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ORIGINAL PAPER

The variety generated by semi-Heyting chains

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Abstract The purpose of this paper was to investigate the structure of semi-Heyting chains and the variety $\mathcal{CSH}$ generated by them. We determine the number of non-isomorphic n-element semi-Heyting chains. As a contribution to the study of the lattice of subvarieties of $\mathcal{CSH}$, we investigate the inclusion relation between semi-Heyting chains. Finally, we provide equational bases for $\mathcal{CSH}$ and for the subvarieties of $\mathcal{CSH}$ introduced in [5].

Keywords Heyting algebras · Varieties · Semi-Heyting algebras

1 Introduction and preliminaries

In [5], Sankappanavar introduced a new equational class of algebras, which he called "*semi-Heyting Algebras*", as an abstraction of Heyting algebras. This variety includes Heyting algebras and shares with them some rather strong properties. For example, the variety of semi-Heyting is arithmetical; semi-Heyting algebras are pseudocomplemented distributive lattices, and their congruences are determined by filters.

J. M. Cornejo dedicates this work to his wife, Carina Foresi.

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The purpose of this paper was to investigate the properties of semi-Heyting chains and the structure of the variety $\mathcal{CSH}$ generated by all semi-Heyting chains. In [5], Sankappanavar posed the following problem: for $\mathcal{V}$ a subvariety of semi-Heyting algebras and n a natural number, if $f(\mathcal{V}, n)$ denotes the number of non-isomorphic n-element semi-Heyting chains in $\mathcal{V}$, find a formula for $f(\mathcal{V}, n)$. In Sect. 2, we prove some results on semi-Heyting chains and find $f(\mathcal{SH}, n)$, i.e., we determine the number of non-isomorphic structures of semi-Heyting algebra that can be defined over an n-element chain. Section 3 is devoted to investigate the behavior of the subalgebras of a given semi-Heyting chain. Finally, we find equational bases for many subvarieties of $\mathcal{CSH}$ in Sect. 4, solving the problems 14.6–14.10 posed in [5].

We start by recalling some definitions and results. For basic notation and results, the reader is referred to [1–4].

Definition 1.1 An algebra $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

(SH1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1;
(SH2) $x \wedge (x \rightarrow y) \approx x \wedge y$;
(SH3) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$;
(SH4) $x \rightarrow x \approx 1$.

We will denote by $\mathcal{SH}$ the variety of semi-Heyting algebras.

Semi-Heyting algebras are pseudocomplemented distributive lattices, with the pseudocomplement given by $x^* = x \rightarrow 0$ (see [5]).

Lemma 1.2 [5] *Let* $\mathbf{L} \in \mathcal{SH}$ *and* $a, b \in L$.

(a) *If* $a \rightarrow b = 1$ *then* $a \leq b$.
(b) *If* $a \leq b$ *then* $a \leq a \rightarrow b$.
(c) $a = b$ *if and only if* $a \rightarrow b = b \rightarrow a = 1$.
(d) $1 \rightarrow a = a$.

Proof From $a \to b = 1$ and (SH3), we get $a \wedge 1 = a \wedge b$, i.e., $a = a \wedge b$, and we have (a). For (b), by (SH3) and $a \leq b$ it follows that $a = a \wedge (a \to b) \leq a \to b$. Property (c) is clear. To prove (d), observe that $a = 1 \wedge a = 1 \wedge (1 \to a) = 1 \to a$. □

Congruences on semi-Heyting algebras are determined by filters. Besides, we have the following characterization of subdirectly irreducible algebras in $\mathcal{SH}$ (see [5, Theorem 7.5])

Theorem 1.3 *Let* $\mathbf{L} \in \mathcal{SH}$ *with* $|\mathbf{L}| \geq 2$. *The following are equivalent*:

(a) $\mathbf{L}$ *is subdirectly irreducible*.
(b) $\mathbf{L}$ *has a unique coatom*.

Observe that as a consequence of this theorem, if $\mathbf{L}$ is subdirectly irreducible, then $1 \in L$ is $\vee$-irreducible.

2 Semi-Heyting chains

We say that $\mathbf{L} \in \mathcal{SH}$ is a *semi-Heyting chain* if the lattice reduct of $\mathbf{L}$ is totally ordered. In this section, we prove some results on semi-Heyting chains, and we determine a formula for the number of semi-Heyting structures that can be defined on an n-element chain. We generalize the results of Sankappanavar in [5] for chains with 2, 3, and 4 elements.

The following lemma states that part of the operation table of $\to$ in a semi-Heyting chain is uniquely determined.

Lemma 2.1 [5] *Let* $\mathbf{L}$ *be a semi-Heyting chain*, $a, b \in L$. *If* $a < b$ *then* $b \to a = a$.

Proof From $b \wedge a = b \wedge (b \to a)$ and $a < b$, we obtain $a = b \wedge (b \to a)$. As L is a chain, $a = b$ or $a = b \to a$, so $a = b \to a$. □

The next two lemmas are useful when we have to verify if a given chain is a semi-Heyting algebra.

Lemma 2.2 *Let* $\langle L, \vee, \wedge, \to, 0, 1\rangle$ *be a totally ordered bounded lattice* L *with a binary operation* $\to$ *such that if* $a, b \in L$ *with* $a < b$ *then* $b \to a = a$. *The following conditions are equivalent*:

(1)

  (a) $x \to x \approx 1$;
  (b) $x \wedge (x \to y) \approx x \wedge y$;

(2)

  (a) $x \to x \approx 1$;
  (b) If $x < y$ then $x \wedge (x \to y) = x \wedge y$.

Proof We only have to prove that $(2) \Rightarrow (1)$. We want to prove that $a \wedge (a \to b) = a \wedge b$ for every $a, b \in L$. If $a < b$, by (2) (b), $a \wedge (a \to b) = a \wedge b$. If $a = b$ by (2)(a), $a \wedge (a \to a) = a \wedge 1 = a = a \wedge a$. Finally, if $a > b$, by hypothesis $a \to b = b$ and so, $a \wedge (a \to b) = a \wedge b$. □

Lemma 2.3 *Let* $\langle L, \vee, \wedge, \to, 0, 1\rangle$ *be a totally ordered bounded lattice* L *with a binary operation* $\to$ *such that if* $a, b \in L$ *with* $a < b$ *then* $b \to a = a$. *The following conditions are equivalent*

(1) $\langle L, \vee, \wedge, \to, 0, 1\rangle \in \mathcal{SH}$;
(2)

  (a) $x \to x \approx 1$;
  (b) $x \wedge (x \to y) \approx x \wedge y$;
  (c) If $y < x < z$, then $x \wedge (y \to x) = x \wedge (y \to z)$.

