wedge_{y\in U}[(1 - R(x, y)) \vee A(y)]$ and $S_L^{\overline{f}(A)}(x|\alpha) = S_U^{\overline{f}(A)}(x|\alpha) = S_L^{\overline{f}(A)}(x|1) = \vee_{y\in U}[R(x, y) \wedge A(y)]$ hold for any $x \in U$ and $\alpha \in [0, 1]$.

That is, for any $\alpha \in [0, 1]$, we have $(\underline{f}(A))_\alpha = (\underline{f}(A))_1 = \int_{x\in U} \{\wedge_{y\in U}[(1 - R(x, y)) \vee A(y)]\}/x$ and $(\overline{f}(A))_\alpha = (\overline{f}(A))_1 = \int_{x\in U} \{\vee_{y\in U}[R(x, y) \wedge A(y)]\}/x$.

Hence, $\underline{f}(A) = \int_{x\in U} \{\wedge_{y\in U}[(1 - R(x, y)) \vee A(y)]\}/x$ and $\overline{f}(A) = \int_{x\in U} \{\vee_{y\in U}[R(x, y) \wedge A(y)]\}/x$. This completes the poof of the theorem. □

3.2 Properties of the general type-2 fuzzy rough approximation operators

In this part, we discuss some basic properties of general type-2 fuzzy rough approximation operators.

Theorem 3.3 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A \in F_2(U)$, the following properties hold:*

1. $(\underline{f}(A^c))^c = \overline{f}(A)$
2. $(\overline{f}(A^c))^c = \underline{f}(A)$

Proof (1) For any $x \in U$ and $\alpha \in [0, 1]$,

$$\begin{aligned} S_L^{\underline{f}(A^c)}(x|\alpha) &= \mathop{\wedge}_{y\in U} \left[\left(1 - S_U^R((x, y)|\alpha)\right) \vee S_L^{A^c}(y|\alpha) \right] \\ &= \mathop{\wedge}_{y\in U} \left[\left(1 - S_U^R((x, y)|\alpha)\right) \vee \left(1 - S_U^A(y|\alpha)\right) \right] \\ &= \mathop{\wedge}_{y\in U} \left[1 - S_U^R((x, y)|\alpha) \wedge S_U^A(y|\alpha) \right] \\ &= 1 - \mathop{\vee}_{y\in U} \left[S_U^R((x, y)|\alpha) \wedge S_U^A(y|\alpha) \right] \\ &= 1 - S_U^{\overline{f}(A)}(x|\alpha). \end{aligned}$$

Similarly, $S_U^{\underline{f}(A^c)}(x|\alpha) = 1 - S_L^{\overline{f}(A)}(x|\alpha)$.

Thus, $(\underline{f}(A^c))_\alpha = \int_{x\in U} [\int_{u\in[S_L^{\underline{f}(A^c)}(x|\alpha), S_U^{\underline{f}(A^c)}(x|\alpha)]} u]/x = \int_{x\in U} [\int_{u\in[1-S_U^{\overline{f}(A)}(x|\alpha), 1-S_L^{\overline{f}(A)}(x|\alpha)]} u]/x$.

According to Theorem 2.2, we have $((\underline{f}(A^c))^c)_\alpha = ((\underline{f}(A^c))_\alpha)^c$.

Hence, $((\underline{f}(A^c))^c)_\alpha = ((\underline{f}(A^c))_\alpha)^c = \int_{x\in U} [\int_{u\in[) S_L^{\overline{f}(A)}} (x|\alpha), S_U^{\overline{f}(A)}(x|\alpha)]u]/x = (\overline{f}(A))_\alpha$.

We can obtain $(\underline{f}(A^c))^c = \overline{f}(A)$.

(2) The proof procedure is similar to (1). □

Theorem 3.4 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A, B \in F_2(U)$, if $A \subseteq B$, then the following properties hold:*

1. $\underline{f}(A) \subseteq \underline{f}(B)$
2. $\overline{f}(A) \subseteq \overline{f}(B)$

Proof (1) Since $A \subseteq B$, we have that $S_L^A(x|\alpha) \le S_L^B(x|\alpha)$ and $S_U^A(x|\alpha) \le S_U^B(x|\alpha)$ hold for any $\alpha \in [0, 1]$ and $\forall x \in U$. Hence, $S_L^{\underline{f}(A)}(x|\alpha) = \wedge_{y\in U}[(1 - S_U^R((x, y)|\alpha)) \vee S_L^A(y|\alpha)] \le \wedge_{y\in U}[(1 - S_U^R((x, y)|\alpha)) \vee S_L^B(y|\alpha)] = S_L^{\underline{f}(B)}(x|\alpha)$. Similarly, $S_U^{\underline{f}(A)}(x|\alpha) \le S_U^{\underline{f}(B)}(x|\alpha)$.

