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ORIGINAL PAPER

A generalization of the Chebyshev type inequalities for Sugeno integrals

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Abstract In this paper, we give a generalization of the Chebyshev type inequalities for Sugeno integral with respect to non-additive measures. The main results of this paper generalize most of the inequalities for Sugeno integral obtained by many researchers. Also, some conclusions are drawn and some problems for further investigations are given.

Keywords Nonadditive measure · Sugeno integral · Chebyshev's inequality · Minkowski's inequality · Hölder's inequality

1 Introduction

The theory of nonadditive measures and integrals was introduced by Sugeno (1974) as a tool for modeling nondeterministic problems. Sugeno integral is a useful tool in several theoretical and applied statistics (see Fig. 1). In decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility. The use of the Sugeno integral can be envisaged from two points of view: decision under uncertainty and multi-criteria decision-making (Dubois et al. 1998). Sugeno integral is analogous to Lebesgue integral which has been studied by many authors, including Pap (1995), Ralescu and Adams (1980), and Wang and Klir (1992), among others. Román-Flores et al. (2007, 2008a, b) started the studies of inequalities for Sugeno integral, and then followed by the authors (Agahi and Yaghoobi 2010; Agahi et al. 2010; Mesiar and Ouyang 2009; Ouyang and Fang 2008; Ouyang et al. 2008, 2010).

Problem Under what conditions does the inequality

$$\left((S)\int (f\bigstar g)^{\alpha}\,\mathrm{d}\mu\right)^{\lambda} \geq \left((S)\int f^{\beta}\,\mathrm{d}\mu\right)^{\upsilon} \bigstar \left((S)\int g^{\gamma}\,\mathrm{d}\mu\right)^{\tau} \tag{1.1}$$

or its reverse hold for an arbitrary fuzzy measure-based type fuzzy integral μ and a binary operation $\bigstar\colon [0,\infty)^2 \to [0,\infty)$?

In this contribution, we address this question. This is a generalization of the Chebyshev inequalities that appear in the papers (Flores-Franulič and Román-Flores 2007; Ouyang et al. 2008; Mesiar and Ouyang 2009). Mesiar and Ouyang (2009) considered Chebyshev type inequality (1.1) for $\alpha = \lambda = \beta = \upsilon = \gamma = \tau = 1$.

In general, any integral inequality can be a very strong tool for applications. In particular, when we think an integral operator as a predictive tool then an integral inequality can be very important in measuring and dimensioning such processes.

The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarization of

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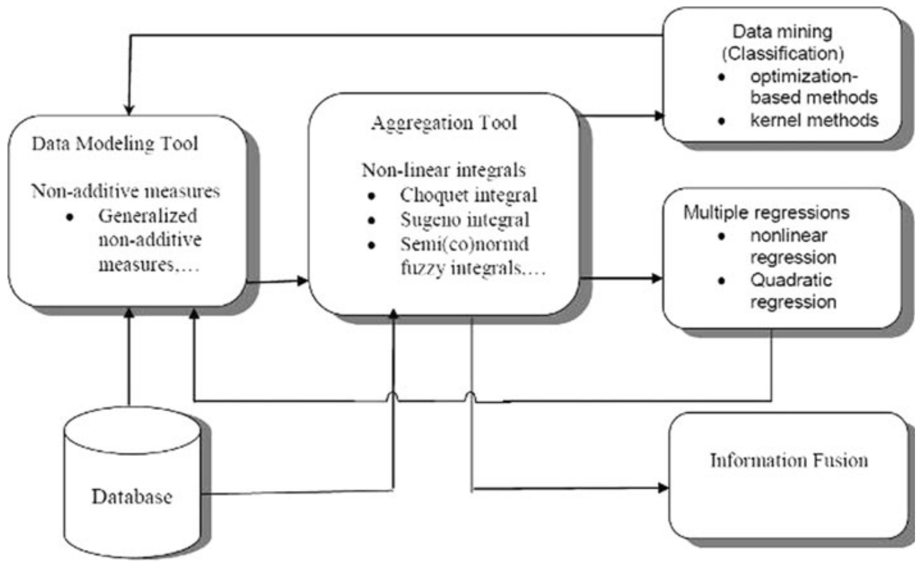

Fig. 1 Framework of Data Mining with Non-Additive Measures

some previous known results. In Sect. 3, we will focus on a generalization of Chebyshev type inequalities for Sugeno integrals. Finally, some conclusions are given.

2 Preliminaries

In this section, we recall some basic definitions and previous results which will be used in the sequel.

