Soft Comput (2014) 18:85–94
DOI 10.1007/s00500-013-1035-5

METHODOLOGIES AND APPLICATION

An efficient similarity measure for intuitionistic fuzzy sets

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Published online: 7 April 2013

Abstract We introduce a new methodology for measuring the degree of similarity between two intuitionistic fuzzy sets. The new method is developed on the basis of a distance defined on an interval by the use of convex combination of endpoints and also focusing on the property of *min* and *max* operators. It is shown that among the existing methods, the proposed method meets all the well-known properties of a similarity measure and has no counter-intuitive examples. The validity and applicability of the proposed similarity measure is illustrated with two examples known as pattern recognition and medical diagnosis.

Keywords Intuitionistic fuzzy sets · Similarity measures · Distance measure · Pattern recognition

1 Introduction

In the last few decades, several extensions of fuzzy sets (Zadeh 1965) have been proposed and developed by many researchers. Among various generalizations of fuzzy sets such as L-fuzzy sets (Goguen 1967), interval-valued fuzzy sets (Turksen 1986) and vague sets (Bustince and Burillo 1996), intuitionistic fuzzy sets (IFS) have gained more attention from researchers. This attention is due to the consistency of IFS in modeling many real life situations where hesitation exists, such as fuzzy decision making (Szmidt and Kacprzyk 1996), fuzzy pattern recognition (Pedrycz 1997) and market prediction. As a significant content in fuzzy mathematics, the research on the similarity measure between IFSs has received more attention. Similarity measures have been widely applied in many fields such as multicriteria decision-making (Szmidt and Kacprzyk 2005), group decision (Xu and Chen 2007), grey relational analysis (Wei and Lan 2008), pattern recognition (Li and Cheng 2002), image processing (Pal and King 1981), and cluster analysis (Yao and Dash 2000). Since the similarity measures of IFSs have been applied to many real-world situations, it is naturally required to have an efficient similarity measure with no counter-intuitive examples.

In recent years, some definitions of similarity measures for IFSs have been proposed. Atanassov (1999), Szmidt and Kacprzyk (2000) proposed several methods for measuring the degree of similarity between IFSs based on the well known Hamming distance, Euclidean distance and their normalized counterparts. Based on the extension of the Hausdorff distance and L_p metric, Hung and Yang (2007) proposed some methods to calculate the degree of similarity between IFSs. The methods of Chen, Hong and Kim, Fan and Zhangyan, Yanhong et al., Dengfeng and Chuntian, Mitchell, Zhizhen and Pengfei who put forward the concept of similarity measure for IFSs have been summarized and discussed by Li et al. (2007). Recently, Wang and Xin (2005), Huang et al. (2005), Hung and Yang (2007), and Ye (2011) have established several methods which are described briefly later in Sect. 4.

Later, it will be observed that all the papers discussed above may not work as desired because they cannot meet all or most of the well-known properties of a similarity measure. With this point of view and the need to overcome the shortcomings of the existing methods, we

Communicated by G. Acampora.

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develop a new similarity measure that contains more information of IFSs. Indeed, the proposed similarity measure is defined based on the convex combination of the endpoints of the interval which restricts the membership degree of an IFS. Moreover, to avoid all the cons cases that the existing methods have, the proposed method focuses also on the property of *min* and *max* operators.

The present paper is organized as follows: Background on the intuitionistic fuzzy sets (IFSs) is presented in Sect. 2. In Sect. 3, a new distance is defined on the interval numbers based on convex combination of endpoints and then it is extended to IFSs by deriving a distance measure. Then, the desired similarity measure is established by combining the mentioned distance measure and a mapping defined based on *min* and *max* operators. Section 4 is devoted to review briefly the existing similarity measures for IFSs and they are compared with the desired method. In Sect. 5, two examples known as pattern recognition and medical diagnosis are brought to illustrate the validity and applicability of the new method. Conclusion is drown in Sect. 6 Finally, the detailed discussions of the main results which are stated in the last part of Sect. 3 can be found in "Appendix".

2 Preliminaries

In this section, we briefly describe the basic definitions and notions of IFSs and similarity measure for IFSs. Throughout this article, we use $X = \{x_1, x_2,..., x_n\}$ to denote the discourse set.

Definition 1 (See Zadeh 1965) A fuzzy set (FS) A_{FS} in X is defined as $A_{FS} = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ is the membership function of the fuzzy set A_{FS} and $\mu_A(x)$ is the degree of membership of $x \in X$ in A_{FS}.

Definition 2 (See Atanassov 1999) An intuitionistic fuzzy set (IFS) A_{IFS} in X is defined as $A_{IFS} = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ are such that $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$. The number $\mu_A(x)$ and $\nu_A(x)$ represent respectively the degree of membership and nonmembership of x in A_{IFS}.

We denote all the IFSs in X by $IFS(X)$.

Definition 3 (See Atanassov 1999) The complement of A_{IFS}, denoted by A^c_{IFS}, is defined as $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$.

Remark 1 (See e.g. Li et al. 2007) From the above definitions, it is obvious that the membership degree of any A_{IFS} has been restricted by the interval $[\mu_A(x), 1 - \nu_A(x)]$, where $[\mu_A(x), 1 - \nu_A(x)] \subseteq [0, 1]$. If $\mu_A(x) = 1 - \nu_A(x)$, this implies that we know x precisely. In this case A_{IFS} degenerates into a fuzzy set. If $\mu_A(x) = 0$ and $\nu_A(x) = 1$ (or $\mu_A(x) = 1$ and $\nu_A(x) = 0$), A_{IFS} degenerates into a crisp set.

Definition 4 (See Atanassov 1999) Let $A_{IFS}, B_{IFS} \in IFS(X)$. We define $A_{IFS} \subseteq B_{IFS}$ if and only if $\mu_A(x) \le \mu_B(x)$ and $1 - \nu_A(x) \le 1 - \nu_B(x)$, for each $x \in X$.

The main properties of a similarity measure on IFSs,

$$S : IFS(X) \times IFS(X) \to [0, 1]$$

sometimes considered as axiomatic requirements, are as the following: (see e.g. Li et al. 2007)

(P1) $S(A_{IFS}, B_{IFS}) \in [0, 1]$;
(P2) $S(A_{IFS},B_{IFS}) = 1$ if and only if $A_{IFS} = B_{IFS}$;
(P3) $S(A_{IFS}, B_{IFS}) = S(B_{IFS}, A_{IFS})$;
(P4) $S(A_{IFS}, C_{IFS}) \le S(A_{IFS}, B_{IFS})$ and $S(A_{IFS}, C_{IFS}) \le S(B_{IFS}, C_{IFS})$ if $A_{IFS} \subseteq B_{IFS} \subseteq C_{IFS}$;
(P5) $S(A_{IFS}, A^c_{IFS}) = 0$ if A_{IFS} is a crisp set.

3 New similarity measure for IFSs

In this portion, we describe the construction of a new similarity measure for IFSs using the convex combination of the endpoints of the interval which restricts the membership degree of an IFS.

