Soft Comput (2011) 15:867–876
DOI 10.1007/s00500-010-0554-6

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Generalized intuitionistic fuzzy geometric aggregation operator and its application to multi-criteria group decision making

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Published online: 25 February 2010

Abstract In general, for multi-criteria group decision making problem, there exist inter-dependent or interactive phenomena among criteria or preference of experts, so that it is not suitable for us to aggregate them by conventional aggregation operators based on additive measures. In this paper, based on fuzzy measures a generalized intuitionistic fuzzy geometric aggregation operator is investigated for multiple criteria group decision making. First, some operational laws on intuitionistic fuzzy values are introduced. Then, a generalized intuitionistic fuzzy ordered geometric averaging (GIFOGA) operator is proposed. Moreover, some of its properties are given in detail. It is shown that GIFOGA operator can be represented by special t-norms and t-conorms and is a generalization of intuitionistic fuzzy ordered weighted geometric averaging operator. Further, an approach to multiple criteria group decision making with intuitionistic fuzzy information is developed where what criteria and preference of experts often have inter-dependent or interactive phenomena among criteria or preference of experts is taken into account. Finally, a practical example is provided to illustrate the developed approaches.

Keywords Multi-criteria group decision making · Fuzzy measures · Intuitionistic fuzzy sets · Geometric aggregation operator

1 Introduction

In many complex decision making problems, the decision information provided by a decision maker is often imprecise or uncertain due to time pressure, lack of data, or the decision maker's limited attention and information processing capabilities. Since fuzzy set, whose basic component is only a membership function, was introduced (Zadeh 1965), in the following several decades, fuzzy set theory has been applied successfully in decision making field. Intuitionistic fuzzy set (IFS)(Atanassov 1986), an extension of Zadeh's fuzzy sets, has a prominent characteristic: it assigns to each element a membership degree and a non-membership degree. Intuitionistic fuzzy set has been proven to be highly useful to deal with uncertainty and vagueness, and it is a very suitable tool to be used to describe the imprecise or uncertain decision information. Recently, different decision making problems based on IFS have received a great deal of attention. Gau and Buehrer (1993) introduced the vague set, which is an equivalence of IFS (Bustine and Burillo 1996). Later, based on vague sets, Chen and Tan (1994), and Hong and Choi (2000) utilized the minimum and maximum operations to develop some approximate technique for handling multi-attribute decision making problems under fuzzy environment. Szmidt and Kacprzyk (1996) used IFSs to solve group decision making problems. Atanassov et al. (2005) proposed an intuitionistic fuzzy interpretation of multi-person multi-criteria decision making. Li (2005) investigated multi-attribute decision-making using IFSs and constructed several linear programming models to generate optimal weights for criteria. Xu and Yager (2006) developed some geometric aggregation operators based on IFSs to multiple attribute decision making. Liu and Wang (2007) introduced the intuitionistic fuzzy point operators, and defined a series of new score functions for the multi-attribute decision making problems based on intuitionistic fuzzy point operators and evaluation function. By linear programming model, Lin et al. (2007)

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presented a new method for handling multi-criteria fuzzy decision-making problems based on IFSs. Wei (2009) proposed dynamic intuitionistic fuzzy weighted geometric operator for dynamic multiple attribute decision making.

Although there has been great progress in multicriteria decision making with IFSs, most of these works are based on the assumption that the criteria (attribute) and preferences of decision makers are independent, and the aggregation operators are linear operators based on additive measures, which is characterized by an independence axiom (Keeney and Raiffa 1976; Wakker 1999). In the real problems there is a phenomenon where there exists some degree of inter-dependent or interactive characteristics between criteria (Grabisch 1995; Grabisch et al. 2000). Expert's subjective preference always shows non-linearity. There usually are interactive phenomena among preference of experts. The independence axiom generally cannot be satisfied. To overcome this limitation, it is more reasonable to use a non-additive measure instead of traditional additive measure operators to approximate people's evaluation processes for decision making problems. Sugeno (1974) introduced the concept of non-additive measure (fuzzy measure), which only makes a monotonicity instead of additivity property. It was used to model interaction phenomena (Ishii and Sugeno 1985; Roubens 1996; Grabisch 1996; Kojadinovic 2002) and deal with decision making problems (Grabisch 1995, 1997; Grabisch et al. 2000; O-nisawa et al. 1986). In this paper, based on fuzzy measure we develop a generalized intuitionistic fuzzy geometric aggregation operator for multi-criteria group decision making which takes into account the inter-dependent or interactive characteristics of the criteria and preference.

In order to do that, the paper is organized as follows. In Sect. 2, we review the IFSs. In Sect. 3, we introduce fuzzy measure. A generalized intuitionistic fuzzy ordered geometric averaging operator is proposed, and some of its properties are investigated in detail. In Sect. 4, the multi-criteria group decision making procedure based on the generalized intuitionistic fuzzy ordered geometric averaging operator is presented under intuitionistic fuzzy environment. In Sect. 5, an example is given to illustrate the concrete application of the method. Finally, conclusions are made in Sect. 6.

2 Intuitionistic fuzzy sets

First, let us first review some basic concepts related to intuitionsitic fuzzy set. Atanassov (1986) generalized the concept of fuzzy set and defined the intuitionsitic fuzzy set as follows.

Let X be an ordinary finite non-empty set. An IFS in X is an expression A given by

$$A = \{\langle x, t_A(x), f_A(x)\rangle | x \in X\} \quad (1)$$

where t_A: $X \to [0,1]$, f_A: $X \to [0,1]$ with the condition $0 \le t_A(x) + f_A(x) \le 1$, for all x in X. The numbers $t_A(x)$ and $f_A(x)$ denote, respectively, the degree of membership and the degree of non-membership of element x in set A.

For each IFS A in X, if $\pi_A(x) = 1 - t_A(x) - f_A(x)$, $\forall x \in X$, then $\pi_A(x)$ is called the degree of indeterminacy of x to A. Especially, if $\pi_A(x) = 1 - t_A(x) - f_A(x) = 0$, $\forall x \in X$, the IFS A is reduced to a fuzzy set. Clearly, a prominent characteristic of IFS is that it assigns to each element a membership degree, a non-membership degree and a hesitation degree. Accordingly, IFS is a very suitable tool to be used to deal with decision problems with imprecise or uncertain information.

For computational convenience, in this paper, we call $(t_A(x), f_A(x))$ an intuitionistic fuzzy value. Let Ω be the set of all intuitionistic fuzzy values on x.