Proof

$(1) \Rightarrow (2)$. As $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle \in \mathcal{SH}, \mathbf{L}$ satisfies (2)(a) and (2)(b). Let $a, b, c \in L$ such that $b < a < c$. Then $a \wedge (b \to c) = a \wedge [(a \wedge b) \to (a \wedge c)] = a \wedge (b \to a)$.
$(2) \Rightarrow (1)$. It is enough to prove (SH3). Let $a, b, c \in L$, and suppose that $c < b$.

If $a < c$, then $a \wedge b = a \wedge c = a$. So $a \wedge (b \to c) = a \wedge c = a$ and $a \wedge [(a \wedge b) \to (a \wedge c)] = a \wedge a = a$. If either $c < a < b$ or $c < b < a$, then $a \wedge c < a \wedge b$. Thus, $a \to [(a \wedge b) \to (a \wedge c)] = a \wedge a = a$.
The other cases are similar. □

Lemma 2.4 *Let* $\mathbf{L}$ *be a semi-Heyting chain. The following conditions are equivalent*:

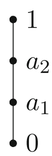
(a) $0 \to a = 0$ *for some* $a \in L$ *with* $a \neq 0$.
(b) $0 \to b = 0$ *for every* $b \in L$ *with* $b \neq 0$.

Proof Suppose that $0 \to a = 0$ for some $a \in L$, $a \neq 0$. $0 = a \wedge 0 = a \wedge (0 \to a) = a \wedge ((0 \wedge a) \to (1 \wedge a)) = a \wedge (0 \to 1)$. As $a \neq 0$ and L is a chain, $0 \to 1 = 0$. Let $b \in L$ such that $b \neq 0$. $0 = b \wedge 0 = b \wedge (0 \to 1) \overset{(SH3)}{=} b \wedge (0 \to b)$. As $b \neq 0$ and L is a chain, $0 \to b = 0$. □

In particular, if $0 \to 1 = 0$, then $0 \to b = 0$ for every $b \in L$, $b \neq 0$.

In order to determine the number of semi-Heyting algebras that can be defined on an n-element chain, let us consider first the following example, since it contains the main ideas underlying in the procedure.

Let $\mathbf{L}$ be a 4-element lattice, $L = \{0, a_1, a_2, 1\}$, with $0 = a_0 < a_1 < a_2 < a_3 = 1$. Consider a binary operation $\to : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$ in such a way that $\langle L, \wedge, \vee, \to, 0, 1\rangle$ is a semi-Heyting chain. From Lemma 2.1, we know that for $x, y \in L$, if $x < y$ then $y \to x = x$. Consequently, the lower half under the main diagonal, including that diagonal, of the operation table of $\to$ of $\mathbf{L}$ is uniquely determined. So, it remains to fill in the upper half of the table, i.e., the elements $x \to y$ for $x < y$.



$\to$	0	a_1	a_2	1
0	1	?	?	?
a_1	0	1	?	?
a_2	0	a_1	1	?
1	0	a_1	a_2	1

Observe that if $x \leq y$ then $x \leq x \to y$ (Lemma 1.2).

If $0 \to 1 = 0$, then by Lemma 2.4, $0 \to a_1 = 0 \to a_2 = 0$. So, the first row in the table is

$\to$	0	a_1	a_2	1
0	1	0	0	0

If $0 \to 1 = a_1$, then $0 \to a_2 = a_2$ and $0 \to a_1 \in [a_1)$, where $[x) = \{y \in L : x \leq y\}$.

Indeed, $a_1 = a_2 \wedge a_1 = a_2 \wedge (0 \to 1) = a_2 \wedge (0 \to a_2)$, so $0 \to a_2 = a_1$; and $a_1 = a_1 \wedge (0 \to 1) \overset{(SH3)}{=} a_1 \wedge (0 \to a_1)$, i.e., $0 \to a_1 \geq a_1$.

So in this case, the first row of the table is

$\to$	0	a_1	a_2	1
0	1	$\geq a_1$	a_1	a_1

In a similar way, it can be seen that if $0 \to 1 = a_2$ or $0 \to 1 = 1$, then $0 \to a_2 \in [a_2)$ and $0 \to a_1 \in [a_1)$. So we have

$\to$	0	a_1	a_2	1
0	1	$\geq a_1$	$\geq a_2$	$\geq a_2$

Then we can write for the first row

$$(0 \to 1, 0 \to a_2, 0 \to a_1) = \begin{cases} (0,0,0), & \\ (a_1,a_1,z), & \text{with } z \in [a_1) \\ (x,y,z), & \text{with } x,y \in [a_2), z \in [a_1), \end{cases}$$

i.e., $(0 \to 1, 0 \to a_2, 0 \to a_1) \in \{\{0\} \times \{0\} \times \{0\}\} \cup \{\{a_1\} \times \{a_1\} \times [a_1)\} \cup \{\{a_2\} \times [a_2) \times [a_1)\} \cup \{\{1\} \times [a_2) \times [a_1)\}$.

For the second and third rows, we would have the following possibilities, determined by $a_1 \to 1$ and $a_2 \to 1$, respectively:

Observe that the behavior of each row is independent of the others, and that, for $x < y$, the element $x \to y$ depends on the element $x \to 1$.

The following lemma shows that in every semi-Heyting chain, the operation table of $\to$ behaves as in the previous example.

Lemma 2.5 *Let* $\mathbf{L}$ *be a semi-Heyting chain and let* $a, b, c \in L$, $a \neq 1$. *If* $a \to 1 = b$ *then for* $c > a$

$$\begin{cases} a \to c = b & \text{if } b < c; \\ a \to c \in [c) & \text{if } b \geq c. \end{cases}$$

Proof Suppose that $b < c$. From $a \to 1 = b$ we get $c \wedge (a \to 1) = c \wedge b = b$, i.e., $c \wedge [(c \wedge a) \to (c \wedge 1)] = b$. Thus, $c \wedge (a \to c) = b$. As L is a chain and $b < c$, then $b = a \to c$.

The case $b \geq c$ is similar. □

Let $\mathbf{L}_n = \langle L_n, \wedge, \vee, 0, 1\rangle$ be an n-element totally ordered lattice:

$0 = a_0 < a_1 < \cdots < a_{n-2} < a_{n-1} = 1$.