Let us consider the interval value $[\mu_A(x_i), 1 - \nu_A(x_i)]$ of $A_{IFS} \in IFS(X)$. We define for any $x_i \in X$,

$$\chi_j(A_{IFS}(x_i)) = \left(1 - \frac{j}{m}\right)\mu_A(x_i) + \frac{j}{m}(1 - \nu_A(x_i)), \quad j = 0, 1, ..., m, \tag{1}$$

where the convex combination of lower and upper bound values of the membership degree of $A_{IFS}(x_i)$ indicates that $\chi_j(A_{IFS}(x_i))$ stands for any point (if $m \to \infty$) in the interval $[\mu_A(x_i), 1 - \nu_A(x_i)]$.

Taking into account the above formulae, a distance between two IFSs $A_{IFS}, B_{IFS} \in IFS(X)$ is defined by the following expression

$$d_{IFS}(A_{IFS}, B_{IFS}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{m+1}\sum_{j=0}^{m}[\chi_j(A_{IFS}(x_i)) - \chi_j(B_{IFS}(x_i))]^2\right)}. \tag{2}$$

Theorem 1 *Let $X = \{x_1, x_2,...,x_n\}$ denote the universe of discourse. Then the mapping $d_{IFS} : IFS(X) \times IFS(X) \to R^+ \cup \{0\}$ given by (2) is metric, that is, for any IFSs A_{IFS}, B_{IFS} and C_{IFS}, it holds*

(i) $d_{IFS}(A_{IFS}, B_{IFS}) \in [0, 1]$;
(ii) $d_{IFS}(A_{IFS}, B_{IFS}) = d_{IFS}(B_{IFS}, A_{IFS})$;

(iii) $d_{IFS}(A_{IFS},B_{IFS}) = 0$ *if and only if* $A_{IFS} = B_{IFS}$;
(iv) $d_{IFS}(A_{IFS}, C_{IFS}) \le d_{IFS}(A_{IFS}, B_{IFS}) + d_{IFS}(B_{IFS}, C_{IFS})$.

Proof Proof of properties (i), (ii) and (iv) is obvious.

To prove the property (iii), we proceed as follows.

Without loss of the generality, we suppose that $X = \{x_1 = x\}$. By the definition (2) of the distance measure d_{IFS}, it can be seen that

$$d_{IFS}(A_{IFS}, B_{IFS}) = 0,$$
$$\textit{iff} \quad \chi_j(A_{IFS}(x)) = \chi_j(B_{IFS}(x)), \quad j = 0, 1, \ldots, m,$$
$$\textit{iff} \quad \left(1 - \frac{j}{m}\right)\mu_A(x) + \frac{j}{m}(1 - \nu_A(x)) = \left(1 - \frac{j}{m}\right)\mu_B(x) + \frac{j}{m}(1 - \nu_B(x)), \quad j = 0, 1, \ldots, m.$$

For the index j there are two cases to consider: (i) $j = 0$ and (ii) $j = m$. In the former case we obtain $\mu_A(x) = \mu_B(x)$, and in the latter case we get $1 - \nu_A(x) = 1 - \nu_B(x)$. This implies that $A_{IFS} = \{\langle x, \mu_A(x), \nu_A(x)\rangle\} = \{\langle x, \mu_B(x), \nu_B(x)\rangle\} = B_{IFS}$. □

Now, we are in a position to introduce a new similarity measure between IFSs.

Theorem 2 *Let* H:[0, 1]→ [0, 1] *be a strictly monotone decreasing real function, and* d_{IFS} *be the distance of IFSs, given by (2). Then for any* A_{IFS}, $B_{IFS} \in IFS(X)$

$$S^d_{IFS}(A_{IFS}, B_{IFS}) = \frac{H(d_{IFS}(A_{IFS}, B_{IFS})) - H(1)}{H(0) - H(1)}, \tag{3}$$

is a similarity measure of IFSs A_{IFS} *and* B_{IFS}.

Proof We need to show that S^d_{IFS} satisfies the properties **(P1)–(P5)**.

Without loss of the generality, suppose that $X = \{x_1 = x\}$.

Proof of properties **(P1)** and **(P3)** is obvious.

To prove the property **(P2)**, we obtain from the definition of S^d_{IFS} that

$$S^d_{IFS}(A_{IFS}, B_{IFS}) = 1, \quad \textit{iff} \quad d_{IFS}(A_{IFS}, B_{IFS}) = 0.$$

Now, from the property (iii) in Theorem 1 and the latter term, it results that

$$S^d_{IFS}(A_{IFS}, B_{IFS}) = 1, \quad \textit{iff} \quad A_{IFS} = B_{IFS}. \qquad \text{[Proved]}$$

The proof of **(P4)** is given as follows. If $A_{IFS} \subseteq B_{IFS} \subseteq C_{IFS}$, then from Definition 4 we obtain

$$\mu_A(x) \le \mu_B(x) \le \mu_C(x),$$
$$1 - \nu_A(x) \le 1 - \nu_B(x) \le 1 - \nu_C(x),$$

and so for any $j = 0, 1, \ldots, m$,

$$\chi_j(A_{IFS}(x)) \le \chi_j(B_{IFS}(x)) \le \chi_j(C_{IFS}(x)).$$

We see immediately that for any $j = 0, 1, \ldots, m$,

$$[\chi_j(A_{IFS}(x)) - \chi_j(B_{IFS}(x))]^2 \le [\chi_j(A_{IFS}(x)) - \chi_j(C_{IFS}(x))]^2,$$
$$[\chi_j(B_{IFS}(x)) - \chi_j(C_{IFS}(x))]^2 \le [\chi_j(A_{IFS}(x)) - \chi_j(C_{IFS}(x))]^2,$$

which result in

$$d_{IFS}(A_{IFS}, B_{IFS}) \le d_{IFS}(A_{IFS}, C_{IFS}),$$
$$d_{IFS}(B_{IFS}, C_{IFS}) \le d_{IFS}(A_{IFS}, C_{IFS}).$$

Since the function $H(.)$ is strictly monotone decreasing, it is easily verified that

$$S^d_{IFS}(A_{IFS}, C_{IFS}) \le S^d_{IFS}(A_{IFS}, B_{IFS}),$$
$$S^d_{IFS}(A_{IFS}, C_{IFS}) \le S^d_{IFS}(B_{IFS}, C_{IFS}). \qquad \text{[Proved]}$$

In order to prove the property **(P5)**, we assume that A_{IFS} is a crisp set, that is, $A_{IFS} = \{\langle x, 1, 0\rangle\}$ (or $A_{IFS} = \{\langle x, 0, 1\rangle\}$). In this regard, the complement set A^c_{IFS} is defined as $A^c_{IFS} = \{\langle x, 0, 1\rangle\}$ (or $A^c_{IFS} = \{\langle x, 1, 0\rangle\}$). Hence, $d_{IFS}(A_{IFS}, A^c_{IFS}) = 1$ which implies that $S^d_{IFS}(A_{IFS}, A^c_{IFS}) = 0$. □

Remark 2 Due to Theorem 2, one can develop different formulas to calculate the similarity measures between IFSs by choosing different strictly monotone decreasing real function $H : [0, 1] \to [0, 1]$, for instance, $H(x) = 1 - x$, $H(x) = e^{-x}$, $H(x) = \frac{1}{1+x}$, and $H(x) = 1 - x^2$.