As usual we denote by $\mathbb{R}$ the set of real numbers. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Let $\mathbb{N}$ denote the set of all positive integers and $\overline{\mathbb{R}_+}$ denote $[0, +\infty]$. Throughout this paper, we fix the measurable space $(X, \mathcal{F})$, and all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 2.1 (Ralescu and Adams 1980) A set function $\mu : \mathcal{F} \to \overline{\mathbb{R}_+}$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$;
(FM2) $A \subset B$ implies $\mu(A) \le \mu(B)$;
(FM3) $A_1 \subset A_2 \subset \cdots$ implies $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$; and
(FM4) $A_1 \supset A_2 \supset \cdots$ and $\mu(A_1) < +\infty$ imply $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

When μ is a fuzzy measure, the triple $(X, \mathcal{F}, \mu)$ then is called a fuzzy measure space.

Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, by $\mathcal{F}_+(X)$ we denote the set of all nonnegative measurable functions $f: X \longrightarrow [0, \infty)$ with respect to $\mathcal{F}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. Let f be a non-negative real-valued function defined on X, we will denote the set $\{x \in X | f(x) \ge \alpha\}$ by F_α for $\alpha \ge 0$. Clearly, F_α is nonincreasing with respect to α, i.e., $\alpha \le \beta$ implies $F_\alpha \supseteq F_\beta$.

Definition 2.2 (Pap 1995; Sugeno 1974; Wang and Klir 1992) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $A \in \mathcal{F}$, the Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined as

$$(S)\int_A f \,\mathrm{d}\mu = \bigvee_{\alpha \ge 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When $A = X$, then

$$(S)\int_X f \,\mathrm{d}\mu = (S)\int f \,\mathrm{d}\mu = \bigvee_{\alpha \ge 0} (\alpha \wedge \mu(F_\alpha)).$$

It is well known that Sugeno integral is a type of nonlinear integral (Mesiar and Mesiarová 2008), i.e., for general case,

$$(S)\int (af + bg) \,\mathrm{d}\mu = a\left((S)\int f \,\mathrm{d}\mu\right) + b\left((S)\int g \,\mathrm{d}\mu\right)$$

does not hold. Some basic properties of Sugeno integral are summarized in Pap (1995), Wang and Klir (1992), we cite some of them in the next theorem.

Theorem 2.3 (Pap 1995; Wang and Klir 1992) *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, then*

(i) $\mu(A \cap F_\alpha) \ge \alpha \Longrightarrow (S)\int_A f \,\mathrm{d}\mu \ge \alpha$;
(ii) $\mu(A \cap F_\alpha) \le \alpha \Longrightarrow (S)\int_A f \,\mathrm{d}\mu \le \alpha$;
(iii) $(S)\int_A f \,\mathrm{d}\mu < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap F_\gamma) < \alpha$;
(iv) $(S)\int_A f \,\mathrm{d}\mu > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap F_\gamma) > \alpha$;
(v) *If* $\mu(A) < \infty$, *then* $\mu(A \cap F_\alpha) \ge \alpha \Longleftrightarrow (S)\int_A f \,\mathrm{d}\mu \ge \alpha$;
(vi) *If* $f \le g$, *then* $(S)\int f \,\mathrm{d}\mu \le (S)\int g \,\mathrm{d}\mu$.

Ouyang and Fang (2008) proved the following result which generalized the corresponding one in Román-Flores et al. (2007).

Lemma 2.4 *Let m be the Lebesgue measure on $\mathbb{R}$ and let $f:[0,\infty)\to[0,\infty)$ be a nonincreasing function. If $(S)\int_0^a f\,\mathrm{d}m = p$, then*

$$f(p-) \geq (S)\int_0^a f\,\mathrm{d}m = p$$

for all $a \geq 0$, where $f(p-) = \lim_{x\to p^-} f(x)$.

Moreover, if $p<a$ and f is continuous at p, then $f(p-) = f(p) = p$.

Notice that if m is the Lebesgue measure and f is nonincreasing, then $f(p-)\geq p$ implies $(S)\int_0^a f\,\mathrm{d}m \geq p$ for any $a \geq p$. In fact, the monotonicity of f and the fact $f(p-)\geq p$ imply that $[0,p)\subset F_p$. Thus, $m([0,a]\cap F_p)\geq m([0,a]\cap[0,p)) = m([0,p)) = p$. Now, by Theorem 2.3(i), we have $(S)\int_0^a f\,\mathrm{d}m \geq p$.

Based on Lemma 2.4, Ouyang et al. (2008) proved some Chebyshev type inequalities and their united form (Mesiar and Ouyang 2009). Notice that when proving these Theorems, the following lemma, which is derived from the transformation theorem for Sugeno integrals (see Wang and Klir 1992), plays a fundamental role.

