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ORIGINAL PAPER

Remarks and corrections to the triangular approximations of fuzzy numbers using α-weighted valuations

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Abstract A recent paper was dedicated to find the nearest fuzzy triangular approximations of a fuzzy number by using α-weighted valuations. We prove, by simple examples, that the results of approximations are not always triangular fuzzy numbers and that in fact they are not fuzzy sets. We give a correct solution of the problem of approximation in a more general case, and we study the properties of identity, additivity, translation invariance, scale invariance, and monotonicity of the new approximation operator.

Keywords Fuzzy number · Triangular fuzzy number · Approximation

1 Introduction

Several researchers focused on the calculus of the nearest triangular or trapezoidal approximation of a fuzzy number, with or without some additional conditions (Abbasbandy and Amirfakhrian 2006; Abbasbandy and Asady 2004; Ban 2008; Delgado et al. 1998; Grzegorzewski and Mrówka 2007; Nasibov and Peker 2007; Yeh 2007, 2008, etc.).

Abbasbandy et al. (2010) used an application of Yager and Filev's (1998) formulation to obtain the nearest triangular fuzzy number to a general fuzzy number, with α-weighted valuations. Two approximation operators are given; the second one is a generalization of the first one. Because some necessary conditions are not imposed, the results of approximation are not always triangular fuzzy numbers; in fact, they are not fuzzy sets. In Sect. 3 we prove, by simple examples, that the results in Abbasbandy et al. (2010) are incomplete. Section 4 contains a correct and complete solution of the problem of nearest triangular fuzzy number of a fuzzy number using α-weighted valuations, in a more general case, when the weighted function is $f(\alpha) = (1-\alpha)^q$, with $q \in [0, +\infty)$. The method is based on the well-known Karush–Kuhn–Tucker theorem proposed to be used in this topic by Grzegorzewski and Mrówka (2007). The properties of this new approximation operator are discussed in Sect. 5. It is invariant to translations, scale invariant, non-monotonic (but with a property of monotonicity with respect to symmetric fuzzy numbers), with the property of identity and without the property of additivity.

2 Preliminaries

The definitions and notations in Abbasbandy et al. (2010) are used.

Defintition 1 A fuzzy number is a fuzzy set on the real line $u : \mathbb{R} \to [0, 1]$ which satisfies

(i) u is upper semicontinuous;
(ii) $u(x) = 0$ outside some interval $[c, d]$;
(iii) There are real numbers a, b such that $c \le a \le b \le d$ and

1. $u(x)$ is monotonic increasing on $[c, a]$;
2. $u(x)$ is monotonic decreasing on $[b, d]$;
3. $u(x) = 1$, $a \le x \le b$.

If u is a fuzzy number and we consider

$$[u]_\alpha = \{s \in \mathbb{R} : u(s) \ge \alpha\}, \quad 0 < \alpha \le 1$$

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and

$$[u]_0 = \overline{\cup_{\alpha\in(0,1]}[u]_\alpha},$$

then $[u]_\alpha$ is a closed bounded interval. We denote $[u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0,1]$ and an equivalent parametric definition of a fuzzy number is the following:

Definition 2 A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha), \overline{u}(\alpha), 0 \le \alpha \le 1$, which satisfies the following requirements:

(i) $\underline{u}(\alpha)$ is a bounded monotonic increasing left continuous function;
(ii) $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function;
(iii) $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1$.

The interval $[u]_0 = [\underline{u}(0), \overline{u}(0)]$ is called the support and the interval $[u]_1 = [\underline{u}(1), \overline{u}(1)]$ is called the core of the fuzzy number u.

We denote by $F(\mathbb{R})$ the set of all fuzzy numbers.

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D_2^2(u,v) = \int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^2 d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 d\alpha \quad (1)$$

is the distance between u and v.

Triangular fuzzy numbers are fuzzy numbers characterized by ordered triples $u = (u_l, u_c, u_r) \in \mathbb{R}^3$ with $u_l \le u_c \le u_r$ such that

$$\begin{aligned}[u]_\alpha &= [\underline{u}(\alpha), \overline{u}(\alpha)] \\ &= [u_l + (u_c - u_l)\alpha, u_r - (u_r - u_c)\alpha], \quad \alpha \in [0,1].\end{aligned}$$

Let $u, v \in F(\mathbb{R})$,

$$[u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)],$$
$$[v]_\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)], \quad \alpha \in [0,1]$$

and $\lambda \in \mathbb{R}$. The sum $u + v$ and the scalar multiplication $\lambda \cdot u$ are introduced by

$$[u+v]_\alpha = [\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha)], \quad \alpha \in [0,1]$$

and

$$[\lambda \cdot u]_\alpha = \begin{cases} [\lambda\underline{u}(\alpha), \lambda\overline{u}(\alpha)], & \text{if } \lambda \ge 0, \\ [\lambda\overline{u}(\alpha), \lambda\underline{u}(\alpha)], & \text{if } \lambda < 0, \end{cases} \quad \alpha \in [0,1].$$

In the case of the triangular fuzzy numbers (u_l, u_c, u_r) and (v_l, v_c, v_r) we get

$$(u_l, u_c, u_r) + (v_l, v_c, v_r) = (u_l + v_l, u_c + v_c, u_r + v_r)$$

and

$$\lambda \cdot (u_l, u_c, u_r) = \begin{cases} (\lambda u_l, \lambda u_c, \lambda u_r), & \text{if } \lambda \ge 0, \\ (\lambda u_r, \lambda u_c, \lambda u_l), & \text{if } \lambda < 0. \end{cases}$$