For every two intuitionistic fuzzy values A and B the following operations and relations are valid:

1. $A = B$ if and only if $t_A(x) = t_B(x)$ and $f_A(\mathrm{x}) = f_B(x)$ for all $x \in X$; (2)

2. $A \le B$ if and only if $t_A(x) \le t_B(x)$ and $f_A(\mathrm{x}) \ge f_B(x)$ for all $x \in X$. (3)

However, for some intuitionistic fuzzy values, (3) is not satisfied in some situations. So it cannot be used to compare these intuitionistic fuzzy values. In the following, we use a score function and an accuracy function of intuitionistic fuzzy values for the comparison between two intuitionistic fuzzy values (Xu and Yager 2006).

Definition 1 Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = \left(t_{\tilde{b}}, f_{\tilde{b}}\right)$ be two intuitionistic fuzzy values, $S(\tilde{a}) = t_{\tilde{a}} - f_{\tilde{a}}$ and $S\left(\tilde{b}\right) = t_{\tilde{b}} - f_{\tilde{b}}$ be the score functions of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a}) = t_{\tilde{a}} + f_{\tilde{a}}$ and $H\left(\tilde{b}\right) = t_{\tilde{b}} + f_{\tilde{b}}$ be the accuracy functions of $\tilde{a}$ and $\tilde{b}$, respectively.

If $S(\tilde{a}) < S\left(\tilde{b}\right)$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$;

If $S(\tilde{a}) = S\left(\tilde{b}\right)$, then

1. If $H(\tilde{a}) < H\left(\tilde{b}\right)$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$;
2. If $H(\tilde{a}) = H\left(\tilde{b}\right)$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$.

According to De et al. (2000) and Xu and Yager (2006), we introduce some operational laws on intuitionistic fuzzy values as follows:

Definition 2 Let $\tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values; then

1. $\tilde{a} \otimes \tilde{b} = \left(t_{\tilde{a}} t_{\tilde{b}}, f_{\tilde{a}} + f_{\tilde{b}} - f_{\tilde{a}} f_{\tilde{b}}\right)$,
2. $\tilde{a}^{\lambda} = \left((t_{\tilde{a}})^{\lambda}, 1 - (1 - f_{\tilde{a}})^{\lambda}\right), \lambda > 0$.

For these operational laws of Definition 2, there are the following properties (Xu and Yager 2006):

Proposition 1 *Let $\tilde{a} = (t_{\tilde{a}}, f_a)$ and $\tilde{b} = (t_{\tilde{b}}, f_{\tilde{b}})$ be two intuitionistic fuzzy values, and let $\tilde{c} = \tilde{a} \otimes \tilde{b}$, $\tilde{d} = \tilde{a}^{\lambda}$, $\forall \lambda$, λ_1, $\lambda_2 > 0$. Then both $\tilde{c}$ and $\tilde{d}$ are also intuitionistic fuzzy values; furthermore,*

$$\tilde{a} \otimes \tilde{b} = \tilde{b} \otimes \tilde{a}; \; \left(\tilde{a} \otimes \tilde{b}\right)^{\lambda} = \tilde{a}^{\lambda} \otimes \tilde{b}^{\lambda}; \; \tilde{a}^{\lambda_1 + \lambda_2} = \tilde{a}^{\lambda_1} \otimes \tilde{a}^{\lambda_2}.$$

According to Definition 2, Xu and Yager (2006) extended weighted geometric averaging operator and ordered weighted geometric averaging operator to IFSs, and defined intuitionistic fuzzy weighted geometric averaging (IFWGA) and intuitionistic fuzzy ordered weighted geometric averaging (IFOWGA) operator.

Definition 3 Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1,\ldots,n)$ be a collection of intuitionistic fuzzy values on X. An intuitionistic fuzzy weighted geometric averaging (IFWGA) operator of dimension n is a mapping IFWGA: $\Omega^n \to \Omega$, and

$$\mathrm{IFWGA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (\tilde{a}_1)^{w_1} \otimes (\tilde{a}_2)^{w_2} \otimes \cdots \otimes (\tilde{a}_n)^{w_n}$$

where $w = (w_1, w_2,\ldots,w_n)^T$ is the exponential weighting vector of $\tilde{a}_i$ $(i = 1, 2,\ldots,n)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore

$$\mathrm{IFWGA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} (t_{\tilde{a}_i})^{w_i}, 1 - \prod_{i=1}^{n} (1 - f_{\tilde{a}_i})^{w_i}\right).$$

Definition 4 Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X. An IFOWGA operator of dimension n is a mapping IFOWGA: $\Omega^n \to \Omega$, that has associated with it an exponential weighting vector $w = (w_1, w_2,\ldots,w_n)^T$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$\mathrm{IFOWGA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\tilde{a}_{(1)}\right)^{w_1} \otimes \left(\tilde{a}_{(2)}\right)^{w_2} \otimes \cdots \otimes \left(\tilde{a}_{(n)}\right)^{w_n}$$

where $(\cdot)$ indicates is a permutation on X such that $\tilde{a}_{(1)} \le \tilde{a}_{(2)} \le \cdots \le \tilde{a}_{(n)}$. Furthermore,

$$\mathrm{IFOWGA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} \left(t_{\tilde{a}_{(i)}}\right)^{w_i}, 1 - \prod_{i=1}^{n} \left(1 - f_{\tilde{a}_{(i)}}\right)^{w_i}\right).$$

3 Generalized intuitionistic fuzzy ordered geometric averaging operator

3.1 Fuzzy measure

For traditional additive aggregation operators, such as the weighted arithmetic mean or OWA (Yager 1988) operator, each criteria $i \in N$ (N denotes a criteria set) is given a weight $w_i \in [0, 1]$ representing the importance of this criteria in the decision, and the sum of all w_i $(i = 1, 2,\ldots n)$ amount to one. But it does not define a weight on each combination of criteria. In real decision problems, since there are often interdependent or interactive phenomena among criteria, the overall importance of a criterion $i \in N$ is not solely determined by itself i, but also by all other criteria T, $i \in T$. Suppose that $w(i)$ denotes the importance degree of i, we may have $w(i) = 0$, suggesting that element is unimportant, but it may happen that for many subsets $T \subseteq N$, $w(T \cup i)$ is much greater than $w(T)$, suggesting that i is actually an important element in the decision. In 1974, Sugeno (1974) introduced the concept of fuzzy measure (non-additive measure), which only makes a monotonicity instead of additivity property. For real decision making problems, fuzzy measure define a weight on not only each criteria but also each combination of criteria, and the sum of every w_i $(i = 1, 2,\ldots n)$ does not equal to one. Thus it is used as a powerful tool for modeling interaction phenomena in decision making.