Lemma 2.5 *Let $(S)\int_A f\,\mathrm{d}\mu = p$. Then $\forall\, r\geq p, (S)\int_A f\,\mathrm{d}\mu = (S)\int_0^r \mu(A\cap F_\alpha)\,\mathrm{d}m$, where m is the Lebesgue measure.*

Recall that two functions $f,g: X\to R$ are said to be comonotone if for all $(x,y)\in X^2$, $(f(x)-f(y))(g(x)-g(y))\geq 0$. Clearly, if f and g are comonotone, then for all non-negative real numbers p,q either $F_p\subset G_q$ or $G_q\subset F_p$. Indeed, if this assertion does not hold, then there are $x\in F_p\backslash G_q$ and $y\in G_q\backslash F_p$. That is,

$$f(x)\geq p, g(x)<q \quad \text{and} \quad f(y)<p, g(y)\geq q,$$

and hence $(f(x)-f(y))(g(x)-g(y))<0$, contradicts with the comonotonicity of f and g. Notice that comonotone functions can be defined on any abstract space.

In Flores-Franulič and Román-Flores (2007), a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. (2008). Furthermore, Chebyshev type inequalities for fuzzy integral were proposed in a rather general form by Mesiar and Ouyang (2009). In fact, they proved the following result:

Theorem 2.6 *Let $f,g\in\mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_A f\,\mathrm{d}\mu$ and $(S)\int_A g\,\mathrm{d}\mu$ are finite. And let $\star:[0,\infty)^2\to[0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f,g are comonotone, then the inequality*

$$(S)\int_A f\star g\,\mathrm{d}\mu \geq \left((S)\int_A f\,\mathrm{d}\mu\right)\star\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{2.1}$$

holds.

It is known that

$$(S)\int_A f\star g\,\mathrm{d}\mu \leq \left((S)\int_A f\,\mathrm{d}\mu\right)\star\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{2.2}$$

where f,g are comonotone functions whenever $\star\geq\max$ (for a similar result, see Ouyang and Mesiar 2009a), it is of great interest to determine the operator $\star$ such that

$$(S)\int_A f\star g\,\mathrm{d}\mu = \left((S)\int_A f\,\mathrm{d}\mu\right)\star\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{2.3}$$

holds for any comonotone functions f,g, and for any fuzzy measure μ and any measurable set A. Ouyang et al. (2009) and Ouyang and Mesiar (2009b) proved that there are only 18 operators such that (2.3) holds, including the four well-known operators: minimum, maximum, first projection (PF), if $x\star y = x$ for each pair (x,y)) and last projection (PL), if $x\star y = y$ for each pair (x,y)).

Recently, Agahi and Yaghoobi (2010) proved a Minkowski type inequality for monotone functions and arbitrary fuzzy measure-based Sugeno integrals.

Theorem 2.7 *Let μ be an arbitrary fuzzy measure on $[0,a]$ and $f,g:[0,a]\to[0,\infty]$ be two real-valued measurable functions such that $(S)\int_0^a (f+g)^s\,\mathrm{d}\mu\leq 1$. If f,g are both non-decreasing (non-increasing) functions, then the inequality*

$$\left((S)\int_0^a (f+g)^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \leq \left((S)\int_0^a f^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} + \left((S)\int_0^a g^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \tag{2.4}$$

holds for all $1\leq s<\infty$.

3 Main results

The main results of this paper are as follows

Theorem 3.1 *Let $f,g\in\mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A f^\beta\,\mathrm{d}\mu\leq 1$ and $(S)\int_A g^\gamma\,\mathrm{d}\mu\leq 1$. Let $\star:[0,\infty)^2\to[0,\infty)$ be continuous and non-*

decreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality

$$\left((S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu\right)^{\lambda} \geq \left((S)\int_A f^{\beta}\,\mathrm{d}\mu\right)^{\upsilon} \bigstar \left((S)\int_A g^{\gamma}\,\mathrm{d}\mu\right)^{\tau} \tag{3.1}$$

holds for all $\alpha,\beta,\gamma,\lambda,\upsilon,\tau\in(0,\infty), 0<\alpha\lambda\leq 1, 1\leq\beta\upsilon<\infty, 1\leq\gamma\tau<\infty, \lambda\leq\tau,\upsilon$.