Let $\to : \mathbf{L}_n \times \mathbf{L}_n \to \mathbf{L}_n$ be a function such that $\langle L_n, \wedge, \vee, \to, 0, 1\rangle \in \mathcal{SH}$.

Motivated by the previous example, we are going to introduce the following sets:

For $a_j \in L_n$, $0 \leq j \leq n-1$, and for $1 \leq k \leq n-1$,

$$E_k^j = \begin{cases} \{a_j\} & \text{if } \ k < n-j \\ [a_{n-k}) & \text{if } \ k \geq n-j \end{cases}$$

and for $a_i \leq a_j$,

$$E_{a_j}^{a_i} = \prod_{k=1}^{n-i-1} E_k^j$$

Finally, let F_i be the set

$$F_i = \bigcup_{j=i}^{n-1} E_{a_j}^{a_i} \quad \text{with} \quad 0 \leq i < n-1,$$

where F_i would represent the row in the table that corresponds to the elements of the form $a_i \to y$ with $a_i < y$.

In the previous example ($n = 4$),

$$\begin{aligned} F_0 = \bigcup_{j=0}^{3} E_{a_j}^{a_0} &= E_{a_0}^{a_0} \cup E_{a_1}^{a_0} \cup E_{a_2}^{a_0} \cup E_{a_3}^{a_0} = \{E_1^0 \times E_2^0 \times E_3^0\} \\ &\cup \{E_1^1 \times E_2^1 \times E_3^1\} \cup \\ &\{E_1^2 \times E_2^2 \times E_3^2\} \cup \{E_1^3 \times E_2^3 \times E_3^3\} \\ &= \{\{a_0\} \times \{a_0\} \times \{a_0\}\} \cup \{\{a_1\} \times \{a_1\} \times [a_1)\} \\ &\cup \{\{a_2\} \times [a_2) \times [a_1)\} \cup \{\{a_3\} \times [a_2) \times [a_1)\}, \end{aligned}$$

$\to$	0	a_1	a_2	1
a_1	0	1	a_1	a_2

$\to$	0	a_1	a_2	1
a_1	0	1	$\geq a_2$	$\geq a_2$

$\to$	0	a_1	a_2	1
a_2	0	a_1	1	$\geq a_2$

We have $(0 \to 1, 0 \to a_2, 0 \to a_1) \in F_0$, $(a_1 \to 1, a_1 \to a_2) \in F_1$ and $a_2 \to 1 \in F_2$ and we will write $((0 \to 1, 0 \to a_2, 0 \to a_1), (a_1 \to 1, a_1 \to a_2), a_2 \to 1) \in F_0 \times F_1 \times F_2$.

This motivates the introduction of the following set F that would represent the upper half above the main diagonal of the table:

$$F = \prod_{i=0}^{n-2} F_i.$$

Lemma 2.6 $(a_i \to 1; a_i \to a_{n-2}, \ldots, a_i \to a_{i+1}) \in F_i$.

Proof As $a_i \le 1$, then $a_i \to 1 \ge a_i$. Suppose that $a_i \to 1 = a_k$ with $i \le k \le n-1$. We will see that $(a_i \to 1; a_i \to a_{n-2}, \ldots, a_i \to a_{i+1}) \in E_{a_k}^{a_i}$. By Lemma 2.5,

$$a_i \to a_j \begin{cases} = a_k & \text{if} \quad k<j \\ \ge a_j & \text{if} \quad k \ge j \quad \text{with} \quad j > i \end{cases}. \tag{1}$$

We will see that $a_i \to a_j \in E_{n-j}^k$ with $i+1 \le j \le n-1$. If $n-j < n-k$, then $j > k$. Hence, by Eq. 1, $a_i \to a_j = a_k$ and, consequently, $a_i \to a_j \in E_{n-j}^k$. If $n-j \ge n-k$, then $j \le k$. Thus, by Eq. 1, $a_i \to a_j \ge a_j$, and then $a_i \to a_j \in [a_{n-(n-j)})$. So, $a_i \to a_j \in E_{n-j}^k$ and $(a_i \to 1; a_i \to a_{n-2}, \ldots, a_i \to a_{i+1}) \in E_{a_k}^{a_i}$. Therefore, $(a_i \to 1; a_i \to a_{n-2}, \ldots, a_i \to a_{i+1}) \in F_i$. □

Notation 2.7 We denote $\alpha_\to(i) = (a_i \to 1; a_i \to a_{n-2}, \ldots, a_i \to a_{i+1})$ with $0 \le i \le n-2$.

Corollary 2.8 $(\alpha_\to(0), \alpha_\to(1), \ldots, \alpha_\to(n-2)) \in F$.

Now we are going to construct an implication from a given element $x \in F$.

Consider the set F and $x \in F$. Then $x = (x(0), x(1), \ldots, x(n-2))$, where $x(i) \in F_i$ for $0 \le i \le n-2$. Now, for each i, there exists j_i with $i \le j_i \le n-1$ such that $x(i) \in E_{a_{j_i}}^{a_i}$. Hence, $x(i) = (x(i)_{j_i}(1), x(i)_{j_i}(2), \ldots, x(i)_{j_i}(n-i-1))$ with $x(i)_{j_i}(k) \in E_k^{j_i}$, $1 \le k \le n-i-1$. We define in L_n an operation $\Rightarrow$ in the following way:

$$a_r \Rightarrow a_l = \begin{cases} a_l & \text{if} \quad r > l \\ a_{n-1} & \text{if} \quad r = l \\ x_{j_r}(r)(n-l) & \text{if} \quad r < l \end{cases}$$

The following lemma proves that $\Rightarrow$ is a semi-Heyting implication.

Lemma 2.9 $\langle L_n, \wedge, \vee, \Rightarrow, 0, 1\rangle \in \mathcal{SH}$.

Proof First observe that if $r<l$, then $1 \le n-l \le n-r-1$. So $\Rightarrow$ is well defined.

Now, we will see that $\Rightarrow$ it is an implication of semi-Heyting algebras.

By definition of $\Rightarrow, L \models x \Rightarrow x \approx 1$.

Let us see that $L \models x \wedge (x \Rightarrow y) \approx x \wedge y$. Let $x, y \in L$ with $x<y$. Then $x = a_r$ and $y = a_l$ with $0 \le r, l \le n-1$ and $r<l$. Thus, $a_r \Rightarrow a_l = x(r)_{j_r}(n-l)$ that belongs to

$$E_{n-l}^{j_r} = \begin{cases} \{a_{j_r}\} & \text{if} \quad n-l < n-j_r \\ [a_{n-(n-l)}) & \text{if} \quad n-l \ge n-j_r \end{cases}.$$

As $r \le j_r$ and $r<l$, $a_r \wedge (a_r \Rightarrow a_l) = a_r = a_r \wedge a_l$. Hence, $x \wedge (x \Rightarrow y) = x \wedge y$. By Lemma 2.2, $L \models x \wedge (x \Rightarrow y) \approx x \wedge y$.