A class of representative values for a fuzzy number, so-called valuations, was introduced by Yager (1999)

$$\mathrm{Val}(u) = \frac{\int_0^1 \mathrm{Average}([u]_\alpha) f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}, \quad (2)$$

where f is a mapping from $[0,1]$ to $[0,1]$ and

$$\mathrm{Average}([u]_\alpha) = \frac{\underline{u}(\alpha) + \overline{u}(\alpha)}{2}.$$

If $f(\alpha) = \alpha^q$, with $q \to \infty$, then the valuation is

$$\mathrm{Val}(u) = \mathrm{Average}([u]_1) = \frac{\underline{u}(1) + \overline{u}(1)}{2}, \quad (3)$$

that is the average of the core. If $f(\alpha) = (1-\alpha)^q$, with $q \to \infty$, then the valuation is

$$\mathrm{Val}(u) = \mathrm{Average}([u]_0) = \frac{\underline{u}(0) + \overline{u}(0)}{2}, \quad (4)$$

that is the average of the support (Abbasbandy et al. 2010).

The following version of the well-known Karush–Kuhn-Tucker theorem is useful in the proof of the main result of the paper.

Theorem 1 (Rockafellar 1970, pp. 281–283) *Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable functions. Then $\overline{x}$ solves the convex programming problem*

$$\begin{aligned}&\text{Minimize} \quad f(x) \\ &\text{s.t.} g_i(x) \le b_i, i \in \{1,\ldots,m\}\end{aligned} \quad (5)$$

if and only if there exists $\mu_i, i \in \{1,\ldots,m\}$, such that

(i) $\bigtriangledown f(\overline{x}) + \sum_{i=1}^{m} \mu_i \bigtriangledown g_i(\overline{x}) = 0$;
(ii) $g_i(\overline{x}) - b_i \le 0$;
(iii) $\mu_i \ge 0$;
(iv) $\mu_i(b_i - g_i(\overline{x})) = 0$.

3 Incomplete results and examples

Abbasbandy et al. (2010) obtained a triangular approximation $T(u) = (t_l(u), t_c(u), t_r(u))$ of a general fuzzy number u by a minimization technique as follows:

(i) The core $t_c(u)$ is determined by minimizing the quantity

$$S = (\mathrm{Val}(u) - \mathrm{Val}(T(u)))^2$$

such that $\mathrm{Val}(u)$ is given by (2) with $f(\alpha) = \alpha^q, q \ge 0$, then q approaches infinity such that to place more emphasis on the 1-level set.

(ii) The support $[t_l(u), t_r(u)]$ is obtained by solving the mathematical programming problem

$$\text{Minimize} \quad P = D_1^2(u, T(u)) + w D_2^2(u, T(u)) \quad (6)$$

where

$$D_1(u, T(u)) = |\mathrm{Val}(u) - \mathrm{Val}(T(u))|, \tag{7}$$

D_2 is given by (1) and $w > 0$ is a large number which is determined by the decision maker. In the calculus $f(\alpha) = (1-\alpha)^q, q \geq 0$, then q approaches infinity to place more emphasis on the support.

The following result is given:

Theorem 2 *The nearest triangular fuzzy number* $T(u) = (t_l(u), t_c(u), t_r(u))$

to a fuzzy number u exists, is unique, and it is given by

$$t_l(u) = \frac{1}{6+4w}\left(6B - 2wt_c(u) - 9\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (9+12w)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right) \tag{8}$$

$$t_c(u) = \mathrm{Average}([u]_1) = \frac{\underline{u}(1) + \overline{u}(1)}{2}, \tag{9}$$

$$t_r(u) = \frac{1}{6+4w}\left(6B - 2wt_c(u) - 9\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (9+12w)\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right), \tag{10}$$

where

$$B = \frac{\underline{u}(0) + \overline{u}(0)}{2}.$$

The authors proved $t_l(u) \leq t_r(u)$, for every fuzzy number u (Abbasbandy et al. 2010, Theorem 4.2). Nevertheless, the inequalities $t_c(u) < t_l(u)$ and $t_r(u) < t_c(u)$ are possible, as the following examples prove, such that (8)–(10) do not always give a triangular fuzzy number.

Example 1 Let us consider $w = 10$, as in the examples given in Abbasbandy et al. (2010) and the fuzzy number u given by $\underline{u}(\alpha) = \sqrt{\alpha}$ and $\overline{u}(\alpha) = 92 - 90\sqrt{\alpha}, \alpha \in [0,1]$. According to (8)–(10) we obtain

$$t_l(u) = \frac{206}{115} > \frac{3}{2} = t_c(u).$$

Example 2 If v is the fuzzy number given by $\underline{v}(\alpha) = 98\sqrt{\alpha}$ and $\overline{v}(\alpha) = 100 - \sqrt{\alpha}, \alpha \in [0,1]$ and $w = 10$ then (8)–(10) imply

$$t_c(v) = \frac{197}{2} > \frac{11276}{115} = t_r(v).$$

In the same paper (Abbasbandy et al. 2010) the aforementioned method is generalized by considering instead of valuation $\mathrm{Val}(u)$ the real number $\mathrm{Val}_p(u)$ associated with the fuzzy number u,

$$\mathrm{Val}_p(u) = p\frac{\int_0^p \mathrm{Average}([u]_\alpha)\alpha^q \mathrm{d}\alpha}{\int_0^p \alpha^q \mathrm{d}\alpha} + (1-p)\frac{\int_p^1 \mathrm{Average}([u]_\alpha)(1-\alpha)^q \mathrm{d}\alpha}{\int_p^1 (1-\alpha)^q \mathrm{d}\alpha}.$$