Definition 5 A fuzzy measure on X is a set functions μ: $2^X \to [0, 1]$, satisfying the following conditions:

1. $\mu(\phi) = 0$, $\mu(X) = 1$;
2. If $A, B \in P(X)$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$.

If the universal set X is infinite, it is necessary to add an extra axiom of continuity (Wang and Klir 1992). However, in actual practice, it is enough to consider the finite universal set. $\mu(S)$ can be viewed as the grade of subjective importance of decision criteria S. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined. However, in order to determine fuzzy measures on criteria set $N = \{1, 2, \ldots, n\}$, we generally need to find $2^n - 2$ values for n criteria only; values $\mu(\phi)$ and $\mu(X)$ are always equal to 0 and 1, respectively. So the evaluation model obtained becomes quite complex. To avoid the problems with computational complexity, λ-fuzzy measure g, defined on 2^X, was proposed by Sugeno (1974), which satisfies the following λ-rule:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B), \quad (4)$$

where $\lambda > -1$ for all $A, B \in P(X)$ and $A \cap B = \phi$.

In Eq. 4, $\lambda = 0$ indicates that g is called an additive fuzzy measure and there is no interaction between A and B. $\lambda \neq 0$ indicates that λ-fuzzy measure g is non-additive and there is interaction between A and B. If $\lambda > 0$, then $g(A \cup B) > g(A) + g(B)$, which implies that the set $\{A, B\}$ has multiplicative effect. If $\lambda < 0$, then $g(A \cup B) < g(A) + g(B)$, which implies that the set $\{A, B\}$ has substitutive effect. By parameter λ the interaction between criteria can be represented. If X is a finite set, then $\cup_{i=1}^{n} x_i = X$. The λ-fuzzy measure g satisfies following Eq. 5:

$$g(X) = g\left(\bigcup_{i=1}^{n} x_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{n}[1 + \lambda g(x_i)] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i=1}^{n} g(x_i) & \text{if } \lambda = 0, \end{cases} \quad (5)$$

where $x_i \cap x_j = \phi$ for all $i, j = 1, 2,\ldots,n$ and $i \neq j$. It can be noted that $g(x_i)$ for a subset with a single element x_i is called a fuzzy density, and can be denoted as $g_i = g(x_i)$. Especially for every subset $A \in P(X)$, we have

$$g(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i \in A}[1 + \lambda g_i] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i \in A} g_i & \text{if } \lambda = 0. \end{cases} \quad (6)$$

Based on Eq. 5, the value λ of can be uniquely determined from $g(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g_i). \quad (7)$$

3.2 Generalized intuitionistic fuzzy geometric operator

Information aggregation is an essential process and is also an important research topic in the field of information fusion. According to Definitions 3 and 4, it is known that these operations can only be used to deal with criteria independent arguments where a prominent characteristic is that the sum of weight of each criteria amount to one. In order to take interdependent or interactive phenomena among criteria into account, in the following, based on fuzzy measure, we first define the notion of generalized intuitionistic fuzzy geometric aggregation operator.

Definition 6 Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. A generalized intuitionistic fuzzy ordered geometric averaging (GIFOGA) operator of dimension n based on fuzzy measure is a mapping GIFOGA: $\Omega^n \to \Omega$ such that

$$\text{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\tilde{a}_{(1)}\right)^{\mu(A_{(1)}) - \mu(A_{(2)})} \otimes \left(\tilde{a}_{(2)}\right)^{\mu(A_{(2)}) - \mu(A_{(3)})} \otimes \cdots \otimes \left(\tilde{a}_{(n)}\right)^{\mu(A_{(n)}) - \mu(A_{(n+1)})} \quad (8)$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i),\ldots,(n))$, $A_{(n+1)} = \phi$.

Theorem 1 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. Then their aggregated value by using GIFOGA operator is also an intuitionistic fuzzy value; furthermore,*

$$\text{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n}\left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)}) - \mu(A_{(i+1)})}, 1 - \prod_{i=1}^{n}\left(1 - f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)}) - \mu(A_{(i+1)})}\right) \quad (9)$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$. and $A_{(i)} = ((i),\ldots,(n))$, $A_{(n+1)} = \phi$.

Proof The first result follows immediately from Definition 6 and Proposition 1. In the following, we prove Eq. 9 by using mathematical induction on n. Let $(\cdot)$ indicate a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i),\ldots,(n))$, $A_{(n+1)} = \phi$. Since $A_{(i+1)} \subset A_{(i)}$ $i = 1, .2,\ldots,n$, $\mu(A_{(i+1)}) \leq \mu(A_{(i)})$.

For $n = 2$, according to Definition 2, we have

$$\left(\tilde{a}_{(1)}\right)^{\mu(A_{(1)}) - \mu(A_{(2)})} = \left(\left(t_{\tilde{a}_{(1)}}\right)^{\mu(A_{(1)}) - \mu(A_{(2)})}, 1 - \left(1 - f_{\tilde{a}_{(1)}}\right)^{\mu(A_{(1)}) - \mu(A_{(2)})}\right),$$

$$\left(\tilde{a}_{(2)}\right)^{\mu(A_{(2)}) - \mu(A_{(3)})} = \left(\left(t_{\tilde{a}_{(2)}}\right)^{\mu(A_{(2)}) - \mu(A_{(3)})}, 1 - \left(1 - f_{\tilde{a}_{(2)}}\right)^{\mu(A_{(2)}) - \mu(A_{(3)})}\right).$$

Since

$$\tilde{a}_1 \otimes \tilde{a}_2 = (t_{\tilde{a}_1} t_{\tilde{a}_2}, f_{\tilde{a}_1} + f_{\tilde{a}_2} - f_{\tilde{a}_1} f_{\tilde{a}_2}) = (t_{\tilde{a}_1} t_{\tilde{a}_2}, 1 - (1 - f_{\tilde{a}_1})(1 - f_{\tilde{a}_2})),$$

then

$$\text{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2) = \left(\tilde{a}_{(1)}\right)^{(\mu(A_{(1)}) - \mu(A_{(2)}))} \otimes \left(\tilde{a}_{(2)}\right)^{(\mu(A_{(2)}) - \mu(A_{(3)}))} = \left(\prod_{i=1}^{2}\left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)}) - \mu(A_{(i+1)})}, 1 - \prod_{i=1}^{2}\left(1 - f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)}) - \mu(A_{(i+1)})}\right)$$