It is long but computational to prove that $L \models x \wedge (y \Rightarrow z) \approx x \wedge [(x \wedge y) \Rightarrow (x \wedge z)]$.

By Lemma 2.3, $\Rightarrow$ is a semi-Heyting implication. □

From the previous results, we have the following:

Theorem 2.10 *Let* $S_n = \{\to : L_n^2 \to L_n | \langle L_n, \wedge, \vee, \to, 0, 1\rangle \in \mathcal{SH}\}$. *There exists a bijective correspondence between* S_n and F.

Proof We define $\alpha : S_n \to F$ as follows: $\alpha(\to) = (\alpha_\to(0), \alpha_\to(1), \ldots, \alpha_\to(n-2))$ for each $\to \in S_n$. By Corollary 2.8, $\alpha(\to) \in F$. By Lemma 2.9, α is onto.

Let us prove that α is injective. Let $\to_1, \to_2 \in S_n$ such that $\alpha(\to_1) = \alpha(\to_2)$. By Lemma 2.1, if $a_r > a_l$ then $a_r \to_1 a_l = a_l = a_r \to_2 a_l$. If $a_r = a_l$, $a_r \to_1 a_l = 1 = a_r \to_2 a_l$. As $\alpha(\to_1) = \alpha(\to_2)$, $(\alpha_{\to_1}(0), \alpha_{\to_1}(1), \ldots, \alpha_{\to_1}(n-2)) = (\alpha_{\to_2}(0), \alpha_{\to_2}(1), \ldots, \alpha_{\to_2}(n-2))$. Hence, $\alpha{\to_1}(i) = \alpha{\to_2}(i)$ for all i with $0 \le i < n-1$. Then $a_i \to_1 1 = a_i \to_2 1$, $a_i \to_1 a_{n-2} = a_i \to_2 a_{n-2}, \ldots, a_i \to_1 a_{i+1} = a_i \to_2 a_{i+1}$ for all i with $0 \le i < n-1$. Hence, $\to_1 = \to_2$, and α is injective. □

Our next objective is to determine the cardinal of the set F.

Lemma 2.11 $E_{a_j}^{a_i} \cap E_{a_k}^{a_i} = \emptyset$ *with* $j \ne k$.

Proof Let $x \in E_{a_j}^{a_i} \cap E_{a_k}^{a_i}$ and suppose that $j<k$. As $x \in E_{a_j}^{a_i}$ then $x = (x(1), x(2), \ldots, x(n-i-1))$, with $x(m) \in E_m^j$, $1 \le m \le n-i-1$. As $x \in E_{a_k}^{a_i}$ then $x = (z(1), z(2), \ldots, z(n-i-1))$ with $z(m) \in E_m^k$, $1 \le m \le n-i-1$. Now, if $1 < n-j$, $x(1) = a_j$ and if $1 \ge n-j$, then $x(1) = 1$. If $1 < n-k$, $z(1) = a_k$ and if $1 \ge n-k$, then $z(1) = 1$. Moreover, as $j<k$, $n-j > n-k$. If $1 < n-k < n-j$, $x(1) = a_j$ and $z(1) = a_k$ a contradiction since $x(1) = z(1)$. The other cases are similar. Consequently, $E_{a_j}^{a_i} \cap E_{a_k}^{a_i} = \emptyset$. □

Lemma 2.12 $|F| = \prod_{i=0}^{n-2} \left[1 + (n-i-1)! \sum_{j=i+1}^{n-1} \frac{1}{(n-j-1)!}\right]$

Proof By Lemma 2.11, $|F_i| = \sum_{j=i}^{n-1} \left|E_{a_j}^{a_i}\right|$ with $\left|E_{a_j}^{a_i}\right| = \left|E_1^j\right| \cdot \left|E_2^j\right| \cdots \left|E_{n-i-1}^j\right|$.

$$\left|E_k^j\right| = \begin{cases} 1 & \text{if} \quad k < n-j \\ n-1-(n-k)+1 & \text{if} \quad k \ge n-j \end{cases}$$
$$= \begin{cases} 1 & \text{if} \quad k < n-j \\ k & \text{if} \quad k \ge n-j \end{cases}.$$

Hence,
$$\left|E_{a_j}^{a_i}\right| = \begin{cases} (n-j) \cdot (n-j+1) \cdots (n-i-1) & \text{if} \quad j > i \\ 1 & \text{if} \quad j = i \end{cases}.$$

So,

$$|F_i| = 1 + \sum_{j=i+1}^{n-1} (n-j)\cdot(n-j+1)\cdots(n-i-1)$$
$$= 1 + \sum_{j=i+1}^{n-1} \frac{(n-i-1)!}{(n-j-1)!}$$
$$= 1 + (n-i-1)! \sum_{j=i+1}^{n-1} \frac{1}{(n-j-1)!}.$$

Thus, $|F| = \prod_{i=0}^{n-2}\left[1 + (n-i-1)! \sum_{j=i+1}^{n-1} \frac{1}{(n-j-1)!}\right]$. □

By Lemma 2.12 and Theorem 2.10, it follows immediately the next corollary:

Corollary 2.13 *There are*

$$\prod_{i=0}^{n-2}\left[1 + (n-i-1)! \sum_{j=i+1}^{n-1} \frac{1}{(n-j-1)!}\right]$$

non-isomorphic n-element semi-Heyting chains, for $n \geq 2$.

For $n = 2, 3, 4$ this formula gives 2, 10, and 160, respectively, which coincide with the numbers determined by Sankappanavar in [5]. For $n = 5$, there are 10,400 non-isomorphic semi-Heyting chains with five elements.

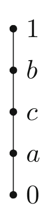
3 Isomorphic subalgebras of a semi-Heyting chain

When one investigates the lattice of subvarieties of $\mathcal{CSH}$, it is important to characterize the subalgebras of a finite semi-Heyting chain and to study the inclusion relation between chains in $\mathcal{CSH}$.

In this section, we consider the following related problem, which will show the complexity of the general problem. Given an n-element chain $\mathbf{L}$, we wish to know when an $(n+1)$-element semi-Heyting chain $\mathbf{L}'$ contains a subalgebra isomorphic to $\mathbf{L}$, and the number of semi-Heyting chains $\mathbf{L}'$ satisfying this condition.