Hereafter, we consider $H : [0, 1] \to [0, 1]$ given by $H(x) = 1 - x$. Hence, the corresponding similarity measure of IFSs A_{IFS} and B_{IFS} is defined as follows:

$$S^d_{IFS}(A_{IFS}, B_{IFS}) = 1 - d_{IFS}(A_{IFS}, B_{IFS}) = 1 - \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{m+1}\sum_{j=0}^{m}[\chi_j(A_{IFS}(x_i)) - \chi_j(B_{IFS}(x_i))]^2\right)}. \tag{4}$$

Now, we are interested here to introduce a mapping on $IFS(X) \times IFS(X)$ into [0, 1] which satisfies all the properties **(P1)–(P5)**.

Theorem 3 *Let* $A_{IFS}, B_{IFS} \in IFS(X)$. *The mapping* $S^{mix}_{IFS} : IFS(X) \times IFS(X) \to [0, 1]$, *given by,*

$$S^{mix}_{IFS}(A_{IFS}, B_{IFS}) = \frac{\sum_{i=1}^{n}(\min\{\mu_A(x_i), \mu_B(x_i)\} + \min\{1 - \nu_A(x_i), 1 - \nu_B(x_i)\})}{\sum_{i=1}^{n}(\max\{\mu_A(x_i), \mu_B(x_i)\} + \max\{1 - \nu_A(x_i), 1 - \nu_B(x_i)\})}, \tag{5}$$

satisfies the properties **(P1)–(P5)** *for any IFSs* A_{IFS} *and* B_{IFS}.

Proof Without loss of the generality, suppose that $X = \{x_1 = x\}$. Proof of properties **(P1)** and **(P3)** is obvious.

To prove the property **(P2)**, we obtain from the definition of S_{IFS}^{mix} that $S_{IFS}^{mix}(A_{IFS},B_{IFS}) = 1$ if and only if $\frac{\min\{\mu_A(x),\mu_B(x)\}+\min\{1-\nu_A(x),1-\nu_B(x)\}}{\max\{\mu_A(x),\mu_B(x)\}+\max\{1-\nu_A(x),1-\nu_B(x)\}} = 1$, if and only if $\min\{\mu_A(x), \mu_B(x)\} = \max\{\mu_A(x), \mu_B(x)\}$ and $\min\{1-\nu_A(x), 1-\nu_B(x)\} = \max\{1-\nu_A(x), 1-\nu_B(x)\}$. This implies that $A_{IFS} = \{\langle x, \mu_A(x), \nu_A(x)\rangle\} = \{\langle x, \mu_B(x), \nu_B(x)\rangle\} = B_{IFS}$ and hence

$$S_{IFS}^{mix}(A_{IFS}, B_{IFS}) = 1, \quad \textit{iff} \quad A_{IFS} = B_{IFS}. \qquad \text{[Proved]}$$

The proof of **(P4)** is given as follows. If $A_{IFS} \subseteq B_{IFS} \subseteq C_{IFS}$, then from Definition 4 the following results can be deduced.

$$\mu_A(x) \le \mu_B(x) \le \mu_C(x), \tag{6}$$

$$1-\nu_A(x) \le 1-\nu_B(x) \le 1-\nu_C(x). \tag{7}$$

The monotonicity conditions of (6) and (7) ensure that

$$\begin{aligned}&\mu_A(x) = \min\{\mu_A(x), \mu_C(x)\},\\ &\quad 1-\nu_A(x) = \min\{1-\nu_A(x), 1-\nu_C(x)\},\end{aligned} \tag{8}$$

$$\begin{aligned}&\mu_C(x) = \max\{\mu_A(x), \mu_C(x)\},\\ &\quad 1-\nu_C(x) = \max\{1-\nu_A(x), 1-\nu_C(x)\},\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\mu_A(x) = \min\{\mu_A(x), \mu_B(x)\},\\ &\quad 1-\nu_A(x) = \min\{1-\nu_A(x), 1-\nu_B(x)\},\end{aligned} \tag{10}$$

$$\begin{aligned}&\mu_B(x) = \max\{\mu_A(x), \mu_B(x)\},\\ &\quad 1-\nu_B(x) = \max\{1-\nu_A(x), 1-\nu_B(x)\}.\end{aligned} \tag{11}$$

Once again from (6), we have

$$\frac{\mu_A(x)+(1-\nu_A(x))}{\mu_C(x)+(1-\nu_C(x))} \le \frac{\mu_A(x)+(1-\nu_A(x))}{\mu_B(x)+(1-\nu_B(x))}.$$

By the use of (8)–(11) together with the latter relation, the following result is immediate

$$\frac{\min\{\mu_A(x),\mu_C(x)\}+\min\{1-\nu_A(x),1-\nu_C(x)\}}{\max\{\mu_A(x),\mu_C(x)\}+\max\{1-\nu_A(x),1-\nu_C(x)\}} \le \frac{\min\{\mu_A(x),\mu_B(x)\}+\min\{1-\nu_A(x),1-\nu_B(x)\}}{\max\{\mu_A(x),\mu_B(x)\}+\max\{1-\nu_A(x),1-\nu_B(x)\}},$$

that is,

$$S_{IFS}^{mix}(A_{IFS}, C_{IFS}) \le S_{IFS}^{mix}(A_{IFS}, B_{IFS}).$$

By a similar reasoning and with the help of

$$\frac{\mu_A(x)+(1-\nu_A(x))}{\mu_C(x)+(1-\nu_C(x))} \le \frac{\mu_B(x)+(1-\nu_B(x))}{\mu_C(x)+(1-\nu_C(x))},$$

which is ensured by the validity of (6), we find that

$$\frac{\min\{\mu_A(x),\mu_C(x)\}+\min\{1-\nu_A(x),1-\nu_C(x)\}}{\max\{\mu_A(x),\mu_C(x)\}+\max\{1-\nu_A(x),1-\nu_C(x)\}} \le \frac{\min\{\mu_B(x),\mu_C(x)\}+\min\{1-\nu_B(x),1-\nu_C(x)\}}{\max\{\mu_B(x),\mu_C(x)\}+\max\{1-\nu_B(x),1-\nu_C(x)\}},$$

that is,

$$S_{IFS}^{mix}(A_{IFS}, C_{IFS}) \le S_{IFS}^{mix}(B_{IFS}, C_{IFS}). \qquad \text{[Proved]}$$

In order to prove the property **(P5)**, we assume that A_{IFS} is a crisp set, that is, $A_{IFS} = \{\langle x, 1, 0\rangle\}$ (or $A_{IFS} = \{\langle x, 0, 1\rangle\}$). In this regard, the complement set A_{IFS}^c is defined as $A_{IFS}^c = \{\langle x, 0, 1\rangle\}$ (or $A_{IFS}^c = \{\langle x, 1, 0\rangle\}$). Hence, $S_{IFS}^{mix}(A_{IFS}, A_{IFS}^c) = \frac{\min\{\mu_A(x),\mu_{A^c}(x)\}+\min\{1-\nu_A(x),1-\nu_{A^c}(x)\}}{\max\{\mu_A(x),\mu_{A^c}(x)\}+\max\{1-\nu_A(x),1-\nu_{A^c}(x)\}} = \frac{\min\{1,0\}+\min\{1,0\}}{\max\{1,0\}+\max\{1,0\}} = 0.$ □

It should be noted that we do not claim here the mapping S_{IFS}^{mix} is a proper similarity measure for IFSs, especially from a point of view of decision making.