When $q \to +\infty$ the authors obtained the triangular fuzzy number $T^*(u) = (t_l^*(u), t_c^*(u), t_r^*(u))$ which is the nearest to u, as follows:

$$t_l^*(u) = \frac{-1}{6p^2 - 12p + 4w + 6} \times \left\{ 6\mathrm{Val}_p(u)(p-1) + t_c(u)(-6p^2 + 6p + 2w) + \int_0^1 \underline{u}(\alpha)(1-\alpha)(-9p^2 + 18p - 9 - 12w)\mathrm{d}\alpha + \int_0^1 \overline{u}(\alpha)(1-\alpha)(9p^2 - 18p + 9)\mathrm{d}\alpha \right\},$$

$$t_c^*(u) = t_c(u) = \frac{\underline{u}(1) + \overline{u}(1)}{2},$$

$$t_r^*(u) = \frac{-1}{6p^2 - 12p + 4w + 6} \times \left\{ 6\mathrm{Val}_p(u)(p-1) + t_c(u)(-6p^2 + 6p + 2w) + \int_0^1 \overline{u}(\alpha)(1-\alpha)(-9p^2 + 18p - 9 - 12w)\mathrm{d}\alpha + \int_0^1 \underline{u}(\alpha)(1-\alpha)(9p^2 - 18p + 9)\mathrm{d}\alpha \right\}.$$

Unfortunately, even if

$$t_r^*(u) - t_l^*(u) = \frac{18p^2 - 36p + 18 + 12w}{6p^2 - 12p + 4w + 6} \times \left(\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \right) \geq 0,$$

for every fuzzy number u, the inequalities $t_c^*(u) < t_l^*(u)$ and $t_r^*(u) < t_c^*(u)$ are possible (see the below example) such that $T^*(u)$ is not always a fuzzy number.

Example 3 Let $w = 10, p = \frac{1}{2}$ and u be the fuzzy number given by

$$\underline{u}(\alpha) = \alpha^2,$$
$$\overline{u}(\alpha) = c - \alpha^2(c-1), \alpha \in [0,1],$$

where $c > \frac{836}{3}$. Because

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{1}{12},$$

$$\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{5c+1}{12},$$

$$\mathrm{Val}_{\frac{1}{2}}(u) = \frac{c+2}{4} + \frac{(c-2)(q+1)}{4(q+2)} + \frac{(2-c)(q+1)}{8(q+3)} \overset{q\to\infty}{\to} \frac{3c+2}{8}$$

we get

$$t_c^*(u) = 1$$

and

$$t_l^*(u) = \frac{3c-172}{664} > 1.$$

If $w = 5, p, u$ as above and $c > \frac{436}{3}$ we obtain

$$t_c^*(u) = 1$$

and

$$t_l^*(u) = \frac{3c-92}{344} > 1.$$

4 Main result and examples

In this section, we note the triangular fuzzy number $T_q(u)$ which is the nearest to fuzzy number u in the sense of (6) by $(t_{l,q}(u), t_{c,q}(u), t_{r,q}(u))$, or $(t_{l,q}, t_{c,q}, t_{r,q})$ if no confusion is possible. To give a complete solution of the problem we must impose

$$t_{l,q} \le t_{c,q} \le t_{r,q}.$$

Because (Abbasbandy et al. 2010)

$$\mathrm{Val}(T_q(u)) = \frac{t_{c,q}}{q+2} + \frac{q+1}{2(q+2)}(t_{l,q} + t_{r,q})$$

when $f(\alpha) = (1-\alpha)^q, q \ge 0$, the problem becomes to minimize the function

$$P(t_{l,q}, t_{r,q}) = \left(\mathrm{Val}(u) - \frac{t_{c,q}}{q+2} - \frac{(q+1)(t_{l,q}+t_{r,q})}{2(q+2)}\right)^2 + w\int_0^1 (\underline{u}(\alpha) - (t_{c,q} - (1-\alpha)(t_{c,q} - t_{l,q})))^2\mathrm{d}\alpha + w\int_0^1 (\overline{u}(\alpha) - (t_{c,q} + (1-\alpha)(t_{r,q} - t_{c,q})))^2\mathrm{d}\alpha, \quad (11)$$

such that

$$t_{l,q} \le t_{c,q} \quad (12)$$

and

$$t_{c,q} \le t_{r,q}, \quad (13)$$

where $\mathrm{Val}(u)$ is given by (2) with $f(\alpha) = (1-\alpha)^q, q \ge 0$ and

$$t_{c,q} = t_c = \frac{\underline{u}(1) + \overline{u}(1)}{2}. \quad (14)$$

The main result of the paper is

Theorem 3 *Let* $u, [u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0,1]$, *be a fuzzy number and*

$$[t_{l,q}(u), t_{r,q}(u)] = [t_{l,q}, t_{r,q}]$$

the support of the fuzzy number $T_q(u) = (t_{l,q}, t_{c,q}, t_{r,q})$, *the nearest triangular fuzzy number of* u *with respect to* (6) *such that* $t_{c,q} = t_c$ *is given by* (9), *the weighted value Val is calculated by* (2) *with* $f(\alpha) = (1-\alpha)^q, q \ge 0$ *and* $w > 0$ *is a number which is determined by the decision maker.*