Consider the following example. Let $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ be a 4-element semi-Heyting chain, $L : 0 < a < b < 1$. We want to know the number of 5-element semi-Heyting chains $\mathbf{L}'$ such that $\mathbf{L}$ is (isomorphic to) a subalgebra of $\mathbf{L}'$. Let $\mathbf{L}'$ be a 5-element semi-Heyting chain such that $\mathbf{L} \in \mathbb{IS}(\mathbf{L}')$ and assume that $L' = L \cup \{c\}$.

Suppose first that $0 < a < c < b < 1$. We have just to consider the elements $x \rightarrow c$ for $x < c$ and the elements $c \rightarrow y$ for $c < y$.



	0	a	c	b	1
0			?		
a			?		
c				?	?
b					
1					

If $0 \rightarrow b \leq a$, then $c \wedge (0 \rightarrow c) = c \wedge (0 \rightarrow b) = 0 \rightarrow b$. Since $\mathbf{L}$ is a chain, $0 \rightarrow c = 0 \rightarrow b$.

If $0 \rightarrow b \geq b$, then $c \wedge (0 \rightarrow c) = c \wedge (0 \rightarrow b) = c$. So, $0 \rightarrow c \geq c$, i.e.,

$0 \rightarrow c = 0 \rightarrow b$ if $0 \rightarrow b \leq a$ and $0 \rightarrow c \geq c$ if $0 \rightarrow b \geq b$.

Similarly, if $a \rightarrow b = a$ then $a \rightarrow c = a \rightarrow b$, and if $a \rightarrow b \geq b$ then $a \rightarrow c \geq c$, i.e.,

$a \rightarrow c = a \rightarrow b$ if $a \rightarrow b = a$ and $a \rightarrow c \geq c$ if $0 \rightarrow b \geq b$.

Besides, $c \rightarrow 1 \geq c$ and

$c \rightarrow b = c \rightarrow 1$ if $c \rightarrow 1 = c$ and $c \rightarrow b \geq b$ if $c \rightarrow 1 \geq b$.

So, we conclude that

$(c \rightarrow 1, c \rightarrow b) \in (\{c\} \times \{c\}) \cup (\{b\} \times [b)) \cup (\{1\} \times [b))$.

In a similar way, we should consider the cases $0 < a < b < c < 1$ and $0 < c < a < b < 1$.

In general, let $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ be an n-element semi-Heyting chain, $n \geq 3$, with

$L : 0 = a_0 < a_1 < \cdots < a_{n-2} < a_{n-1} = 1.$

For a given $0 \leq i \leq n-2$, let $\langle L^i, \wedge, \vee, 0, 1\rangle$ be an $(n+1)$-element chain with $L^i = L \cup \{b_i\}$, $b_i \notin L$, and

$L^i : 0 = a_0 < a_1 < \cdots < a_i < b_i < a_{i+1} < \cdots < a_{n-1} = 1.$

We want to find conditions on an implication $\rightarrow : L^i \times L^i \rightarrow L^i$ in order for $\mathbf{L}$ to be a subalgebra of $\mathbf{L}^i = \langle L^i, \wedge, \vee, \rightarrow, 0, 1\rangle$ and to determine the number of implication operations that satisfy those conditions.

Consider the following sets

(a) $F_j^k = \begin{cases} \{a_k\} & \text{if} \quad k < j \\ [a_j) & \text{if} \quad k \geq j \end{cases}$

(b) $F_k^i = \{a_k\} \times \prod_{j=i+1}^{n-2} F_j^k$

(c) $F_0^i = \{b_i\} \times \prod_{j=i+1}^{n-2} \{b_i\}$

(d) $E_j^i = \begin{cases} \{a_j \rightarrow a_{i+1}\} & \text{if} \quad a_j \rightarrow a_{i+1} \leq a_i \\ [b_i) & \text{if} \quad a_j \rightarrow a_{i+1} > a_i \end{cases}$

(e) $E^i = \prod_{j=0}^{i} E_j^i$

(f) $G^i = E^i \times \left[\left(\bigcup_{k=i+1}^{n-1} F_k^i\right) \cup F_0^i\right]$

Lemma 3.1 *Let* $\mathbf{L}$ *and* $\mathbf{L}^i$ *be as defined earlier. Then*

$((a_0 \rightarrow b_i, a_1 \rightarrow b_i, \ldots, a_i \rightarrow b_i), (b_i \rightarrow 1, b_i \rightarrow a_{i+1}, b_i \rightarrow a_{i+2}, \ldots, b_i \rightarrow a_{n-2})) \in G^i.$

Proof For $0 \leq j \leq i$, $a_j < a_{i+1}$, and then by Lemma 1.2, $a_j \to a_{i+1} \geq a_j$. If $a_j \to a_{i+1} = a_l$ with $j \leq l \leq i$, as $b_i > a_l$, by Lemma 2.5 $a_j \to b_i = a_l = a_j \to a_{i+1}$. If $a_j \to a_{i+1} = a_l$ with $i+1 \leq l \leq n-1$, by Lemma 2.5 $a_j \to b_i \geq b_i$. Then $a_j \to b_i \in E^i_j$. Consequently, $(a_0 \to b_i, a_1 \to b_i, \ldots, a_i \to b_i) \in E^i$. Moreover, as $b_i < 1$, by Lemma 1.2, $b_i \to 1 \geq b_i$. Hence, $b_i \to 1 = b_i$ or $b_i \to 1 = a_k$ with $i+1 \leq k \leq n-1$.

Suppose that $b_i \to 1 = b_i$. Consider j with $i+1 \leq j \leq n-2$. By Lemma 2.5, $b_i \to a_j = b_i$. Then $(b_i \to 1, b_i \to a_{i+1}, \ldots, b_i \to a_{n-2}) \in F^i_0$.

If $b_i \to 1 = a_k$ with $i+1 \leq k \leq n-1$, consider j with $i+1 \leq j \leq n-2$. If $j \leq k$, by Lemma 2.5, $b_i \to a_j \geq a_j$. If $j > k$, by Lemma 2.5, $b_i \to a_j = a_k$. Then $(b_i \to 1, b_i \to a_{i+1}, \ldots, b_i \to a_{n-2}) \in F^i_k$.