We can obtain $\underline{f}(A) \subseteq \underline{f}(B)$.

(2) The proof procedure is similar to (1). □

Theorem 3.5 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A, B \in F_2(U)$, the following properties hold:*

1. $\underline{f}(A \cap B) = \underline{f}(A) \cap \underline{f}(B)$
2. $\overline{f}(A \cup B) = \overline{f}(A) \cup \overline{f}(B)$.

Proof (1) For any $\alpha \in [0, 1]$ and $\forall x \in U$,

$$\begin{aligned} & S_L^{\underline{f}(A\cap B)}(x|\alpha) \\ &= \mathop{\wedge}_{y\in U} \left[\left(1 - S_U^R((x, y)|\alpha)\right) \vee S_L^{A\cap B}(y|\alpha) \right] \\ &= \mathop{\wedge}_{y\in U} \left\{ \left(1 - S_U^R((x, y)|\alpha)\right) \vee \left[S_L^A(y|\alpha) \wedge S_L^B(y|\alpha) \right] \right\} \\ &= \left\{ \mathop{\wedge}_{y\in U} \left[\left(1 - S_U^R((x, y)|\alpha)\right) \vee S_L^A(y|\alpha) \right] \right\} \\ &\quad \wedge \left\{ \mathop{\wedge}_{y\in U} \left[\left(1 - S_U^R((x, y)|\alpha)\right) \vee S_L^B(y|\alpha) \right] \right\} \\ &= S_L^{\underline{f}(A)}(x|\alpha) \wedge S_L^{\underline{f}(B)}(x|\alpha). \end{aligned}$$

Similarly, $S_U^{\underline{f}(A\cap B)}(x|\alpha) = S_L^{\underline{f}(A)}(x|\alpha) \wedge S_L^{\underline{f}(B)}(x|\alpha)$. Thus, $\underline{f}(A \cap B) = \underline{f}(A) \cap \underline{f}(B)$.

(2) The proof procedure is similar to (1). □

4 Connections between approximation operators and special general type-2 fuzzy relations

In previous section, some properties of general type-2 fuzzy rough approximation operators have been discussed. However, some properties of general type-2 fuzzy rough approximation operators may be relative to special general type-2 fuzzy relations. In this section, the relationships between special properties and special general type-2 fuzzy relations will be constructed.

Definition 4.1 Let U be a nonempty universe and R be a general type-2 fuzzy relation on U, then:

1. if for any $x \in U$ and $\alpha \in [0,1]$, $S_L^R((x,x)|\alpha) = S_U^R((x,x)|\alpha) = 1$, then R is defined as a reflexive general type-2 fuzzy relation on U.
2. if for any $x, y \in U$ and $\alpha \in [0,1]$, $S_L^R((x,y)|\alpha) = S_L^R((y,x)|\alpha)$, and $S_U^R((x,y)|\alpha) = S_U^R((y,x)|\alpha)$, then R is defined as a symmetric general type-2 fuzzy relation on U.
3. if for any $x, y \in U$ and $\alpha \in [0,1]$, $S_L^R((x,y)|\alpha) \geq \vee_{z\in U}[S_L^R((x,z)|\alpha) \wedge S_L^R((z,y)|\alpha)]$ and $S_U^R((x,y)|\alpha) \geq \vee_{z\in U}[S_U^R((x,z)|\alpha) \wedge S_U^R((z,y)|\alpha)]$, then R is defined as a transitive general type-2 fuzzy relation on U.

If R is reflexive, symmetric and transitive, then R is called general type-2 fuzzy similarity relation on U.

For any $y \in U$, a type-2 fuzzy singleton set 1_y and its complement $1_{U-\{y\}}$ are, respectively, defined as follows:

$$u_{1_y}(x) = \begin{cases} 1/1 & x = y \\ 1/0 & x \neq y \end{cases}, \quad u_{1_{U-\{y\}}}(x) = \begin{cases} 1/0 & x = y \\ 1/1 & x \neq y \end{cases}.$$

Based the above definition, we can obtain that $S_L^{1_y}(x|\alpha) = S_U^{1_y}(x|\alpha) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$ and $S_L^{1_{U-\{y\}}}(x|\alpha) = S_U^{1_{U-\{y\}}}(x|\alpha) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ hold for any $\alpha \in [0,1]$ and $x \in U$,

Theorem 4.1 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A, B \in F_2(U)$, the following statements are equivalent:*

1. *R is reflexive*
2. *$\underline{f}(A) \subseteq A$*
3. *$A \subseteq \overline{f}(A)$.*

Proof (1) $\Rightarrow$ (2)

If R is reflexive, then for any $\alpha \in [0,1]$ and $x \in U$, there always holds that $S_L^R((x,x)|\alpha) = S_U^R((x,x)|\alpha) = 1$.