(i) *If*

$$\frac{q+1}{q+2}\mathrm{Val}(u) - \left(w + \frac{q+1}{q+2}\right)t_c - \frac{3(q+1)^2}{2(q+2)^2}\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \left(2w + \frac{3(q+1)^2}{2(q+2)^2}\right)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \le 0 \quad (15)$$

and

$$\frac{q+1}{q+2}\mathrm{Val}(u) - \left(w + \frac{q+1}{q+2}\right)t_c + \left(2w + \frac{3(q+1)^2}{2(q+2)^2}\right) \times \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \frac{3(q+1)^2}{2(q+2)^2}\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \ge 0 \quad (16)$$

then

$$t_{l,q} = \frac{3(q+2)^2}{3(q+1)^2 + 2w(q+2)^2}\left\{\frac{q+1}{q+2}\mathrm{Val}(u) - \left(\frac{1}{3}w + \frac{q+1}{(q+2)^2}\right)t_c - \frac{3(q+1)^2}{2(q+2)^2}\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \left(2w + \frac{3(q+1)^2}{2(q+2)^2}\right)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\} \quad (17)$$

and

$$t_{r,q}=\frac{3(q+2)^2}{3(q+1)^2+2w(q+2)^2}\left\{\frac{q+1}{q+2}\mathrm{Val}(u)-\left(\frac{1}{3}w+\frac{q+1}{(q+2)^2}\right)t_c+\left(2w+\frac{3(q+1)^2}{2(q+2)^2}\right)\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\frac{3(q+1)^2}{2(q+2)^2}\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\};$$

(ii) *If*

$$\frac{q+1}{q+2}\mathrm{Val}(u)-\left(w+\frac{q+1}{q+2}\right)t_c+\left(2w+\frac{3(q+1)^2}{2(q+2)^2}\right)\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\frac{3(q+1)^2}{2(q+2)^2}\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha>0 \tag{19}$$

then

$$t_{l,q}=t_c$$

and

$$t_{r,q}=\frac{6(q+2)^2}{3(q+1)^2+4w(q+2)^2}\left\{\frac{q+1}{q+2}\mathrm{Val}(u)+2w\times\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\left(\frac{(q+1)(q+3)}{2(q+2)^2}+\frac{1}{3}w\right)t_c\right\};$$

(iii) *If*

$$\frac{q+1}{q+2}\mathrm{Val}(u)-\left(w+\frac{q+1}{q+2}\right)t_c+\left(2w+\frac{3(q+1)^2}{2(q+2)^2}\right)\times\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\frac{3(q+1)^2}{2(q+2)^2}\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha<0 \tag{20}$$

then

$$t_{l,q}=\frac{6(q+2)^2}{3(q+1)^2+4w(q+2)^2}\left\{\frac{q+1}{q+2}\mathrm{Val}(u)+2w\times\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\left(\frac{(q+1)(q+3)}{2(q+2)^2}+\frac{1}{3}w\right)t_c\right\}$$

and

$$t_{r,q}=t_c.$$

Proof If $f\equiv P$ in (11) and

$$g_1(t_{l,q},t_{r,q})=t_{l,q}-t_{c,q},$$
$$g_2(t_{l,q},t_{r,q})=t_{c,q}-t_{r,q}$$

then the hypothesis of convexity and differentiability in the Karush–Kuhn–Tucker theorem are satisfied. Because

$$\begin{aligned}\nabla P(t_{l,q},t_{r,q})&=\left(\frac{\partial P}{\partial t_{l,q}},\frac{\partial P}{\partial t_{r,q}}\right)\\&=\Bigg(-\frac{q+1}{q+2}\mathrm{Val}(u)+\frac{q+1}{(q+2)^2}t_c+wt_c+\frac{(q+1)^2}{2(q+2)^2}\times(t_{l,q}+t_{r,q})-2w\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\frac{2}{3}w(t_c-t_{l,q}),\\&\quad-\frac{q+1}{q+2}\mathrm{Val}(u)+\frac{q+1}{(q+2)^2}t_c+wt_c+\frac{(q+1)^2}{2(q+2)^2}\times(t_{l,q}+t_{r,q})-2w\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha+\frac{2}{3}w(t_{r,q}-t_c)\Bigg)\end{aligned}$$

$$\nabla g_1(t_{l,q},t_{r,q})=\left(\frac{\partial g_1}{\partial t_{l,q}},\frac{\partial g_1}{\partial t_{r,q}}\right)=(1,0),$$
$$\nabla g_2(t_{l,q},t_{r,q})=\left(\frac{\partial g_2}{\partial t_{l,q}},\frac{\partial g_2}{\partial t_{r,q}}\right)=(0,-1)$$

and $b_1=b_2=0$, the conditions (i)–(iv) in Theorem 1, with respect to the minimization problem (11)–(13) become

$$-\frac{q+1}{q+2}\mathrm{Val}(u)+\frac{q+1}{(q+2)^2}t_c+wt_c+\frac{(q+1)^2}{2(q+2)^2}(t_{l,q}+t_{r,q})-2w\int_0^1\underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha-\frac{2}{3}w(t_c-t_{l,q})+\mu_1=0, \tag{21}$$