Therefore, $(b_i \to 1, b_i \to a_{i+1}, \ldots, b_i \to a_{n-2}) \in \bigcup_{k=i+1}^{n-1} F^i_k \cup F^i_0$. Thus, $((a_0 \to b_i, a_1 \to b_i, \ldots, a_i \to b_i), (b_i \to 1, b_i \to a_{i+1}, b_i \to a_{i+2}, \ldots, b_i \to a_{n-2})) \in G^i$. □

Now we will construct an implication from a given element of the set G^i.

Consider now the set G^i. Let $\alpha \in G^i$. $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \in E^i$ and $\alpha_2 \in \left(\bigcup_{k=i+1}^{n-1} F^i_k\right) \cup F^i_0$. As $\alpha_1 \in E^i$, then $\alpha_1 = (\alpha_1(0), \alpha_1(1), \ldots, \alpha_1(i))$ with $\alpha_1(j) \in E^i_j$ for every j, $0 \leq j \leq i$. As $\alpha_2 \in \left(\bigcup_{k=i+1}^{n-1} F^i_k\right) \cup F^i_0$, then $\alpha_2 \in F^i_{k_0}$ for any k_0 with $i+1 \leq k_0 \leq n-1$ ó $\alpha_2 \in F^i_0$. Hence, $\alpha_2 = (\alpha_2(n-1), \alpha_2(i+1), \alpha_2(i+2), \ldots, \alpha_2(n-2))$. If $\alpha_2 \in F^i_{k_0}$ for any k_0 with $i+1 \leq k_0 \leq n-1$, $\alpha_2(n-1) = a_{k_0}$ and $\alpha_2(j) \in F^{k_0}_j$ with $i+1 \leq j \leq n-2$. If $\alpha_2 \in F^i_0$, $\alpha_2(n-1) = b_i$ and $\alpha_2(j) = b_i$ with $i+1 \leq j \leq n-2$. In L^i we define an operation $\Rightarrow$ as:

$$x \Rightarrow y = \begin{cases} x \to y & \text{if} \quad x, y \in L \\ 1 & \text{if} \quad x = y \\ y & \text{if} \quad (x,y) = (a_j, b_i) \quad \text{with} \quad i+1 \leq j \leq n-1 \\ y & \text{if} \quad (x,y) = (b_i, a_j) \quad \text{with} \quad 0 \leq j \leq i \\ \alpha_1(j) & \text{if} \quad (x,y) = (a_j, b_i) \quad \text{with} \quad 0 \leq j \leq i \\ \alpha_2(j) & \text{if} \quad (x,y) = (b_i, a_j) \quad \text{with} \quad i+1 \leq j \leq n-1 \end{cases}$$

Lemma 3.2 *Let* $\mathbf{L}^i = \langle L^i, \wedge, \vee, \Rightarrow, 0, 1 \rangle$. *Then* $\mathbf{L}^i \in \mathcal{SH}$ *and, consequently*, $\mathbf{L}$ *is a subalgebra of* $\mathbf{L}^i$.

Proof Clearly, $\mathbf{L}^i \models x \Rightarrow x \approx 1$.

We will see that $\mathbf{L}^i \models x \wedge (x \Rightarrow y) \approx x \wedge y$. Let $a, b \in L^i$ with $a < b$.

If $a, b \in L$, then $a \wedge (a \Rightarrow b) = a \wedge (a \to b) = a \wedge b$.

Suppose that $a = b_i$, $b = a_r$ with $i+1 \leq r \leq n-1$ and $\alpha_2 \in F^i_{k_0}$ with $i+1 \leq k_0 \leq n-1$. Then $b_i \wedge (b_i \Rightarrow a_r) = b_i \wedge \alpha_2(r)$. If $r = n-1$, $b_i \wedge (b_i \Rightarrow 1) = b_i \wedge \alpha_2(n-1) = b_i \wedge a_{k_0} = b_i = b_i \wedge 1$. If $r < n-1$, $b_i \Rightarrow a_r = \begin{cases} a_{k_0} & \text{if} \quad k_0 < r \\ x & \text{if} \quad k_0 \geq r \quad \text{with} \quad x \geq a_r \end{cases}$.

Consequently, $b_i \wedge (b_i \Rightarrow a_r) = \begin{cases} b_i & \text{if} \quad k_0 < r \\ b_i & \text{if} \quad k_0 \geq r \end{cases} = b_i \wedge a_r$

Suppose now that $a = b_i$, $b = a_r$ with $i+1 \leq r \leq n-1$ and $\alpha_2 \in F^i_0$. In this case $b_i \wedge (b_i \Rightarrow a_r) = b_i \wedge \alpha_2(r) = b_i \wedge b_i = b_i = b_i \wedge a_r$.

Finally, if $a = b_i$ and $b = a_r$ with $0 \leq r \leq i$. $a_r \wedge (a_r \Rightarrow b_i) = a_r \wedge \alpha_1(r)$. Thus,

$$\alpha_1(r) = \begin{cases} a_r \to a_{i+1} & \text{if} \quad a_r \to a_{i+1} \leq a_i \\ x & \text{if} \quad a_r \to a_{i+1} > a_i \quad \text{with} \quad x \geq b_i \end{cases},$$

so,

$$a_r \wedge \alpha_1(r) = \begin{cases} a_r \wedge (a_r \to a_{i+1}) & \text{if} \quad a_r \to a_{i+1} \leq a_i \\ a_r \wedge x & \text{if} \quad a_r \to a_{i+1} > a_i \quad \text{with} \quad x \geq b_i \end{cases}.$$

Hence, $a_r \wedge (a_r \Rightarrow b_i) = a_r \wedge \alpha_1(r) = a_r = a_r \wedge b_i$.

In a similar way, it can be proved that $\mathbf{L}^i \models x \wedge (y \Rightarrow z) \approx x \wedge [(x \wedge y) \Rightarrow (x \wedge z)]$.

By Lemma 2.3, $\mathbf{L}^i_{\to} \in \mathcal{SH}$. □

Remark 3.3 From the previous lemma, it follows that for every n-element semi-Heyting chain $\mathbf{L}$ there exists an $(n+1)$-element semi-Heyting chain $\mathbf{L}'$ such that $\mathbf{L}$ is a subalgebra of $\mathbf{L}'$.

From the previous results, we can establish the following correspondence:

Theorem 3.4 *Let* S_i *be the set of operations* $\to: L^i \times L^i \to L^i$ *such that* $\mathbf{L}^i = \langle L^i, \wedge, \vee, \to, 0, 1 \rangle \in \mathcal{SH}$ *and* $\mathbf{L}$ *is a subalgebra of* $\mathbf{L}^i$. *Then there exists a bijective correspondence between* S_i *and* G^i.