That is, for $n = 2$, Eq. 9 holds. Suppose that if for $n = k$, the Eq. 9 holds, i.e.,

$$\mathrm{GIFOGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_k) = \left(\prod_{i=1}^{k}\left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-\prod_{i=1}^{k}\left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)$$

Then, for $n = k + 1$; according to Definition 6, we have

$$\begin{aligned}&\mathrm{GIFOGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_k,\tilde{a}_{k+1})\\ &= \mathrm{GIFOGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_k)\otimes(\tilde{a}_{(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\\ &= \left(\prod_{i=1}^{k}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\left(t_{\tilde{a}_{(k+1)}}\right)^{\mu(A_{(k+1)})-\mu(A_{(k+2)})},\right.\\ &\left.1-\prod_{i=1}^{k}\left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\left(1-f_{\tilde{a}_{(k+1)}}\right)^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\right)\\ &= \left(\prod_{i=1}^{k+1}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-\prod_{i=1}^{k+1}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).\end{aligned}$$

That is, for $n = k + 1$, Eq. 9 still holds.

Therefore, for all n, Eq. 9 always holds, which completes the proof of Theorem 1.

Remark 1 Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ and $\tilde{b}_i = \left(t_{\tilde{b}_i}, f_{\tilde{b}_i}\right)$ $(i = 1, 2,\ldots,n)$, $T_P\left(t_{\tilde{a}_i}, t_{\tilde{b}_i}\right) = t_{\tilde{a}_i}t_{\tilde{b}_i}$, $S_P\left(f_{\tilde{a}_i}, f_{\tilde{b}_i}\right) = f_{\tilde{a}_i} + f_{\tilde{b}_i} - f_{\tilde{a}_i}f_{\tilde{b}_i}$. Since $t_{\tilde{a}_i}, f_{\tilde{a}_i}, t_{\tilde{b}_i}, f_{\tilde{b}_i} \in [0, 1]$, then $T_P\left(t_{\tilde{a}_i}, t_{\tilde{b}_i}\right)$ is one of the basic t-norms, called the product T_P (Klement et al. 2000), which satisfies the following properties: (boundary) $T_P(0, 0) = 0$, $T_P(x, 1) = x$; (monotonicity) $T_P(x, y) \le T_P(x, z)$ whenever $y \le z$; (commutativity) $T_P(x, y) = T_P(y, x)$; (associativity) $T_P(x, T_P(y, z)) = T_P(T_P(x, y), \text{where } x, y, z \in [0, 1]$. $S_P\left(f_{\tilde{a}_i}, f_{\tilde{b}_i}\right)$ is one of the basic t-conorms, called the probabilistic sum S_P (Klement et al. 2000), which satisfies the boundary, i.e., $S_P(1, 1) = 1$, $T_P(x, 0) = x$, monotonicity, commutativity, and associativity. The associativity of t-norms and t-conorms allows us to extend the product T_P and probabilistic sum S_P in a unique way to an n-ary operation to $[0, 1]^n$ by induction:

$$T_P(x_1, x_2, \ldots, x_n) = \underset{i=1}{\overset{n}{\mathrm{T_P}}}\, x_i = T_P\left(\underset{i=1}{\overset{n-1}{\mathrm{T_P}}}\, x_i, x_n\right) = \prod_{i=1}^{n} x_i,$$

$$\begin{aligned}S_P(y_1, y_2, \ldots, y_n) &= \underset{i=1}{\overset{n}{\mathrm{S_P}}}\, y_i = S_P\left(\underset{i=1}{\overset{n-1}{\mathrm{S_P}}}\, y_i, y_n\right)\\ &= 1-\prod_{i=1}^{n}(1-y_i).\end{aligned}$$

Assume that $x_i = \left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $y_i = 1 - \left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, then

$$\begin{aligned}&\mathrm{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\\ &= (T_P(x_1, x_2, \ldots, x_n), S_P(y_1, y_2, \ldots, y_n)).\end{aligned}$$

From Theorem 1, the following property can be directly obtained.

Proposition 2 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If all $\tilde{a}_i$ $(i = 1, 2,\ldots,n)$ are equal, that is, for all i, $\tilde{a}_i = \tilde{a} = (t_{\tilde{a}}, f_{\tilde{a}})$, then*

$$\mathrm{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}.$$

Proposition 3 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ and $\tilde{b}_i = \left(t_{\tilde{b}_i}, f_{\tilde{b}_i}\right)$ $(i = 1, 2,\ldots,n)$ be two collections of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. $(\cdot)$ indicates a permutation such that $\tilde{a}_{(1)} \le \cdots \le \tilde{a}_{(n)}$ and $\tilde{b}_{(1)} \le \cdots \le \tilde{b}_{(n)}$. If $\tilde{a}_{(i)} \le \tilde{b}_{(i)}$ for all i, that is, $t_{\tilde{a}_{(i)}} \le t_{\tilde{b}_{(i)}}$ and $f_{\tilde{a}_{(i)}} \ge f_{\tilde{b}_{(i)}}$, then*

$$\mathrm{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \mathrm{GIFOGA}_\mu\left(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\right).$$

Proof Since $A_{(i+1)} \subseteqq A_{(i)}$, then $\mu(A_{(i)}) - \mu(A_{(i+1)}) \ge 0$. For all i, $t_{\tilde{a}_{(i)}} \le t_{\tilde{b}_{(i)}}$ and $f_{\tilde{a}_{(i)}} \ge f_{\tilde{b}_{(i)}}$, we have

$$\prod_{i=1}^{n}\left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \le \prod_{i=1}^{n}\left(t_{\tilde{b}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$\begin{aligned}1-\prod_{i=1}^{n}\left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\ge 1\\ -\prod_{i=1}^{n}\left(1-f_{\tilde{b}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}.\end{aligned}$$

According to Theorem 1 and (3), we have

$$\mathrm{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \mathrm{GIFOGA}_\mu\left(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\right).$$

Proposition 4 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $\tilde{a}^- = (\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i}))$, $\tilde{a}^+ =$ $(\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i}))$,* then

$$\tilde{a}^- \le \mathrm{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}^+.$$