Thus, $S_L^{\underline{f}(A)}(x|\alpha) = \wedge_{y\in U}[(1 - S_U^R((x,y)|\alpha)) \vee S_L^A(y|\alpha)] = \{\wedge_{y\in U, y\neq x}[(1 - S_U^R((x,y)|\alpha)) \vee S_L^A(y|\alpha)]\} \wedge [(1 - S_U^R((x,x)|\alpha)) \vee S_L^A(x|\alpha)] = \{\wedge_{y\in U, y\neq x}[(1 - S_U^R((x,y)|\alpha)) \vee S_L^A(y|\alpha)]\} \wedge S_L^A(x|\alpha) \leq S_L^A(x|\alpha)$. Similarly, we can derive $S_U^{\underline{f}(A)}(x|\alpha) \leq S_U^A(x|\alpha)$.

Thus, $\underline{f}(A) \subseteq A$.

(2) $\Rightarrow$ (3)

We can obtain the conclusion according to Theorem 3.3.

(3) $\Rightarrow$ (1)

For any $y \in U$, let $A = 1_y$, thus $S_L^A(x|\alpha) = S_U^A(x|\alpha) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

If $A \subseteq \overline{f}(A)$, then $S_L^A(x|\alpha) \leq S_L^{\overline{f}(A)}(x|\alpha)$ and $S_U^A(x|\alpha) \leq S_U^{\overline{f}(A)}(x|\alpha)$ hold for any $\alpha \in [0,1]$ and $x \in U$.

Hence, $1 \leq S_L^{\overline{f}(A)}(y|\alpha)$ and $1 \leq S_U^{\overline{f}(A)}(y|\alpha)$.

We can obtain $1 = S_L^{\overline{f}(A)}(y|\alpha)$ and $1 = S_U^{\overline{f}(A)}(y|\alpha)$.

Therefore, $\vee_{z\in U}[S_L^R((y,z)|\alpha) \wedge S_L^A(z|\alpha)] = \vee_{z\in U}[S_U^R((y,z)|\alpha) \wedge S_U^A(z|\alpha)] = 1$.

That is to say, $\{\vee_{z\in U, z\neq y}[S_L^R((y,z)|\alpha) \wedge S_L^A(z|\alpha)]\} \vee [S_L^R((y,y)|\alpha) \wedge S_L^A(y|\alpha)] = 1$ and $\{\vee_{z\in U, z\neq y}[S_U^R((y,z)|\alpha) \wedge S_U^A(z|\alpha)]\} \vee [S_U^R((y,y)|\alpha) \wedge S_U^A(y|\alpha)] = 1$.

Thus, $S_L^R((y,y)|\alpha) = S_U^R((y,y)|\alpha) = 1$.

We can conclude that R is reflexive. □

Theorem 4.2 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A, B \in F_2(U)$, the following statements are equivalent:*

1. *R is symmetric*
2. *For any $\alpha \in [0,1]$ and $x, y \in U$, $S_L^{\overline{f}(1_x)}(y|\alpha) = S_L^{\overline{f}(1_y)}(x|\alpha)$ and $S_U^{\overline{f}(1_x)}(y|\alpha) = S_U^{\overline{f}(1_y)}(x|\alpha)$ hold.*
3. *For any $\alpha \in [0,1]$ and $x, y \in U$, $S_L^{\underline{f}(1_{U-\{y\}})}(x|\alpha) = S_L^{\underline{f}(1_{U-\{x\}})}(y|\alpha)$ and $S_U^{\underline{f}(1_{U-\{y\}})}(x|\alpha) = S_U^{\underline{f}(1_{U-\{x\}})}(y|\alpha)$ hold.*

Proof (1) $\Leftrightarrow$ (2)

$$\begin{aligned} S_L^{\overline{f}(1_x)}(y|\alpha) &= \vee_{z\in U}\left[S_L^R((y,z)|\alpha) \wedge S_L^{1_x}(z|\alpha)\right] \\ &= \left\{\vee_{z\in U, z\neq x}\left[S_L^R((y,z)|\alpha) \wedge S_L^{1_x}(z|\alpha)\right]\right\} \\ &\quad \vee \left[S_L^R((y,x)|\alpha) \wedge S_L^{1_x}(x|\alpha)\right] \\ &= 0 \vee S_L^R((y,x)|\alpha) = S_L^R((y,x)|\alpha). \end{aligned}$$

Similarly, $S_L^{\overline{f}(1_y)}(x|\alpha) = S_L^R((x,y)|\alpha)$.

In addition, $S_U^{\overline{f}(1_x)}(y|\alpha) = \vee_{z\in U}[S_U^R((y,z)|\alpha) \wedge S_U^{1_x}(z|\alpha)] = \{\vee_{z\in U, z\neq x}[S_U^R((y,z)|\alpha) \wedge S_U^{1_x}(z|\alpha)]\} \vee [S_U^R((y,x)|\alpha) \wedge S_U^{1_x}(x|\alpha)] = 0 \vee S_U^R((y,x)|\alpha) = S_U^R((y,x)|\alpha)$. Similarly, $S_U^{\overline{f}(1_y)}(x|\alpha) = S_U^R((x,y)|\alpha)$.