Proof Let $\alpha,\beta,\gamma,\lambda,\upsilon,\tau\in(0,\infty)$, $\left((S)\int f^{\beta}\,\mathrm{d}\mu\right)^{\upsilon}=p\leq 1$ and $\left((S)\int g^{\gamma}\,\mathrm{d}\mu\right)^{\tau}=q\leq 1$. Theorem 2.3 (v) implies that

$$(S)\int_A f^{\beta}\,\mathrm{d}\mu = p^{\frac{1}{\upsilon}} \Longrightarrow \mu\left(A\cap\left\{f^{\beta}\geq p^{\frac{1}{\upsilon}}\right\}\right) \geq p^{\frac{1}{\upsilon}} \Longrightarrow \mu\left(A\cap F_{p^{\frac{1}{\beta\upsilon}}}\right)\geq p^{\frac{1}{\upsilon}}, \tag{3.2}$$

$$(S)\int_A g^{\gamma}\,\mathrm{d}\mu = q^{\frac{1}{\tau}} \Longrightarrow \mu\left(A\cap\left\{g^{\gamma}\geq q^{\frac{1}{\tau}}\right\}\right) \geq q^{\frac{1}{\tau}} \Longrightarrow \mu\left(A\cap G_{q^{\frac{1}{\gamma\tau}}}\right)\geq q^{\frac{1}{\tau}}, \tag{3.3}$$

where $F_{\alpha}=\{x|f(x)\geq\alpha\}$ and $G_{\alpha}=\{x|g(x)\geq\alpha\}$. Since $0<\alpha\lambda\leq 1, 1\leq\beta\upsilon<\infty, 1\leq\gamma\tau<\infty, \lambda\leq\tau,\upsilon$ and $\bigstar:[0,\infty)^2\to[0,\infty)$ is continuous and nondecreasing in both arguments and bounded from above by minimum, then we have

$$(p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}})^{\alpha}\geq(p\bigstar q)^{\alpha}\geq(p\bigstar q)^{\frac{1}{\lambda}},$$

$$\begin{aligned}\mu\left(A\cap H_{p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}}}\right) &\geq \mu\left(A\cap F_{p^{\frac{1}{\beta\upsilon}}}\cap G_{q^{\frac{1}{\gamma\tau}}}\right)\\ &=\min\left(\mu\left(A\cap F_{p^{\frac{1}{\beta\upsilon}}}\right),\mu\left(A\cap G_{q^{\frac{1}{\gamma\tau}}}\right)\right)\\ &\geq p^{\frac{1}{\upsilon}}\wedge q^{\frac{1}{\tau}}\geq p^{\frac{1}{\lambda}}\wedge q^{\frac{1}{\lambda}},\end{aligned}$$

where $H_{\alpha}=\{x|f(x)\bigstar g(x)\geq\alpha\}$. Hence

$$\begin{aligned}\left((S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu\right)^{\lambda} &\geq \left((p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}})^{\alpha}\wedge\mu(A\cap\{x|\,(f\bigstar g)^{\alpha}(x)\right.\\ &\qquad \left.\geq(p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}})^{\alpha}\})\right)^{\lambda}\\ &=\left((p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}})^{\alpha}\wedge\mu\left(A\cap H_{(p^{\frac{1}{\beta\upsilon}}\bigstar q^{\frac{1}{\gamma\tau}})}\right)\right)^{\lambda}\\ &\geq\left((p\bigstar q)^{\frac{1}{\lambda}}\wedge\left(p^{\frac{1}{\lambda}}\wedge q^{\frac{1}{\lambda}}\right)\right)^{\lambda}\\ &=p\bigstar q.\end{aligned}$$

This completes the proof. □

The following example shows that the conditions of $(S)\int_A f^{\beta}\,\mathrm{d}\mu\leq 1$ and $(S)\int_A g^{\gamma}\,\mathrm{d}\mu\leq 1$ in Theorem 3.1 are inevitable.

Example 3.2 Let $A=[0,4]$, $f(x)=g(x)=\sqrt{x}$, $\beta=\gamma=\frac{1}{3}$, $\upsilon=\tau=3$, $\alpha=\frac{1}{2}$, $\lambda=2$ and m be the Lebesgue measure. A straightforward calculus shows that

$$\begin{aligned}\text{(i)}\quad (S)\int_0^4 f^{\frac{1}{3}}(x)\,\mathrm{d}m &= (S)\int_0^4 g^{\frac{1}{3}}(x)\,\mathrm{d}m = \bigvee_{\alpha\in[0,1.2599]}\left[\alpha\wedge m\left(\left\{\left(\sqrt{x}\right)^{\frac{1}{3}}\geq\alpha\right\}\right)\right]\\ &=\bigvee_{\alpha\in[0,1.2599]}\left[\alpha\wedge\left(4-\alpha^6\right)\right]=1.18805,\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad (S)\int_0^4 (f\wedge g)^{\frac{1}{2}}(x)\,\mathrm{d}m &= (S)\int_0^4 f^{\frac{1}{2}}(x)\,\mathrm{d}m = \bigvee_{\alpha\in[0,1.4142]}\left[\alpha\wedge m\left(\left\{\left(\sqrt{x}\right)^{\frac{1}{2}}\geq\alpha\right\}\right)\right]\\ &=\bigvee_{\alpha\in[0,1.4142]}\left[\alpha\wedge\left(4-\alpha^4\right)\right]=1.28378.\end{aligned}$$