$$-\frac{q+1}{q+2}\mathrm{Val}(u)+\frac{q+1}{(q+2)^2}t_c+wt_c+\frac{(q+1)^2}{2(q+2)^2}(t_{l,q}+t_{r,q})-2w\int_0^1\overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha+\frac{2}{3}w(t_{r,q}-t_c)-\mu_2=0, \tag{22}$$

$$\mu_1(t_c-t_{l,q})=0, \tag{23}$$

$$\mu_2(-t_c+t_{r,q})=0, \tag{24}$$

$$\mu_1\ge 0, \tag{25}$$

$$\mu_2\ge 0, \tag{26}$$

$$t_{l,q} - t_c \le 0, \quad (27)$$

$$-t_{r,q} + t_c \le 0. \quad (28)$$

(i) If $\mu_1 = \mu_2 = 0$ then (21) and (22) imply $t_{l,q}$ and $t_{r,q}$ are given by (17) and (18). Conditions (25), (26) are satisfied and (27) and (28) are equivalent to (15) and (16), respectively.

(ii) If $\mu_1 \neq 0$ and $\mu_2 = 0$ then the solution of the system (21)–(24) is given by

$$t_{l,q} = t_c,$$

$$t_{r,q} = \frac{6(q+2)^2}{3(q+1)^2 + 4w(q+2)^2} \left\{ \frac{q+1}{q+2} \mathrm{Val}(u) + 2w \times \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \left(\frac{(q+1)(q+3)}{2(q+2)^2} + \frac{1}{3}w \right) t_c \right\},$$

$$\mu_1 = \frac{4w(q+2)^2}{3(q+1)^2 + 4w(q+2)^2} \left\{ \frac{q+1}{q+2} \mathrm{Val}(u) - \left(w + \frac{q+1}{q+2} \right) t_c + \left(2w + \frac{3(q+1)^2}{2(q+2)^2} \right) \times \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \frac{3(q+1)^2}{2(q+2)^2} \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \right\},$$

$$\mu_2 = 0.$$

Conditions (26) and (27) are satisfied. If (19), which is equivalent to $\mu_1 > 0$ is satisfied, then

$$t_{r,q} - t_c = \frac{3}{2w}\mu_1 + 3 \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - 3 \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha > 0,$$

that is (28) is verified too.

(iii) If $\mu_1 = 0$ and $\mu_2 \neq 0$ then the solution of the system (21)–(24) is given by

$$t_{l,q} = \frac{6(q+2)^2}{3(q+1)^2 + 4w(q+2)^2} \left\{ \frac{q+1}{q+2} Val(u) + 2w \times \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \left(\frac{(q+1)(q+3)}{2(q+2)^2} + \frac{1}{3}w \right) t_c \right\},$$

$$t_{r,q} = t_c,$$

$$\mu_1 = 0,$$

$$\mu_2 = -\frac{4w(q+2)^2}{3(q+1)^2 + 4w(q+2)^2} \left\{ \frac{q+1}{q+2} \mathrm{Val}(u) - \left(w + \frac{q+1}{q+2} \right) t_c - \frac{3(q+1)^2}{2(q+2)^2} \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \left(2w + \frac{3(q+1)^2}{2(q+2)^2} \right) \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \right\}.$$

Conditions (25) and (28) are satisfied. If (20), which is equivalent to $\mu_2 > 0$ is satisfied, then

$$t_{l,q} - t_c = -\frac{3}{2w}\mu_2 + 3 \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - 3 \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha < 0,$$

that is (27) is verified too.

(iv) If $\mu_1 \neq 0$ and $\mu_2 \neq 0$, then the solution of the system (21)–(24) is given by

$$t_{l,q} = t_c,$$

$$t_{r,q} = t_c,$$

$$\mu_1 = \frac{q+1}{q+2} \mathrm{Val}(u) - \left(w + \frac{q+1}{q+2} \right) t_c + 2w \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha,$$

$$\mu_2 = -\frac{q+1}{q+2} \mathrm{Val}(u) + \left(w + \frac{q+1}{q+2} \right) t_c - 2w \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha.$$

Because

$$\mu_1 + \mu_2 = 2w \left\{ \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \right\} \le 0$$

(25) and (26) are not satisfied; therefore we have not a solution of the problem in this case.

In what follows we prove that to any fuzzy number we can apply one and only one of the above situations (i)–(iii). Let us denote

$$\Omega_1 = \{u \in F(\mathbb{R}) : \text{case } (i) \text{ is applicable to } u\},$$

$$\Omega_2 = \{u \in F(\mathbb{R}) : \text{case } (ii) \text{ is applicable to } u\},$$

$$\Omega_3 = \{u \in F(\mathbb{R}) : \text{case } (iii) \text{ is applicable to } u\},$$

$$M = \{u \in F(\mathbb{R}) : u \text{ satisfies } (15)\},$$

$$N = \{u \in F(\mathbb{R}) : u \text{ satisfies } (16)\}.$$

Then

$$\Omega_1 = M \cap N,$$

$$\Omega_2 = M^c$$

and

$$\Omega_3 = N^c;$$

therefore

$$\Omega_1 \cap \Omega_2 = \Omega_1 \cap \Omega_3 = \emptyset$$

and

$$\Omega_1 \cup \Omega_2 \cup \Omega_3 = F(\mathbb{R}).$$

If $u \in \Omega_2 \cap \Omega_3$, then u satisfies both relations (19) and (20). We immediately obtain

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha > 0,$$

which contradicts $\underline{u}(\alpha) \le \overline{u}(\alpha)$, for every $\alpha \in [0,1]$; therefore $\Omega_2 \cap \Omega_3 = \emptyset$.