Although, in some cases we do not make any difference among quite different situations, for example, for $A_{IFS} = \{\langle x, 1, 0\rangle\}$ and quite different IFSs $B_{IFS} = \{\langle x, 0.4, 0.4\rangle\}$ and $C_{IFS} = \{\langle x, 0.3, 0.3\rangle\}$ we get $S_{IFS}^{mix}(A_{IFS}, B_{IFS}) = S_{IFS}^{mix}(A_{IFS}, C_{IFS}) = 0.5$, but making use of the mapping S_{IFS}^{mix} as a part of the following similarity measure S_{IFS} enhances the distinguishability of the new similarity measure S_{IFS}.

Taking into account the mappings S_{IFS}^{d} and S_{IFS}^{mix} given by (4) and (5), respectively, we can obtain a similarity measure on IFSs as follows.

Theorem 5 *Let $A_{IFS}, B_{IFS} \in IFS(X)$. If $S_{IFS}^{d}: IFS(X) \times IFS(X) \to [0,1]$ and $S_{IFS}^{mix}: IFS(X) \times IFS(X) \to [0,1]$ are the mappings given by (4) and (5), respectively. Then, the mapping $S_{IFS}: IFS(X) \times IFS(X) \to [0,1]$ given by*

$$S_{IFS}(A_{IFS}, B_{IFS}) = \frac{1}{2}\left(S_{IFS}^{d}(A_{IFS}, B_{IFS}) + S_{IFS}^{mix}(A_{IFS}, B_{IFS})\right), \tag{12}$$

is a similarity measure of IFSs A_{IFS} and B_{IFS}.

Proof We are required to show that S_{IFS} satisfies the properties **(P1)–(P5)**. Due to Theorem 2 and Theorem 3, it follows that S_{IFS}^{d} and S_{IFS}^{mix} satisfy the properties **(P1)–(P5)** and therefore we get immediately the properties **(P1)–(P5)** are fulfilled for S_{IFS}, too. □

Here, we would like to give the reasons why S_{IFS} should be considered?

1. The formulae of the proposed similarity measure given by (12) requires no complicated computation.
2. By the fact that "the more the information that the similarity measure focuses on, the more powerful its distinguishability", we adopted the convex combination of the endpoints of the interval $[\mu_{A_{IFS}}, 1-\nu_{A_{IFS}}]$ as follows

$$\chi_j(A_{IFS}(x_i)) = \left(1-\frac{j}{m}\right)\mu_A(x_i) + \frac{j}{m}(1-\nu_A(x_i)), \quad j = 0, 1, \ldots, m,$$

to define S_{IFS}^{d} as a term of S_{IFS}. From mathematical point of view, the larger the value of the parameter m, the more precise the degree of similarity of IFSs.

This follows from the fact that S_{IFS} is a monotonically increasing function as the parameter m increases (See Theorem 7 in "Appendix").

3. The existence of some cons cases might be resulted from the circumstance that the membership-degree interval [μ, 1 − ν] has equal endpoints. To avoid such cons cases, we make use of S_{IFS}^{mix} given by (5) to enhance the distinguishability of S_{IFS}.
4. From the comparison, one can easily observe that S_{IFS} has no counter-intuitive cases, specially for sufficiently large value of m, that the existing similarity measures have. Furthermore, S_{IFS} satisfies all the well-known properties **(P1)–(P5)**.

4 Comparisons of similarity measures on IFSs

Here, to illustrate and compare our proposed similarity measure S_{IFS} with the existing similarity measures, we recall all the methods analyzed by Li et al. (2007), and the other methods suggested by Wang and Xin (2005), Huang et al. (2005), Hung and Yang (2007), and Ye (2011).

Consider two IFSs $A_{IFS}, B_{IFS} \in IFS(X)$, where $X = \{x_1, x_2,\ldots,x_n\}$. The similarity measures analyzed by Li et al. (2007) are briefly described as follows:

- Chen's measure

$$S_C(A_{IFS}, B_{IFS}) = 1 - \frac{\sum_{i=1}^{n} |S_A(x_i) - S_B(x_i)|}{2n}, \tag{13}$$

where $S_A(x_i) = \mu_A(x_i) - \nu_A(x_i)$ and $S_B(x_i) = \mu_B(x_i) - \nu_B(x_i)$.

- Hong and Kim's measure

$$S_H(A_{IFS}, B_{IFS}) = 1 - \frac{\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)}{2n}. \tag{14}$$

- Fan and Zhangyan's measure

$$S_L(A_{IFS}, B_{IFS}) = 1 - \frac{\sum_{i=1}^{n} |S_A(x_i) - S_B(x_i)|}{4n} - \frac{\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)}{4n}. \tag{15}$$

- Yanhong et al. measure

$$S_O(A_{IFS}, B_{IFS}) = 1 - \sqrt{\frac{\sum_{i=1}^{n} ((\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)}{2n}}. \tag{16}$$

- Dengfeng and Chuntian's measure

$$S_{DC}(A_{IFS}, B_{IFS}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} |\psi_A(x_i) - \psi_B(x_i)|^p}{n}}, \tag{17}$$

where $\psi_A(x_i) = \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}$ and $\psi_B(x_i) = \frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}$.

- Mitchell's measure

$$S_{HB}(A_{IFS}, B_{IFS}) = \frac{1}{2}(\rho_\mu(A_{IFS}, B_{IFS}) + \rho_\nu(A_{IFS}, B_{IFS})), \tag{18}$$

where $\rho_\mu(A_{IFS}, B_{IFS}) = S_{DC}(\mu_A(x_i), \mu_B(x_i))$ and $\rho_\nu(A_{IFS}, B_{IFS}) = S_{DC}(1 - \nu_A(x_i), 1 - \nu_B(x_i))$.