Therefore,

$$\begin{aligned}1.6481 &= \left((S)\int_0^4 (f\wedge g)^{\frac{1}{2}}(x)\,\mathrm{d}m\right)^2 = \left((S)\int_0^4 f^{\frac{1}{2}}(x)\,\mathrm{d}m\right)^2\\ &<\left((S)\int_0^4 f^{\frac{1}{3}}\,\mathrm{d}m\right)^3\wedge\left((S)\int_0^4 g^{\frac{1}{3}}\,\mathrm{d}m\right)^3\\ &=\left((S)\int_0^4 f^{\frac{1}{3}}\,\mathrm{d}m\right)^3=1.6769,\end{aligned}$$

which violates Theorem 3.1.

Remark 3.3 We can use an example in Mesiar and Ouyang (2009) to show the necessity of the comonotonicity of f, g, and so we omit it here.

The following example shows that the condition of $\bigstar\leq\min$ in Theorem 3.1 cannot be omitted.

Example 3.4 Let $X\in[0,1]$, $f(x)=g(x)=1$ and $\bigstar=+$. Then

$$(S)\int(f+g)\,\mathrm{d}m=1\quad\text{and}\quad\left((S)\int f\,\mathrm{d}m\right)=\left((S)\int g\,\mathrm{d}m\right)=1,$$

where m denotes the Lebesgue measure on $\mathbb{R}$. But

$$(S)\int(f+g)\,\mathrm{d}m=1<(S)\int f\,\mathrm{d}m+(S)\int f\,\mathrm{d}m=2,$$

which violates Theorem 3.1.

The following example shows that the conditions of $0<\alpha\lambda\leq 1, 1\leq \beta\upsilon<\infty, 1\leq\gamma\tau<\infty$ and $\lambda\leq\tau,\upsilon$ in Theorem 3.1 are inevitable.

Example 3.5 *Let* $X\in[0,1], \beta=\gamma=\frac{1}{2}, \upsilon=\alpha=\lambda=1$, $\tau=2$ *and* m *be the Lebesgue measure. Let* f,g *be two real valued functions defined as* $f(x)=x$ *and* $g(x)\equiv 1$ *for all* $x\in[0,1]$ *and* $\bigstar$ *be the standard product. A straightforward calculus shows that*

(i) $$\left((S)\int_0^1 (f\bigstar g)^\alpha\, \mathrm{d}m\right)^\lambda = (S)\int_0^1 f(x)\,\mathrm{d}m = \bigvee_{\alpha\in[0,1]}[\alpha\wedge m(\{x\geq\alpha\})] = \bigvee_{\alpha\in[0,1]}[\alpha\wedge(1-\alpha)]=0.5,$$

(ii) $$\left((S)\int_0^1 f^\beta\,\mathrm{d}m\right)^\upsilon = (S)\int_0^1 f^{\frac{1}{2}}(x)\,\mathrm{d}m = \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge m\left(\left\{\sqrt{x}\geq\alpha\right\}\right)\right] = \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\left(1-\alpha^2\right)\right]=0.61803,$$

(iii) $$\left((S)\int_0^1 g^\gamma\,\mathrm{d}m\right)^\tau = \left((S)\int_0^1 g^{\frac{1}{2}}(x)\,\mathrm{d}m\right)^2 = (S)\int_0^1 \mathrm{d}m = 1.$$

Therefore,

$$\left((S)\int_0^1 (f\bigstar g)^\alpha\,\mathrm{d}\mu\right)^\lambda = 0.5 < \left((S)\int_0^1 f^\beta\,\mathrm{d}\mu\right)^\upsilon \bigstar\left((S)\int_0^1 g^\gamma\,\mathrm{d}\mu\right)^\tau = 0.61803\times 1 = 0.61803,$$

which violates Theorem 3.1.

We get an inequality related to the Minkowski type whenever $\alpha=\beta=\gamma=s$ and $\lambda=\upsilon=\tau=\frac{1}{s}$.

Corollary 3.6 (Ouyang et al. 2010) *Let* $f,g\in\mathcal{F}_+(X)$ *and* μ *be an arbitrary fuzzy measure such that* $(S)\int_A f^s\,\mathrm{d}\mu\leq 1$ *and* $(S)\int_A g^s\,\mathrm{d}\mu\leq 1$. *And let* $\bigstar\colon[0,\infty)^2\to[0,\infty)$ *be continuous and nondecreasing in both arguments and bounded from above by minimum. If* f,g *are comonotone, then the inequality*

$$\left((S)\int_A (f\bigstar g)^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \geq \left((S)\int_A f^s\,\mathrm{d}\mu\right)^{\frac{1}{s}}\bigstar\left((S)\int_A g^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \tag{3.4}$$

holds for all $0<s<\infty$.