Proof We define $\alpha : S_i \to G^i$ as $\alpha(\to) = ((a_0 \to b_i, a_1 \to b_i, \ldots, a_i \to b_i), (b_i \to 1, b_i \to a_{i+1}, b_i \to a_{i+2}, \ldots, b_i \to a_{n-2}))$. By Lemmas 3.1 and 3.2, α is well defined, and it is onto. The injectivity is left to the reader. □

Now we want to determine the cardinal of the set G^i.

Lemma 3.5 *Let* $i+1 \leq k, k' \leq n-1$ *and* $0 \leq i \leq n-2$. *Then* $F^i_k \cap F^i_{k'} = \emptyset$ *if* $k \neq k'$.

Proof Let $\alpha \in F^i_k \cap F^i_{k'}$ and suppose that $k < k'$. As $\alpha \in F^i_k$, $\alpha = (\alpha_1, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{n-2})$ with $\alpha_1 = a_k$ and $\alpha_j \in F^k_j$ with $i+1 \leq j \leq n-2$. As $\alpha \in F^i_{k'}$, $\alpha = (\alpha_1, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{n-2})$ with $\alpha_1 = a_{k'}$ and $\alpha_j \in F^{k'}_j$ with $i+1 \leq j \leq n-2$. Hence, $a_k = a_{k'}$. Then $k = k'$ a contradiction. □

Lemma 3.6

$$\left|G^i\right| = \prod_{j=0}^{i} A_j^i\left(\sum_{k=i+1}^{n-2}\frac{(n-(i+1)-1)!}{(n-(i+1)-(k-(i+1)+2))!} + (n-(i+1)-1)!+1\right),$$

where

$$A_j^i = \begin{cases} 1 & \text{if} \quad a_j \to a_{i+1} \le a_i \\ n-i & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases}.$$

Proof We have that $F_j^k = \begin{cases} \{a_k\} & \text{if} \quad k < j \\ [a_j) & \text{if} \quad k \ge j \end{cases}$ and then

$$\left|F_j^k\right| = \begin{cases} 1 & \text{if} \quad k<j \\ n-2-j+1 & \text{if} \quad k \ge j \end{cases} = \begin{cases} 1 & \text{if} \quad k<j \\ n-j-1 & \text{if} \quad k \ge j \end{cases}$$

Hence, for $k \neq n-1$,

$$\prod_{j=i+1}^{n-2}\left|F_j^k\right| = \frac{(n-(i+1)-1)!}{(n-(i+1)-(k-(i+1)+2))!}.$$

If $k = n-1$, $\prod_{j=i+1}^{n-2}\left|F_j^k\right| = (n-(i+1)-1)!$.

Besides, as

$$E_j^i = \begin{cases} \{a_j \to a_{i+1}\} & \text{if} \quad a_j \to a_{i+1} \le a_i \\ [b_i) & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases},$$

then $|E_j^i| = \begin{cases} 1 & \text{if} \quad a_j \to a_{i+1} \le a_i \\ |[b_i)| & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases}$ with $|[b_i)| = n-1-(i+1)+2 = n-1-i-1+2 = n-i$.

Let

$$A_j^i = \begin{cases} 1 & \text{if} \quad a_j \to a_{i+1} \le a_i \\ n-i & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases}.$$

Then $G^i = E^i \times \left[\left(\bigcup_{k=i+1}^{n-1} F_k^i\right) \cup F_0^i\right]$, by Lemma 3.5

$$\left|G^i\right| = \prod_{j=0}^{i} A_j^i\left(\sum_{k=i+1}^{n-2}\frac{(n-(i+1)-1)!}{(n-(i+1)-(k-(i+1)+2))!} + (n-(i+1)-1)!+1\right).$$

□

From Lemma 3.6 and Theorem 3.4 the next corollary follows.

Corollary 3.7 *The number of non-isomorphic algebras* $\mathbf{L}^i$ *is given by the number*:

$$\prod_{j=0}^{i} A_j^i\left(\sum_{k=i+1}^{n-2}\frac{(n-(i+1)-1)!}{(n-(i+1)-(k-(i+1)+2))!} + (n-(i+1)-1)!+1\right)$$

where

$$A_j^i = \begin{cases} 1 & \text{if} \quad a_j \to a_{i+1} \le a_i \\ n-i & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases}.$$

Corollary 3.8 *The number of non-isomorphic* $(n+1)$-*element semi-Heyting chains* $\mathbf{L}^i$ *such that* $\mathbf{L}$ *is a subalgebra of* $\mathbf{L}^i$ *is given by*

$$\sum_{i=0}^{n-2}\left[\prod_{j=0}^{i} A_j^i\left(\sum_{k=i+1}^{n-2}\frac{(n-(i+1)-1)!}{(n-(i+1)-(k-(i+1)+2))!} + (n-(i+1)-1)!+1\right)\right]$$

where

$$A_j^i = \begin{cases} 1 & \text{if} \quad a_j \to a_{i+1} \le a_i \\ n-i & \text{if} \quad a_j \to a_{i+1} > a_i \end{cases}$$

It remains to consider the case $n = 2$.

Let $\mathbf{2}$ and $\overline{\mathbf{2}}$ denote the two-element semi-Heyting chains whose operation $\to$ satisfies $0 \to 1 = 1$ and $0 \to 1 = 0$, respectively. It is easy to see that $\mathbf{2}$ is a Heyting algebra (which is also a Boolean algebra), while $\overline{\mathbf{2}}$ is not.

Let $\mathbf{L}$ be either $\mathbf{2}$ or $\overline{\mathbf{2}}$. Let $\mathbf{L}' = \langle L', \wedge, \vee, 0, 1\rangle$ be a chain with $L' : 0 < b < 1$ and $b \notin L$. If $\mathbf{L}$ is a subalgebra of $\mathbf{L}'$, then we have just to determine the elements $0 \to b$ and $b \to 1$. From Lemma 2.5, if $0 \to 1 = 0$ then $0 \to b = 0$ and if $0 \to 1 = 1$ then $0 \to b \ge b$. In addition, $b \to 1 \ge b$.

So we have

Corollary 3.9 *There are two non-isomorphic* 3-*element semi-Heyting chains that contain* $\mathbf{2}$ *as a subalgebra and* four *non-isomorphic* 3-*element semi-Heyting chains that contain* $\overline{\mathbf{2}}$ *as a subalgebra*.