Thus, $S_L^{\overline{f}(1_x)}(y|\alpha) = S_L^{\overline{f}(1_y)}(x|\alpha)$ and $S_U^{\overline{f}(1_x)}(y|\alpha) = S_U^{\overline{f}(1_y)}(x|\alpha)$ hold $\Leftrightarrow$ $S_L^R((x, y)|\alpha) = S_L^R((y, x)|\alpha)$, and $S_U^R((x, y)|\alpha) = S_U^R((y, x)|\alpha)$ hold.

That is to say, (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3)

Since $1_y = (1_{U-\{y\}})^c$ for any $y \in U$, according to Theorem 3.3, we have that $S_L^{\underline{f}(1_{U-\{y\}})}(x|\alpha) = 1 - S_U^{\overline{f}(1_y)}(x|\alpha)$, $S_L^{\underline{f}(1_{U-\{x\}})}(y|\alpha) = 1 - S_U^{\overline{f}(1_x)}(y|\alpha)$, $S_U^{\underline{f}(1_{U-\{y\}})}(x|\alpha) = 1 - S_L^{\overline{f}(1_y)}(x|\alpha)$, and $S_U^{\underline{f}(1_{U-\{x\}})}(y|\alpha) = 1 - S_L^{\overline{f}(1_x)}(y|\alpha)$ hold.

That is, (2) $\Leftrightarrow$ (3). □

Theorem 4.3 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any $A, B \in F_2(U)$, the following statements are equivalent:*

1. *R is transitive*
2. $\overline{f}(\overline{f}(A)) \subseteq \overline{f}(A)$
3. $\underline{f}(A) \subseteq \underline{f}(\underline{f}(A))$.

Proof (1) $\Rightarrow$ (2)

For any $\alpha \in [0, 1]$ and $x \in U$,

$$\begin{aligned}
&S_L^{\overline{f}(\overline{f}(A))}(x|\alpha)\\
&= \bigvee_{y\in U} \left[S_L^R((x, y)|\alpha) \wedge S_L^{\overline{f}(A)}(y|\alpha)\right]\\
&= \bigvee_{y\in U} \left\{S_L^R((x, y)|\alpha) \wedge \bigvee_{z\in U} \left[S_L^R((y, z)|\alpha) \wedge S_L^A(z|\alpha)\right]\right\}\\
&= \bigvee_{y\in U} \bigvee_{z\in U} \left[S_L^R((x, y)|\alpha) \wedge S_L^R((y, z)|\alpha) \wedge S_L^A(z|\alpha)\right]\\
&= \bigvee_{z\in U} \left\{\bigvee_{y\in U} \left[S_L^R((x, y)|\alpha) \wedge S_L^R((y, z)|\alpha)\right] \wedge S_L^A(z|\alpha)\right\}
\end{aligned}$$

Since R is transitive, we have $S_L^R((x, z)|\alpha) \geq \vee_{z\in U}[S_L^R((x, y)|\alpha) \wedge S_L^R((y, z)|\alpha)]$.

Hence, $S_L^{\overline{f}(\overline{f}(A))}(x|\alpha) \leq \vee_{z\in U}[S_L^R((x, z)|\alpha) \wedge S_L^A(z|\alpha)] \leq S_L^{\overline{f}(A)}(x|\alpha)$.

Similarly, we can obtain $S_U^{\overline{f}(\overline{f}(A))}(x|\alpha) \leq S_U^{\overline{f}(A)}(x|\alpha)$.

Therefore, $\overline{f}(\overline{f}(A)) \subseteq \overline{f}(A)$.

(2) $\Rightarrow$ (1)

For any $\alpha \in [0, 1]$ and $x, y, z \in U$,

$$\begin{aligned}
&S_L^{\overline{f}(1_z)}(x|\alpha)\\
&= \bigvee_{y\in U} [S_L^R((x, y)|\alpha) \wedge S_L^{1_z}(y|\alpha)]\\
&= \left\{\bigvee_{y\in U, y\neq z} \left[S_L^R((x, y)|\alpha) \wedge S_L^{1_z}(y|\alpha)\right]\right\}\\
&\quad \vee \left[S_L^R((x, z)|\alpha) \wedge S_L^{1_z}(z|\alpha)\right]\\
&= 0 \vee S_L^R((x, z)|\alpha) = S_L^R((x, z)|\alpha).
\end{aligned}$$

Similarly, $S_L^{\overline{f}(1_z)}(y|\alpha) = S_L^R((y, z)|\alpha)$, and $S_U^{\overline{f}(1_z)}(x|\alpha) = S_U^R((x, z)|\alpha)$.