Because the larger q the more emphasis we give to the lower level sets when $f(\alpha) = (1-\alpha)^q$ and the aim is to obtain the support of $T_q(u)$, the case $q \to \infty$ is considered in Abbasbandy et al. (2010). We get

$$\mathrm{Val}(u) = \mathrm{Average}([u]_0) = \frac{\underline{u}(0) + \overline{u}(0)}{2}$$

and the following important consequence of Theorem 3.

Corollary 1 (i) *If*

$$\underline{u}(0) + \overline{u}(0) + (4w+3)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - (1+w) \times (\underline{u}(1) + \overline{u}(1)) - 3\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \le 0 \quad (29)$$

and

$$\underline{u}(0) + \overline{u}(0) + (4w+3)\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - (1+w) \times (\underline{u}(1) + \overline{u}(1)) - 3\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \ge 0 \quad (30)$$

then

$$t_{l,\infty} = \frac{1}{6+4w}\left\{3(\underline{u}(0) + \overline{u}(0)) - w(\underline{u}(1) + \overline{u}(1)) - 9\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + (9+12w)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}, \quad (31)$$

$$t_{r,\infty} = \frac{1}{6+4w}\left\{3(\underline{u}(0) + \overline{u}(0)) - w(\underline{u}(1) + \overline{u}(1)) + (9+12w)\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - 9\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\}; \quad (32)$$

(ii) *If*

$$\underline{u}(0) + \overline{u}(0) + (4w+3)\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - (1+w)(\underline{u}(1) + \overline{u}(1)) - 3\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha > 0$$

then

$$t_{l,\infty} = \frac{1}{2}(\underline{u}(1) + \overline{u}(1)),$$

$$t_{r,\infty} = \frac{1}{3+4w}\left\{3(\underline{u}(0) + \overline{u}(0)) - \left(\frac{3}{2} + w\right) \times (\underline{u}(1) + \overline{u}(1)) + 12w\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\};$$

(iii) *If*

$$\underline{u}(0) + \overline{u}(0) + (4w+3)\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - (1+w)(\underline{u}(1) + \overline{u}(1)) - 3\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha < 0$$

then

$$t_{l,\infty} = \frac{1}{3+4w}\left\{3(\underline{u}(0) + \overline{u}(0)) - \left(\frac{3}{2} + w\right) \times (\underline{u}(1) + \overline{u}(1)) + 12w\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha\right\},$$

$$t_{r,\infty} = \frac{1}{2}(\underline{u}(1) + \overline{u}(1)).$$

Remark 1 Corollary 1 completes Theorem 2, the main result in Abbasbandy et al. (2010). The triangular fuzzy number given in Corollary 1, (i), under conditions (29) and (30), is even the entity, which is not always a fuzzy number, in Theorem 2.

Example 4 The case (i) in Theorem 3 is applicable to any symmetric fuzzy number. Indeed, the symmetry of u implies

$$t_c - \underline{u}(\alpha) = \overline{u}(\alpha) - t_c, \quad \forall \alpha \in [0,1].$$

Then

$$\mathrm{Val}(u) = t_c$$

and (15) and (16) are both equivalent to

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha - \frac{t_c}{2} \leq 0$$

which is true. In addition, by applying (17) and (18) we obtain

$$t_{l,q} = -\frac{1}{2}t_c + 3\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha \tag{33}$$

and

$$t_{r,q} = -\frac{1}{2}t_c + 3\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha, \tag{34}$$

that is the triangular approximation is symmetric too.

Example 5 If $w = 10, q = 1$ and u is the fuzzy number given by

$$\underline{u}(\alpha) = \alpha^2,$$
$$\overline{u}(\alpha) = 4 - 2\alpha^2, \quad \alpha \in [0,1]$$

then

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{1}{12},$$
$$\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{11}{6},$$
$$\mathrm{Val}(u) = \frac{23}{12}$$

and (15), (16) are satisfied. Theorem 3, (*i*) is applicable and

$$T_1(u) = \left(-\frac{1}{2}, \frac{3}{2}, \frac{19}{4}\right).$$

Example 6 Case (ii) in Corollary 1 is applicable to fuzzy number u in Example 1 ($w = 10$) and

$$T_\infty(u) = \left(\frac{3}{2}, \frac{3}{2}, \frac{5763}{86}\right).$$

Example 7 Case (iii) in Corollary 1 is applicable to fuzzy number v in Example 2 ($w = 10$) and

$$T_\infty(v) = \left(\frac{2341}{86}, \frac{197}{2}, \frac{197}{2}\right).$$

5 Properties

In this section we discuss some important properties of the triangular approximation given in Theorem 3: identity, invariance to translations, scale invariance, additivity, monotonicity. They were suggested to be important in Grzegorzewski and Mrówka (2005) and studied in Abbasbandy et al. (2010) for the operator given by Theorem 2.