- Zhizhen and Pengfei's measures

$$S_e^p(A_{IFS}, B_{IFS}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\phi_\mu(x_i) + \phi_\nu(x_i))^p}{n}}, \tag{19}$$

where $\phi_\mu(x_i) = \frac{|\mu_A(x_i) - \mu_B(x_i)|}{2}$ and $\phi_\nu(x_i) = \frac{|(1-\nu_A(x_i)) - (1-\nu_B(x_i))|}{2}$.

$$S_s^p(A_{IFS}, B_{IFS}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\varphi_{s1}(x_i) + \varphi_{s2}(x_i))^p}{n}}, \tag{20}$$

where $\varphi_{s1}(x_i) = \frac{|m_{A1}(x_i) - m_{B1}(x_i)|}{2}$, $\varphi_{s2}(x_i) = \frac{|m_{A2}(x_i) - m_{B2}(x_i)|}{2}$, $m_{A1}(x_i) = \frac{(\mu_A(x_i) + m_A(x_i))}{2}$, $m_{B1}(x_i) = \frac{(\mu_B(x_i) + m_B(x_i))}{2}$, $m_{A2}(x_i) = \frac{(1-\nu_A(x_i) + m_A(x_i))}{2}$, $m_{B2}(x_i) = \frac{(1-\nu_B(x_i) + m_B(x_i))}{2}$, $m_A(x_i) = \frac{(1-\nu_A(x_i) + \mu_A(x_i))}{2}$, $m_B(x_i) = \frac{(1-\nu_B(x_i) + \mu_B(x_i))}{2}$.

$$S_h^p(A_{IFS}, B_{IFS}) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\eta_1(i) + \eta_2(i) + \eta_3(i))^p}{3n}}, \tag{21}$$

where, $\eta_1(i) = \phi_\mu(x_i) + \phi_\nu(x_i)$, (see S_e^p), or $\eta_1(i) = \varphi_{s1}(x_i) + \varphi_{s2}(x_i)$, (see S_s^p), $\eta_2(i) = |\psi_A(x_i) - \psi_B(x_i)|$, (see S_{DC}), $\eta_3(i) = \max\{l_A(i), l_B(i)\} - \min\{l_A(i), l_B(i)\}$, where, $l_A(i) = \frac{(1-\nu_A(x_i) - \mu_A(x_i))}{2}$, $l_B(i) = \frac{(1-\nu_B(x_i) - \mu_B(x_i))}{2}$.

- Hung and Yang's measures

$$S_{HY}^1(A_{IFS}, B_{IFS}) = 1 - d_H(A_{IFS}, B_{IFS}), \tag{22}$$

$$S_{HY}^2(A_{IFS}, B_{IFS}) = \frac{e^{-d_H(A_{IFS}, B_{IFS})} - e^{-1}}{1 - e^{-1}}, \tag{23}$$

$$S_{HY}^3(A_{IFS}, B_{IFS}) = \frac{1 - d_H(A_{IFS}, B_{IFS})}{1 + d_H(A_{IFS}, B_{IFS})}, \tag{24}$$

where $d_H(A_{IFS}, B_{IFS}) = \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\}$.

The other existing similarity measures are described by the following forms:

- Wang and Xin's measure (2005)

$$S_{WX}(A_{IFS},B_{IFS}) = 1-\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|}{4}\right.$$
$$\left.+\frac{\max\{|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|\}}{2}\right\}, \quad (25)$$

- Huang et al. measures (2005)

$$S_{H1}(A_{IFS},B_{IFS}) = 1-\frac{1}{n}\frac{\sum_{i=1}^{n}2(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)}{2+(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)}, \quad (26)$$

$$S_{H2}(A_{IFS},B_{IFS}) = 1-\frac{2\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)}{2n+\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)}, \quad (27)$$

- Hung and Yang's measure (2007)

$$S^{p}_{HY}(A_{IFS},B_{IFS}) = 1-\frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|^{p}+|\nu_A(x_i)-\nu_B(x_i)|^{p})^{\frac{1}{p}}, \quad (28)$$

- Ye's measure (2011)

$$S^{C}_{Y}(A_{IFS},B_{IFS}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i)+\nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i)+\nu_A^2(x_i)}\sqrt{\mu_B^2(x_i)+\nu_B^2(x_i)}}. \quad (29)$$

With the help of some counter-intuitive examples, we will show that the similarity measures mentioned above are not fit so well (see Tables 1, 2). Meanwhile, our proposed method bears no such drawbacks. This superiority can be easily found from the last rows of Table 2. Keeping in mind that each similarity measure may have different counter-intuitive examples, but it suffices to give one example for each formula in Table 1.

Table 1 Similarity measures and their counter-intuitive example

Counter-intuitive example	Similarity measure
Example 1	Eq. (13): $S_c(A_{IFS},B_{IFS})=1$ (unreasonable)
$A_{IFS}=\{\langle x,0,0\rangle\}, B_{IFS}=\{\langle x,0.5,0.5\rangle\}$	
Example 2	Eq. (14): $S_H(A_{IFS}, B_{IFS}) = S_H(C_{IFS}, D_{IFS})$ (unreasonable)
$A_{IFS}=\{\langle x,0.3,0.3\rangle\}, B_{IFS}=\{\langle x,0.4,0.4\rangle\}$	
$C_{IFS}=\{\langle x,0.3,0.4\rangle\}, D_{IFS}=\{\langle x,0.4,0.3\rangle\}$	
Example 3	Eq. (14): $S_H(A_{IFS}, B_{IFS}) = S_H(C_{IFS}, B_{IFS})$ (unreasonable)
$A_{IFS}=\{\langle x,1,0\rangle\}, B_{IFS}=\{\langle x,0,0\rangle\}$	
$C_{IFS}=\{\langle x,0.5,0.5\rangle\}$	
Example 4	Eq. (15): $S_L(A_{IFS}, B_{IFS}) = S_L(A_{IFS}, C_{IFS})$ (unreasonable)
$A_{IFS}=\{\langle x,0.4,0.2\rangle\}, B_{IFS}=\{\langle x,0.5,0.3\rangle\}$	
$C_{IFS}=\{\langle x,0.5,0.2\rangle\}$	
Same as Example 2	Eq. (16): $S_O(A_{IFS}, B_{IFS}) = S_O(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 1	Eq. (17): $S_{DC}(A_{IFS}, B_{IFS}) = 1$ (unreasonable)
Same as Example 2	Eq. (18): $S_{HB}(A_{IFS}, B_{IFS}) = S_{HB}(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 2	Eq. (19): $S^p_e(A_{IFS}, B_{IFS}) = S^p_e(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 4	Eq. (20): $S^p_s(A_{IFS}, B_{IFS}) = S^p_s(A_{IFS}, C_{IFS})$ (unreasonable)
Example 5	Eq. (21): $S^p_h(A_{IFS}, B_{IFS}) = S^p_h(A_{IFS}, C_{IFS})$ (unreasonable)
$A_{IFS}=\{\langle x,0.3,0.7\rangle\}, B_{IFS}=\{\langle x,0.4,0.6\rangle\}$	
$C_{IFS}=\{\langle x,0.2,0.8\rangle\}$	
Same as Example 4	Eq. (22): $S^1_{HY}(A_{IFS}, B_{IFS}) = S^1_{HY}(A_{IFS}, C_{IFS})$ (unreasonable)
Same as Example 4	Eq. (23): $S^2_{HY}(A_{IFS}, B_{IFS}) = S^2_{HY}(A_{IFS}, C_{IFS})$ (unreasonable)
Same as Example 4	Eq. (24): $S^3_{HY}(A_{IFS}, B_{IFS}) = S^3_{HY}(A_{IFS}, C_{IFS})$ (unreasonable)
Same as Example 2	Eq. (25): $S_{WX}(A_{IFS}, B_{IFS}) = S_{WX}(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 2	Eq. (26): $S_{H1}(A_{IFS}, B_{IFS}) = S_{H1}(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 2	Eq. (27): $S_{H2}(A_{IFS}, B_{IFS}) = S_{H2}(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 2	Eq. (28): $S^p_{HY}(A_{IFS}, B_{IFS}) = S^p_{HY}(C_{IFS}, D_{IFS})$ (unreasonable)
Same as Example 3	Eq. (29): $S^C_Y(A_{IFS}, B_{IFS}) = S^C_Y(C_{IFS}, B_{IFS})$ (unreasonable)