Proof For any $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1,\ldots,n)$, it is obvious that $\tilde{a}^- = (\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i}))$, and $\tilde{a}^+ = (\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i}))$ are intuitionistic fuzzy values. Let $(\cdot)$ indicate a permutation such that $\tilde{a}_{(1)} \le \cdots \le \tilde{a}_{(n)}$. Since $A_{(i+1)} \subseteqq A_{(i)}$, $g(A_{(i)}) - g(A_{(i+1)}) \ge 0$. For any $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$,

$$\min_i(t_{\tilde{a}_i}) \le t_{\tilde{a}_{(i)}} \le \max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i}) \le f_{\tilde{a}_{(i)}} \le \max_i(f_{\tilde{a}_i}).$$

So we have

$$\begin{aligned}\prod_{i=1}^{n}\min_i(t_{\tilde{a}_i})^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\le \prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\\ &\le \prod_{i=1}^{n}\max_i(t_{a_i})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\end{aligned}$$

$$\begin{aligned}&1-\prod_{i=1}^{n}(1-\min_i(f_{\tilde{a}_i}))^{\mu(A_{(i)})-\mu(A_{(i+1)})} \le 1-\prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\\ &\le 1-\prod_{i=1}^{n}(1-\max_i(f_{\tilde{a}_i}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \text{i.e.,}\end{aligned}$$

$$\min_i (t_{\tilde{a}_i})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \max_i (t_{\tilde{a}_i})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$1-(1-\min_i(f_{\tilde{a}_i}))^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leq 1 - \prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq 1-(1-\max_i(f_{\tilde{a}_i}))^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}.$$

Thus,

$$\min_i(t_{\tilde{a}_i}) \leq \prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i}) \leq 1 - \prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \max_i(f_{\tilde{a}_i}).$$

According to (3), we have

$$(\min_i(t_{\tilde{a}_i}), \max_i(f_{\tilde{a}_i})) \leq \text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq (\max_i(t_{\tilde{a}_i}), \min_i(f_{\tilde{a}_i})),$$

that is, $\tilde{a}^- \leq \text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+$.

Proposition 5 *Let* $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ *be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If* $\tilde{s} = (t_{\tilde{s}}, f_{\tilde{s}})$ *is an intuitionistic fuzzy value, then*

$$\text{GIFOGA}_\mu(\tilde{a}_1 \otimes \tilde{s}, \tilde{a}_2 \otimes \tilde{s}, \ldots, \tilde{a}_n \otimes \tilde{s}) = \text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \otimes \tilde{s}.$$

Proof Since for any i $(i = 1, 2,\ldots,n)$, $\tilde{a}_i \otimes \tilde{s} = (t_{\tilde{a}_i}t_{\tilde{s}}, f_{\tilde{a}_i} + f_{\tilde{s}} - f_{\tilde{a}_i}f_{\tilde{s}}) = (t_{\tilde{a}_i}t_{\tilde{s}}, 1-(1-f_{\tilde{a}_i})(1-f_{\tilde{s}}))$.

According to Theorem 1, we have

$$\begin{aligned}
&\text{GIFOGA}_\mu(\tilde{a}_1 \otimes \tilde{s}, \tilde{a}_2 \otimes \tilde{s}, \ldots, \tilde{a}_n \otimes \tilde{s}) \\
&= \left(\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}}t_{\tilde{s}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-\prod_{i=1}^{n}((1-f_{\tilde{a}_{(i)}})(1-f_{\tilde{s}}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right) \\
&= \left((t_{\tilde{s}})^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \left(1-(1-f_{\tilde{s}})\right)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}\prod_{i=1}^{n}\left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu\left(A_{(i+1)}\right)}\right) \\
&= \left(t_{\tilde{s}}\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \left(1-(1-f_{\tilde{s}})\prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)\right).
\end{aligned}$$

According to Definition 2, we have

$$\text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \otimes \tilde{s} = \left(t_{\tilde{s}}\prod_{i=1}^{n}\left(t_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-(1-f_{\tilde{s}})\prod_{i=1}^{n}\left(1-f_{\tilde{a}_{(i)}}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right).$$

Thus

$$\text{GIFOGA}_\mu(\tilde{a}_1 \otimes \tilde{s}, \tilde{a}_2 \otimes \tilde{s}, \ldots, \tilde{a}_n \otimes \tilde{s}) = \text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \otimes \tilde{s}.$$

Proposition 6 *Let* $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ *be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If* $r > 0$, *then*

$$\text{GIFOGA}_\mu((\tilde{a}_1)^r, (\tilde{a}_2)^r, \ldots, (\tilde{a}_n)^r) = \left(\text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\right)^r.$$

Proof According to Definition 2, for any i $(i = 1, 2,\ldots,n)$ and $r > 0$ we have

$$(\tilde{a}_i)^r = ((t_{\tilde{a}_i})^r, 1-(1-f_{\tilde{a}_i})^r).$$

According to Theorem 1, we have

$$\begin{aligned}
&\text{GIFOGA}_\mu((\tilde{a}_1)^r, (\tilde{a}_2)^r, \ldots, (\tilde{a}_n)^r) \\
&= \left(\prod_{i=1}^{n}((t_{\tilde{a}_{(i)}})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-\prod_{i=1}^{n}((1-f_{\tilde{a}_{(i)}})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right) \\
&= \left(\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}, 1-\prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\text{GIFOGA}_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\right)^r \\
&= \left(\left(\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r, 1-\left(\prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r\right) \\
&= \left(\prod_{i=1}^{n}(t_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}, 1-\prod_{i=1}^{n}(1-f_{\tilde{a}_{(i)}})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right).
\end{aligned}$$

Thus

$$\mathrm{GIFOGA}_{\mu}((\tilde{a}_1)^r, (\tilde{a}_2)^r, \ldots, (\tilde{a}_n)^r) = \left(\mathrm{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\right)^r.$$

According to Proposition 5 and Proposition 6, we can obtain the following corollary:

Corollary 1 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $r > 0$ and $\tilde{s} = (t_{\tilde{s}}, f_{\tilde{s}})$ is an intuitionistic fuzzy value, then*

$$\mathrm{GIFOGA}_{\mu}((\tilde{a}_1)^r \otimes \tilde{s}, (\tilde{a}_2)^r \otimes \tilde{s}, \ldots, (\tilde{a}_n)^r \otimes \tilde{s}) = \left(\mathrm{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\right)^r \otimes \tilde{s}.$$