Also, we get an inequality related to the Hölder type whenever $\alpha=\lambda=1, \beta=p, \gamma=q, \upsilon=\frac{1}{p}, \tau=\frac{1}{q}$.

Corollary 3.7 *Let* $f,g\in\mathcal{F}_+(X)$ *and* μ *be an arbitrary fuzzy measure such that* $(S)\int_A f^p\,\mathrm{d}\mu\leq 1$ *and* $(S)\int_A g^q\,\mathrm{d}\mu\leq 1$. *And let* $\bigstar\colon[0,\infty)^2\to[0,\infty)$ *be continuous and nondecreasing in both arguments and bounded from above by minimum. If* f,g *are comonotone, then the inequality*

$$(S)\int_A (f\bigstar g)\,\mathrm{d}\mu \geq \left((S)\int_A f^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\bigstar\left((S)\int_A g^q\,\mathrm{d}\mu\right)^{\frac{1}{q}} \tag{3.5}$$

holds for all $p,q\in(0,1]$.

Let $\alpha=\beta=\gamma=\lambda=\upsilon=\tau=1$, then we get the Chebyshev inequality.

Corollary 3.8 (Mesiar and Ouyang 2009; Ouyang and Mesiar 2009a) *Let* $f,g\in\mathcal{F}_+(X)$ *and* μ *be an arbitrary fuzzy measure such that* $(S)\int_A f\,\mathrm{d}\mu\leq 1$ *and* $(S)\int_A g\,\mathrm{d}\mu\leq 1$. *And let* $\bigstar\colon[0,\infty)^2\to[0,\infty)$ *be continuous and nondecreasing in both arguments and bounded from above by minimum. If* f,g *are comonotone, then the inequality*

$$(S)\int_A (f\bigstar g)\,\mathrm{d}\mu \geq \left((S)\int_A f\,\mathrm{d}\mu\right)\bigstar\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{3.6}$$

holds.

Let $g(x)\equiv 1, \beta=\gamma=\tau=\upsilon=1, \alpha=\frac{1}{\lambda}$ and $\bigstar$ be the standard product, then we have the following result.

Corollary 3.9 (Román-Flores et al. 2008a, b) *If* $f\colon[0,1]\to[0,\infty)$ *is a measurable function, then the inequality*

$$(S)\int_0^1 f^\alpha\,\mathrm{d}\mu \geq \left((S)\int_0^1 f\,\mathrm{d}\mu\right)^\alpha \tag{3.7}$$

holds for all $1\leq\alpha<\infty$.

Let $g(x)\equiv 1, \alpha=\gamma=\tau=\lambda=1, \beta=\frac{1}{\upsilon}$ and $\bigstar$ be the standard product, then we have the following result.

Corollary 3.10 *If* $f\colon[0,1]\to[0,\infty)$ *is a measurable function, then the inequality*

$$\left((S)\int_0^1 f\,\mathrm{d}\mu\right)^{\beta} \geq \left((S)\int_0^1 f^{\beta}\,\mathrm{d}\mu\right) \tag{3.8}$$

holds for all $0 < \beta \leq 1$.

Lemmas 2.4 and 2.5 help us to reach the following result.

Theorem 3.11 *Let* $f, g \in \mathcal{F}_+(X)$ *and* μ *be an arbitrary fuzzy measure such that* $(S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu \leq 1$. *Let* $\bigstar\colon [0,\infty)^2 \to [0,\infty)$ *be continuous and nondecreasing in both arguments and bounded from below by maximum. If* f, g *are comonotone, then the inequality*

$$\left((S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu\right)^{\lambda} \leq \left((S)\int_A f^{\beta}\,\mathrm{d}\mu\right)^{\upsilon} \bigstar \left((S)\int_A g^{\gamma}\,\mathrm{d}\mu\right)^{\tau} \tag{3.9}$$

holds for all $\alpha, \beta, \gamma, \lambda, \upsilon, \tau \in (0,\infty), 1 \leq \alpha\lambda < \infty, 0 < \beta\upsilon \leq 1, 0 < \gamma\tau \leq 1, \lambda \geq \tau, \upsilon$.