Theorem 4 *If $u = (a, b, c)$ is a triangular fuzzy number then $T_q(u) = u$.*

Proof Because

$$\underline{u}(\alpha) = a + \alpha(b-a),$$
$$\overline{u}(\alpha) = c - \alpha(c-b), \quad \alpha \in [0,1]$$

we get

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{2a+b}{6},$$
$$\int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{b+2c}{6}$$

and

$$\mathrm{Val}(u) = \frac{b}{q+2} + \frac{q+1}{2(q+2)}(a+c).$$

Conditions (15) and (16) are equivalent to

$$\left(\frac{2}{3}w + \frac{(q+1)^2}{(q+2)^2}\right)(a-b) \leq 0$$

and

$$\left(\frac{2}{3}w + \frac{(q+1)^2}{(q+2)^2}\right)(c-b) \geq 0,$$

respectively, that is case (i) in Theorem 3 is applicable to any triangular fuzzy number. In addition, (17) and (18) give us $t_{l,q}(u) = a$ and $t_{r,q}(u) = c$.

The property of additivity is not satisfied for the approximation operator in Theorem 3:

Example 8 If u is the fuzzy number in Example 1 and v is the fuzzy number in Example 2, taking into account the results in Examples 6 and 7 we obtain

$$T_\infty(u) + T_\infty(v) = \left(\frac{1235}{43}, 100, \frac{7117}{43}\right).$$

Because

$$(\underline{u+v})(\alpha) = 99\sqrt{\alpha},$$
$$(\overline{u+v})(\alpha) = 192 - 91\sqrt{\alpha}, \quad \alpha \in [0,1]$$

and

$$\int_0^1 (\underline{u+v})(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{132}{5},$$
$$\int_0^1 (\overline{u+v})(\alpha)(1-\alpha)\mathrm{d}\alpha = \frac{1076}{15}$$

we get that the case (i) in Corollary 1 is applicable to fuzzy number $u + v$ and

$$T_\infty(u+v) = \left(\frac{3340}{115}, 100, \frac{3796}{23}\right).$$

As a conclusion, $T_\infty(u) + T_\infty(v) \neq T_\infty(u+v)$.

Nevertheless, the property of invariance to translations and a property of partial additivity can be proved.

Theorem 5 *The nearest triangular approximation given by Theorem 3 is invariant to translation, i.e.,*

$$T_q(u+z) = T_q(u) + z$$

for all $z \in \mathbb{R}$ and $u \in F(\mathbb{R})$.

Proof Let $z \in \mathbb{R}$ and $u \in F(\mathbb{R})$. By the definition of addition,

$$(\underline{u+z})(\alpha) = \underline{u}(\alpha) + z$$

and

$$(\overline{u+z})(\alpha) = \overline{u}(\alpha) + z,$$

for every $\alpha \in [0,1]$. We get

$$\mathrm{Val}(u+z) = \mathrm{Val}(u) + z,$$
$$t_c(u+z) = t_c(u) + z,$$
$$\int_0^1 (\underline{u+z})(\alpha)(1-\alpha)\mathrm{d}\alpha = \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \frac{z}{2}$$

and

$$\int_0^1 (\overline{u+z})(\alpha)(1-\alpha)\mathrm{d}\alpha = \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \frac{z}{2}.$$

We immediately obtain that u satisfies one of (15), (16), (19) or (20) if and only if the fuzzy number $u + z$ satisfies the same condition. In addition, in every case (i)–(iii) in Theorem 3 we obtain

$$t_{l,q}(u+z) = t_{l,q}(u) + z,$$
$$t_{r,q}(u+z) = t_{r,q}(u) + z,$$

therefore

$$T_q(u+z) = T_q(u) + z.$$

With the notations in the proof of the Theorem 3 we present

Theorem 6 *If* $u, v \in \Omega_i (i \in \{1,2,3\})$ then

$$T_q(u) + T_q(v) = T_q(u+v).$$

Proof If $u, v \in \Omega_i$ then $u + v \in \Omega_i$, for every $i \in \{1,2,3\}$, because the definition of addition implies

$$\mathrm{Val}(u+v) = \mathrm{Val}(u) + \mathrm{Val}(v),$$
$$t_c(u+v) = t_c(u) + t_c(v),$$
$$\int_0^1 (\underline{u+v})(\alpha)(1-\alpha)\mathrm{d}\alpha = \int_0^1 \underline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \int_0^1 \underline{v}(\alpha)(1-\alpha)\mathrm{d}\alpha$$

and

$$\int_0^1 (\overline{u+v})(\alpha)(1-\alpha)\mathrm{d}\alpha = \int_0^1 \overline{u}(\alpha)(1-\alpha)\mathrm{d}\alpha + \int_0^1 \overline{v}(\alpha)(1-\alpha)\mathrm{d}\alpha.$$

In addition,

$$t_{l,q}(u+v) = t_{l,q}(u) + t_{l,q}(v)$$

and

$$t_{r,q}(u+v) = t_{r,q}(u) + t_{r,q}(v),$$

in every case (i)–(iii) in Theorem 3.

The scale invariance is an important property too.

Theorem 7 *The nearest triangular approximation given by Theorem 3 is scale invariant, i.e.,*

$$T_q(\lambda \cdot u) = \lambda \cdot T_q(u),$$

for all $\lambda \in \mathbb{R}\backslash\{0\}$ *and* $u \in F(\mathbb{R})$.