Table 2 The demonstration table of counter-intuitive examples being visible in bold

	1	2	3	4	5	6
$A_{IFS} = \{\langle x, \mu_A, \nu_A \rangle\}$	$\{\langle x, 0.3, 0.3 \rangle\}$	$\{\langle x, 0.3, 0.4 \rangle\}$	$\{\langle x, 1, 0 \rangle\}$	$\{\langle x, 0.5, 0.5 \rangle\}$	$\{\langle x, 0.4, 0.2 \rangle\}$	$\{\langle x, 0.4, 0.2 \rangle\}$
$B_{IFS} = \{\langle x, \mu_B, \nu_B \rangle\}$	$\{\langle x, 0.4, 0.4 \rangle\}$	$\{\langle x, 0.4, 0.3 \rangle\}$	$\{\langle x, 0, 0 \rangle\}$	$\{\langle x, 0, 0 \rangle\}$	$\{\langle x, 0.5, 0.3 \rangle\}$	$\{\langle x, 0.5, 0.2 \rangle\}$
S_C	**1**	0.9	0.5	**1**	**1**	0.95
S_H	**0.9**	**0.9**	**0.5**	**0.5**	**0.9**	0.95
S_L	**0.95**	0.9	0.5	0.75	**0.95**	**0.95**
S_O	**0.9**	**0.9**	0.3	0.5	**0.9**	0.93
S_{DC}	**1**	0.9	0.5	**1**	**1**	0.95
S_{HB}	**0.9**	**0.9**	**0.5**	**0.5**	**0.9**	0.95
S_e^p	**0.9**	**0.9**	**0.5**	**0.5**	0.9	0.95
S_s^p	**0.95**	0.9	0.5	0.75	**0.95**	**0.95**
S_h^p	**0.93**	0.933	0.5	0.67	**0.93**	0.95
S_{HY}^1	**0.9**	**0.9**	**0**	0.5	**0.9**	**0.9**
S_{HY}^2	**0.85**	**0.85**	**0**	0.38	**0.85**	**0.85**
S_{HY}^3	**0.82**	**0.82**	**0**	0.33	**0.82**	**0.82**
S_{WX}	**0.9**	**0.9**	0.25	0.5	**0.9**	0.95
S_{H1}	**0.8182**	**0.8182**	**0.3333**	**0.3333**	**0.8182**	0.9048
S_{H2}	**0.8182**	**0.8182**	**0.3333**	**0.3333**	**0.8182**	0.9048
S_{HY}^p	**0.8586**	**0.8586**	**0**	0.2929	0.8586	0.9
S_Y^C	**1**	0.96	**0**	**0**	0.9971	0.9965
S_{IFS}, $m = 6$	0.8758	0.8591	0.4495	0.5000	0.8897	0.9315
S_{IFS}, $m = 7$	0.8764	0.8591	0.4512	0.5030	0.8903	0.9317
S_{IFS}, $m = 8$	0.8768	0.8591	0.4524	0.5053	0.8908	0.9318

5 Applications of S_{IFS} in practice

In this section, to illustrate the efficiency of the proposed similarity measure and to compare its results with that of some methods, we apply S_{IFS} with $m = 5$ to two examples borrowed from Liu (2005), Vlachos and Sergiadis (2007) and Ye (2011).

Example 1 (Pattern recognition) Let $X = \{x_1, x_2, x_3\}$. Consider three known patterns C_1, C_2 and C_3 which are represented by the following IFSs, respectively,

$$C_1 = \{\langle x_1, 1.0, 0.0 \rangle, \langle x_2, 0.8, 0.0 \rangle, \langle x_3, 0.7, 0.1 \rangle\},$$

$$C_2 = \{\langle x_1, 0.8, 0.1 \rangle, \langle x_2, 1.0, 0.0 \rangle, \langle x_3, 0.9, 0.0 \rangle\},$$

$$C_3 = \{\langle x_1, 0.6, 0.2 \rangle, \langle x_2, 0.8, 0.0 \rangle, \langle x_3, 1.0, 0.0 \rangle\}.$$

The aim here is to classify an unknown pattern

$$Q = \{\langle x_1, 0.5, 0.3 \rangle, \langle x_2, 0.6, 0.2 \rangle, \langle x_3, 0.8, 0.1 \rangle\},$$

in one of the above-mentioned classes C_1, C_2 and C_3.

In order to proceed, we use the criteria

$$\max_{1 \le i \le 3} \{S_{IFS}(C_i, Q)\},$$

where $S_{IFS}(C_1, Q) = 0.7708$, $S_{IFS}(C_2, Q) = 0.7739$, $S_{IFS}(C_3, Q) = \mathbf{0.8378}$, give rise to that the pattern Q should be classified in C_3. This result is exactly matching with that obtained in (Liu 2005) and (Ye 2011).

Example 2 (Medical diagnosis) Suppose that the universe of discourse is to be a set of symptoms $X = \{x_1$ (*Temperature*), x_2 (*Headache*), x_3 (*Stomach pain*), x_4 (*Cough*), x_5 (*Chest pian*)$\}$. Consider a set of diagnosis $Q = \{Q_1$(*Viral fever*), Q_2(*Malaria*), Q_3(*Typhoid*), Q_4(*Stomach problem*), Q_5(*Chest problem*)$\}$ whose elements are represented by the following IFSs, respectively,

$$Q_1 = \{\langle x_1, 0.4, 0.0 \rangle, \langle x_2, 0.3, 0.5 \rangle, \langle x_3, 0.1, 0.7 \rangle, \langle x_4, 0.4, 0.3 \rangle, \langle x_5, 0.1, 0.7 \rangle\},$$

$$Q_2 = \{\langle x_1, 0.7, 0.0 \rangle, \langle x_2, 0.2, 0.6 \rangle, \langle x_3, 0.0, 0.9 \rangle, \langle x_4, 0.7, 0.0 \rangle, \langle x_5, 0.1, 0.8 \rangle\},$$

$$Q_3 = \{\langle x_1, 0.3, 0.3 \rangle, \langle x_2, 0.6, 0.1 \rangle, \langle x_3, 0.2, 0.7 \rangle, \langle x_4, 0.2, 0.6 \rangle, \langle x_5, 0.1, 0.9 \rangle\},$$

$$Q_4 = \{\langle x_1, 0.1, 0.7 \rangle, \langle x_2, 0.2, 0.4 \rangle, \langle x_3, 0.8, 0.0 \rangle, \langle x_4, 0.2, 0.7 \rangle, \langle x_5, 0.2, 0.7 \rangle\},$$

$$Q_5 = \{\langle x_1, 0.1, 0.8 \rangle, \langle x_2, 0.0, 0.8 \rangle, \langle x_3, 0.2, 0.8 \rangle, \langle x_4, 0.2, 0.8 \rangle, \langle x_5, 0.8, 0.1 \rangle\}.$$