Proposition 7 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and g be a λ-fuzzy measure on X. if $\lambda \neq 0$, then*

$$\mathrm{GIFOGA}_{g}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} (t_{\tilde{a}_{(i)}})^{g_{(i)} \prod_{j=i+1}^{n}[1+\lambda g_{(j)}]}, 1 - \prod_{i=1}^{n} (1 - f_{\tilde{a}_{(i)}})^{g_{(i)} \prod_{j=i+1}^{n}[1+\lambda g_{(j)}]}\right).$$

If $\lambda = 0$, that is, g is an additive measure, then

$$\mathrm{GIFOGA}_{g}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} (t_{\tilde{a}_{(i)}})^{g_{(i)}}, 1 - \prod_{i=1}^{n} (1 - f_{\tilde{a}_{(i)}})^{g_{(i)}}\right).$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i),\ldots,(n))$, $A_{(n+1)} = \phi$.

Proof Let $(\cdot)$ indicate a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i),\ldots,(n))$, $A_{(n+1)} = \phi$.

Since $A_{(i)} = \{(i),\ldots,(n)\}$ and $A_{(i+1)} = \{(i+1),\ldots,(n)\}$, according to Eq. 6, if $\lambda \neq 0$, then

$$g(A_{(i)}) - g(A_{(i+1)}) = g_{(i)} \cdot \left\{1 + \lambda \cdot \frac{1}{\lambda}\left(\prod_{j=i+1}^{n}[1 + \lambda g_{(j)}] - 1\right)\right\} = g_{(i)} \cdot \left(\prod_{j=i+1}^{n}[1 + \lambda g_{(j)}]\right).$$

Thus according to Eq. 9, we have

$$\mathrm{GIFOGA}_{g}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} (t_{\tilde{a}_{(i)}})^{g_{(i)} \prod_{j=i+1}^{n}[1+\lambda g_{(j)}]}, 1 - \prod_{i=1}^{n} (1 - f_{\tilde{a}_{(i)}})^{g_{(i)} \prod_{j=i+1}^{n}[1+\lambda g_{(j)}]}\right).$$

If $\lambda = 0$, then $g(A_{(i)}) - g(A_{(i+1)}) = g_{(i)}$. So we have

$$\mathrm{GIFOGA}_{g}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\prod_{i=1}^{n} (t_{\tilde{a}_{(i)}})^{g_{(i)}}, 1 - \prod_{i=1}^{n} (1 - f_{\tilde{a}_{(i)}})^{g_{(i)}}\right).$$

In the following, we will find some relations between GIFOGA and IFOWGA (or GIFOGA and IFWGA) operators. According to Proposition 7 and Definition 3, the following conclusion is easily obtained:

Proposition 8 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X and μ be a fuzzy measure on X. If μ is an additive fuzzy measure, then*

$$\mathrm{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathrm{IFWGA}_{w}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$$

where w_i is the weight index of $\tilde{a}_{(i)}$ $(i = 1, 2,\ldots,n)$, $w_i = \mu_{(i)}$. In particular, if $\mu_{(i)} = 1/n$, then GIFOGA operator reduces to an intuitionistic fuzzy geometric operator.

Moreover, one can readily see that IFWGA is a GIFOGA operator which has an additive fuzzy measure μ:

$$\mu(A) = \sum_{i \in A} w_i, (A \subseteq X).$$

Theorem 2 *If μ is an additive fuzzy measure, then there exists $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$ such that* $\mathrm{IFWGA}_{w}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathrm{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$.

Suppose that IFOWGA operator has associated with it an exponential weighting vector $w = (w_1, w_2,\ldots,w_n)^T$, according to Theorem 1 and Definition 4, it is easily seen that IFOWGA operator will be equivalent to a GIFOGA operator, where fuzzy measure μ associated to the GIFOGA is given by

$$\mu(S) = \sum_{i=n-s+1} w_i (S \subseteq X, S \neq \phi).$$

Conversely, the GIFOGA operator will be equivalent to the IFOWGA operator that has associated with it an exponential weighting vector $w = (w_1, w_2,\ldots,w_n)^T$, $w_{n-s} = \mu(S \cup i) - \mu(S)$, $i \in X$, $S \subseteqq X \setminus i$.

Theorem 3 *Let $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ $(i = 1, 2,\ldots,n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. The following assertions are equivalent:*

1. *For any $A, B \in P(X)$ such that $|A| = |B|$, we have $\mu(A) = \mu(B)$.*
2. *There exists an exponential weighting vector $w = (w_1, w_2,\ldots,w_n)$ such that*

$$\mathrm{GIFOGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathrm{IFOWGA}_{w}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n).$$

3. *GIFOGA is a symmetric function.*

Proof The proof is similar to that of Proposition 4.1 in Marichal (2002). Here we do not duplicate it.

From Theorems 2 and 3, it is easily known that the GIFOGA operator generalizes both IFWGA and IFOWGA operators. The IFWGA and IFOWGA operators are two special cases of GIFOGA operator.

4 Multi-criteria group decision making with GIFOGA operator

Multi-criteria group decision-making problems generally involve the following two phases: (1) Aggregation phase: It combines individual criteria value of each alternative given by experts to obtain an overall value for each alternative; (2) Exploitation phase: It orders over value to obtain the best alternative(s). In some real-life situations, decision makers or experts may not possess a precise or sufficient level of knowledge of the problem, or are unable to discriminate explicitly the degree to which one alternative is better than the other. Generally, multi-criteria group decision making problem includes uncertain and imprecise data and information. We first describe the multi-criteria group decision making problems under intuitionistic fuzzy environment.

Let $E = \{e_1, e_2,\ldots,e_r\}$ be the set of the experts involved in the decision process, $A = (a_1, a_2,\ldots,a_m$ the set of the considered alternatives, and $C = (c_1, c_2,\ldots,c_n)$ be the set of the criteria used for evaluating the alternatives.