In addition, $S_L^{\overline{f}(\overline{f}(1_z))}(x|\alpha) = \vee_{y\in U}[S_L^R((x, y)|\alpha) \wedge S_L^{\overline{f}(1_z)}(y|\alpha)] = \vee_{y\in U}[S_L^R((x, y)|\alpha) \wedge S_L^R((y, z)|\alpha)]$. Similarly, we can obtain $S_U^{\overline{f}(\overline{f}(1_z))}(x|\alpha) = \vee_{y\in U}[S_U^R((x, y)|\alpha) \wedge S_U^R((y, z)|\alpha)]$.

Since $\overline{f}(\overline{f}(A)) \subseteq \overline{f}(A)$, we have $S_L^{\overline{f}(\overline{f}(1_z))}(x|\alpha) \leq S_L^{\overline{f}(1_z)}(x|\alpha)$, and $S_U^{\overline{f}(\overline{f}(1_z))}(x|\alpha) \leq S_U^{\overline{f}(1_z)}(x|\alpha)$.

Thus, $\vee_{y\in U}[S_L^R((x, y)|\alpha) \wedge S_L^R((y, z)|\alpha)] \leq S_L^R((x, z)|\alpha)$ and $\vee_{y\in U}[S_U^R((x, y)|\alpha) \wedge S_U^R((y, z)|\alpha)] \leq S_U^R((x, z)|\alpha)$. That is, R is transitive.

(2) $\Leftrightarrow$ (3)

We can obtain the conclusion according to Theorem 3.3. □

5 Axiomatic characterization of general type-2 fuzzy rough sets

In axiomatic approach of general type-2 fuzzy rough sets, the primitive notion is a system $(F_2(U), \cap, \cup, c, L, H)$, where $L, H : F_2(U) \to F_2(U)$ are operators from $F_2(U)$ to $F_2(U)$. In this section, we show that general type-2 fuzzy rough approximation operators can be characterized by sets of axioms.

Here we first define a constant type-2 fuzzy set $\beta^{\leftrightarrow} = \int_{x\in U} u_{\overset{\leftrightarrow}{\beta}}(x)/x = \int_{x\in U} \beta/x$, where β is secondary membership function. That is to say, the secondary membership functions of constant type-2 fuzzy set $\overset{\leftrightarrow}{\beta}$ in each of main variables are the same, i.e., β. Obviously, $S_L^{\overset{\leftrightarrow}{\beta}}(x|\alpha) = S_L^{\overset{\leftrightarrow}{\beta}}(y|\alpha) = l(\alpha)$ and $S_U^{\overset{\leftrightarrow}{\beta}}(x|\alpha) = S_U^{\overset{\leftrightarrow}{\beta}}(y|\alpha) = r(\alpha)$ hold for any $\alpha \in [0, 1]$ and $x, y \in U$, where $l(\alpha)$ and $r(\alpha)$ are functions related to α.

In the following, for any $\alpha \in [0, 1]$, $x \in U$ and $A \in F_2(U)$, we define two special type-2 fuzzy sets, which are denoted by $\overleftrightarrow{S_L^A(x|\alpha)}$ and $\overleftrightarrow{S_U^A(x|\alpha)}$, respectively, and satisfy $S_L^{\overleftrightarrow{S_L^A(x|\alpha)}}(y|\eta) = S_L^A(x|\eta)$, $S_U^{\overleftrightarrow{S_L^A(x|\alpha)}}(y|\eta) = S_U^A(x|\eta)$, $S_L^{\overleftrightarrow{S_U^A(x|\alpha)}}(y|\eta) = S_L^A(x|\eta)$ and $S_U^{\overleftrightarrow{S_U^A(x|\alpha)}}(y|\eta) = S_U^A(x|\eta)$ for any $\eta \in [0, 1]$ and $y \in U$. Clearly, $\overleftrightarrow{S_L^A(x|\alpha)}$ and $\overleftrightarrow{S_U^A(x|\alpha)}$ are constant type-2 fuzzy sets.

Definition 5.1 Let $L, H : F_2(U) \to F_2(U)$ be two operators. They are referred to as dual operators if for all $A \in F_2(U)$,

(L1) $(L(A^c))^c = H(A)$; (H1) $(H(A^c))^c = L(A)$.