Proof Let $\alpha, \beta, \gamma, \lambda, \upsilon, \tau \in (0,\infty)$ and $(S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu = r \leq 1$. Theorem 2.3 (v) implies that:

$$(S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu = r \Longrightarrow \mu(A \cap \{x|(f\bigstar g)(x) \geq r^{\frac{1}{\alpha}}\}) \geq r. \tag{3.10}$$

Denote $A(\alpha) = \mu(A \cap \{x|f^{\beta}(x) \geq \alpha\}), B(\alpha) = \mu(A \cap \{x|g^{\gamma}(x) \geq \alpha\})$ and $C(\alpha) = \mu(A \cap \{x|(f\bigstar g)^{\alpha}(x) \geq \alpha\})$. By Lemma 2.5, we have $(S)\int_A (f\bigstar g)^{\alpha}\,\mathrm{d}\mu = (S)\int_0^1 C(\alpha)\,\mathrm{d}m$, therefore, it is sufficient to prove

$$\left((S)\int_0^1 C(\alpha)\,\mathrm{d}m\right)^{\lambda} \leq \left((S)\int_0^1 A(\alpha)\,\mathrm{d}m\right)^{\upsilon} \bigstar \left((S)\int_0^1 B(\alpha)\,\mathrm{d}m\right)^{\tau}.$$

Let $p = (S)\int_0^1 A(\alpha)\,\mathrm{d}m$ and $q = (S)\int_0^1 B(\alpha)\,\mathrm{d}m$. Without loss of generality, let $p, q < 1$. Since $A(\alpha)$ and $B(\alpha)$ are non-increasing with respect to α and m is a Lebesgue measure, by Lemma 2.4 (moreover part), we have the following equalities:

$$A(p-) = p, B(q-) = q. \tag{3.11}$$

Now, on the contrary suppose

$$r^{\lambda} > p^{\upsilon}\bigstar q^{\tau}. \tag{3.12}$$

Since $1 \leq \alpha\lambda < \infty, 0 < \beta\upsilon \leq 1, 0 < \gamma\tau \leq 1, \lambda \geq \tau, \upsilon$, then (3.12) implies that

$$r^{\frac{1}{\alpha}} > p^{\frac{1}{\beta}}\bigstar q^{\frac{1}{\gamma}}, \tag{3.13}$$

$$r > (p^{\upsilon}\bigstar q^{\tau})^{\frac{1}{\lambda}} \geq (p^{\lambda}\bigstar q^{\lambda})^{\frac{1}{\lambda}}. \tag{3.14}$$

For each $\varepsilon > 0$, by the monotonicity of $\bigstar$ and using (3.13) we have

$$\begin{aligned}
&\mu\left(A \cap \left\{x|(f\bigstar g)(x) \geq r^{\frac{1}{\alpha}}\right\}\right) \\
&\leq \mu\left(A \cap \left\{x|(f\bigstar g)(x) > p^{\frac{1}{\beta}}\bigstar q^{\frac{1}{\gamma}}\right\}\right) \\
&\leq \mu\left(A \cap \left(\left\{x|f(x) \geq p^{\frac{1}{\beta}}\right\} \cup \left\{x|g(x) \geq q^{\frac{1}{\gamma}}\right\}\right)\right) \\
&\leq \mu\left(A \cap \left(\left\{x|f^{\beta}(x) \geq p - \varepsilon\right\} \cup \left\{x|g^{\gamma}(x) \geq q - \varepsilon\right\}\right)\right).
\end{aligned} \tag{3.15}$$

Letting $\varepsilon \to 0$, by the continuity of $\bigstar$ and (3.11) we have

$$\begin{aligned}
r &\leq \lim_{\varepsilon\to 0} \mu\left(A \cap \left(\left\{x|f^{\beta}(x) \geq p - \varepsilon\right\} \cup \left\{x|g^{\gamma}(x) \geq q - \varepsilon\right\}\right)\right) \\
&= \lim_{\varepsilon\to 0}(\max(A(p-\varepsilon), B(q-\varepsilon))) \\
&= \max(p, q) \leq (p^{\lambda}\bigstar q^{\lambda})^{\frac{1}{\lambda}},
\end{aligned}$$

which is a contradiction to (3.14). Hence $r^{\lambda} \leq p^{\upsilon}\bigstar q^{\tau}$ and the proof is completed. □

Remark 3.12 We can use the same examples in Agahi et al. (2010) to show the necessities of $\bigstar \geq \max$ and the comonotonicity of f, g, and so we omit them here.

The following example shows that the conditions of $1 \leq \alpha\lambda < \infty, 0 < \beta\upsilon \leq 1, 0 < \gamma\tau \leq 1$ and $\lambda \geq \tau, \upsilon$ in Theorem 3.11 are inevitable.