The aim here is to assign a patient

$$P = \{\langle x_1, 0.8, 0.1 \rangle, \langle x_2, 0.6, 0.1 \rangle, \langle x_3, 0.2, 0.8 \rangle, \langle x_4, 0.6, 0.1 \rangle, \langle x_5, 0.1, 0.6 \rangle\},$$

to one of the above-mentioned diagnosis Q_1, Q_2, Q_3, Q_4 and Q_5.

We now proceed by considering the criteria

$$\max_{1\le i\le 5}\{S_{IFS}(P,Q_i)\},$$

where $S_{IFS}(P,Q_1)=\mathbf{0.7654}$, $S_{IFS}(P,Q_2)=0.7580$,

$S_{IFS}(P,Q_3)=0.7187$, $S_{IFS}(P,Q_4)=0.4482$,

$S_{IFS}(P,Q_5)=0.3809$.

This gives rise to that the proper diagnosis for the patient P is Q_1(Viral fever). Here the result is exactly matching with that obtained in Vlachos and Sergiadis (2007) and Ye (2011).

6 Conclusion

This article presents a new similarity measure for intuitionistic fuzzy sets by making use of the convex combination of endpoints of the membership-degree interval and also focusing on the property of *min* and *max* operators. Among the existing methods, the proposed method seems to be more suitable for real cases and more valuable because of considering more information of IFSs. The proposed similarity measure enriches the theories and methods for measuring the degree of similarity between intuitionistic fuzzy sets.

Acknowledgments The author thanks the editor-in-chief professor Antonio Di Nola and the referees for their helpful suggestions which improved the presentation of the paper.

Appendix

In this section we prove the main results stated in the last part of Sect. 3. First we prove a key theorem.

Theorem 6 *Let $A_{IFS}, B_{IFS}\in IFS(X)$. The parametric distance $d_{IFS}: IFS(X)\times IFS(X)\to[0,1]$ given by (2) is monotonically decreasing as the parameter m increases.*

Proof Without loss of the generality, we assume that $X=\{x_1=x\}$ and $A_{IFS}, B_{IFS}\in IFS(X)$ are respectively represented by the intervals $[a_1, a_2]$ and $[b_1, b_2]$ where $a_1=\mu_A(x)$, $a_2=1-\nu_A(x)$, $b_1=\mu_B(x)$, $b_2=1-\nu_B(x)$. With the latter in mind, we can now restate $d_{IFS}(A_{IFS}, B_{IFS})$ in parametric form as follows

$$d_{IFS}^{(m)}(A_{IFS},B_{IFS})=\sqrt{\frac{1}{m+1}\sum_{j=0}^{m}[\chi_j(A_{IFS}(x))-\chi_j(B_{IFS}(x))]^2},$$

where $\chi_j(A_{IFS}(x))=\left(1-\frac{j}{m}\right)a_1+\frac{j}{m}a_2,\quad j=0,1,\dots,m,$

$\chi_j(B_{IFS}(x))=\left(1-\frac{j}{m}\right)b_1+\frac{j}{m}b_2,\quad j=0,1,\dots,m.$

As a first step toward the general case, we first show that

$$d_{IFS}^{(1)}(A_{IFS},B_{IFS})\ge d_{IFS}^{(2)}(A_{IFS},B_{IFS}),$$

for any $A_{IFS}, B_{IFS}\in IFS(X)$. By the definition of the parametric distance $d_{IFS}^{(m)}$, one gets

$$\begin{aligned}&(d_{IFS}^{(1)}(A_{IFS},B_{IFS}))^2-(d_{IFS}^{(2)}(A_{IFS},B_{IFS}))^2\\&=\frac{1}{2}[(a_1-b_1)^2+(a_2-b_2)^2]-\frac{1}{3}[(a_1-b_1)^2\\&\quad+((a_1+\frac{a_2-a_1}{2})-(b_1+\frac{b_2-b_1}{2}))^2+(a_2-b_2)^2].\end{aligned}$$

In this and subsequent results, it is notationally convenient to set

$$\alpha=a_1-b_1,$$
$$\beta=a_2-b_2.$$

With the use of the above notations, the following result is obtained

$$\alpha+k\frac{\beta-\alpha}{m}=\left(a_1+k\frac{a_2-a_1}{m}\right)-\left(b_1+k\frac{b_2-b_1}{m}\right),\quad k=0,1,\dots,m.$$

Thus, with the above setting in mind, we find that

$$\begin{aligned}&(d_{IFS}^{(1)}(A_{IFS},B_{IFS}))^2-(d_{IFS}^{(2)}(A_{IFS},B_{IFS}))^2\\&=\frac{1}{2}[\alpha^2+\beta^2]-\frac{1}{3}\left[\alpha^2+\left(\alpha+\frac{\beta-\alpha}{2}\right)^2+\beta^2\right]\\&=\frac{1}{6}\left[3\alpha^2+3\beta^2-2\alpha^2-2\left(\frac{\alpha+\beta}{2}\right)^2-2\beta^2\right]\\&=\frac{1}{6}\left[\alpha^2+\beta^2-2\left(\frac{\alpha+\beta}{2}\right)^2\right]\\&=\frac{1}{12}(\alpha-\beta)^2\ge 0,\end{aligned}$$

completing the proof of $d_{IFS}^{(1)}(A_{IFS}, B_{IFS})\ge d_{IFS}^{(2)}(A_{IFS}, B_{IFS})$.

We are now ready to prove the general case where the parameter m is a natural number.