In the following, we shall utilize the GIFOGA operator to propose an approach to multiple criteria group decision making with intuitionistic fuzzy information, which involves the following steps:

Step 1 As for every alternative a_i $(i = 1,\ldots,m)$, each expert e_k $(k = 1,\ldots,r)$ is invited to express their individual evaluation or preference according to each criteria c_j $(j = 1,\ldots,n)$ by a intuitionistic fuzzy value $\tilde{a}_{ij}^k = (t_{\tilde{a}_{ij}^k}, f_{\tilde{a}_{ij}^k})$ $(i = 1,\ldots,m;\ \ j = 1,\ldots,n,\ \ k = 1,\ldots,r)$. Then we can obtain a decision making matrix as follows:

$$R^k = \begin{pmatrix} \tilde{a}_{11}^k, \tilde{a}_{12}^k, \ldots, \tilde{a}_{1n}^k \\ \tilde{a}_{21}^k, \tilde{a}_{22}^k, \ldots, \tilde{a}_{2n}^k \\ \cdots\cdots\cdots \\ \tilde{a}_{m1}^k, \tilde{a}_{m2}^k, \cdots, \tilde{a}_{mn}^k \end{pmatrix}.$$

Step 2 Confirm fuzzy density $g_i = g(c_i)$ of each criteria and $g_i = g(e_i)$ of each expert. According to Eq. 7, parameter λ_1 of criteria and λ_2 of expert can be determined, respectively.

Step 3 By (3) or Definition 1, $\tilde{a}_{ij}^k$ in i-th line of R^k is reordered such that $\tilde{a}_{i(1)}^k \le \tilde{a}_{i(2)}^k \le \cdots \le \tilde{a}_{i(n)}^k$.Using the GIFOGA operator

$$\tilde{a}_i^k = \mathrm{GIFOGA}_g(\tilde{a}_{i1}^k, \ldots, \tilde{a}_{in}^k) = \left(\prod_{j=1}^{n} (t_{\tilde{a}_{i(j)}^k})^{g(c_{(j)}) \prod_{t=j+1}^{n} [1+\lambda_1 g(c_{(t)})]}, \; 1 - \prod_{j=1}^{n} (1 - f_{\tilde{a}_{i(j)}^k})^{g(c_{(j)}) \prod_{t=j+1}^{n} [1+\lambda_1 g(c_{(t)})]} \right)$$

aggregate all $\tilde{a}_{ij}^k = (t_{\tilde{a}_{ij}^k}, f_{\tilde{a}_{ij}^k})$ $(j = 1, 2,\ldots,n)$ in the ith line of the intuitionistic fuzzy decision matrix R^k into the overall values $\tilde{a}_i^k = (t_{\tilde{a}_i^k}, f_{\tilde{a}_i^k})$ $(i = 1,\ 2,\ldots,m,\ k = 1,\ 2,\ldots,r)$.

Step 4 Similar to Step 4, all $\tilde{a}_i^k$ $(k = 1,\ 2,\ldots,r)$ is reordered such that $\tilde{a}_i^{(k)} \le \tilde{a}_i^{(k+1)}$.. Using GIFOGA operator aggregates all $\tilde{a}_i^k$ $(k = 1,\ 2,\ldots,r)$ into a collective overall values $\tilde{a}_i$ of alternative a_i.

$$\tilde{a}_i = \mathrm{GIFOGA}_g(\tilde{a}_i^1, \ldots, \tilde{a}_i^r) = \left(\prod_{k=1}^{r} (t_{\tilde{a}_i^{(k)}})^{g(e_{(k)}) \prod_{t=k+1}^{r} [1+\lambda_2 g(e_{(t)})]}, \; 1 - \prod_{k=1}^{r} (1 - f_{\tilde{a}_i^{(k)}})^{g(e_{(k)}) \prod_{t=k+1}^{r} [1+\lambda_2 g(e_{(t)})]} \right).$$

Step 5 According to the collective overall values $\tilde{a}_i = (t_{\tilde{a}_i}, f_{\tilde{a}_i})$ of alternatives a_i $(i = 1,\ 2,\ldots,m)$, we rank alternative a_i $(i = 1, 2,\ldots,m)$, then to select the best one.

5 Numerical example

In this section, a group decision making problem is concerned with a manufacturing company which wants to select the best global supplier according to the core competencies of suppliers. Now suppose that there are four suppliers (a_1, a_2, a_3, a_4) whose core competencies are evaluated by means of the following four criteria (c_1, c_2, c_3, c_4): the level of technology innovation (c_1); the control ability of flow (c_2); the ability of management (c_3); the level of service (c_4).

Now there are three experts $E = \{e_1, e_2, e_3\}$ who are invited to evaluate the core competencies of four candidates under these five criteria. For expert e_k $(k = 1, 2, 3)$, the evaluated value of supplier a_i $(i = 1, 2, 3)$ with respect to c_j $(j = 1, 2, 3, 4)$ can be expressed by intuitionistic fuzzy value $\tilde{a}_{ij}^k = (t_{\tilde{a}_{ij}^k}, f_{\tilde{a}_{ij}^k})$. The intuitionistic fuzzy decision matrix $R^k = (\tilde{a}_{ij})_{4\times4}$ $(k = 1, 2, 3)$ can be gotten as listed in Tables 1, 2, and 3.

To get the best supplier(s), the following steps are involved:

Step 1 We first determine fuzzy density of criteria and expert, and their λ parameter, respectively.

Table 1 Intuitionistic fuzzy decision matrix R^1

	c_1	c_2	c_3	c_4
a_1	(0.3, 0.5)	(0.5, 0.3)	(0.7, 0.2)	(0.2, 0.1)
a_2	(0.4, 0.3)	(0.3, 0.4)	(0.8, 0.1)	(0.5, 0.2)
a_3	(0.4, 0.1)	(0.5, 0.4)	(0.5, 0.3)	(0.6, 0.2)
a_4	(0.6, 0.1)]	(0.2, 0.5)	(0.4, 0.2)	(0.7, 0.1)

Table 2 Intuitionistic fuzzy decision matrix R^2

	c_1	c_2	c_3	c_4
a_1	(0.4, 0.3)	(0.3, 0.5)	(0.6, 0.2)	(0.3, 0.3)
a_2	(0.5, 0.4)	(0.2, 0.6)	(0.8, 0.1)	(0.5, 0.3)
a_3	(0.4, 0.3)	(0.3, 0.5)	(0.5, 0.1)	(0.7, 0.1)
a_4	(0.6, 0.3)	(0.2, 0.6)	(0.6, 0.2)	(0.8, 0.1)

Table 3 Intuitionistic fuzzy decision matrix R^3

	c_1	c_2	c_3	c_4
a_1	(0.3, 0.5)	(0.1, 0.4)	(0.7, 0.1)	(0.4, 0.2)
a_2	(0.4, 0.3)	(0.5, 0.4)	(0.7, 0.2)	(0.3, 0.5)
a_3	(0.3, 0.2)	(0.2, 0.3)	(0.4, 0.4)	(0.6, 0.3)
a_4	(0.5, 0.1)	(0.3, 0.4)	(0.4, 0.3)	(0.5, 0.2)

Suppose that $g(c_1) = 0.40$, $g(c_2) = 0.25$, $g(c_3) = 0.37$, $g(c_4) = 0.20$, according to Eq. 7, the λ of criteria can be determined: $\lambda_1 = -0.44$.