Example 3.13 Let $X \in [0,1], \beta = \gamma = 2, \upsilon = \alpha = \lambda = 1$, $\tau = \frac{1}{2}$ and m be the Lebesgue measure. Let f, g be two real valued functions defined as $f(x) = g(x) = \frac{1}{2}$ for all $x \in [0,1]$ and $\bigstar = +$. A straightforward calculus shows that

$$(i)\ \left((S)\int_0^1 (f\bigstar g)^{\alpha}\,\mathrm{d}m\right)^{\lambda} = (S)\int_0^1 (f+g)\,\mathrm{d}m = (S)\int_0^1 \mathrm{d}m = 1,$$

$$(ii)\ \left((S)\int_0^1 f^{\beta}\,\mathrm{d}m\right)^{\upsilon} = (S)\int_0^1 f^2(x)\,\mathrm{d}m = \frac{1}{4},$$

$$(iii)\ \left((S)\int_0^1 g^{\gamma}\,\mathrm{d}m\right)^{\tau} = \left((S)\int_0^1 g^2(x)\,\mathrm{d}m\right)^{\frac{1}{2}} = \frac{1}{2}.$$

Therefore,

$$\left((S)\int_0^1 (f\bigstar g)^\alpha \, d\mu\right)^\lambda = 1 > \left((S)\int_0^1 f^\beta \, d\mu\right)^\upsilon$$
$$\bigstar\left((S)\int_0^1 g^\gamma \, d\mu\right)^\tau = \frac{1}{4}+\frac{1}{2}=\frac{3}{4},$$

which violates Theorem 3.11.

Let $\alpha=\beta=\gamma=s$ and $\lambda=\upsilon=\tau=\frac{1}{s}$, then we get the Minkowski inequality.

Corollary 3.14 (Agahi et al. 2010) *Let $f,g\in\mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A(f\bigstar g)^s\, d\mu\le 1$. Let $\bigstar\colon[0,\infty)^2\to[0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f,g are comonotone, then the inequality*

$$\left((S)\int_A (f\bigstar g)^s \, d\mu\right)^{\frac{1}{s}} \le \left((S)\int_A f^s \, d\mu\right)^{\frac{1}{s}}\bigstar\left((S)\int_A g^s \, d\mu\right)^{\frac{1}{s}} \tag{3.16}$$

holds for all $0<s<\infty$.

Remark 3.15 Note that for any subnorm M, we have $M(x,y)\le\min(x,y)$, so Ineq. (3.16) does not work when $\bigstar$ is a subnorm. However, Ineq. (3.16) works whenever $\bigstar$ is a t-conorm.

Let $\alpha=\lambda=1,\beta=p,\gamma=q,\upsilon=\frac{1}{p},\tau=\frac{1}{q}$, then we get the Hölder inequality:

Corollary 3.16 *Let $f,g\in\mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A(f\bigstar g)\, d\mu\le 1$. Let $\bigstar\colon[0,\infty)^2\to[0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f,g are comonotone, then the inequality*

$$(S)\int_A (f\bigstar g) \, d\mu \le \left((S)\int_A f^p \, d\mu\right)^{\frac{1}{p}}\bigstar\left((S)\int_A g^q \, d\mu\right)^{\frac{1}{q}} \tag{3.17}$$

holds for all $p,q\in[1,\infty)$.

Remark 3.17 Let $\bigstar$ be continuous and nondecreasing. If $\bigstar|_{[0,1]^2}$ is a triangular subnorm (Klement et al. 2000), then Ineq. (3.17) works for any comonotone functions f,g with $(S)\int_A f\, d\mu\le 1$ and $(S)\int_A g\, d\mu\le 1$.

Let $\alpha=\beta=\gamma=\lambda=\upsilon=\tau=1$, then we get the following result.

Corollary 3.18 (Ouyang and Mesiar 2009a) *Let $f,g\in\mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A(f\bigstar g)\, d\mu\le 1$. Let $\bigstar\colon[0,\infty)^2\to[0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f,g are comonotone, then the inequality*

$$(S)\int_A (f\bigstar g) \, d\mu \le \left((S)\int_A f \, d\mu\right)\bigstar\left((S)\int_A g \, d\mu\right)$$

holds.

4 Conclusions and problems for further investigation

In this paper, we have investigated a generalization of Chebyshev type inequalities for Sugeno integrals. More precisely, sufficient conditions under which the inequality

$$\left((S)\int (f\bigstar g)^\alpha \, d\mu\right)^\lambda \ge \left((S)\int f^\beta \, d\mu\right)^\upsilon\bigstar\left((S)\int g^\gamma \, d\mu\right)^\tau \tag{4.1}$$

or its reverse hold for an arbitrary fuzzy measure-based type fuzzy integral μ and a binary operation $\bigstar\colon[0,\infty)^2\to[0,\infty)$ are given.

Open Problem 1 Are there any operators such that the inequalities (4.1) and/or its reverse become equalities?

Open Problem 2 Under what conditions, does the inequality (4) or its reverse hold for seminormed fuzzy integrals or semiconormed fuzzy integrals (Suárez García and Gil Álvarez 1986)?

We will address the problem 2 in the near future.

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