Proof Let λ be a non-zero real number and $u \in F(\mathbb{R})$. If $\lambda > 0$ then the proof is immediate because u verifies (15), (16), (19) or (20) if and only if $\lambda \cdot u$ verifies the same condition. In addition,

$$\mathrm{Val}(\lambda \cdot u) = \lambda \mathrm{Val}(u),$$
$$t_c(\lambda \cdot u) = \lambda t_c(u),$$

then

$$t_{l,q}(\lambda \cdot u) = \lambda t_{l,q}(u)$$

and

$$t_{r,q}(\lambda \cdot u) = \lambda t_{r,q}(u),$$

in every case (i)–(iii) in Theorem 3.

In the case $\lambda < 0$ we also obtain

$$\mathrm{Val}(\lambda \cdot u) = \lambda \mathrm{Val}(u),$$
$$t_c(\lambda \cdot u) = \lambda t_c(u)$$

and it is easy to prove that u satisfies (15) [(16), (19), (20)] if and only $\lambda \cdot u$ satisfies (16) [(15), (20), (19), respectively]. As a conclusion, with the notations in Theorem 3, $u \in \Omega_1$ if and only if $\lambda \cdot u \in \Omega_1$, $u \in \Omega_2$ if and only if $\lambda \cdot u \in \Omega_3$ and $u \in \Omega_3$ if and only if $\lambda \cdot u \in \Omega_2$. In each case (i)–(iii), Theorem 3 we obtain

$$t_{l,q}(\lambda \cdot u) = \lambda t_{r,q}(u)$$

and

$$t_{r,q}(\lambda \cdot u) = \lambda t_{l,q}(u),$$

that is, $T_q(\lambda \cdot u) = \lambda \cdot T_q(u)$.

Let us consider the relation between fuzzy numbers given by

$$v \subseteq u \quad \text{iff } \underline{u}(\alpha) \leq \underline{v}(\alpha), \quad \overline{v}(\alpha) \leq \overline{u}(\alpha), \quad \forall \alpha \in [0,1],$$

where $[u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$, $[v]_\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in [0,1]$. The approximation operator given by Theorem 3 is not invariant with respect to $\subseteq$ (see the below examples), but a similar result with Theorem 5.4 in Abbasbandy et al. (2010) can be proved.

Example 9 If v is the triangular fuzzy number $(1, \frac{3}{2}, 2)$ and u is the fuzzy number given in Example 1, then $v \subseteq u$. According to Example 6 and Theorem 4 ($w = 10$)

$$T_\infty(u) = \left(\frac{3}{2}, \frac{3}{2}, \frac{5763}{86}\right),$$

and

$$T_\infty(v) = \left(1, \frac{3}{2}, 2\right),$$

that is, $T_\infty(v) \nsubseteq T_\infty(u)$.

The case $q = +\infty$ is not an exception, as the following example proves:

Example 10 We consider $w = 10$ and $q = 5$. If u is given by $\underline{u}(\alpha) = \sqrt{\alpha}$ and $\overline{u}(\alpha) = 432 - 430\sqrt{\alpha}$ and v as above, then $v \subseteq u$ and $T_5(v) = (1, \frac{3}{2}, 2)$. We get

$$t_c(u) = \frac{3}{2},$$

$$\int_0^1 \underline{u}(\alpha)(1-\alpha)d\alpha = \frac{4}{15}$$

$$\int_0^1 \overline{u}(\alpha)(1-\alpha)d\alpha = \frac{304}{3}$$

and

$$\mathrm{Val}(u) = \frac{1000}{7}.$$

The fuzzy number u satisfies the condition (19) and Theorem 3, (ii) implies

$$T_5(u) = \left(\frac{3}{2}, \frac{3}{2}, \frac{315077}{1034}\right),$$

that is, $T_5(v) \nsubseteq T_5(u)$.

Theorem 8 *If $u, v \in F(\mathbb{R})$ are symmetric and*

$$\underline{u}(1) + \overline{u}(1) = \underline{v}(1) + \overline{v}(1)$$

then $u \subseteq v$ implies

$$T_q(u) \subseteq T_q(v).$$

Proof Taking into account $t_c(u) = t_c(v)$ and (33), (34) we get

$$\begin{aligned} t_{l,q}(u) &= -\frac{1}{2}t_c(u) + 3\int_0^1 \underline{u}(\alpha)(1-\alpha)d\alpha \\ &\geq -\frac{1}{2}t_c(v) + 3\int_0^1 \underline{v}(\alpha)(1-\alpha)d\alpha \\ &= t_{l,q}(v) \end{aligned}$$

and

$$\begin{aligned} t_{r,q}(u) &= -\frac{1}{2}t_c(u) + 3\int_0^1 \overline{u}(\alpha)(1-\alpha)d\alpha \\ &\leq -\frac{1}{2}t_c(v) + 3\int_0^1 \overline{v}(\alpha)(1-\alpha)d\alpha \\ &= t_{r,q}(v), \end{aligned}$$

that is, $T_q(u) \subseteq T_q(v)$.

6 Conclusion

The main result of the paper is Theorem 3. It gives a triangular approximation of a fuzzy number using α-weighted valuations and corrects the recent results in Abbasbandy et al. (2010). Some properties of the triangular approximation operator are discussed. To compare the present method of approximation with other methods, some additional properties, like continuity, will be studied in future articles.

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