For given m and from definition of the parametric distance $d_{IFS}^{(m)}$, we have

$$(d_{IFS}^{(m)}(A_{IFS},B_{IFS}))^2-(d_{IFS}^{(m+1)}(A_{IFS},B_{IFS}))^2=\frac{1}{m+1}\left[\alpha^2+\left(\alpha+\frac{\beta-\alpha}{m}\right)^2+\cdots+\left(\alpha+(m-1)\frac{\beta-\alpha}{m}\right)^2+\beta^2\right]$$

$$-\frac{1}{m+2}\left[\alpha^2+\left(\alpha+\frac{\beta-\alpha}{m+1}\right)^2+\cdots+\left(\alpha+(m-1)\frac{\beta-\alpha}{m+1}\right)^2+\left(\alpha+(m)\frac{\beta-\alpha}{m+1}\right)^2+\beta^2\right]=\frac{1}{(m+1)(m+2)}$$

$$\left\{(m+2)\left[\alpha^2+\left(\frac{(m-1)\alpha+\beta}{m}\right)^2+\left(\frac{(m-2)\alpha+2\beta}{m}\right)^2+\cdots+\left(\frac{\alpha+(m-1)\beta}{m}\right)^2+\beta^2\right]\right.$$

$$\left.-(m+1)\left[\alpha^2+\left(\frac{(m)\alpha+\beta}{m+1}\right)^2+\left(\frac{(m-1)\alpha+2\beta}{m+1}\right)^2+\cdots+\left(\frac{2\alpha+(m-1)\beta}{m+1}\right)^2+\left(\frac{\alpha+(m)\beta}{m+1}\right)^2+\beta^2\right]\right\}$$

$$=\frac{1}{(m+1)(m+2)}\left\{\alpha^2+\beta^2+\frac{(m+2)}{m^2}[((m-1)\alpha+\beta)^2+((m-2)\alpha+2\beta)^2+\cdots+(\alpha+(m-1)\beta)^2]\right.$$

$$\left.-\frac{1}{(m+1)}[((m)\alpha+\beta)^2+((m-1)\alpha+2\beta)^2+\cdots+(2\alpha+(m-1)\beta)^2+(\alpha+(m)\beta)^2]\right\}$$

$$=\frac{1}{(m+1)(m+2)}\left\{\frac{(m+2)}{m^2}([(m-1)^2+(m-2)^2+\cdots+1]\alpha^2+[1+2^2+\cdots+(m-1)^2]\beta^2+2[1(m-1)\right.$$

$$+2(m-2)+\cdots+(m-1)1]\alpha\beta)-\frac{1}{(m+1)}([m^2+(m-1)^2+\cdots+1]\alpha^2+[1+2^2+\cdots+(m-1)^2+m^2]\beta^2$$

$$\left.+2[1(m)+2(m-1)+\cdots+(m)1]\alpha\beta)\right\}=\frac{1}{(m+1)(m+2)}\left\{\frac{m+2}{6m}\alpha^2+\frac{m+2}{6m}\beta^2-\frac{m+2}{3m}\alpha\beta\right\}$$

$$=\frac{1}{(m+1)6m}(\alpha-\beta)^2\geq 0,$$

completing the proof of $d_{IFS}^{(m)}(A_{IFS}, B_{IFS}) \geq d_{IFS}^{(m+1)}(A_{IFS}, B_{IFS})$. □

Corollary 1 *Let $A_{IFS}, B_{IFS} \in IFS(X)$. The parametric similarity measure $S_{IFS}^{d(m)} : IFS(X) \times IFS(X) \to [0,1]$ given by (4) is a monotone increasing function of the parameter m.*

Proof The proof is concluded by taking definition of $S_{IFS}^{d(m)}$ and Theorem 6 into account. □

Theorem 7 *Let $A_{IFS}, B_{IFS} \in IFS(X)$. If $S_{IFS}^{d} : IFS(X) \times IFS(X) \to [0,1]$ and $S_{IFS}^{mix} : IFS(X) \times IFS(X) \to [0,1]$ are the mappings given by (4) and (5), respectively. Then, the sequence of parametric similarity measures $S_{IFS}^{(m)} : IFS(X) \times IFS(X) \to [0,1]$ given by (12) which can be restated as*

$$S_{IFS}^{(m)}(A_{IFS},B_{IFS})=\frac{1}{2}(S_{IFS}^{d(m)}(A_{IFS},B_{IFS})+S_{IFS}^{mix}(A_{IFS},B_{IFS})),$$

is a convergent sequence on [0,1].

Proof Since the parametric similarity measure $S_{IFS}^{d(m)} : IFS(X) \times IFS(X) \to [0,1]$ given by (4) is a monotone increasing function of the parameter m (by Corollary 1 and since S_{IFS}^{mix} is not dependant on the choice of m), we deduce that the parametric similarity measure $S_{IFS}^{(m)} : IFS(X) \times IFS(X) \to [0,1]$ is a monotone increasing function of the parameter m, too. This incorporating with the boundedness of $S_{IFS}^{(m)}$ (by the property **(P1)** where $S_{IFS}^{(m)}(A_{IFS},B_{IFS}) \leq 1$ for any $A_{IFS}, B_{IFS} \in IFS(X)$) will immediately lead to the convergence property of $S_{IFS}^{(m)}$. □

The earlier result shows that to have a more precise comparison we need to choose m sufficiently large. This finding is confirmed and illustrated by the graph in Fig. 1 where the curves C1–C6 show the behavior of $S_{IFS}^{(m)}$ applied to each pair of IFSs given in columns 1–6 of Table 2, respectively, as the parameter m increases from 1 to 50.

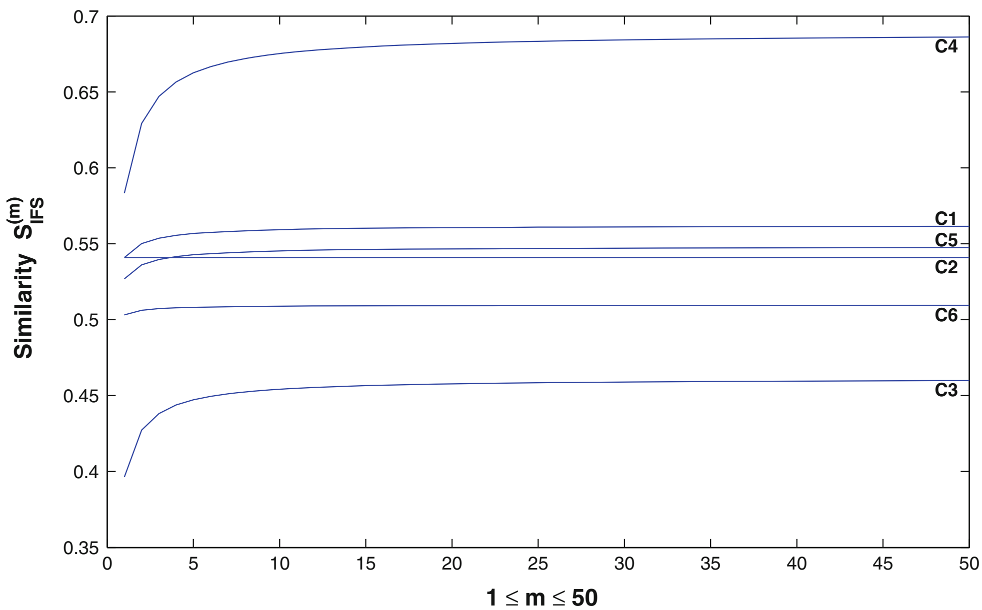

Fig. 1 Graphical illustration of the convergence property of $S_{IFS}^{(m)}$ applied to IFSs given in Table 2

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