Suppose that $g(e_1) = 0.40$, $g(e_2) = 0.40$, $g(e_3) = 0.40$. Then λ of expert can be determined: $\lambda_2 = -0.44$.

Step 2 For intuitionistic fuzzy decision matrix R^1, according to Definition 1, the evaluated value $\tilde{a}_{ij}^1$of supplier a_i is reordered such that $\tilde{a}_{i(j)}^1 \leq \tilde{a}_{i(j+1)}^1$, then utilize the GIFOGA operator

$$\tilde{a}_i^1 = \text{GIFOGA}_g\left(\tilde{a}_{i1}^1, \tilde{a}_{i2}^1, \tilde{a}_{i3}^1, \tilde{a}_{i4}^1\right) = \left(\prod_{j=1}^{4} \left(t_{\tilde{a}_{i(j)}^k}\right)^{g(c_{(j)}) \prod_{t=j+1}^{4}[1+\lambda_1 g(c_{(t)})]}, \right.$$
$$\left. 1 - \prod_{j=1}^{4} \left(1 - f_{\tilde{a}_{i(j)}^k}\right)^{g(c_{(j)}) \prod_{t=j+1}^{4}[1+\lambda_1 g(c_{(t)})]} \right)$$

to aggregate $\tilde{a}_{ij}^1$ (j = 1,2,3,4) corresponding to supplier a_i:

$\tilde{a}_1^1 = (0.43,\ 0.30), \tilde{a}_2^1 = \ (0.51, 0.23), \tilde{a}_3^1 = (0.48, 0.23), \tilde{a}_4^1 = (0.47, 0.21).$

Similarly, for Tables 2 and 3, we have, respectively,

$\tilde{a}_1^2 = (0.43,\ 0.30), \tilde{a}_2^2 = (0.52, 0.33), \tilde{a}_3^2 = (0.46,\ 0.24), \tilde{a}_4^2 = (0.54, 0.29).$

$\tilde{a}_1^3 = (0.36, 0.31), \tilde{a}_2^3 = (0.50,\ 0.32), \tilde{a}_3^3 = (0.35, 0.30), \tilde{a}_4^3 = (0.43, 0.23).$

Step 3 For supplier a_1, we reorder $\tilde{a}_1^k$ (k = 1, 2, 3) such that $\tilde{a}_1^{(k)} \leq \tilde{a}_1^{(k+1)}$. Using the GIFOGA operator aggregates $\tilde{a}_1^k$ (k = 1, 2, 3) into a collective overall values $\tilde{a}_1$:

$$\tilde{a}_1 = \text{GIFOGA}_g\left(\tilde{a}_1^1, \tilde{a}_1^2, \tilde{a}_1^3\right) = \left(\prod_{k=1}^{3} \left(t_{\tilde{a}_1^{(k)}}\right)^{g(e_{(k)}) \prod_{t=k+1}^{3}[1+\lambda_2 g(e_{(t)})]}, \right.$$
$$\left. 1 - \prod_{k=1}^{3} \left(1 - f_{\tilde{a}_1^{(k)}}\right)^{g(e_{(k)}) \prod_{t=j+1}^{3}[1+\lambda_2 g(e_{(t)})]} \right) = (0.41, 0.30).$$

Similar to supplier a_1, for a_2, a_3, a_4, we have, respectively,$\tilde{a}_2 = (0.51, 0.28), \tilde{a}_3 = (0.43, 0.25), \tilde{a}_4 = (0.48, 0.25)$.

Step 4 According to collective values $\tilde{a}_i$ of supplier a_i (i = 1, 2, 3, 4), we can obtain that

$\tilde{a}_2 > \tilde{a}_4 > \tilde{a}_3 > \tilde{a}_1.$

Thus the order of four suppliers is a_2, a_4, a_3, a_1. Hence the best supplier is a_2.

If we suppose that criteria and preference of experts are independent, then the above approach is reduced a traditional aggregation approach by means of intuitionistic fuzzy geometric operator. In fact, there are interactive phenomena among these four criteria. Since experts invited to provide assessment information of every criteria on each suppliers usually come from same or similar fields, they have similar knowledge, social status, and preference. So we must consider interactive phenomena among these criteria and experts. To solve this issue, we apply the approach in Sect. 4 to the ranking and selection of the best one(s). From the above analysis, the major advantage over traditional approach is that the developed approach considers interactive phenomena among criteria and preference of experts, which approximates to the truth of real decision making problems, and does not lose information in the process of aggregation. So the result of decision making accords with the real situation.

6 Conclusions

Being a generalization of fuzzy sets, the IFSs give us an additional possibility to represent imperfect knowledge. This allows us to use more flexible ways to simulate real decision situations. In this paper, based on operational laws on intuitionistic fuzzy values, we have developed a GIFOGA operator for multiple criteria group decision making, where interactions phenomena among the decision criteria and preference of experts are considered. Some of its properties are investigated in detail. It is shown that the GIFOGA operator generalizes both the IFOWGA operator and IFWGA operator. Finally, an example is given to illustrate the multi-criteria group decision making process. The proposed method differs from previous approaches for multi-criteria group decision-making by the following: the proposed method uses IFS theory rather than fuzzy set theory, and the interaction phenomena among criteria or preference of experts is taken into account, which makes it more feasible and practical than other traditional aggregation operators for real decision making problems.

Acknowledgments The author is very grateful to the Co-Guest-Editor, Professor Hepu Deng and the anonymous referees for their insightful and constructive comments and suggestions, which have been very helpful in improving the paper. This work was supported by a grant from the Funds for Creative Research Groups of China (No. 70921001), the National Natural Science Foundation of China (Nos. 70801064, 70771010, and 70631004), the Ph.D. Programs Foundation of Ministry of Education of China (No. 200805331059), and the Philosophy and Social Science Foundation of Hunan Province, China (No. 08YBA021).

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