Lemma 5.1 *Let (U, R) be a general type-2 fuzzy approximation space, $\underline{f}$ and $\overline{f}$ be lower and upper general type-2 fuzzy rough approximation operators about (U, R), for any*

$A \in F_2(U)$ and constant type-2 fuzzy set $\overleftrightarrow{\beta}$, the following statements hold:

1. $\overline{f}(A \cap \overleftrightarrow{\beta}) = \overline{f}(A) \cap \overleftrightarrow{\beta}$
2. $\underline{f}(A \cup \overleftrightarrow{\beta}) = \underline{f}(A) \cup \overleftrightarrow{\beta}$

Proof (1) For any $\alpha \in [0,1]$ and $x \in U$,

$$\begin{aligned}
S_L^{\overline{f}(A\cap\overleftrightarrow{\beta})}(x|\alpha) &= \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^{A\cap\overleftrightarrow{\beta}}(y|\alpha)\right]\\
&= \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^A(y|\alpha) \wedge S_L^{\overleftrightarrow{\beta}}(y|\alpha)\right]\\
&= \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^A(y|\alpha) \wedge l(\alpha)\right]\\
&= \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^A(y|\alpha)\right] \wedge l(\alpha)\\
&= S_L^{\overline{f}(A)}(x|\alpha) \wedge S_L^{\overleftrightarrow{\beta}}(x|\alpha).
\end{aligned}$$

Similarly, we can obtain $S_U^{\overline{f}(A\cap\overleftrightarrow{\beta})}(x|\alpha) = S_U^{\overline{f}(A)}(x|\alpha) \wedge S_U^{\overleftrightarrow{\beta}}(x|\alpha)$.

Thus, $\overline{f}(A \cap \overleftrightarrow{\beta}) = \overline{f}(A) \cap \overleftrightarrow{\beta}$.

(2) The proof procedure is similar to (1). □

Lemma 5.2 *For any $A \in F_2(U)$, $\alpha \in [0,1]$ and $y \in U$, the following statements hold:*

1. $A = \bigcap_{y\in U}[1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y|\alpha)}]$
2. $A = \bigcap_{y\in U}[1_{U-\{y\}} \cup \overleftrightarrow{S_U^A(y|\alpha)}]$
3. $A = \bigcup_{y\in U}[1_y \cap \overleftrightarrow{S_L^A(y|\alpha)}]$
4. $A = \bigcup_{y\in U}[1_y \cap \overleftrightarrow{S_U^A(y|\alpha)}]$

Proof (1) For any $\eta \in [0,1]$ and $x \in U$,

$$\begin{aligned}
&S_L^{\bigcap_{y\in U}[1_{U-\{y\}}\cap\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta)\\
&= \bigwedge_{y\in U} S_L^{[1_{U-\{y\}}\cup\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta)\\
&= \bigwedge_{y\in U}\left[S_L^{1_{U-\{y\}}}(x|\eta) \vee S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x|\eta)\right]\\
&= \bigwedge_{y\in U}\left[S_L^{1_{U-\{y\}}}(x|\eta) \vee S_L^A(y|\eta)\right]\\
&= \left\{\bigwedge_{y\in U, y\neq x}\left[S_L^{1_{U-\{y\}}}(x|\eta) \vee S_L^A(y|\eta)\right]\right\}\\
&\quad \wedge \left[S_L^{1_{U-\{x\}}}(x|\eta) \vee S_L^A(x|\eta)\right]\\
&= S_L^A(x|\eta).
\end{aligned}$$

Similarly, $S_U^{\bigcap_{y\in U}[1_{U-\{y\}}\cap\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta) = S_U^A(x|\eta)$.

Thus, $A_\eta = (\bigcap_{y\in U}[1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y|\alpha)}])_\eta$.

That is, $A = \bigcap_{y\in U}[1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y|\alpha)}]$.

(2) The proof procedure is similar to (1).

(3) For any $\eta \in [0,1]$ and $x \in U$,

$$\begin{aligned}
&S_L^{\bigcup_{y\in U}[1_y\cap\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta)\\
&= \bigvee_{y\in U} S_L^{[1_y\cap\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta)\\
&= \bigvee_{y\in U}\left[S_L^{1_y}(x|\eta) \wedge S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x|\eta)\right]\\
&= \bigvee_{y\in U}\left[S_L^{1_y}(x|\eta) \wedge S_L^A(y|\eta)\right]\\
&= \left\{\bigvee_{y\in U, y\neq x}\left[S_L^{1_y}(x|\eta) \wedge S_L^A(y|\eta)\right]\right\}\\
&\quad \vee \left[S_L^{1_x}(x|\eta) \wedge S_L^A(x|\eta)\right]\\
&= S_L^A(x|\eta).
\end{aligned}$$

Similarly, $S_U^{\bigcup_{y\in U}[1_y\cap\overleftrightarrow{S_L^A(y|\alpha)}]}(x|\eta) = S_U^A(x|\eta)$.

Thus, $A = \bigcup_{y\in U}[1_y \cap \overleftrightarrow{S_L^A(y|\alpha)}]$.