Theorem 3.10 *Let* $n \ge 3$. *There is an* $(n+1)$-*element semi-Heyting chain that does not have an n-element subalgebra.*

Proof Let $L' : b_0 < b_1 < \cdots < b_{n-1} < b_n$. We define $\to$: $L' \times L' \to L'$ by means of

$$b_i \to b_j = \begin{cases} b_{i+1} & \text{if} \quad i<j \\ 1 & \text{if} \quad i=j \\ b_j & \text{if} \quad i>j \quad \text{with} \quad 0 \le i,j \le n \end{cases}$$

It is easy to prove that $\to$ is a semi-Heyting implication.

Suppose that there exists a subalgebra $\mathbf{L}$ of $\mathbf{L}'$ and $|\mathbf{L}| = n$. Then there exists $0 < k < n$ such that $L = L' \setminus \{b_k\}$. Now, $b_k = b_{k-1} \to b_{k+1}$, and then $b_k \in L$, a contradiction. □

4 Equational basis for $\mathcal{CSH}$

In this section, we give equational bases for the variety $\mathcal{CSH}$ and some of its subvarieties.

Lemma 4.1 *Let* **L** *be a semi-Heyting chain. Then* **L** *satisfies the following identity*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1 \quad \text{(Ch)}$$

Proof Let $a, b \in L$. As L is a chain, $a \leq b$ or $b < a$. If $a \leq b$, by Lemma 1.2, $a \leq a \to b$. Hence, $a \vee (a \to b) = a \to b$, so $(a \vee (a \to b)) \to (a \to b) = (a \to b) \to (a \to b) = 1$. If $b < a$, $b = b \wedge a$, and then $b \to (a \wedge b) = b \to b = 1$. □

Lemma 4.2 *Let* $\mathbb{V}$ *be the subvariety of* $\mathcal{SH}$ *defined by (Ch). If* $\mathbf{L} \in \mathbb{V}$ *is subdirectly irreducible, then* **L** *is a chain.*

Proof Let $\mathbf{L} \in \mathbb{V}$ subdirectly irreducible and let $a, b \in L$. Since L satisfies (Ch), $((a \vee (a \to b)) \to (a \to b)) \vee (b \to (a \wedge b)) = 1$. As **L** is subdirectly irreducible, 1 is $\vee$-irreducible. Thus, $(a \vee (a \to b)) \to (a \to b) = 1$ or $b \to (a \wedge b) = 1$.

Suppose that $(a \vee (a \to b)) \to (a \to b) = 1$. Then $a \vee (a \to b) \leq a \to b$, and thus, $a \leq a \vee (a \to b) \leq a \to b$. Hence, $a \wedge b = a \wedge (a \to b) = a$, so $a \leq b$.

If $b \to (a \wedge b) = 1$, then $b \leq a \wedge b$, and thus, $b \leq a$.

Hence, **L** is a chain. □

From Lemmas 4.1 and 4.2, we have the following:

Theorem 4.3 *An equational basis for* $\mathcal{CSH}$ *relative to* $\mathcal{SH}$ *is given by*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1 \quad \text{(Ch)}.$$

Let $\mathcal{C}_n$ denote the subvariety of $\mathcal{SH}$ generated by all the n-element chains, $n \geq 2$.

It is easily seen that the following theorem holds.

Theorem 4.4 *An equational basis for* $\mathcal{C}_2$ *is given by*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1 \quad \text{(Ch)}$$

and

$$x \vee x^* \approx 1.$$

Observe that $\mathcal{C}_2$ is the subvariety generated by the algebras **2** and $\overline{\mathbf{2}}$, which have the 2-element chain as their lattice reduct and whose operation $\to$ satisfies $0 \to 1 = 1$ and $0 \to 1 = 0$, respectively.

Theorem 4.5 *An equational basis for* $\mathcal{C}_n$ *with* $n \geq 3$ *is given by the identities*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1 \quad \text{(Ch)}$$

and

$$\bigvee_{i=1}^{n-1} (x_i \vee x_i^*) \vee \bigvee_{j=1; j<i}^{n-1} (x_i \to x_j) \approx 1 \quad (\mathrm{H_n})$$

Proof Let **L** be an n-element semi-Heyting chain, $L: 0 < a_1 < a_2 < \cdots < a_{n-2} < 1$. By Theorem 4.3, **L** satisfies (Ch).

Let us prove that **L** satisfies $(\mathrm{H_n})$. Let $z_1, z_2, \ldots, z_{n-1} \in L$. If $z_k \in \{0, 1\}$ for some k, $1 \leq k \leq n-1$, then $z_k \vee z_k^* = 1$. Suppose that $z_k \notin \{0, 1\}$ for every k, $1 \leq k \leq n-1$, i.e., $z_k \in \{a_1, a_2, \ldots, a_{n-2}\}$ for every k. Then there exists $j < i$ such that $z_i = z_j$, and so $z_i \to z_j = 1$. Hence, **L** satisfies $(\mathrm{H_n})$.

Let $\mathbb{V}$ be the subvariety of $\mathcal{SH}$ defined by (Ch) and $(\mathrm{H_n})$ and consider a subdirectly irreducible algebra $\mathbf{L} \in \mathbb{V}$. By Theorem 4.3, $\mathbf{L} \in \mathcal{CSH}$, and by Lemma 4.2, L is a chain. Suppose that exists $a_1, a_2, \ldots, a_{n-2}, a_{n-1} \in L$ such that $0 < a_1 < a_2 < \cdots < a_{n-2} < a_{n-1} < 1$. Then $\bigvee_{i=1}^{n-1} (a_i \vee a_i^*) = \bigvee_{i=1}^{n-1} (a_i \vee 0) = a_{n-1}$. By hypothesis, $a_{n-1} \vee \bigvee_{i,j=1; j<i}^{n-1} (a_i \to a_j) = 1$, and as 1 is $\vee$-irreducible, $a_i \to a_j = 1$ for some $j < i$. Hence, $a_i \leq a_j$, a contradiction. Thus, $|L| \leq n$, and consequently $\mathbf{L} \in \mathcal{C}_n$. □

As an immediate corollary of Theorem 4.3 we can determine equational bases for the following subvarieties of $\mathcal{CSH}$ introduced in [5]: $\mathcal{CFTT}(0 \to 1 \approx 1)$, $\mathcal{CFTD}((0 \to 1)^* \approx 0)$, quasi-Heyting chains $(y \leq x \to y)$, Heyting chains $((x \wedge y) \to x \approx 1)$, $\mathcal{CFTF}(0 \to 1 \approx 0)$ and conmutative semi-Heyting chains $(x \to y \approx y \to x)$.

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