(4) The proof procedure is similar to (3). □

Theorem 5.1 *Suppose that $L, H : F_2(U) \to F_2(U)$ are two dual operators. Then there exists a general type-2 fuzzy relation R on U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L and H satisfy the axioms: for all $A, B \in F_2(U)$ and any constant type-2 fuzzy set $\overleftrightarrow{\beta}$,*

(L2) $L(A \cap B) = L(A) \cap L(B)$

(L3) $L(A \cup \overleftrightarrow{\beta}) = L(A) \cup \overleftrightarrow{\beta}$

Proof "⇒" It follows immediately from Theorem 3.5 and Lemma 5.1.

"⇐" Using L, we can define a general type-2 fuzzy relation R on U by $R = \cup_{\alpha\in[0,1]} \alpha/R_\alpha$, where $R_\alpha = \int_{(x,y)\in U\times U} [\int_{u\in[S_L^R((x,y)|\alpha), S_U^R((x,y)|\alpha)]} u]/x$, $1 - S_L^R((x,y)|\alpha) = S_U^{L(1_{U-\{y\}})}(x|\alpha)$ and $1 - S_U^R((x,y)|\alpha) = S_L^{L(1_{U-\{y\}})}(x|\alpha)$.

For any $\alpha \in [0,1]$ and $x \in U$,

$$\begin{aligned}
&S_L^{\underline{f}(A)}(x|\alpha)\\
&= \bigwedge_{y\in U}\left[\left(1 - S_U^R((x,y)|\alpha)\right) \vee S_L^A(y|\alpha)\right]\\
&= \bigwedge_{y\in U}\left[S_L^{L(1_{U-\{y\}})}(x|\alpha) \vee S_L^A(y|\alpha)\right]\\
&= \bigwedge_{y\in U}\left[S_L^{L(1_{U-\{y\}})}(x|\alpha) \vee S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x|\alpha)\right]
\end{aligned}$$

$$= \bigwedge_{y\in U} S_L^{L(1_{U-\{y\}})\cup \overleftrightarrow{S_L^A(y|\alpha)}}(x|\alpha)$$

$$= \bigwedge_{y\in U} S_L^{L(1_{U-\{y\}}\cup \overleftrightarrow{S_L^A(y|\alpha)})}(x|\alpha) \quad \text{(According to (L3))}$$

$$= S_L^{\bigcap_{y\in U}[L(1_{U-\{y\}}\cup \overleftrightarrow{S_L^A(y|\alpha)})]}(x|\alpha)$$

$$= S_L^{L(\bigcap_{y\in U}[1_{U-\{y\}}\cup \overleftrightarrow{S_L^A(y|\alpha)}])}(x|\alpha) \quad \text{(According to (L2))}$$

$$= S_L^{L(A)}(x|\alpha) \quad \text{(According to Lemma 5.2)}$$

Similarly, we can obtain $S_U^{\underline{f}(A)}(x|\alpha) = S_U^{L(A)}(x|\alpha)$.

Thus, $L(A) = \underline{f}(A)$.

We have $H(A) = \overline{f}(A)$ according to Theorem 3.3. □

Theorem 5.2 *Suppose that $L, H : F_2(U) \to F_2(U)$ are two dual operators. Then there exists a general type-2 fuzzy relation R on U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L and H satisfy the axioms: for all $A, B \in F_2(U)$ and any constant type-2 fuzzy set $\overleftrightarrow{\beta}$,*

(H2) $H(A \cup B) = H(A) \cup H(B)$

(H3) $H(A \cap \overleftrightarrow{\beta}) = H(A) \cap \overleftrightarrow{\beta}$

Proof "⇒" It follows immediately from Theorem 3.5 and Lemma 5.1.

"⇐" Using H, we can define a general type-2 fuzzy relation R on U by $R = \cup_{\alpha\in[0,1]}\alpha/R_\alpha$, where $R_\alpha = \int_{(x,y)\in U\times U}[\int_{u\in[S_L^R((x,y)|\alpha),S_U^R((x,y)|\alpha)]}u]/x$, $S_L^R((x,y)|\alpha) = S_L^{H(1_y)}(x|\alpha)$ and $S_U^R((x,y)|\alpha) = S_U^{H(1_y)}(x|\alpha)$.

For any $\alpha \in [0,1]$ and $x \in U$,

$$S_L^{\overline{f}(A)}(x|\alpha)$$

$$= \bigvee_{y\in U}\left[S_L^R((x,y)|\alpha) \wedge S_L^A(y|\alpha)\right]$$

$$= \bigvee_{y\in U}\left[S_L^{H(1_y)}(x|\alpha) \wedge S_L^A(y|\alpha)\right]$$

$$= \bigvee_{y\in U}\left[S_L^{H(1_y)}(x|\alpha) \wedge S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x|\alpha)\right]$$

$$= \bigvee_{y\in U}\left[S_L^{H(1_y)\cap \overleftrightarrow{S_L^A(y|\alpha)}}(x|\alpha)\right]$$

$$= \bigvee_{y\in U}\left[S_L^{H(1_y\cap \overleftrightarrow{S_L^A(y|\alpha)})}(x|\alpha)\right] \quad \text{(According to (H3))}$$

$$= \left[S_L^{\bigcup_{y\in U} H(1_y\cap \overleftrightarrow{S_L^A(y|\alpha)})}(x|\alpha)\right]$$

$$= \left[S_L^{H(\bigcup_{y\in U}[1_y\cap \overleftrightarrow{S_L^A(y|\alpha)}])}(x|\alpha)\right] \quad \text{(According to (H2))}$$

$$= S_L^{H(A)}(x|\alpha) \quad \text{(According to Lemma 5.2)}$$

Similarly, $S_U^{\overline{f}(A)}(x|\alpha) = S_U^{H(A)}(x|\alpha)$.

Thus, $H(A) = \overline{f}(A)$.

We have $L(A) = \underline{f}(A)$ according to Theorem 3.3. □

Definition 5.2 Let $L, H : F_2(U) \to F_2(U)$ be a pair of dual operators. If L satisfies axioms (L2) and (L3) or equivalently, H satisfies axioms (H2) and (H3), then the system $(F_2(U), \cap, \cup, c, L, H)$ is referred to as a general type-2 fuzzy rough set algebra, and L and H are referred to as general type-2 fuzzy approximation operators.

Theorem 5.3 *Suppose that $L, H : F_2(U) \to F_2(U)$ are a pair of dual general type-2 fuzzy approximation operators, i.e., L satisfies axioms (L1), (L2) and (L3), or H satisfies axioms (H1), (H2) and (H3). Then there exists a reflexive general type-2 fuzzy relation Ron U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L and H satisfy the axioms:*

(L4) $L(A) \subseteq A$

(H4) $A \subseteq H(A)$

Proof "⇒" It follows immediately from Theorem 4.1

"⇐" It follows immediately from Theorems 4.1, 5.1 and 5.2. □

Theorem 5.4 *Suppose that $L, H : F_2(U) \to F_2(U)$ are a pair of dual general type-2 fuzzy approximation operators. Then there exists a symmetric general type-2 fuzzy relation R on U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L and H satisfy the axioms:*

(L5) For any $\alpha \in [0,1]$ and $x, y \in U$, $S_L^{L(1_{U-\{y\}})}(x|\alpha) = S_L^{L(1_{U-\{x\}})}(y|\alpha)$ and $S_U^{L(1_{U-\{y\}})}(x|\alpha) = S_U^{L(1_{U-\{x\}})}(y|\alpha)$ hold.

(H5) For any $\alpha \in [0,1]$ and $x, y \in U$, $S_L^{H(1_x)}(y|\alpha) = S_L^{H(1_y)}(x|\alpha)$ and $S_U^{H(1_x)}(y|\alpha) = S_U^{H(1_y)}(x|\alpha)$ hold.

Proof "⇒" It follows immediately from Theorem 4.2.

"⇐" It follows immediately from Theorems 4.2, 5.1 and 5.2. □

Theorem 5.5 *Suppose that $L, H : F_2(U) \to F_2(U)$ are a pair of dual general type-2 fuzzy approximation operators. Then there exists a transitive general type-2 fuzzy relation R on U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L and H satisfy the axioms:*

(L6) $L(A) \subseteq L(L(A))$

(H6) $H(H(A)) \subseteq H(A)$

Proof "⇒" It follows immediately from Theorem 4.3.

"⇐" It follows immediately from Theorems 4.3, 5.1 and 5.2. □

Theorem 5.6 *Suppose that $L, H : F_2(U) \to F_2(U)$ are a pair of dual general type-2 fuzzy approximation operators.*

Then there exists a similarity general type-2 fuzzy relation R on U such that for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ if and only if L satisfies the axioms (L4)–(L 6) and H satisfies the axioms (H4)–(H6).

Proof "⇒" It follows immediately from Theorems 4.1, 4.2 and 4.3.

"⇐" It follows immediately from Theorems 4.1, 4.2, 4.3, 5.1 and 5.2. □

6 Conclusions

A general framework for the study of general type-2 fuzzy rough sets has been developed by combining rough set theory with general type-2 fuzzy set theory. Since he α-plane representation method of general type-2 fuzzy sets has been extensively studied and can reduce the computational workload, and we therefore study general type-2 fuzzy rough sets by using α-plane representation method in which both constructive and axiomatic approaches are considered. Based on an arbitrary general type-2 fuzzy relation, a pair of upper and lower general type-2 fuzzy rough approximation operators have been derived and the properties of them have been investigated. The connections between special general type-2 fuzzy relations and general type-2 fuzzy rough upper and lower approximation operators are examined. We also provide axiomatics to fully characterize the general type-2 fuzzy rough approximation operators. Further research will concentrate on the applications to data analysis of the proposed general type-2 fuzzy rough set model.

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